Inhomogeneous circular law for correlated matrices

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\textbf{Abstract}

We consider non-Hermitian random matrices $X \in \mathbb{C}^{n \times n}$ with general decaying correlations between their entries. For large $n$, the empirical spectral distribution is well approximated by a deterministic density, expressed in terms of the solution to a system of two coupled non-linear $n \times n$ matrix equations. This density is interpreted as the Brown measure of a linear combination of free circular elements with matrix coefficients on a non-commutative probability space. It is radially symmetric, real analytic in the radial variable and strictly positive on a disk around the origin in the complex plane with a discontinuous drop to zero at the edge. The radius of the disk is given explicitly in terms of the covariances of the entries of $X$. We show convergence down to local spectral scales just slightly above the typical eigenvalue spacing with an optimal rate of convergence.

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1. Introduction

Many random matrix models exhibit a strong concentration of measure phenomenon; their empirical eigenvalue distributions are well approximated by deterministic measures as their sizes tend to infinity. For Hermitian matrices, the simplest and most prominent example is the celebrated semicircle law for Wigner ensembles with independent and identically distributed (i.i.d.) entries above the diagonal [53]. Girko’s circular law [29,12] is its non-Hermitian analogue.\footnote{We refer to the survey [16] for a complete account of the history of the circular law until the minimal moment assumptions in [51].} For matrices $X = (x_{ij})_{i,j=1}^{n}$ with centred i.i.d. entries, unrestricted by symmetry and with normalisation $\mathbb{E} |x_{ij}|^2 = \frac{1}{n}$, it asserts convergence of the eigenvalue distribution to the uniform probability measure on the unit disk in the complex plane.

Establishing similar concentration results and identifying the limiting spectral density while simultaneously relaxing the two basic assumptions of identical distributions and independence of the entries has since been the focus of many works in random matrix theory. When the entries are independently drawn from different distributions, their variance profile $s_{ij} = \mathbb{E} |x_{ij}|^2$ becomes an additional parameter of the model that determines...
the density through the nonlinear Dyson equation for \( n \) unknowns. Since in general no explicit formula for its solution is available, analysing the characteristic properties of the spectral density has attracted considerable attention.

In the Hermitian case, convergence of the empirical spectral measure is well established \([9,32,47]\) and a classification of the degree of regularity of the asymptotic density as well as of its possible singularities has been given \([1]\). Even when the independence of matrix entries is dropped and local correlations with sufficient decay are considered this classification persists \([6]\) and concentration of the spectral measure has been proven in broad generality \([10,13,19,26,30,35,41,43,46]\).

There are far fewer results on the existence and characteristics of limiting spectral densities for non-Hermitian matrices since their spectral instability makes such questions more challenging compared to the Hermitian situation. For random matrices \( X \) with centred, independent entries and a general variance profile, the convergence of the spectral measure of \( X \) to a rotationally symmetric, continuous limiting density \( \sigma \) was shown in \([24]\), and independently in \([5]\) on all mesoscopic scales in the bulk spectrum under stronger assumptions on the variance profile and regularity of the entry distribution. The extension of convergence on mesoscopic scales to the spectral edges and optimal control of the spectral radius was achieved in \([7]\). These three papers avoided the requirement of identical variances imposed earlier.

In the present paper we also depart from the independence assumption on the entries. We consider a large class of centred non-Hermitian random matrices \( X \in \mathbb{C}^{n \times n} \) with general decaying correlations among their entries. Throughout this class, the limiting spectral density \( \sigma \) is determined solely by the covariances between the matrix entries and has the following properties: (i) the density is rotationally symmetric around zero, (ii) its support is a disk centred at the origin, (iii) the density is real analytic as a function of the radial variable inside the disk and has a jump at its boundary.

The analyticity is a new result even when the entries of \( X \) are independent (apart from the explicitly known circular law case). In this case, the other properties are known \([5]\). Remarkably, the support of \( \sigma \) is always connected in the non-Hermitian case, in the independent as well as the correlated setup. This is in sharp contrast to the Hermitian case, where the support can be disconnected even for matrices with centred, independent entries and a variance profile \([2]\).

The class of random matrices \( X \) we consider here contains finite-dimensional approximations of linear combinations of free circular elements with matrix coefficients on a non-commutative probability space. These are non-normal analogs of operator-valued semicircular elements introduced in \([52]\) (see also \([49]\)). For such linear combinations, one is interested in their Brown measure, a generalisation of the spectral measure of normal operators to general operators in a finite von Neumann algebra. It was introduced in \([20]\) and revived in \([34]\). Since then significant attention has been given to determining the Brown measure and understanding its properties for specific classes of non-normal operators, see e.g. \([14,15,25,33,34,37]\). In the present work, we prove that the Brown measure of these matrix-valued circular elements has the properties (i), (ii), (iii) listed
above. In previous works addition of or multiplication with an \( R \)-diagonal element (see [40] for the definition) and its invariance under unitary transformations was crucial in order to introduce generic directionality into the model. In contrast our model and its ensuing analysis are generically non-isotropic due to the matrix coefficients.

Convergence of the eigenvalue density to a limiting measure is commonly expressed by showing that for each ball with fixed diameter on the scale of the entire spectrum the fraction of eigenvalues in it agrees asymptotically with the mass assigned to this ball by the limiting measure. Such global law is refined to a local law, showing convergence on mesoscopic scales, by allowing the diameter to decrease with \( n \) as long as it stays slightly above the typical eigenvalue spacing. We now review some previous results on local laws for non-Hermitian random matrices with independent entries. A bulk local law for random matrices with centred, independent entries of identical variances was shown in [17]. Additionally requiring the first three moments of the entry distribution to match a standard Gaussian, the local law including the edge was established in [50] and in [18]. The third moment matching condition for the edge local law was then removed in [54].

For the bulk local law, the assumption of identical variances was dropped in [5]. In this situation, the limiting density differs substantially from the circular law. Under weaker moment assumptions and asymptotically identical variances, a bulk local law with the circular law as limiting density was shown in [31]. In the setup of [5], the edge local law was proven in [7].

The availability of a local law has wide ranging implications for the spectral analysis of any random matrix model. In the present paper, we apply it to exclude eigenvalues away from the support of the limiting spectral density \( \sigma \), i.e. with high probability all eigenvalues concentrate on a disk around the origin whose radius is determined by the covariances of the matrix entries. We also obtain the complete isotropic delocalisation of all eigenvectors associated to the bulk eigenvalues. Furthermore, local laws have been a key ingredient in the study of more refined eigenvalue statistics. In the non-Hermitian i.i.d. setup they have been crucially used in the proofs of universality of bulk and edge eigenvalues with a four moment matching condition in [50], edge universality with two matching moments in [22] and the central limit theorem for linear statistics in [21,23].

Non-Hermitian random matrices without any symmetry constraint also play an important role in various applications. In particular, they are used to model connectivities in food webs and neural networks [4,38,42,48]. Since understanding the stability properties of such systems requires precise knowledge of the eigenvalue locations of the associated random matrix model, our work contributes to this line of research by allowing the correlation among the connectivities to depend on underlying geometric structures.

The analysis of the eigenvalue density of a non-Hermitian random matrix \( X \) is commonly reduced via Girko’s Hermitization trick [29] to the study of the family of Hermitian matrices

\[
H_\zeta := \begin{pmatrix}
0 & X - \zeta \\
(X - \zeta)^* & 0
\end{pmatrix}
\] (1.1)
with spectral parameter \( \zeta \in \mathbb{C} \). Consequently the main task is to control the resolvent 
\( G(\zeta, \eta) := (H_\zeta - i\eta)^{-1} \) of \( H_\zeta \) on the imaginary axis via its deterministic approximation 
\( M = M(\zeta, \eta) \) that solves the associated matrix Dyson equation (MDE)

\[
-M^{-1} = i\eta + \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix} + \mathcal{S}[M].
\] (1.2)

This equation has a unique solution for every \( \zeta \in \mathbb{C} \) and \( \eta > 0 \) if \( \text{Im} \ M = \frac{1}{2\pi}(M - M^*) \) is required to be positive definite \([36]\). Here, \( \mathcal{S} \) is a linear map on \( \mathbb{C}^{2n \times 2n} \) defined through

\[
\mathcal{S}[R] := \begin{pmatrix} \mathbb{E}[XR_{22}X^*] & 0 \\ 0 & \mathbb{E}[X^*R_{11}X] \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}
\] (1.3)

for any deterministic matrix \( R \in \mathbb{C}^{2n \times 2n} \) with \( n \times n \)-blocks \( R_{11}, R_{12}, R_{21}, R_{22} \). The operator \( \mathcal{S} \) captures the covariances between the entries of \( X \).

The main tool developed in the present paper is a precise stability result for the non-linear high dimensional matrix equation (1.2). From \([3,26]\), we know that \( G = G(\zeta, \eta) \) satisfies a perturbed version of (1.2) with a small error term when \( n \) becomes large. Thus, \( G \) is close to \( M \) if stability of (1.2) against small perturbations is controlled. Moreover, the limiting spectral density \( \sigma \) for \( X \) is obtained as a derivative of \( \text{Im} \ M \) with respect to \( |\zeta| \), where \( \zeta \) is the spectral parameter of \( X \). Thus, any analysis of \( \sigma \) also requires stability of (1.2).

In previous works, the matrix structure of (1.2) was crucially simplified due to more restrictive assumptions on \( X \). If the entries of \( X \) are independent, then (1.2) reduces to a vector-valued equation for the diagonal of \( M \) and, thus, the Dyson equation is formulated on the commutative algebra of diagonal matrices. For identical variances, all diagonal entries of \( M \) coincide, yielding a single scalar equation.

In the matrix setup, a general version of (1.2) and its stability have been studied in \([3]\) under a strong irreducibility condition on \( \mathcal{S} \), which is called flatness. However, \( \mathcal{S} \) as defined in (1.3) does not fulfill this flatness condition due to its special block structure, making the equation inherently unstable. This issue was overcome in \([5]\) for vector Dyson equation, i.e. when the entries of \( X \) are independent. Owing to the commutative structure of this vector case, an additional symmetry of \( M \) could be exploited to obtain the stability against perturbations respecting this symmetry.

The analysis in the present situation necessitates tackling, at the same time, both main challenges from \([3]\) and \([5]\), the non-commutativity of the MDE and the instability due to the specific block structure, respectively. The genuinely non-commutative structure of the MDE is a major obstacle throughout the entire argument requiring the introduction of appropriately symmetrized objects, which are much more complicated than their counterparts in the commutative setup. To resolve the instability we perform a non-linear transformation of the MDE that allows to restrict the analysis to the manifold of perturbations that respect the additional symmetry of \( M \). This transformation
is also applicable in the context of other non-normal models, e.g. non-Hermitian polynomials in several non-commutative variables. Furthermore, it is crucial to show that $\sigma$ is a real analytic function of $|\zeta|^2$.

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### 2. Main results

#### 2.1. Correlated random matrices

Let $X \in \mathbb{C}^{n \times n}$ be a random matrix with centred, $\mathbb{E} x_{ij} = 0$, entries. For the index set we write

$$[n] := \{1, \ldots, n\}. $$

Within our main results we will refer to the following assumptions on the entries of $X$. Some of them are stated in terms of the covariances between the entries of $X$. These covariances are encoded in the two operators $S, S^* : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ on the space of $n \times n$-matrices, defined through

$$SA := \mathbb{E} XAX^*, \quad S^*A := \mathbb{E} X^*AX,$$

(2.1)

**A1** *Finite moments:* All moments of the entries of $\sqrt{n}X$ are finite, i.e. there is a sequence of positive constants $C_\nu$ such that

$$\mathbb{E} |x_{ij}|^\nu \leq C_\nu n^{-\nu/2},$$

(2.2)

for all $i, j \in [n]$ and $\nu \in \mathbb{N}$.

**A2** *Decay of correlation:* The index set $[n]$ is equipped with a pseudo-metric $d$ that satisfies for a fixed $p \in \mathbb{N}$ the sub-$p$-dimensional volume growth condition

$$\{|j \in [n] : d(i, j) \leq \tau\} \leq C_\tau^p, \quad \tau \geq 1, \ i \in [n],$$

(2.3)

with a constant $C > 0$. Furthermore, the correlations among the entries of $\sqrt{n}X$ decay in the product metric $d \times d$ on $[n]^2$ faster than any power law, i.e. there is a sequence of positive constants $C_\nu$ such that

$$\text{Cov}(f_1(\sqrt{n}X), f_2(\sqrt{n}X)) \leq \frac{C_\nu \|f_1\|_2 \|f_2\|_2}{1 + d \times d(\text{supp} f_1, \text{supp} f_2)^\nu}, \quad \nu \in \mathbb{N},$$

(2.4)

for any two measurable functions $f_i : \mathbb{C}^{A_i} \to \mathbb{C}$ with $\|f_i\|_2^2 := \mathbb{E}|f_i(\sqrt{n}X)|^2 < \infty$, where $A_i = \text{supp} f_i \subset [n]^2$. 
A3 Flatness: There is a constant $c > 0$ such that for any two deterministic vectors $x,y \in \mathbb{C}^n$ we have

$$
\mathbb{E} |\langle x, Xy \rangle|^2 \geq \frac{c}{n} \|x\|^2 \|y\|^2,
$$

(2.5)

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the standard Euclidean scalar product and norm on $\mathbb{C}^n$, respectively.

A4 Smallest singular value: For each $\varepsilon > 0$ and $\nu \in \mathbb{N}$, there is $C_{\varepsilon, \nu} > 0$ such that

$$
P(s_{\text{min}}(X - \zeta) \leq e^{-n^\varepsilon}) \leq C_{\varepsilon, \nu} n^{-\nu}
$$

(2.6)

for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{C}$. Here, $s_{\text{min}}(X - \zeta)$ denotes the smallest singular value of $X - \zeta$.

A4′ Bounded conditional density: There are $q \in (1, \infty]$ and $\kappa > 0$ such that, for each pair $(i,j) \in [n]^2$, there is a probability density $\psi_{ij} \in L^q(\mathbb{C})$ (or $\psi_{ij} \in L^q(\mathbb{R})$) if $X \in \mathbb{R}^{n \times n}$ which satisfies $\mathbb{E}\|\psi_{ij}\|_q \leq n^\kappa$ and

$$
P(\sqrt{n}x_{ij} \in B | (x_{kl})_{(k,l) \in [n]^2 \setminus \{(i,j)\}}) = \int_B \psi_{ij}(z)dz
$$

(2.7)

almost surely for all measurable $B \subset \mathbb{C}$ (or $B \subset \mathbb{R}$).

A5 There is $c > 0$ such that the spectral radius $\varrho(S)$ of $S$ satisfies $\varrho(S) \geq c$. Moreover, there is an $n$-independent monotonically decreasing function $f: (0, \infty) \to (0, \infty)$ such that

$$
\|(\tau - S)^{-1}\| \leq f(\tau)/\tau
$$

(2.8)

for all $\tau > \varrho(S)$ and for all $n \in \mathbb{N}$.

We remark that Assumption A4′ implies Assumption A4 as shown in Proposition 2.10 below. Moreover, in Section 2.3, we explain how some assumptions can be relaxed (see in particular Remark 2.11 for weaker versions of A2) and examples satisfying the assumptions listed above.

The $n$-independent constants appearing in Assumptions A1–A5 will be called model parameters and while many constants in the following depend on these parameters, we consider them as fixed and often do not explicitly mention this dependence.

Remark 2.1. The monotonicity of $f(\tau)$ in Assumption A5 is not a restriction since multiplying the right-hand side of (2.8) by $\tau$ yields a monotonically decreasing function.
Furthermore, Assumption A3 implies Assumption A5 (cf. Lemma C.3). The weaker Assumption A5 is imposed to exclude eigenvalues away from the support of the asymptotic spectral density of X, while Assumption A3 is imposed to guarantee convergence of the empirical eigenvalue distribution to this density in the spectral bulk.

The first main result is that, with very high probability, X does not have any eigenvalues away from the disk of radius $\sqrt{\varrho(S)}$ centred at the origin. This will be proven in Section 3.1 below.

**Theorem 2.2** (No eigenvalue outliers). Let X satisfy A1, A2 and A5. Then, for every $\nu \in \mathbb{N}$ and $\tau_* > 0$, there exists a constant $C_\nu > 0$ such that

$$
P \left[ |\zeta|^2 \leq \varrho(S) + \tau_* \text{ for all } \zeta \in \text{Spec } X \right] \geq 1 - C_\nu n^{-\nu},$$

uniformly for all $n \in \mathbb{N}$.

The next theorem states that, for large $n$, the empirical spectral distribution $\frac{1}{n} \sum_{\zeta \in \text{Spec } X} \delta_\zeta$ is well approximated by a deterministic probability density $\sigma$ on the complex plane.

**Theorem 2.3** (Global inhomogeneous circular law). Let X satisfy A1 – A4. Then there is a (possibly $n$-dependent) deterministic probability density $\sigma : \mathbb{C} \to [0, \infty)$ such that the empirical spectral distribution of X approaches $\sigma(\zeta) d^2 \zeta$ weakly in probability for $n \to \infty$. That is, for every bounded, continuous function $f : \mathbb{C} \to \mathbb{C}$ and $\varepsilon > 0$, we have

$$
\lim_{n \to \infty} P \left( \left| \frac{1}{n} \sum_{\zeta \in \text{Spec } X} f(\zeta) - \int_{\mathbb{C}} f(\zeta) \sigma(\zeta) d^2 \zeta \right| > \varepsilon \right) = 0.
$$

The proof of Theorem 2.3 will be presented in Section 3.2 below. The density $\sigma$ will be explicitly defined in (2.11) below in terms of the solution to a system of two coupled $n \times n$-matrix equations determined by the operators S and $S^*$ from (2.1). The existence and uniqueness of this solution is stated in the following proposition, whose proof is deferred to the end of Subsection 4.3 below.

**Proposition 2.4** (Existence and uniqueness). Let X satisfy A1–A3, S, $S^*$ be defined as in (2.1) and $\tau \in [0, \varrho)$, where $\varrho = \varrho(S)$ is the spectral radius of S. Then the coupled system of matrix equations

$$
\frac{1}{V_1} = SV_2 + \frac{\tau}{S^* V_1}, \quad \frac{1}{V_2} = S^* V_1 + \frac{\tau}{SV_2},
$$

has a unique solution $V_1(\tau) = V_1, V_2(\tau) = V_2 \in \mathbb{C}^{n \times n}$ such that both $V_i$ are positive definite and satisfy the constraint
This solution can be extended to real analytic functions $V_1, V_2 : (-c, \varrho) \to \mathbb{C}^{n \times n}$ with some $n$-independent constant $c > 0$.

We will refer to (2.9) as the Dyson equation since as we will see later in Section 4 it is equivalent to a Dyson equation that describes the limit of the resolvent of self-adjoint random matrices. Our first theorem expresses the density $\sigma$ in terms of the solution to (2.9) and shows that its support is a disk centred at the origin of the complex plane. It is proven at the end of Section 5.2.

**Theorem 2.5 (Density).** Let $X$ satisfy A1–A3, $V_1$, $V_2$ be the unique positive definite solution of (2.9) with (2.10) and $\varrho = \varrho(S)$. Then the radially symmetric function $\sigma : \mathbb{C} \to \mathbb{R}$ given by

$$
\sigma(\zeta) := \frac{1}{\pi n} \frac{d}{d\tau} \text{Tr} \left[ \frac{\tau}{\tau + (S^*V_1(\tau))(SV_2(\tau))} \right]_{\tau = |\zeta|^2} \times 1(|\zeta|^2 < \varrho),
$$

(2.11)
is non-negative and inherits its analyticity as a function of $\tau = |\zeta|^2$ in the disk $\mathbb{D}_\varrho = \{ \zeta \in \mathbb{C} : |\zeta|^2 < \varrho \}$ from $V_1$ and $V_2$. Furthermore, $\sigma$ is a probability density, $\int_{\mathbb{C}} \sigma(\zeta) d^2\zeta = 1$ and is uniformly bounded and bounded away from zero, i.e. there are $n$-independent constants $c, C > 0$ such that

$$
c \leq \sigma(\zeta) \leq C, \quad \zeta \in \mathbb{D}_\sqrt{\varrho}.
$$

(2.12)
In particular, $\text{supp} \sigma = \overline{\mathbb{D}_\varrho}$ and at the boundary $|\zeta|^2 = \varrho$, the density $\sigma$ has a jump height

$$
\lim_{|\zeta| \uparrow \varrho} \sigma(\zeta) = \frac{1}{\pi \varrho n} \frac{\text{Tr}[S_1S_2]^2}{\text{Tr}[(S_1S_2)^2]},
$$

(2.13)
expressed in terms of the right and left Perron-Frobenius eigenmatrices of $S$, i.e. $SS_2 = \varrho S_2$ and $S^*S_1 = \varrho S_1$.

**Definition 2.6 (Self-consistent density of states).** We call the probability density $\sigma$, defined through (2.11), the self-consistent density of states associated to $S$ or to $X$.

In order to formulate the local law in the spectral bulk we introduce observables around a fixed spectral parameter $\zeta_0 \in \mathbb{C}$ on mesoscopic scales $n^{-\alpha}$ with $\alpha \in (0,1/2)$. For any function $f : \mathbb{C} \to \mathbb{C}$ we define

$$
f_{\zeta_0, \alpha} : \mathbb{C} \to \mathbb{C}, \quad f_{\zeta_0, \alpha}(\zeta) := n^{2\alpha} f(n^\alpha (\zeta - \zeta_0)).
$$
For any $r > 0$, we denote the disk of radius $r$ centred at the origin by $\mathbb{D}_r := \{\zeta \in \mathbb{C} : |\zeta| < r\}$.

**Theorem 2.7** (Local inhomogeneous circular law). Let $X$ be a centred non-Hermitian random matrix satisfying A1–A4. Let $\alpha \in (0,1/2)$, $\varepsilon$, $\tau_* > 0$ and $\nu \in \mathbb{N}$. Then there is a constant $C > 0$ such that

$$
\mathbb{P}\left[ \frac{1}{n} \sum_{\zeta \in \text{Spec } X} f_{\zeta_0,\alpha}(\zeta) - \int_{\mathbb{C}} f_{\zeta_0,\alpha}(\zeta) \sigma(\zeta) d^2\zeta \right] \leq n^{-1+2\alpha+\varepsilon} \|\Delta f\|_{L^1} 
\geq 1 - C n^{-\nu},
$$

(2.14) uniformly for every $n \in \mathbb{N}$, every $\zeta_0 \in \mathbb{C}$ with $|\zeta_0|^2 \leq \varrho(S) - \tau_*$ and for every $f \in C^2_0(\mathbb{C})$ satisfying supp $f \subseteq \mathbb{D}_\varphi$ and $\|\Delta f\|_{L^{1+\beta}} \leq n^D \|\Delta f\|_{L^1}$ with some fixed $\varphi$, $\beta > 0$ and $D \in \mathbb{N}$.

The proof of Theorem 2.7 will be given in Section 6 below. Under the stronger Assumption A4′, the condition $\|\Delta f\|_{L^{1+\beta}} \leq n^D \|\Delta f\|_{L^1}$ in Theorem 2.7 is not necessary (as explained at the beginning of its proof). However, if the eigenvalue distribution has a discrete component, then control on $\|\Delta f\|_{L^1}$ alone may not ensure convergence of the linear statistics of $f$ in the eigenvalues in (2.14) which coincides with the integral of $\Delta f$ against the log-potential of the empirical spectral measure (see (3.14) below).

As a corollary we prove complete delocalisation of the eigenvectors of $X$. In the case of independent entries eigenvector delocalisation was first proven in [45].

**Corollary 2.8** (Isotropic eigenvector delocalisation). Let $X$ satisfy A1–A3. For any $\tau_* > 0$, let $U_{\tau_*}$ denote the set of eigenvectors of $X$ with corresponding eigenvalue in $\mathbb{D}_{\sqrt{\varrho - \tau_*}}$ with $\varrho = \varrho(S)$. Then for any $\varepsilon > 0$ and $\nu \in \mathbb{N}$ there exists a constant $C_{\varepsilon,\nu}$ such that

$$
\mathbb{P}\left[ |\langle v, u \rangle| \leq n^{-1/2+\varepsilon} \|u\| \|v\| \text{ for all } u \in U_{\tau_*} \right] \geq 1 - C_{\varepsilon,\nu} n^{-\nu},
$$

for all $n \in \mathbb{N}$ and all $v \in \mathbb{C}^n$.

Corollary 2.8 will be proven in Section 6.2 below.

### 2.2. Brown measure of matrix-valued circular elements

We now illustrate how the probability density defined in (2.11) is interpreted as the Lebesgue density of the Brown measure associated to a matrix linear combination of circular operators and thus how Theorem 2.5 provides information about this measure. To that end, let $(\mathcal{M}, \tau)$ be a tracial $W^*$-probability space.4 For $\ell \in \mathbb{N}$, free circular

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4 For this and other basic notions in free probability theory, we refer to the recent monograph [39].
elements $c_1, \ldots, c_\ell \in M$ and deterministic matrices $a_1, \ldots, a_\ell \in \mathbb{C}^{n \times n}$, we consider the operator

$$X = \sum_{j=1}^\ell a_j \otimes c_j \in M^{n \times n}. \quad (2.15)$$

We are interested in the spectral distribution of $X$. Since $X$ is non-normal, we consider the Brown measure, a generalisation of the spectral measure for normal operators. The Brown measure $\mu_X$ of $X$ is the unique compactly supported probability measure on $\mathbb{C}$ such that

$$\int \log|\lambda - \zeta| \, \mu_X(d\zeta) = \log D(X - \lambda) \quad (2.16)$$

for all $\lambda \in \mathbb{C}$, where $D$ is the Fuglede-Kadison determinant on $(M^{n \times n}, \langle \cdot \rangle \otimes \tau)$ defined by

$$D(Y) := \lim_{\varepsilon \downarrow 0} \exp(\langle \cdot \rangle \otimes \tau(\log(Y^*Y + \varepsilon)^{1/2})) \in [0, \infty), \quad (2.17)$$

for any $Y \in M^{n \times n}$. The Brown measure was originally introduced in [20] and revived in [34]. The Fuglede-Kadison determinant was first defined in [28]. For an introduction to both of these objects, we refer to [39, Section 11].

In the next result, we express the Brown measure $\mu_X$ of $X$ from (2.15) in terms of the operators $S$ and $S^*$ on $\mathbb{C}^{n \times n}$ defined through

$$S[R] := \sum_{j=1}^\ell a_j R a_j^*, \quad S^*[R] := \sum_{j=1}^\ell a_j^* R a_j \quad (2.18)$$

for any $R \in \mathbb{C}^{n \times n}$. In particular, we identify the support of $\mu_X$ and classify its regularity.

**Proposition 2.9 (Regularity of $\mu_X$).** Let $X \in M^{n \times n}$ be defined as in (2.15). We assume that there are constants $C > c > 0$ such that

$$c\langle R \rangle \leq S[R] \leq C\langle R \rangle, \quad (2.19)$$

for all positive semidefinite $R \in \mathbb{C}^{n \times n}$. Then the Brown measure $\mu_X$ of $X$ is given by

$$\mu_X(d\zeta) = \sigma(\zeta)d^2\zeta, \quad (2.20)$$

where $\sigma$ is defined via (2.11) with $S$ and $S^*$ from (2.18). In particular, the Brown measure of $X$ has all properties of $\sigma$ stated in Theorem 2.5.

The proof of Proposition 2.9 is presented in Section 5.4 below.
2.3. Relaxed assumptions and examples

In this subsection, we explain how the assumptions A1 – A5 are related, how some of them can be relaxed and provide some concrete examples satisfying these assumptions.

The first result shows that A4′ implies A4 and follows directly from Proposition 7.1 below.

**Proposition 2.10 (Smallest singular value of X − ζ).** If X satisfies Assumption A4′, then it also satisfies Assumption A4.

**Remark 2.11 (Relaxing Assumption A2).** We chose to assume a decay of correlation within the matrix X in the form A2 because it is easy to state. However, for our proof it suffices to assume that the decay of correlation (2.4) holds with a fixed power ν > 12p with p ∈ N from (2.3), provided higher order cumulants of the matrix entries of X satisfy a certain compatibility condition. This compatibility condition is [26, equation (3b)], where d is interpreted as the pseudometric from Assumption A2 and W is replaced by √nX. In this case Hζ from (1.1) satisfies [26, Assumptions (C) and (D)] (see also [26, Remark 2.7]). On the other hand, Assumption A2 implies [26, Assumption (C)] and a modified version of [26, Assumption (D)] by a similar argument as was used in [26, Example 2.10]. This is made explicit in Lemma 6.5 below.

In analogy to [26, Example 2.12] we also provide a simple description of our assumptions for the case of Gaussian random matrices while relaxing the polynomial decay of correlations from (2.4) to be of order ν = 2 when d(i, j) = |i − j| is the standard metric on [n].

**Example 2.12 (Results for Correlated Gaussian matrices).** Let X ∈ C^{n×n} be a random matrix with centred Gaussian entries such that

\[ n(|E x_{ij} x_{lk}| + |E x_{ij} x_{lk}|) \leq \frac{C}{|i - l|^2 + |j - k|^2} \]

for all i, j, l, k ∈ [n], as well as E |Tr BX|^2 ≥ c_1 Tr B*B for all B ∈ C^{n×n}, where c, C > 0 are some positive constants. Then the conclusions of Theorem 2.2, Theorem 2.3, Theorem 2.7 and Corollary 2.8 hold for X.

Next, we will formulate a condition for block matrices that ensures Assumption A4′. We denote by E_{ij} ∈ C^{N×N} the matrix whose (i, j)-entry is 1 and whose other entries are zero, that is, E_{ij} = (δ_{ik}δ_{jl})_{k,l∈[N]}. In the following lemma, we write z for a matrix-valued variable z = (z_{γδ})_{γ,δ∈[K]}. We denote by dz integration with respect to all entries of z and d^2 z_{αβ} denotes the omission of the integration over z_{αβ}.
Lemma 2.13 (Block matrices). Let $K \in \mathbb{N}$ be fixed. Let $\{x_{ij} : i, j \in [N]\}$ be a family of independent random matrices in $\mathbb{C}^{K \times K}$ satisfying $\mathbb{E}x_{ij} = 0$ for all $i, j \in [N]$. We assume that, for all $i, j \in [N]$, the matrix $x_{ij} \sqrt{NK}$ has a density $f_{ij}$ on $\mathbb{C}^{K \times K}$, i.e.

$$
P \left( x_{ij} \sqrt{NK} \in B \right) = \int_B f_{ij}(z)dz$$

for all measurable subsets $B \subset \mathbb{C}^{K \times K}$. If there are $q > 0$ and $C > 0$ such that

$$
\int_{\mathbb{C}^{K \times K-1}} \left( \int_{\mathbb{C}} f_{ij}(z)^q d^2z_{\alpha\beta} \right)^{1/q} d^2z_{11}d^2z_{12} \cdots d^2z_{\alpha\beta} \cdots d^2z_{KK} \leq NC\tag{2.21}
$$

for all $\alpha, \beta \in [K]$ then Assumption A4' is satisfied for the block matrix

$$
X = \sum_{i,j \in [N]} x_{ij} \otimes E_{ij} \tag{2.22}
$$

An analogous statement holds when $x_{ij} \sqrt{NK}$ has a density on $\mathbb{R}^{K \times K}$ instead of $\mathbb{C}^{K \times K}$. Lemma 2.13 will be proven in Section 7.1 below.

Notations

Here we introduce some notations that will be used throughout the paper. We start with basic notations for matrices. We equip the space of $d \times d$-matrices with the normalised scalar product

$$
\langle A, B \rangle := \frac{1}{n} \text{Tr} A^* B, \quad A, B \in \mathbb{C}^{d \times d},
$$

corresponding norm $\|A\|_{\text{hs}}^2 := \langle A, A \rangle$ and use the short hand $\langle A \rangle = \frac{1}{d} \text{Tr} A$ for the normalised trace. By $\|A\|$ we denote the operator norm induced by the standard Euclidean metric on $\mathbb{C}^d$. More generally, for linear operators $A : \mathcal{A} \to \mathcal{B}$ from a normed space $\mathcal{A}$ to a normed space $\mathcal{B}$, we indicate the corresponding operator norm by writing $\|A\|_{\mathcal{A} \to \mathcal{B}}$ and simply $\|A\|_\mathcal{A}$ in case $\mathcal{A} = \mathcal{B}$. Since we often work with $2 \times 2$-block matrices having block dimension $n$, we will frequently use the block notation from (1.3), where $\mathbf{R} \in \mathbb{C}^{2n \times 2n}$ and $R_{ij} \in \mathbb{C}^{n \times n}$.

For nonnegative quantities $\phi, \psi$ we use the comparison relation $\phi \lesssim \psi$ whenever $\phi \leq C\psi$ with an $n$-independent constant $C > 0$. This constant is uniform in all parameters except the model parameters from Assumptions A1–A4 and possibly other parameters that are either clearly indicated or obvious from the context. In particular, $C$ is uniform in the spectral parameter $\zeta$ within the domain under consideration. If $c\psi \leq \phi \leq C\psi$ we write $\phi \sim \psi$ and $\phi = \psi + \mathcal{O}(\nu)$ is a short hand for $|\phi - \psi| \lesssim \nu$. We also use the comparison relation for positive definite matrices, where it is interpreted in a quadratic form sense.
3. Inhomogeneous circular law

In this section we prove Theorems 2.2 and 2.3. These proofs will illustrate how Girko’s Hermitization trick translates these questions to Hermitian random matrices which will be analysed via their resolvents and the associated matrix Dyson equation. The proof of Theorem 2.3 is a prototype of the more complicated proof of Theorem 2.7 in Section 6 below.

The fundamental observation due to Girko [29] is that \( \zeta \in \mathbb{C} \) is an eigenvalue of \( X \) if and only if the kernel of \( H_\zeta \) is nontrivial, where the Hermitian matrix \( H_\zeta \) is defined through

\[
H_\zeta := \begin{pmatrix}
0 & X - \zeta \\
(X - \zeta)^* & 0
\end{pmatrix}.
\] (3.1)

The family \( (H_\zeta)_{\zeta \in \mathbb{C}} \) is called the Hermitization of \( X \). All spectral information about the kernel of \( H_\zeta \) is captured by the resolvent \( G = G(\zeta, \eta) \) of \( H_\zeta \) defined by

\[
G(\zeta, \eta) := (H_\zeta - i\eta)^{-1},
\] (3.2)

where \( \zeta \in \mathbb{C} \) and \( \eta > 0 \).

We will see in Proposition 3.7 below that the resolvent \( G \) is well approximated by the matrix \( M = M(\zeta, \eta) \in \mathbb{C}^{2n \times 2n} \) which is the unique solution of the matrix Dyson equation (MDE)

\[
-M^{-1} = i\eta 1 + Z(\zeta, \bar{\zeta}) + \mathcal{S}[M], \quad \eta > 0, \quad \zeta \in \mathbb{C},
\] (3.3)

under the constraint that the imaginary part \( \text{Im} M = \frac{1}{2i}(M - M^*) \) is positive definite.

Here, the matrix-valued function \( Z: \mathbb{C}^2 \to \mathbb{C}^{2n \times 2n} \) and the self-energy operator \( \mathcal{S} \), a linear operator on \( \mathbb{C}^{2n \times 2n} \), are defined through

\[
Z(\zeta, \omega) := \begin{pmatrix}
0 & \zeta \\
\omega & 0
\end{pmatrix}, \quad \mathcal{S} \left[ \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix} \right] := \begin{pmatrix}
S[A_{22}] & 0 \\
0 & S^*[A_{11}]
\end{pmatrix},
\] (3.4)

where all blocks in these matrix representations are of size \( n \times n \) (see (2.1) for the definitions of \( S \) and \( S^* \)). The existence and uniqueness of \( M \) have been shown in [36].

We represent \( M \) in terms of the \( 2 \times 2 \)-block structure corresponding to the right-hand side of (3.3). For this purpose we first introduce the matrices \( V_1, V_2 \in \mathbb{C}^{n \times n} \) which are the unique solution of

\[
\begin{align*}
\frac{1}{V_1(\tau, \eta)} &= \eta + SV_2(\tau, \eta) + \frac{\tau}{\eta + S^*V_1(\tau, \eta)}, \quad (3.5a) \\
\frac{1}{V_2(\tau, \eta)} &= \eta + S^*V_1(\tau, \eta) + \frac{\tau}{\eta + SV_2(\tau, \eta)}, \quad (3.5b)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{V_1(\tau, \eta)} &= \eta + SV_2(\tau, \eta) + \frac{\tau}{\eta + S^*V_1(\tau, \eta)}, \quad (3.5a) \\
\frac{1}{V_2(\tau, \eta)} &= \eta + S^*V_1(\tau, \eta) + \frac{\tau}{\eta + SV_2(\tau, \eta)}, \quad (3.5b)
\end{align*}
\]
for any $\eta > 0$ and $\tau \geq 0$ under the constraint that $V_1$ and $V_2$ are positive definite. We note that (3.5) is a regularised version of the Dyson equation (2.9), used for the definition of $\sigma$ in (2.11), with some regularisation parameter $\eta > 0$. Moreover, we introduce the auxiliary matrix

$$U(\tau, \eta) := \frac{1}{\tau + (\eta + S^*V_1(\tau, \eta))(\eta + SV_2(\tau, \eta))}. \quad (3.6)$$

Then we obtain that

$$M(\zeta, \eta) = \begin{pmatrix} \zeta U(|\zeta|^2, \eta) & -\zeta U(|\zeta|^2, \eta) \\ -\zeta^* U(|\zeta|^2, \eta)^* & \zeta U(|\zeta|^2, \eta) \end{pmatrix} \quad (3.7)$$

since the right-hand side of (3.7) satisfies (3.3) and has a positive definite imaginary part. Thus solving (3.3) for $M$ with positive imaginary part is equivalent to solving (3.5) for positive definite $V_1, V_2$. From (3.7) we easily get that

$$\text{Im} \, M(\zeta, \eta) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}. \quad (3.8)$$

### 3.1. Exclusion of eigenvalues away from the disk

We now prepare the proof of Theorem 2.2. First we note that if $X$ satisfies $A_1$ and $A_2$ then, for all positive definite $R \in \mathbb{C}^{n \times n}$, we have

$$S[R] \lesssim \langle R \rangle, \quad S^*[R] \lesssim \langle R \rangle. \quad (3.9)$$

The next lemma describes the behaviour of $\text{Im} \, M(\zeta, \eta)$ when $|\zeta|^2 > \rho(S)$.

**Lemma 3.1.** Let $X$ satisfy $A_1$, $A_2$ and $A_5$. Then, for every $\delta > 0$, we have

$$\text{Im} \, M(\zeta, \eta) \sim \delta \frac{\eta}{|\zeta|^2} \quad (3.10)$$

for all $\eta \in (0, 1]$ and $\zeta \in \mathbb{C}$ satisfying $|\zeta|^2 \geq \rho(S) + \delta$.

**Proof of Lemma 3.1.** Multiplying (3.5a) with $\eta + S^*[V_1]$ from the left and (3.5b) from the right with $\eta + S[V_2]$ as well as realizing that the resulting right hand sides coincide reveal the identity

$$(\eta + S^*V_1) \frac{1}{V_1} = \frac{1}{V_2}(\eta + SV_2). \quad (3.11)$$

Taking the inverse on both sides of (3.11) and applying this identity to the result of multiplying (3.5a) with $V_1$ from the right and with $\eta + S^*V_1$ from the left yield
\[ \eta + SV_2 = (\eta + SV_2)V_1(\eta + SV_2) + \tau V_2. \]  

We reorganize the terms in (3.12), use that \( \tau - S \) is invertible as \( \tau > \varrho(S) \) and obtain
\[ V_2 = (\tau - S)^{-1}(\eta - (\eta + SV_2)V_1(\eta + SV_2)) \leq \eta(\tau - S)^{-1}. \]
Here, we used in the last step that \( (\tau - S)^{-1} \) is positivity preserving due to the Neumann series and that \( (\eta + SV_2)V_1(\eta + SV_2) \geq 0 \). Therefore, we have shown that
\[ V_2 \leq \eta f(\tau)/\tau \leq \eta f(\varrho(S) + \delta)/\tau \lesssim \eta/\tau \]
for all \( \eta > 0 \) and all \( \tau \geq \varrho(S) + \delta \). Similarly, we get \( V_1 \lesssim \eta/\tau \).

Using \( V_1 + V_2 \lesssim \eta/\tau \) and (3.9) to estimate the right-hand side of (3.12) from above implies
\[ \eta \lesssim \eta^3 + \tau V_2. \]
Hence, \( V_2 \gtrsim \eta/\tau \) and a similar argument yields \( V_1 \gtrsim \eta/\tau \). Owing to (3.8), these estimates and \( \tau = |\zeta|^2 \) complete the proof of Lemma 3.1. \( \square \)

For the upcoming arguments, it is convenient to use the following notion of events that occur with “very high probability”.

**Definition 3.2 (With very high probability).** We say that the (sequence of) events \( (A_n)_{n \in \mathbb{N}} \) occur with very high probability if for every \( \nu > 0 \) there is \( C_\nu > 0 \) such that
\[ \mathbb{P}(A_n) \geq 1 - C_\nu n^{-\nu} \]
for all \( n \in \mathbb{N} \).

The constants \( C_\nu \) in (3.13) will typically depend on the model parameters. Note that an intersection of \( n^C \)-many events holding with very high probability also holds with very high probability.

**Proof of Theorem 2.2.** The theorem will follow from the next lemma and an interpolation argument. As we will see in its proof in Appendix A below, this lemma is a direct consequence of [26, Corollary 2.3] and Lemma 3.1.

**Lemma 3.3 (No eigenvalues of \( H_\zeta \) around zero).** Let \( X \) satisfy \( A1, A2 \) and \( A5 \). If \( \zeta \in \mathbb{C} \) satisfies \( \sqrt{\varrho(S)} + \delta \leq |\zeta|^2 \leq \delta^{-1} \) for some \( \delta \sim 1 \) then there is \( \varepsilon \sim 1 \) such that
\[ \text{Spec}(H_\zeta) \cap (-\varepsilon, \varepsilon) = \emptyset \]
with very high probability.
Since $\text{Spec}(X) = \{ \zeta \in \mathbb{C} : 0 \in \text{Spec}(H_\zeta) \}$ we conclude from Lemma 3.3 that with very high probability $X$ has no eigenvalues in the annulus $A := \{ \zeta : \varrho(S) + \delta < |\zeta|^2 < \delta^{-1} \}$. We will now show that there are no eigenvalues of $X$ outside $\mathbb{D}_\delta^{-1/2}$ either. For this purpose we apply Lemma 3.3 to the Hermitization $tH_{\zeta/t}$ of $tX$ for any $t \in [0, 1]$. We choose a finite subset $Z \subset A$ such that $Z + \mathbb{D}_{n-1}$ covers the entire annulus $A$ and $|Z| \leq n^C$ for some $C > 0$. By a union bound and Lemma 3.3, we find that for any $(t, \zeta) \in n^{-1}[n] \times Z$ with very high probability $\text{Spec}(tH_{\zeta/t}) \cap (-\varepsilon, \varepsilon) = \emptyset$. Thus, by Lipschitz-continuity of $tH_{\zeta/t}$ in $t$ and $\zeta$, we have $0 \notin \bigcup_{t \in [0,1]} \bigcup_{\zeta \in A} \text{Spec}(tH_{\zeta/t})$ with very high probability. In particular, the eigenvalues of each matrix along the interpolation $t \mapsto tX$ between the zero matrix and $X$, that continuously depend on $t$, do not cross the annulus. Therefore, $X$ has the same number of eigenvalues inside the disk with radius $\sqrt{\varrho(S)} + \delta$ as the zero matrix, namely $n$, i.e. it has no eigenvalues outside this disk. \(\Box\)

3.2. Global inhomogeneous circular law

In this section, we prove Theorem 2.3. We first derive the basic formula relating the eigenvalue density of $X$ to the Hermitian matrices $H_\zeta$ defined in (3.1). This approach goes back to Girko [29]. Then we motivate and collect all other ingredients required for the proof of Theorem 2.3.

The starting point is a relation for the averaged linear statistics with a test function $f \in C_0^2(\mathbb{C})$ given by

$$\frac{1}{n} \sum_{\xi \in \text{Spec } X} f(\xi) = \frac{1}{2\pi n} \sum_{\xi \in \text{Spec } X} \int_C \Delta f(\zeta) \log |\zeta - \xi| d^2\zeta, \quad (3.14)$$

where we used in the first step that log is the fundamental solution of the Laplace equation in $\mathbb{R}^2$.

The right-hand side of (3.14) can be expressed purely in terms of the Hermitian matrices $H_\zeta$ since

$$\sum_{\xi \in \text{Spec } X} \log |\xi - \zeta| = \log |\det(X - \zeta)| = \frac{1}{2} \log |\det H_\zeta|. \quad (3.15)$$

The resolvent $G$ contains all spectral information about $H_\zeta$. In particular, $\log |\det H_\zeta|$ is expressed in terms of $G$ via the well-known identity

$$\log |\det H_\zeta| = -2n \int_0^T (\text{Im } G(\zeta, \eta)) d\eta + \log \det (H_\zeta - iT)| \quad (3.16)$$

for any $T > 0$ (see [50] for the use of (3.16) in a similar context). Hence, owing to (3.14), (3.15) and (3.16), it suffices to control $G$ in order to understand the averaged linear
statistics. As indicated in Section 3.1, the resolvent \( G \) will be well approximated by the solution \( M \) of the MDE (3.3) for large \( n \).

We now collect some auxiliary results about \( M \) and \( \sigma \). We will need the following bounds on \( M \) proven at the end of Section 4.1.

**Lemma 3.4 (Bounds on \( M \)).** Let \( X \) satisfy A1 – A3. Then, uniformly for \( \zeta \in \mathbb{C} \) and \( \eta > 0 \), we have
\[
\| M(\zeta, \eta) \| \lesssim \frac{1}{1 + \eta + |\zeta|}. \tag{3.17}
\]

Moreover, for any \( T > 0 \) and \( \zeta \in \mathbb{C} \), we have
\[
\int_0^T \left| \text{Im} M(\zeta, \eta) - \frac{1}{1 + \eta} \right| d\eta \lesssim \min\left\{ T, 1 + \frac{|\zeta|}{T} \right\},
\int_T^\infty \left| \text{Im} M(\zeta, \eta) - \frac{1}{1 + \eta} \right| d\eta \lesssim \frac{1 + |\zeta|}{T}. \tag{3.18}
\]

The self-consistent density of states \( \sigma \) introduced in (2.11) relates to \( \text{Im} M \) in the way expected from (3.14), (3.15) and (3.16) as well as \( G \approx M \). This is the content of the next lemma.

**Lemma 3.5 (**\( \sigma \) as distributional derivative).** Let \( X \) satisfy A1 – A3. Then we have
\[
\int_{\mathbb{C}} f(\zeta) \sigma(\zeta) d^2\zeta = -\frac{1}{2\pi} \int_{\mathbb{C}} \Delta f(\zeta) L(\zeta) d^2\zeta, \quad L(\zeta) := \int_0^\infty \left( \text{Im} \langle M(\zeta, \eta) \rangle - \frac{1}{1 + \eta} \right) d\eta \tag{3.19}
\]
for every \( f \in C_0^\infty(\mathbb{C}) \). The integral in the definition of \( L \) exists in the Lebesgue sense due to (3.18).

Lemma 3.5 in particular shows that \( \Delta L = -2\pi \sigma \) in the sense of distributions, i.e. \( L \) is the logarithmic potential of the probability measure \( \sigma(\zeta) d^2\zeta \). The proof of Lemma 3.5 requires a very detailed analysis of the MDE, (3.3), and its stability properties and will be presented in Section 5.3 below.

To illustrate the basic formula used in the proof of Theorem 2.3 below, we combine the identities (3.14), (3.15), (3.16) and (3.19) and, thus, obtain for any \( T > 0 \) that
\[
\frac{1}{n} \sum_{\xi \in \text{Spec } X} f(\xi) - \int_{\mathbb{C}} f(\xi) \sigma(\zeta) d^2\zeta = \frac{1}{2\pi} \int_{\mathbb{C}} \Delta f(\zeta) \left( \int_0^T \text{Im} \langle M(\zeta, \eta) \rangle - \text{G}(\zeta, \eta) \rangle d\eta \right.
\]
\[
+ \left. \int_T^\infty \left( \text{Im} \langle M(\zeta, \eta) \rangle - \frac{1}{1 + \eta} \right) d\eta + \frac{1}{2n} \log|\det(H_\zeta - iT)| \right) d^2\zeta, \tag{3.20}
\]
where we used that \( \int_{\mathbb{C}} \Delta f(\zeta)d^2 \zeta = 0 \). The terms on the right-hand side of (3.20) will be bounded as follows. The second term is controlled due to the second bound in (3.18) and the third by a simple argument using A1. For the first term, we shall use Proposition 3.7 below and Assumption A4.

For technical reasons, we discretise the integral over \( \zeta \) in (3.14) via Lemma 3.6 below. Afterwards, we apply (3.15) and (3.16) to the discretised expression. Thus, the final proof of Theorem 2.3 does not start from (3.20) directly. For the discretisation of the \( \zeta \)-integral, we apply the sampling method formulated in the following lemma. For \( a = 2 \), it is a special case of [50, Lemma 36], which was used in a similar context in [50].

**Lemma 3.6 (Monte Carlo sampling).** Let \( \Omega \subset \mathbb{C} \) be a bounded subset of positive Lebesgue measure and \( \mu \) the normalized Lebesgue measure on \( \Omega \). Let \( F: \Omega \to \mathbb{C} \) be a function in \( L^a(\mu) \) from some \( a > 1 \). For \( m \in \mathbb{N} \), let \( \xi_1, \ldots, \xi_m \) be independent random variables distributed according to \( \mu \).

Then, for any \( \delta \in (0, 1] \), we have

\[
\mathbb{P} \left( \left| \frac{1}{m} \sum_{i=1}^{m} F(\xi_i) - \int_{\Omega} F d\mu \right| \leq \frac{10^{1/a}}{m^{1-1/a} \delta^{1/a}} \left( \int_{\Omega} F - \int_{\Omega} F d\mu \right)^{1/a} \right) \geq 1 - \delta.
\]

**Proof.** The random variables \( F(\xi_1), \ldots, F(\xi_m) \) are i.i.d. with mean \( \int_{\Omega} F d\mu \). Thus, Proposition B.1 in Appendix B below implies Lemma 3.6. \( \square \)

The next bound on \( G - M \) is the last missing ingredient for the proof of Theorem 2.3.

**Proposition 3.7 (Global law for \( H_\zeta \), averaged version).** Let \( X \) satisfy A1 and A2. Then there is an absolute constant \( K > 0 \) such that for all sufficiently small \( \delta > 0 \) we have

\[
|\langle G(\zeta, \eta) - M(\zeta, \eta) \rangle| \leq \frac{n^{K\delta}}{(1 + \eta^2)n}
\]

(3.21)

with very high probability uniformly for all \( n \in \mathbb{N} \), \( \zeta \in \mathbb{D}_\varphi \) and \( \eta \in [n^{-\delta}, n^{100}] \).

Proposition 3.7 is implied by Proposition A4 below. The former has an important consequence, namely the following bound on the number of eigenvalues of \( H_\zeta \) close to zero. Note that the moduli of the eigenvalues of \( H_\zeta \) are the singular values of \( X - \zeta \). The eigenvalues of \( H_\zeta \) are denoted by \( \lambda_1(\zeta), \ldots, \lambda_{2n}(\zeta) \). Together with Assumption A4, the bound in the next lemma will be used to control the integral in (3.16) for small \( \eta \).

**Lemma 3.8 (Number of small singular values of \( X - \zeta \)).** Let \( X \) satisfy A1 and A2. Then there is \( \delta > 0 \) such that

\[
\#\{i \in [2n] : |\lambda_i(\zeta)| \leq \eta\} \lesssim n\eta
\]
with very high probability uniformly for all \( \eta \in [n^{-\delta}, n^{100}] \) and \( \zeta \in \mathbb{D}_\varphi \) with any fixed \( \varphi > 0 \). Here, the constant \( C_\nu \) implicit in the very high probability notion from Definition 3.2 depends on \( \delta \) and \( \varphi \) as well as the constants in \( \mathbf{A1} \) and \( \mathbf{A2} \) in addition to \( \nu \).

**Proof.** The trace of \( \mathbf{G} \) is bounded by \( n, |\text{Tr} \mathbf{G}| \lesssim n \) with very high probability, for all \( \eta \in [n^{-\delta}, n^{100}] \) due to (3.21) and \( \|\mathbf{M}\| \lesssim 1 \) by (3.17). Therefore, setting \( \Sigma_\eta := \{ i \in [2n] : |\lambda_i(\zeta)| \leq \eta \} \) yields

\[
\frac{\# \Sigma_\eta}{2\eta} \leq \sum_{i \in \Sigma_\eta} \frac{\eta}{\eta^2 + \lambda_i(\zeta)^2} \leq \text{Im Tr} \mathbf{G}(\zeta, \eta) \lesssim n. \quad \Box
\]

We will now conclude Theorem 2.3 from Proposition 3.7 and Lemma 3.8.

**Proof of Theorem 2.3.** We will show below that for all sufficiently small \( \delta > 0 \) we have

\[
\left| \frac{1}{n} \sum_{\zeta \in \text{Spec} X} f(\zeta) - \int_{\mathbb{C}} f(\zeta) \sigma(\zeta) d^2\zeta \right| \leq n^{-\delta} \| \Delta f \|_{L^1} \tag{3.22}
\]

with very high probability uniformly for all \( f \in C_0^2(\mathbb{C}) \) satisfying \( \| \Delta f \|_{L^{1+\beta}} \leq n^D \| \Delta f \|_{L^1} \) and \( \text{supp} f \subseteq \mathbb{D}_\varphi \), where \( \beta > 0, D > 0 \) and \( \varphi > 0 \) are some constants. In (3.22), the constant \( C_\nu \) in the definition (3.13) depends only on \( \delta, \beta, D \) and \( \varphi \) in addition to \( \nu \) and the constants from \( \mathbf{A1} \) and \( \mathbf{A2} \).

Given (3.22), we now explain how Theorem 2.3 follows. Let \( f \in C_0(\mathbb{C}) \) and \( \varepsilon > 0 \). Since \( X \) does not have any eigenvalues outside \( \mathbb{D}_{R+1} \), where \( R = \sqrt{\varphi(S)} \), with very high probability by Theorem 2.2, we assume without loss of generality that \( \text{supp} f \subseteq \mathbb{D}_{R+2} \). As \( R = \sqrt{\varphi(S)} \lesssim 1 \) due to Assumptions \( \mathbf{A1} \) and \( \mathbf{A2} \), we can choose a constant \( \varphi \) such that \( \varphi > R + 2 \). We find \( g \in C_0^2(\mathbb{C}) \) such that \( \| f - g \|_{L^\infty} \leq \varepsilon/2, \text{supp} g \subseteq \mathbb{D}_\varphi \) and \( \| \Delta g \|_{L^2} \lesssim \varepsilon \). Therefore, approximating \( f \) by \( g \) in the statement of Theorem 2.3 and applying (3.22) to \( g \) shows Theorem 2.3.

What remains is proving (3.22). We set \( \Omega = \mathbb{D}_\varphi \). Combining (3.14) and (3.19) as well as using the second bound in (3.18) yield

\[
\frac{1}{n} \sum_{\xi \in \text{Spec} X} f(\xi) - \int_{\mathbb{C}} f(\zeta) \sigma(\zeta) d^2\zeta = \int_{\Omega} F(\zeta) d\mu(\zeta) + O(T^{-1} \| \Delta f \|_{L^1}) \tag{3.23}
\]

for any \( T > 0 \). Here, we denoted by \( \mu \) the normalized Lebesgue measure on \( \Omega \) and introduced

\[
F(\zeta) := \frac{\|\Omega\|}{2\pi} \Delta f(\zeta) h(\zeta), \quad h(\zeta) := \frac{1}{n} \sum_{\xi \in \text{Spec} X} \log|\xi - \zeta| + \int_0^T \left( \text{Im} \mathbf{M}(\zeta, \eta) - \frac{1}{1 + \eta} \right) d\eta. \tag{3.24}
\]
We now apply Lemma 3.6 to the first term on the right-hand side of (3.23). Note that \( \zeta \mapsto \log|\zeta - \zeta| \) lies in \( L^p(\Omega) \) for every \( p \in [1, \infty) \). Hence, owing to the first bound in (3.18) we get that, for any \( p \in [1, \infty) \), \( \|h\|_{L^p(\Omega)} \lesssim_{p} 1 \) uniformly for \( T > 0 \). In particular, the function \( F \) defined in (3.24) is square-integrable on \( \Omega \). Thus, Lemma 3.6 is applicable and choosing \( \delta = n^{-\nu}, \ a = 1 + \beta/2 \) and \( m = n^{(\nu+(D+11)a)/(a-1)} \) shows that

\[
\left| \int Fd\mu - \frac{1}{m} \sum_{i=1}^{m} F(\xi_i) \right| \leq n^{-D-10} \|\Delta f\|_{L^{1+\beta}} \tag{3.25}
\]

with very high probability, where \( \xi_1, \ldots, \xi_m \) are independent random variables distributed according to \( \mu \).

We set \( T = n^{100} \) and now show that for all sufficiently small \( \delta > 0 \) we have

\[
|F(\zeta)| \leq n^{-\delta}|\Delta f(\zeta)| \tag{3.26}
\]

with very high probability uniformly for all \( \zeta \in \Omega \). To that end, we define \( \eta_* := n^{-\delta} \) and

\[
\begin{align*}
 h_1(\zeta) &:= \int_{\eta_*}^{T} \text{Im}(\mathbf{M}(\zeta, \eta) - \mathbf{G}(\zeta, \eta))d\eta, \\
h_2(\zeta) &:= -\int_{0}^{\eta_*} \langle \text{Im} \mathbf{G}(\zeta, \eta) \rangle d\eta, \\
h_3(\zeta) &:= \frac{1}{4n} \sum_{i \in [2n]} \log \left( 1 + \frac{\lambda_i(\zeta)^2}{T^2} \right) - \log \left( 1 + \frac{1}{T} \right), \\
h_4(\zeta) &:= \int_{0}^{\eta_*} \langle \text{Im} \mathbf{M}(\zeta, \eta) \rangle d\eta.
\end{align*}
\]

Using (3.15), (3.16) and \( \int_{0}^{T} \frac{1}{1+\eta} d\eta = \log(1+T) \), it is easy to see that \( h(\zeta) = h_1(\zeta) + h_2(\zeta) + h_3(\zeta) + h_4(\zeta) \).

Next, we establish individual estimates on \( h_1, \ldots, h_4 \) which hold with very high probability. We get \( |h_1(\zeta)| \leq 2n^{-1+K\delta} \) from (3.21) as well as a union bound and a continuity argument in \( \eta \). To estimate \( h_2 \), we write \( \lambda_j \equiv \lambda_j(\zeta) \) and compute

\[
-h_2(\zeta) = \frac{1}{4n} \sum_{j \in [2n]} \log \left( 1 + \frac{\eta_*^2}{\lambda_j^2} \right).
\]

In the following, we will decompose the sum into two regimes, \( \lambda_j \leq \eta_*^{1/2} \) and \( \lambda_j \geq \eta_*^{1/2} \), and estimate each regime separately. For the first regime, Assumption A4 and Lemma 3.8 yield

\[
\frac{1}{4n} \sum_{\lambda_j \leq \eta_*^{1/2}} \log \left( 1 + \frac{\eta_*^2}{\lambda_j^2} \right) \leq C \left( \log n + |\log \min_{j \in [2n]}|\lambda_j|| \right) \frac{\# \{ j \in [2n] : \lambda_j \leq \eta_* \}}{n} \leq n^\epsilon \eta_*,
\]

with very high probability for all \( \epsilon > 0 \) small enough. In the remaining regime, \( \log(1+x) \leq x \) yields

\[
(3.27)
\]
\[
\frac{1}{4n} \sum_{|\lambda_j| \geq \eta_*^{1/2}} \log \left( 1 + \frac{\eta_*^2}{\lambda_j^2} \right) \leq \frac{1}{4n} \sum_{|\lambda_j| \geq \eta_*^{1/2}} \log(1 + \eta_*) \leq \eta_*.
\] (3.28)

By slightly shrinking \(\delta\), these estimates imply \(|h_2(\zeta)| \leq n^{-\delta}\). For \(h_3\), we obtain
\[
|h_3(\zeta)| \leq \frac{1}{4nT^2} \text{Tr}(H_\zeta)^2 + T^{-1} = \frac{1}{2nT^2} \sum_{i,j=1}^{n} (x_{ij} - \zeta \delta_{ij})(x_{ij} - \zeta \delta_{ij}) + T^{-1}
\]
\[
\leq n^\epsilon T^{-2}(1 + |\zeta|^2) + T^{-1},
\]
where we used \(\log(1 + x) \leq x\) in the first step and \(|x_{ij}| < n^{-1/2}\) due to Assumption \(A1\) in the last step. Since \(\zeta \in \Omega\) we conclude \(|h_3(\zeta)| \leq n^{-10}\) with very high probability. Since \(\|M\| \lesssim 1\) due to (3.17) we have \(0 \leq h_4(\zeta) \lesssim \eta_*\). Hence, the proof of (3.26) is complete.

Since \(m\) is at most of polynomial order in \(n\), a union bound over \(\xi_1, \ldots, \xi_m\) and (3.26) yield
\[
\frac{1}{m} \sum_{i=1}^{m} |F(\xi_i)| \leq \frac{n^{-\delta}}{m} \sum_{i=1}^{m} |\Delta f(\xi_i)| \leq n^{-\delta} \|\Delta f\|_{L^1} + n^{-D-10} \|\Delta f\|_{L^{1+\beta}}
\] (3.29)
with very high probability. Here, we applied Lemma 3.6 with \(a = 1 + \beta\) in the last step and used \(\text{supp \, supp \, \subseteq \Omega}\).

Finally, we combine the relation (3.23), the estimates (3.25) and (3.29) as well as \(\|\Delta f\|_{L^{1+\beta}} \leq n^D \|\Delta f\|_{L^1}\) and obtain (3.22). This completes the proof of Theorem 2.3. \(\square\)

4. Dyson equation and its stability

In this section, we analyse the solution \(M\) to the matrix Dyson equation (3.3) and its stability against perturbations \(D\), i.e. we control the solution \(G(D)\) of a perturbed version of the MDE (see (4.34) below) such that \(G(0) = M\). These results are the core of this article as they will be the basis of the proofs of Theorem 2.5 and Lemma 3.5 about the properties of \(\sigma\) as well as the local law for \(H_\zeta\) (cf. Theorem 6.2 below).

The matrix Dyson equation and its stability have been analysed in [3,6]. However, their main regularity and stability results impose the flatness condition (see [3, equation (2.7)]) and (4.2) below) on the self-energy operator \(\mathcal{S}\). This condition is not satisfied by \(\mathcal{S}\) as defined in (3.4). In fact, the special structure of \(\mathcal{S}\), originating from the zero blocks on the diagonal of \(H_\zeta\), poses significant challenges since it leads to an instability in the Dyson equation (3.3) which was not present in [3,6]. Dealing with this instability is the main purpose of this section.

In [5], a similar instability was analysed, but in the simpler setup of a random matrix with independent entries. This setup results in a vector-valued Dyson equation whose formulation on the commutative algebra \(\mathbb{C}^{2n}\) with entry wise multiplication simplifies the analysis compared to the present article. In particular, in the commutative setting of [5] the MDE was formulated on the entire algebra \(\mathbb{C}^{2n}\) and the contribution to the error
term in the unstable direction determined to be sufficiently small to cancel the instability in the \( \eta \to 0 \) limit. The corresponding algebraic manipulations are considerably harder in the non-commutative space \( \mathbb{C}^{2n \times 2n} \). Therefore, we develop a different strategy in the present work. Here, we identify a stable manifold \( \Gamma := G^{-1} E^\perp \subset \mathbb{C}^{2n \times 2n} \), defined as the preimage of a linear hyperspace \( E^\perp \subset \mathbb{C}^{2n \times 2n} \) under the solution map \( G \) to the perturbed MDE, such that \( \Gamma \ni D \mapsto G(D) \in E^\perp \) is stable. Then we implicitly construct a parametrisation \( E^\perp \ni \tilde{D} \mapsto D(\tilde{D}) \in \Gamma \) of this manifold (see (4.38) below) and rewrite the MDE directly on the codimension one subspace \( E^\perp \) (see (4.18) below). In short, we remove the unstable direction from the MDE at the beginning. In addition to removing the need to trace the unstable component of the error matrix, this strategy also implies analyticity of \( \tau \mapsto V_1(\tau), \tau \mapsto V_2(\tau) \) from (2.9) in the bulk (see Proposition 2.4). The ensuing analyticity of \( \sigma \) from (2.11) is a new result even for matrices with independent entries. With the strategy from [5] showing only smoothness already required tracking the unstable direction to all derivative orders (cf. [5, proof of Proposition 2.4]).

In Subsection 4.1 we will establish some basic properties of the solution to (3.5) and hence (3.3). Then we will prove stability of the Dyson equation in the bulk of the spectrum against small perturbations in Theorem 4.3 of Subsection 4.2, using an important technical lemma that will be proven in Subsection 4.3.

Since (3.5) is invariant under the scaling \( \mathcal{S} \to \lambda \mathcal{S}, \eta \to \lambda^{1/2} \eta, \tau \to \lambda \tau \) and \( V_1 \to \lambda^{-1/2} V_1 \) for any \( \lambda > 0 \), we will assume for the rest of the paper that

\[
\rho(\mathcal{S}) = 1. \tag{4.1}
\]

Furthermore, we denote the unit disk in the complex plane centred at the origin by \( \mathbb{D} = \mathbb{D}_1 = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \} \).

4.1. Solution

The first result of this subsection establishes matching upper and lower bounds on the solution of (3.5) in the sense of quadratic forms. For this proposition only the following flatness property of \( \mathcal{S} \) is needed. Due to assumptions \( A^1 \text{–} A^3 \) the operators \( \mathcal{S} \) and \( \mathcal{S}^* \) are both comparable to the normalised trace in the sense of quadratic forms, i.e.

\[
\mathcal{S} A \sim \mathcal{S}^* A \sim \langle A \rangle, \quad A \geq 0, \tag{4.2}
\]

for any positive semi-definite \( A \). In fact, the lower bound \( \mathcal{S} A \gtrsim \langle A \rangle \) is just an equivalent formulation of assumption \( A^3 \), while the upper bound \( \mathcal{S} A \lesssim \langle A \rangle \) is a simple consequence of assumptions \( A^1 \) and \( A^2 \). The same is true for \( \mathcal{S}^* \).

Proposition 4.1 (Behaviour of solution). The solution of (3.5) satisfies

\[
\langle V_1(\tau, \eta) \rangle = \langle V_2(\tau, \eta) \rangle, \tag{4.3}
\]
for all \( \tau \geq 0 \) and \( \eta > 0 \), as well as the bounds

\[
V_1(\tau, \eta) \sim V_2(\tau, \eta) \sim \begin{cases} 
(1 - \tau)^{1/2} + \eta^{1/3}, & \tau \leq 1, \ \eta \leq 1, \\
\frac{\eta}{\tau - 1 + \eta^{2/3}}, & \tau \geq 1, \ \eta \leq 1, \\
\frac{\eta}{\eta^2 + \tau}, & \eta \geq 1.
\end{cases}
\] (4.4)

**Proof.** Throughout this proof, we will refer to some identities from the proof of Lemma 3.1. To see (4.3), we multiply (3.11) with \( V_1 \) from the right and with \( V_2 \) from the left and take the normalized trace.

Now we prove (4.4). First we observe that \( V_1 \) and \( V_2 \) are both comparable to their normalized traces, which coincide as we have just shown, i.e.

\[
V_1 \sim V_2 \sim \langle V_1 \rangle = \langle V_2 \rangle.
\] (4.5)

This is seen directly from the two equations (3.5) since the right hand sides are both comparable to the same multiple of the identity due to (4.2) and (4.3).

Now let \( S_2 \) be the unique positive definite Perron-Frobenius eigenmatrix of \( S \) with normalisation \( \langle S_2 \rangle = 1 \), i.e. \( SS_2 = S_2 \). Because of (4.2) this eigenmatrix satisfies \( S_2 \sim 1 \). We take the scalar product with \( S_2 \) on both sides of (3.12) and get

\[
\eta + \langle S_2, V_1 \rangle = \eta + \langle S_2, S^*V_1 \rangle = \langle S_2, (\eta + S^*V_1)V_2(\eta + S^*V_1) \rangle + \tau \langle S_2, V_1 \rangle.
\] (4.6)

Depending on whether \( \tau \leq 1 \) or \( \tau > 1 \) we either subtract \( \tau \langle S_2, V_1 \rangle \) or \( \langle S_2, V_1 \rangle \) on both sides of (4.6) and use (4.2) as well as \( S_2 \sim 1 \) to see that

\[
\eta + (1 - \tau) \langle V_1 \rangle \sim \langle V_2 \rangle(\eta + \langle V_1 \rangle)^2, \quad \tau \leq 1, \quad (4.7a)
\]
\[
\eta \sim \langle V_2 \rangle(\eta + \langle V_1 \rangle)^2 + (\tau - 1)\langle V_1 \rangle, \quad \tau > 1. \quad (4.7b)
\]

The claim (4.4) is now an immediate consequence of (4.7) and (4.5). \( \Box \)

As a consequence of Proposition 4.1 we can also estimate the singular values of \( U \), defined in (3.6), from above and below. When multiplying (3.5a) with \( \eta + SV_1 \) from the left and (3.5b) with \( \eta + SV_2 \) from the right we see the identities

\[
U = V_1 \frac{1}{\eta + S^*V_1} = \frac{1}{\eta + SV_2} V_2.
\] (4.8)

Furthermore, when we multiply (3.5a) with \( V_1 \) from the left and (3.5b) with \( V_2 \) from the right we see that

\[
1 = V_1(\eta + SV_2) + \tau U = (\eta + S^*V_1)V_2 + \tau U.
\] (4.9)
Multiplying (4.9) by $U$ and using (4.8) also reveals
\[ U = V_1 V_2 + \tau U^2. \] (4.10)

Finally, using (4.4) in (4.8) shows the comparison relation
\[ U^* U \sim \frac{1}{1 + \tau^2 + \eta^4}, \quad \text{(4.11)} \]
uniformly for $\eta > 0$ and $\tau \geq 0$. For future reference we also record the identities
\[
\begin{align*}
V_1 &= \eta(V_1^2 + \tau U U^*) + V_1(S^* V_2) V_1 + \tau U(S^* V_1) U^*, \\
V_2 &= \eta(V_2^2 + \tau U^* U) + V_2(S^* V_1) V_2 + \tau U^*(S V_2) U,
\end{align*}
\]
which result from multiplying (3.5a) from left and right by $V_1$ and (3.5b) by $V_2$ and then using (4.8). As a consequence of (4.4) for $\tau \geq 1$ we can extend $V_i$ continuously to $\eta = 0$ as $V_i(\tau, 0) = 0$. This is summarised in the following corollary whose proof is immediate from the representation of $M$ in (3.7) and the definition of $U$ in (3.6).

**Corollary 4.2 (Extension outside the spectrum).** The functions $V_1, V_2$ admit continuous extensions to $([0, 1] \times (0, \infty)) \cup ([1, \infty) \times [0, \infty))$, that is, to $\eta = 0$ for $\tau \geq 1$. These extensions are still denoted by the same symbols. Similarly, the solution $M$ of the MDE (3.3) from (3.7) can be extended continuously to $(\mathbb{D} \times (0, \infty)) \cup ((\mathbb{C} \setminus \mathbb{D}) \times [0, \infty))$, i.e. to $\eta = 0$ for $\zeta \notin \mathbb{D}$. We still denote the extension by $M$. The extension satisfies
\[ M(\zeta, 0) = \begin{pmatrix} 0 & -1/\zeta \\ -1/\zeta & 0 \end{pmatrix}, \quad \zeta \notin \mathbb{D}. \]

**Proof of Lemma 3.4.** First, we get (3.17) from (3.7), (4.4) and (4.11). Second, both bounds in (3.18) follow directly from the estimate
\[
\|M(\zeta, \eta) - i(1 + \eta)^{-1}\| \lesssim \min\{1, (1 + |\zeta|)\eta^{-2}\},
\]
which holds uniformly for $\zeta \in \mathbb{C}$ and $\eta > 0$ and is shown next. Since $\|M\| \lesssim 1$ by (3.17) we trivially have $\|M(\zeta, \eta) - i(1 + \eta)^{-1}\| \lesssim 1$. Multiplying (3.3) by $i\eta^{-1}M$ and using $\|M\| \lesssim \eta^{-1}$ as well as $\|\mathcal{S}\| \lesssim 1$ (cf. upper bound in (4.2)) imply $\|M(\zeta, \eta) - i\eta^{-1}\| \lesssim (1 + |\zeta|)\eta^{-2}$, i.e. the missing bound in (4.13). This completes the proof of Lemma 3.4. \qed

### 4.2. Stability

In this subsection we will establish stability of the MDE (3.3) and its solution against small perturbations. As indicated at the beginning of the section, (3.3) has an inherent instability due to the structure of $\mathcal{S}$. This instability originates from a single unstable
direction and implies that stability can only be expected with respect to perturbations $\mathbf{D}$ that take values in a manifold of codimension 1 in $\mathbb{C}^{2n \times 2n}$. Through a special choice of coordinates this manifold can be mapped to the orthogonal complement of $\mathbf{E}_- \in \mathbb{C}^{2n \times 2n}$ defined through

$$\mathbf{E}_\pm := \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad (4.14)$$

and thus projected out.

Before we state the stability theorem we introduce a norm that is designed to prove isotropic convergence of the resolvent $\mathbf{G}$ from (3.2) to $\mathbf{M}$, i.e. to prove $|\langle \mathbf{x}, (\mathbf{G} - \mathbf{M})\mathbf{y} \rangle| \to 0$ for any fixed vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{2n}$ in a high moment sense. A similar norm was introduced in [26] for the same purpose and to match the notation to this work we introduce coefficients $\kappa_{\mathbf{R}}(\alpha, \beta)$ with $\alpha, \beta \in [2n]^2$ associated to any linear operator $\mathbf{R} : \mathbb{C}^{2n \times 2n} \to \mathbb{C}^{2n \times 2n}$ via

$$(\mathbf{R})_{ad} = \sum_{b,c \in [2n]} \kappa_{\mathbf{R}}(ab, cd) r_{bc}. \quad (4.15)$$

Through this one to one correspondence between $\mathbf{R}$ and $\kappa_{\mathbf{R}}$ we define $\kappa_c := \kappa_{\mathbf{R}}$. We also recall the following notation from [26]. For an expression $f_{a_1a_2...a_k}$ with indices $a_1, \ldots, a_k$ we write $f_{xa_2...a_k} = \sum a_x f_{aa_2...a_k}$ if an index is averaged against a vector $\mathbf{x}$, and similarly if more than one index is averaged. We also write $f_{a_2...a_k}$ for the vector $(f_{aa_2...a_k})_a$. In particular, $\mathbf{A}_{\mathbf{x}\mathbf{y}} = \sum_{i,j} x_i y_j a_{ij}$ and $\mathbf{A}\mathbf{x} = \mathbf{A}_x$.

Let us now fix two deterministic vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{2n}$ and $K \in \mathbb{N}$. Then for fixed $\eta$ and $\zeta$ writing $\mathbf{M} = \mathbf{M}(\zeta, \eta)$ we recursively define the sets of vectors

$$\begin{align*}
I_0 & := \{ \mathbf{x}, \mathbf{y} \} \cup \{ \mathbf{e}_i : i \in [2n] \}, \\
I_{k+1} & := I_k \cup \{ \mathbf{M}_{\mathbf{u}}, \mathbf{Z}_{\mathbf{u}}, (\mathbf{R}\mathbf{M})_{\mathbf{u}}, (i\eta \mathbf{1} + \mathbf{R}\mathbf{M})^{-1}_{\mathbf{u}} : \mathbf{u} \in I_k \} \\
& \quad \cup \{ \kappa_c(\mathbf{u}i, j) : \mathbf{u} \in I_k, i, j \in [2n] \}.
\end{align*}$$

Here, $\mathbf{e}_a$ denotes the $a$-th standard basis vector in $\mathbb{C}^{2n}$. The $\| \cdot \|_*$-norm is then defined as

$$\begin{align*}
\| \mathbf{A} \|_* & := \| \mathbf{A} \|_{K, \mathbf{x}, \mathbf{y}} := \sum_{0 \leq k < K} n^{-k/2} \| \mathbf{A} \|_{I_k} + n^{-1/2} \max_{\mathbf{u} \in I_K} \frac{\| \mathbf{A}_\mathbf{u} \|}{\| \mathbf{u} \|}, \\
\| \mathbf{A} \|_I & := \max_{\mathbf{u}, \mathbf{v} \in I} \frac{|\mathbf{A}_{\mathbf{u}\mathbf{v}}|}{\| \mathbf{u} \| \| \mathbf{v} \|}.
\end{align*} \quad (4.16)$$

The definition of $\| \cdot \|_*$ is chosen such that the arguments from [26] can be followed directly in the proof of Theorem 6.2 below. The norm is dominated by the standard operator norm, $\| \mathbf{A} \|_* \leq 2 \| \mathbf{A} \|$ and by construction and $\mathbf{A}_2$ it satisfies
\[ \|\mathcal{J}\|_{\|\cdot\| \to 1} \leq 1, \quad \|RA\|_\ast \leq n^{1/2K}\|R\|\|A\|_\ast, \quad \|(\mathcal{J}A)B\|_\ast \leq n^{1/2K}\|A\|_\ast\|B\|_\ast, \] (4.17)

for all \( A, B \in \mathbb{C}^{2n \times 2n} \) and \( R \in \{M, Z, \mathcal{J}M, (i\eta 1 + \mathcal{J}M)^{-1}\} \). The bounds from (4.17) follow exactly as (73), (70b) and (70a) in [26] from A2.

Now we present our main stability theorem. It states that when (3.3) is properly rewritten and restricted to \( \mathbb{E}_+^+ \) it is stable against small perturbations.

**Theorem 4.3 (Stability).** For any sufficiently small \( \delta > 0 \) (depending on model parameters) and any \( \eta \in (0, \delta^3) \), \( \zeta \in \mathbb{C} \) with \( |\zeta|^2 \leq 1 - \delta \) there is a unique function

\[ G : D \times B_1 \to B_2, \]

such that \( G = G(\zeta_1, \zeta_2, \tilde{\eta}, D) \) satisfies the equation

\[ (i\tilde{\eta} 1 + \mathcal{J}G) \left( G + \frac{1}{i\tilde{\eta} 1 + Z(\zeta_1, \zeta_2) + \mathcal{J}G} \right) + D = 0. \] (4.18)

Here, \( D \) is a neighbourhood of \( (\zeta, \bar{\zeta}, \eta) \) in \( \mathbb{C}^3 \), \( B_1 \) a neighbourhood of \( 0 \) in \( \mathbb{C}^{n \times n} \cap \mathbb{E}_+^+ \) and \( B_2 \) a neighbourhood of \( M = M(\zeta, \eta) \) in \( \mathbb{C}^{n \times n} \cap \mathbb{E}_-^+ \). For \( D, B_1 \) and \( B_2 \) we have the following choices. Either

\[ D := (\zeta, \bar{\zeta}, \eta) + (\mathbb{D}_{c_1})^3, \quad B_1 := B_{c_1}^\| \|0\| \cap \mathbb{E}_+^+, \quad B_2 := B_{c_2}^\| \|M\| \cap \mathbb{E}_-^+, \] (4.19)

with \( c_1, c_2 > 0 \) constants, depending only on the model parameters and on \( \delta \), or

\[ D := (\zeta, \bar{\zeta}, \eta) + (\mathbb{D}_{n^{-4/K}})^3, \quad B_1 := B_{n^{-4/K}}^\ast(0) \cap \mathbb{E}_+^+, \quad B_2 := B_{n^{-1/K}}^\ast(M) \cap \mathbb{E}_-^+ \]

for sufficiently large \( n \) (depending on model parameters, \( \delta \) and \( K \)). Here the superscripts indicate with respect to which norm the ball \( B_r(A) \) of radius \( r \) around \( A \) is meant. The function \( G \) is analytic in all variables.

**Proof.** We solve the implicit equation

\[ \mathcal{J}_{\zeta_1, \zeta_2, \tilde{\eta}}[G] + D = 0 \]

for \( G = G(\zeta_1, \zeta_2, \tilde{\eta}, D) \), where

\[ \mathcal{J}_{\zeta_1, \zeta_2, \tilde{\eta}}[G] := (i\tilde{\eta} 1 + \mathcal{J}G) \left( G + \frac{1}{i\tilde{\eta} 1 + Z(\zeta_1, \zeta_2) + \mathcal{J}G} \right). \] (4.20)

Note that \( \mathcal{J}_{\zeta, \bar{\zeta}, \eta}[M] = 0 \) due to (3.3). We will show that \( \mathcal{J} \) is a well-defined bounded holomorphic function on \( D \times B_2 \) with values in \( \mathbb{E}_-^+ \). In particular, we will see that
where the constants hidden in the comparison relation may depend on $\delta$ and $K$ in addition to the model parameters. We will keep this convention until the end of this proof. The theorem then follows from the implicit function theorem, Lemma C.1, and the following bound on the inverse of the derivative $\nabla \mathcal{J}_{\zeta, \zeta_1}$ evaluated at $G = M$:

$$
\|(\nabla|_{G=M} \mathcal{J}_{\zeta, \zeta_1})^{-1}\|_{\bf E} \lesssim 1, \quad \|(\nabla|_{G=M} \mathcal{J}_{\zeta, \zeta_1})^{-1}\|_{\Phi} \lesssim n^{1/K}.
$$

(4.22)

Note that the inverse of the derivative in (4.22) is restricted to the hyperplane $E^\perp$ and the $\|\cdot\|_*$-norm on $E^\perp$ is simply the restriction of the $\|\cdot\|_*$-norm from (4.16) on $\mathbb{C}^{2n \times 2n}$.

To see that $\mathcal{J}_{\zeta, \zeta_1}$ leaves the hyperplane $E^\perp$ invariant we compute

$$
\langle E_-, \mathcal{J}_{\zeta, \zeta_1}[G] \rangle = \left( E_-(i\eta_1 + \mathcal{J} G) \frac{1}{i\eta_1 + Z(\zeta_1, \zeta_2) + \Sigma G} \right) = 0.
$$

(4.23)

Here we used $\langle E_-(\mathcal{J} R)R \rangle = 0$ for any $R \in \mathbb{C}^{2n \times 2n}$ and $G \in E^\perp$ in the first identity and the general fact that by the Schur complement formula

$$
\text{Tr} \left( \begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array} \right)^{-1} \left( \begin{array}{cc} R_{11} & 0 \\ 0 & -R_{22} \end{array} \right) = 0,
$$

for any invertible $2 \times 2$-block matrix with square blocks in the second identity.

In the remainder of the proof we verify (4.21) and (4.22) and thus the assumptions of Lemma C.1. In the following we will frequently use the bounds $M^* \sim 1$ and $\text{Im} \ M \sim -\text{Im} \ M^{-1} \sim 1$ that are a consequence of Proposition 4.1 and (3.7).

The inequality (4.21a) is immediate when $c_1$ is chosen small enough and we apply $\|\cdot\| \lesssim 1$, $\|\Sigma\| \lesssim 1$, as well as the fact that the singular values of $i\eta_1 + Z + \Sigma M = -M^{-1}$ are bounded from above and below. For (4.21b) we in addition employ the bounds from (4.17) with $A = B = \Delta$ and $R = \Sigma M$ as well as $\|A\|_* \leq 2\|A\|$. We leave the details to the reader.

The remaining part of the proof is dedicated to showing (4.22). Differentiating (4.20) with respect to $G$ reveals that the derivative of $\mathcal{J} = \mathcal{J}_{\zeta, \zeta_1}$ evaluated at $G = M$ satisfies

$$
\nabla|_{G=M} \mathcal{J} = -iM\mathcal{L}, \quad \mathcal{M} R := \left( \text{Im} \frac{1}{M} \right) R, \quad \mathcal{L} R := R - M(\Sigma R)M,
$$

(4.24)

where we used $i\eta_1 + \mathcal{J}[M] = i(\eta_1 + \mathcal{J}[\text{Im} M]) = -i\text{Im} (M^{-1})$ (cf. (3.4), (3.3) and (3.7)). By the second bound in (4.17) and because $-\text{Im} (M^{-1}) \sim 1$ by (4.4) we have

$$
\|\mathcal{J}_{\zeta, \zeta_1} - \mathcal{J}[M]\| \lesssim \|\mathcal{J}[M]\| + \|\nabla|_{G=M} \mathcal{J}\| \lesssim \|\mathcal{J}[M]\| + \|\nabla|_{G=M} \mathcal{J}\|.
$$

(4.25)

(4.26)
\[
\|\mathcal{M}^{-1}[R]\|_* \lesssim n^{1/2K}\|R\|_*,
\]
and also \(\|\mathcal{M}^{-1}\| \lesssim \|\text{Im}(M^{-1})^{-1}\| \lesssim 1\). Due to (4.24), in order to show (4.22), it therefore suffices to establish bounds on the inverse of the stability operator \(\mathcal{L}\), namely
\[
\|\mathcal{L}^{-1}\|_{\mathcal{M}^{-1}E^\perp_-} \lesssim 1, \quad \|\mathcal{L}^{-1}\|_{\mathcal{M}^{-1}E^\perp_+} \lesssim n^{1/2K},
\]
where the inverse is understood to be restricted to \(\mathcal{M}^{-1}E^\perp_\pm\) and the \(\|\cdot\|_*\)-norm on the hyperplane \(\mathcal{M}^{-1}E^\perp_\pm\) is simply the restriction of the \(\|\cdot\|_*\)-norm on \(\mathbb{C}^{2n \times 2n}\). The bounds (4.25) are a consequence of the following three lemmas.

**Lemma 4.4 (Resolvent control for \(\mathcal{L}\)).** Let \(\mathcal{L}\) be defined as in (4.24). For any sufficiently small \(\delta > 0\) (depending on model parameters) there is a constant \(\varepsilon \sim \delta\) 1 such that uniformly in \(\eta \in (0, \delta^3)\) and \(\zeta \in \mathbb{C}\) with \(|\zeta|^2 \leq 1 - \delta\) we have the resolvent bound
\[
\sup\left\{\|(\mathcal{L} - \xi)^{-1}\|_{\text{hs}} : \xi \in \mathbb{C}, \xi \notin (2 + D_\varepsilon) \cup (1 + D_{1-\varepsilon}) \cup D_\varepsilon\right\} \lesssim_\delta 1. \tag{4.26}
\]
Furthermore, the \(\varepsilon\)-ball around zero contains a single isolated eigenvalue \(\hat{\lambda} \neq 0\) of \(\mathcal{L}\), i.e.
\[
D_\varepsilon \cap \text{Spec}(\mathcal{L}) = \{\hat{\lambda}\}, \quad |\hat{\lambda}| \lesssim_\delta \eta, \quad \dim \ker(\mathcal{L} - \hat{\lambda})^2 = 1. \tag{4.27}
\]
Approximate right and left eigenvectors corresponding to this isolated eigenvalue of \(\mathcal{L}\) are given by the identities
\[
\mathcal{L} E_- \text{ Im } M = O_{\|\cdot\|}(\eta), \tag{4.28a}
\]
\[
\mathcal{L}^* E_- \text{ Im}(M^{-1}) = -\eta E_-, \tag{4.28b}
\]
which are valid globally for \(\eta > 0\) and \(\zeta \in \mathbb{C}\).

**Lemma 4.5 (Smoothing lemma).** Let \(\mathbb{C}^d\) be equipped with two norms \(\|\cdot\|\#, \|\cdot\|_+\) and \(B \in \mathbb{C}^{d \times d}\) with
\[
\|B\|_\# + \|B\|_\#\rightarrow + \|B\|_+\rightarrow \# \leq C,
\]
for some constant \(C > 0\). Then for \(\xi \notin \text{Spec}(B) \cup \{0\}\) we have
\[
\|(B - \xi)^{-1}\|_\# \leq \frac{1}{|\xi|} + \frac{C}{|\xi|^2}(1 + \|(B - \xi)^{-1}\|_+). \tag{4.29}
\]

**Lemma 4.6 (Twist lemma).** Let \(\mathbb{C}^d\) be equipped with a scalar product \(\langle \cdot, \cdot \rangle\) and a norm \(\|\cdot\|\#\) (not necessarily induced by the scalar product), \(\varepsilon \in (0, 1)\) and \(A \in \mathbb{C}^{d \times d}\) such that
\[ D_\varepsilon \cap \text{Spec } A = \{ \alpha \}. \] We assume that \( \alpha \) is a non-degenerate eigenvalue of \( A \) and \( Aa = \alpha a \) for some \( a \in \mathbb{C}^d \) with \( \|a\|_\# = 1 \). Let
\[
P := -\frac{1}{2\pi i} \oint_{\partial D_\varepsilon} \frac{d\zeta}{A - \zeta} = \langle p, \cdot \rangle a ,
\] (4.29)
with some \( p \in \mathbb{C}^d \) be the corresponding spectral projection and \( b \in \mathbb{C}^d \) a vector such that
\[
|\langle a, b \rangle| \geq 2\varepsilon , \quad |\langle b, w \rangle| \leq \|w\|_\# , \quad \forall w \in \mathbb{C}^d .
\] (4.30)
Suppose that \( A \) has a bounded inverse on the range of \( 1 - P \), i.e.
\[
\|Aw\|_\# \geq \|w\|_\# , \quad \forall w \perp p .
\] (4.31)
Then \( A \) has a bounded inverse when restricted to \( b^\perp \), namely
\[
\|Aw\|_\# \geq \varepsilon \|w\|_\# , \quad \forall w \perp b .
\] (4.32)
Lemma 4.4 is an important technical result that allows to apply analytic perturbation theory to the isolated eigenvalue \( \hat{\lambda} \) of the non-selfadjoint operator \( \mathcal{L} \). Its proof is given in Subsection 4.3 below. The proof of Lemma 4.5 is to simply take the \( \|\cdot\|_\#\)-norm in the identity
\[
\frac{1}{B - \xi} = -\frac{1}{\xi} - \frac{1}{\xi^2} B + \frac{1}{\xi^2} B \frac{1}{B - \xi} .
\]
The proof of Lemma 4.6 is postponed to Appendix C.
To show (4.25) we use that by Lemma 4.4 the spectral projection \( \mathcal{P}_\mathcal{L} \) corresponding to the isolated eigenvalue \( \hat{\lambda} \) of \( \mathcal{L} \) close to zero has rank one and thus the form
\[
\mathcal{P}_\mathcal{L} = -\lim_{\gamma \downarrow 0} \frac{1}{2\pi i} \oint_{\partial D_\gamma} \frac{d\xi}{\mathcal{L} - \lambda - \xi} = \frac{\langle \mathcal{L}_l, \cdot \rangle}{\langle \mathcal{L}_l, \mathcal{L}_r \rangle} \mathcal{L}_r ,
\]
where \( (\mathcal{L} - \hat{\lambda})\mathcal{L}_r = (\mathcal{L} - \hat{\lambda})^*\mathcal{L}_l = 0 \), i.e. \( \mathcal{L}_r \) and \( \mathcal{L}_l \) are the unique (up to normalisation) corresponding right and left eigenvectors of \( \mathcal{L} \), respectively.
Now we extend the resolvent control (4.26) from the \( \|\cdot\|_{\text{hs}} \)-norm to the norms \( \|\cdot\| \) and \( \|\cdot\|_* \), with the help of Lemma 4.5 applied to the choice \( B = \text{Id} - \mathcal{L} \). This is possible because
\[
\|B\|_\# \rightarrow \# + \|B\|_{\text{hs}} \rightarrow \# \|B\|_\# \rightarrow \text{hs} \leq \|M\|^2 \|\mathcal{F}\|_\# \rightarrow \|\cdot\| + \|M\|^4 \|\mathcal{F}\|_{\text{hs}} \rightarrow \|\cdot\| \|\mathcal{F}\|_\# \rightarrow \|\cdot\| ,
\]
where \( \# = * , \|\cdot\| \), and \( \|\mathcal{F}\|_{\text{hs}} \rightarrow \|\cdot\| \) + \( \|\mathcal{F}\|_\# \rightarrow \|\cdot\| \) \( \leq 1 \). In particular, we may use analytic perturbation theory in the \( \|\cdot\|_\#\)-norm and find
\[ \mathbf{L}_r = \mathbf{E}_- \text{Im} \mathbf{M} + O_{\| \cdot \|}(\eta), \quad \mathbf{L}_l = \mathbf{E}_- \text{Im}(\mathbf{M}^{-1}) + O_{\| \cdot \|}(\eta) \]  

(4.33)

according to (4.28). Applying Lemma 4.6 with the choices

\[ A = C \mathcal{L}, \quad a = \frac{\mathbf{L}_r}{\| \mathbf{L}_r \|}, \quad p = \frac{\| \mathbf{L}_r \| - \mathbf{L}_l}{\langle \mathbf{L}_l, \mathbf{L}_r \rangle}, \quad b = c \frac{\mathbf{E}_- \text{Im}(\mathbf{M}^{-1})}{\| \text{Im}(\mathbf{M}^{-1}) \|} \]

shows the invertibility of \( \mathcal{L} \) on \( \mathcal{A}^{-1} \mathcal{E}_- = (\mathbf{E}_- \text{Im}(\mathbf{M}^{-1}))^{\perp} \) in the \( \| \cdot \|_{\#} \)-norm. Here, the positive constants \( c \) and \( C \) are chosen sufficiently small and large, respectively, in order to ensure the assumptions (4.30) and (4.31) of Lemma 4.6. In case of the \( \# = \| \cdot \| \) we have \( c \sim C \sim 1 \) and in the \( \# = * \) case \( c \sim n^{-1/2K} \) and \( C \sim 1 \). The expansion (4.33) is used to ensure that indeed \( |(a, b)| \geq 2 \varepsilon \) as required in (4.30) and (4.31) follows from the resolvent control on \( \mathcal{L} \) in \#-norm. \( \Box \)

**Corollary 4.7 (Perturbations).** Let \( \eta > 0 \) and \( \zeta \in \mathbb{C} \) with \( \eta + |\zeta| - 1 | \geq \delta \) for some fixed \( \delta > 0 \). For any \( \mathbf{D} \in \mathbb{C}^{2n \times 2n} \) and \( \mathbf{G} \in \mathcal{E}_- \) such that \( \| \mathbf{G} - \mathbf{M} \|_* + \| \mathbf{D} \|_* \leq n^{-7/2K} \) (respectively \( \| \mathbf{G} - \mathbf{M} \| + \| \mathbf{D} \| \ll 1 \)) that satisfy the perturbed Dyson equation

\[ -1 = (i\eta \mathbf{1} + \mathbf{Z}(\zeta, \bar{\zeta}) + \mathcal{S}[\mathbf{G}]) \mathbf{G} - \mathbf{D}, \]

(4.34)

the matrix \( \mathbf{G} \) is close to \( \mathbf{M} = \mathbf{M}(\zeta, \eta) \) in the sense that for sufficiently large \( n \) we have

\[ \| \mathbf{G} - \mathbf{M} \|_* \leq \frac{n^{6/2K}}{1 + \eta} \| \mathbf{D} \|_* \quad \left( \text{respectively } \| \mathbf{G} - \mathbf{M} \| \lesssim \frac{1}{1 + \eta} \| \mathbf{D} \| \right). \]

(4.35)

We also introduce the Matrix Dyson equation with general spectral parameter given by

\[ -\mathbf{M}^{-1} = z \mathbf{1} + \mathbf{Z} + \mathcal{S} \mathbf{M} \]

(4.36)

with \( z \in \mathbb{H} := \{ w \in \mathbb{C} : \text{Im } w > 0 \} \) as well as \( \mathcal{S} \) and \( \mathbf{Z} = \mathbf{Z}(\zeta, \bar{\zeta}) \) from (3.4) with \( \zeta \in \mathbb{C} \). There is a unique solution \( \mathbf{M} = \mathbf{M}(\zeta, z) \) to (4.36) under the constraint \( \text{Im } \mathbf{M} \geq 0 \) [36]. Note that (4.36) is the counterpart of (3.3), where the special spectral parameter \( i\eta \in \mathbb{H} \) is replaced by a general \( z \in \mathbb{H} \). In particular, both solutions agree for \( z = i\eta \). To (4.36), we associate the self-consistent density of states \( \rho_\zeta \) of \( \mathbf{H}_\zeta \) defined as the unique probability measure on \( \mathbb{R} \) whose Stieltjes transform is given by

\[ \langle \mathbf{M}(\zeta, z) \rangle = \int_{\mathbb{R}} \frac{\rho_\zeta(d\omega)}{\omega - z} \]

(4.37)

for any \( z \in \mathbb{H} \).

The support of \( \rho_\zeta \) is called the self-consistent spectrum of \( \mathbf{H}_\zeta \). By Corollary A.1 below, \( \text{supp } \rho_\zeta \) is bounded away from zero for any \( \zeta \notin \mathbb{D}_{1+\delta} \) due to \( \textbf{A1} - \textbf{A3} \) and our normalisation (4.1).
Proof. We first consider the regime $\max\{\eta, |\varsigma| - 1\} \geq \delta$. As we will see this corresponds to the regime away from the self-consistent spectrum and can be covered by combining existing results. If $|\varsigma| \geq 1 + \delta$ then Corollary A.1 below implies that $\text{dist}(0, \text{supp } \rho_\varsigma) \gtrsim 1$. Therefore, $\text{dist}(i\eta, \text{supp } \rho_\varsigma) \gtrsim \delta$ in the regime under consideration (this estimate is trivial if $\eta \geq \delta$), i.e. this regime is away from the self-consistent spectrum $\text{supp } \rho_\varsigma$. In particular, we may apply Lemma A.3 below and [26, eq. (70c)] to [26, eq. (69)] and conclude that (4.35) holds if $\max\{\eta, |\varsigma| - 1\} \geq \delta$.

The remaining regime $1 - |\varsigma| \geq \delta$ and $\eta < \delta$ is treated using Theorem 4.3. In this case we rewrite (4.34) in the form

$$\mathcal{J}_G \zeta, \eta[G] + \tilde{D} = 0,$$

where $\mathcal{J} = \mathcal{J}_G \zeta, \eta$ is given in (4.20). We have seen in (4.23) that $\mathcal{J} : \mathbb{E}_1^\perp \to \mathbb{E}_1^\perp$ and thus $\mathcal{J}[G] \in \mathbb{E}_1^\perp$ by the assumption on $G$. In particular, (4.38) also implies $\tilde{D} \in \mathbb{E}_1^\perp$.

By Theorem 4.3 the claim (4.35) now follows from

$$\|\tilde{D}\|_* \lesssim n^{2/K}\|D\|_* \quad (\text{respectively } \|\tilde{D}\| \lesssim \|D\|),$$

because $G$ analytically depends on $\tilde{D}$ and thus $\|G(\tilde{D}) - M\|_* \lesssim n^{3/K}\|\tilde{D}\|_* (\text{respectively } \|G(\tilde{D}) - M\| \lesssim \|\tilde{D}\|)$.

To show (4.39) in case of the $\|\cdot\|_*$-norm we use the MDE (3.3) and a geometric series expansion to write $\tilde{D}$ in the form

$$\tilde{D} = -D - Z \frac{1}{M^{-1} - \mathcal{J}\Delta} D$$

$$= -D - Z \left( \sum_{k=0}^{K} M((\mathcal{J}\Delta)M)^k + \frac{1}{M^{-1} - \mathcal{J}\Delta} ((\mathcal{J}\Delta)M)^{K+1} \right) D,$$

where $\Delta = G - M$. Applying (4.17) we take the $\|\cdot\|_*$-norm on both sides and estimate

$$\|\tilde{D}\|_* \lesssim \|D\|_* + \sum_{k=0}^{K} n^{(k+2)/K}\|\Delta\|_s^k\|D\|_* + n^{1/2}\|\Delta\|_s^{K+1}\|D\|_*,$$

where for the last summand we used that $\|AB\|_* \lesssim n^{1/2}\|A\||\|B\|_*$ for any pair of matrices $A, B$. Owing to the assumption $\|\Delta\|_* \lesssim n^{-7/K}$, this verifies (4.39). \qed

4.3. Resolvent control on $\mathcal{L}$

In this subsection we prove Lemma 4.4 by considering a reduction $\mathcal{L}$ of $\mathcal{L}$ on the space of diagonal block matrices, or equivalently on $\mathbb{C}^{n \times n} \oplus \mathbb{C}^{n \times n}$. We introduce the short hand notation
\(C_A B := ABA, \quad C_A B := ABA, \quad K_A B := A^* BA,\)

as well as the average and scalar product on \(C^{n \times n} \oplus C^{n \times n}\) as

\[
\left\langle \begin{pmatrix} A \\ B \end{pmatrix} \rightangle := \frac{1}{2}(\langle A \rangle + \langle B \rangle), \quad \left\langle \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} \rightangle := \frac{1}{2}(\langle A_1, A_2 \rangle + \langle B_1, B_2 \rangle),
\]

(4.40)

for \(A, B, A_1, B_1, A_2, B_2 \in C^{n \times n}\). We will denote linear operators \(A\) on \(C^{n \times n} \oplus C^{n \times n}\) by the block notation

\[
\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} := \begin{pmatrix} A_{11} A + A_{12} B \\ A_{21} A + A_{22} B \end{pmatrix}.
\]

We split the stability operator \(\mathcal{L}\) into diagonal and off-diagonal contributions,

\[
\mathcal{L} = \mathcal{P}^* \mathcal{L} \mathcal{P} + \text{Id} - \mathcal{P}^* \mathcal{P} + \mathcal{Q}.
\]

(4.41)

Here we introduced the projection and embedding operators

\[
\mathcal{P} : C^{2n \times 2n} \to C^{n \times n} \oplus C^{n \times n}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \mapsto (A_{11}, A_{22}),
\]

(4.42)

and the offdiagonal contribution \(\mathcal{Q} : C^{2n \times 2n} \to C^{2n \times 2n}\) to the stability operator,

\[
\mathcal{Q} A := \begin{pmatrix} 0 & \iota \zeta (V_1 (S A_{22}) U + U (S^* A_{11}) V_2) \\ \iota \zeta (U^* (S A_{22}) V_1 + V_2 (S^* A_{11}) U^*) & 0 \end{pmatrix}.
\]

(4.45)
Similarly to (4.44) we also write

\[ V_\pm := \begin{pmatrix} V_1 \\ \pm V_2 \end{pmatrix}. \]

On the level of the reduced stability operator the result analogous to Lemma 4.4 is the following statement.

**Lemma 4.8** (Resolvent control for \( \mathcal{L} \)). For any sufficiently small \( \delta > 0 \) (depending on model parameters) there is a constant \( \varepsilon \approx \delta \) such that uniformly in \( \eta \in (0, \delta^3) \) and \( \zeta \in \mathbb{C} \) with \( |\zeta|^2 \leq 1 - \delta \) we have the resolvent bound

\[
\sup \left\{ \| (\mathcal{L} - \xi)^{-1} \|_{hs} : \xi \in \mathbb{C}, \xi \notin (2 + \mathbb{D}_\varepsilon) \cup (1 + \mathbb{D}_{1-\varepsilon}) \cup \mathbb{D}_\varepsilon \right\} \lesssim 1. \tag{4.46}
\]

Furthermore, the \( \varepsilon \)-ball around zero contains a single isolated eigenvalue \( \lambda \neq 0 \) of \( \mathcal{L} \), i.e.

\[
\mathbb{D}_\varepsilon \cap \text{Spec}(\mathcal{L}) = \{\lambda\}, \ |\lambda| \lesssim \delta \eta, \ \dim \ker(\mathcal{L} - \lambda)^2 = 1. \tag{4.47}
\]

Approximate right and left eigenvectors corresponding to this isolated eigenvalue of \( \mathcal{L} \) are given by the identities

\[
\mathcal{L} V_- = O\|\cdot\|(\eta), \tag{4.48a}
\]
\[
\mathcal{L}^*\begin{pmatrix} \eta + \mathcal{S}V_2 \\ -\eta - \mathcal{S}^*V_1 \end{pmatrix} = \eta E_-, \tag{4.48b}
\]

which are valid globally for \( \eta > 0 \) and \( \zeta \in \mathbb{C} \).

The proof of Lemma 4.8 requires some preparation. But first we will see how the lemma is used to establish Lemma 4.4.

**Proof of Lemma 4.4.** The identities (4.28) follow from (4.48) because the off-diagonal component \( \mathcal{Q} \) of \( \mathcal{L} \) from (4.45) almost vanishes on the approximate eigenvector. More precisely, \( \mathcal{Q}[E_- \text{ Im } \mathbf{M}] = O(\eta) \) and \( \mathcal{Q}^*[E_- \text{ Im } (\mathbf{M}^{-1})] = 0 \) due to the definition of \( U \) in (4.8).

For \( t \in [0, 1] \) we consider an interpolation \( \mathcal{L}_t := \mathcal{L} - t \mathcal{Q} \) that removes the off-diagonal contribution. With the help of (4.46) we now establish the lower bound

\[
\|(\mathcal{L}_t - \xi \text{Id})\mathbf{R}\|_{hs} = \|(\mathcal{L} - \xi)\mathcal{P}\mathbf{R}\|_{hs} + \|((1 - t)\mathcal{Q} + (1 - \xi)(\text{Id} - \mathcal{P}^* \mathcal{P}))\mathbf{R}\|_{hs}
\]
\[
\geq \|(\mathcal{L} - \xi)^{-1}\|_{hs} + \|1 - \xi\|\|\text{Id} - \mathcal{P}^* \mathcal{P}\|\mathbf{R}\|_{hs} - (1 - t)\|\mathcal{Q}\|_{hs}\|\mathcal{P}\|_{hs}\]
\[
\gtrsim_\delta \|\mathbf{R}\|_{hs}
\]
for any $R \in \mathbb{C}^{2n \times 2n}$ and $\xi$ in the domain where the resolvent is controlled, i.e. $\xi \notin (2 + \mathbb{D}_\varepsilon) \cup (1 + \mathbb{D}_{1-\varepsilon}) \cup \mathbb{D}_\varepsilon$. This finishes the proof of (4.26) with the choice $t = 0$. Furthermore, it shows that no eigenvalues can leave the complement of the domain where the resolvent is controlled along the continuous interpolation. We conclude that the non-degeneracy property (4.27) holds if it can be established for $\mathcal{L}_1 = \mathcal{P}^* \mathcal{L} \mathcal{P} + \text{Id} - \mathcal{P}^* \mathcal{P}$. But $\mathcal{L}_1$ leaves both, the space of diagonal and of off-diagonal block matrices, invariant and acts as $\mathcal{L}$ on the first and as the identity on the latter. Thus (4.27) follows from (4.47).

Finally, the fact that $\hat{\lambda} \neq 0$ follows e.g. from the general result on the weak, i.e. $\eta$-dependent, stability of the Dyson equation from Lemma A.3. □

To prepare the proof of Lemma 4.8 we introduce some auxiliary operators. The purpose of these operators is to allow for a rewriting of the non-Hermitian reduced stability operator $\mathcal{L}$ in terms of Hermitian operators for which spectral information can be turned into norm bounds.

**Definition 4.9.** For any $\eta > 0$ and $\tau \geq 0$ we define the $n \times n$-matrices

$$P := \frac{1}{\sqrt{V_1}} U \frac{1}{\sqrt{V_2}}, \quad K_1 := (1 + \tau P^* P)^{-1/4}, \quad K_2 := (1 + \tau PP^*)^{-1/4},$$

in terms of the solution $V_i = V_i(\tau, \eta)$ to (3.5a) and $U = U(\tau, \eta)$ from (3.6). Furthermore, we define the linear operators $T_{\tau, \eta}, F_{\tau, \eta}, V_{\tau, \eta} : \mathbb{C}^{n \times n} \oplus \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n} \oplus \mathbb{C}^{n \times n}$ through

$$T_{\tau, \eta} := \begin{pmatrix} -C_{K_2}^2 & \tau C_{K_2} K_P C_{K_1} \\ \tau C_{K_1} K_P C_{K_2} & -C_{K_1}^2 \end{pmatrix},$$

as well as

$$F_{\tau, \eta} := \begin{pmatrix} 0 & C_{K_2}^{-1} C_{\sqrt{V_2}} S C_{\sqrt{V_1}} C_{K_2}^1 \\ C_{K_1}^{-1} C_{\sqrt{V_2}} S C_{\sqrt{V_1}} C_{K_2}^1 & 0 \end{pmatrix},$$

and

$$V_{\tau, \eta} := \begin{pmatrix} C_{K_2} C_{\sqrt{V_1}} & 0 \\ 0 & C_{K_1} C_{\sqrt{V_2}} \end{pmatrix}.$$

The matrices from Definition 4.9 allow to rewrite $\mathcal{L}$ through the formula

$$\mathcal{L} = V^{-1}(1 - T F)V.$$  

The following three lemmas list important analytical properties of the operators from Definition 4.9.
Lemma 4.10 (Properties of $P$, $K_1$, $K_2$ and $V$). Fix $\eta > 0$ and $\tau \geq 0$. The matrices $K_1$, $K_2$ and $P$ defined in (4.49) satisfy the identities
\[
K_2^4 = \sqrt{V_1} (\eta + S[V_2]) \sqrt{V_1}, \quad K_1^4 = \sqrt{V_2} (\eta + S^*[V_1]) \sqrt{V_2},
\] (4.54)
as well as the comparison relations
\[
PP^* \sim P^* P \sim \frac{1}{\eta^2 + \rho^2}, \quad K_1^4 \sim K_2^4 \sim (1 + \tau + \eta^2) \rho^2.
\] (4.55)

The operator $V$ from (4.52) is invertible and satisfies
\[
\|V\|_{\text{hs}} \|V^{-1}\|_{\text{hs}} \sim 1.
\] (4.56)

Proof. The identities (4.54) follow from
\[
\frac{1}{\sqrt{V_1}} \frac{1}{1 + \tau PP^*} \frac{1}{\sqrt{V_1}} = \frac{1}{V_1 + \tau UV_2^{-1}U^*} = \eta + S[V_2],
\]
\[
\frac{1}{\sqrt{V_2}} \frac{1}{1 + \tau P^*P} \frac{1}{\sqrt{V_2}} = \frac{1}{V_2 + \tau U^*V_1^{-1}U} = \eta + S^*[V_1],
\] (4.57)
which is easily checked by inserting the definition of $P$ from (4.49) and using (4.8) as well as the Dyson equation (3.5) for $V_1$ and $V_2$. In particular, (4.57) implies the third and fourth relation in (4.55) by the comparison relation for $V_1$ and $V_2$ from (4.4). From these comparison relations for $1 + \tau PP^*$ and $1 + \tau P^*P$ as well as (4.4) the bound (4.56) follows. The first two relations in (4.55) are immediate consequences of the definition of $P$ in (4.49), the identity (4.8) and (4.4). \(\Box\)

Lemma 4.11 (Properties of $\mathcal{F}$). The operator $\mathcal{F}$ defined in (4.51) satisfies the following properties uniformly in $\eta > 0$ and $\tau \geq 0$:

1. It is self-adjoint with respect to the scalar product (4.40) and positivity preserving, i.e.
\[
\mathcal{F}^* = \mathcal{F}, \quad \mathcal{F}[\mathcal{C}_+ \oplus \mathcal{C}_+] \subseteq \mathcal{C}_+ \oplus \mathcal{C}_+,
\] (4.58)
where $\mathcal{C}_+$ denotes the cone of positive definite matrices and $\overline{\mathcal{C}}_+$ its closure.

2. It has a positive spectral radius
\[
\|\mathcal{F}\|_{\text{hs}} \sim \frac{1}{1 + \tau + \eta^2},
\]
and $\pm \|\mathcal{F}\|_{\text{hs}}$ are non-degenerate eigenvalues of $\mathcal{F}$ with unique corresponding eigenvectors $F_{\pm}$ of the form
for some normalized ($\|F_1\|_{hs} = \|F_2\|_{hs} = 1$) matrices $F_1, F_2 \in C_\perp$. Both these matrices are comparable to the identity matrix

$$F_1 \sim 1, \quad F_2 \sim 1.$$  \hfill (4.59)

3. The spectral gap of $F$ is bounded away from zero, i.e. there exists $\varepsilon \sim 1$ such that

$$\operatorname{Spec}(F/\|F\|_{hs}) \subseteq \{-1\} \cup [-1 + \varepsilon, 1 - \varepsilon] \cup \{1\}. \quad (4.60)$$

4. The spectral radius of $F$ is given by the formula

$$1 - \|F\|_{hs} = \frac{\langle F_1, C_{K_2}^{-1} V_1 \rangle + \langle F_2, C_{K_1}^{-1} V_2 \rangle}{2 \langle F_+, V_+ \rangle} \eta \sim \frac{1}{1 + \tau + \eta^2 \rho}. \quad (4.61)$$

5. The eigenvectors $F_\pm$ satisfy

$$F_\pm = \frac{\hat{V}[\hat{V}_\pm]}{\|\hat{V}[\hat{V}_\pm]\|_{hs} + \mathcal{O}_{hs}\left(\frac{1}{1 + \tau + \eta^2 \rho}\right)}.$$  \hfill (4.62)

**Proof.** The self-adjointness of $F$ is clear from its definition (4.51) and the property of being positivity preserving is inherited from the same properties of $S$ (cf. (2.1)). Thus (4.58) holds true.

Properties 2 and 3 now follow from the structure

$$F = \begin{pmatrix} 0 & \hat{F} \\ \hat{F}^* & 0 \end{pmatrix}, \quad \hat{F} = C_{K_2}^{-1} C_{\sqrt{\nu_1}} SC_{\sqrt{\nu_2}} C_{K_1}^{-1},$$

given in (4.51). Thus the spectrum of $F$ is determined by the spectrum of $\hat{F}^* \hat{F}$ through

$$\operatorname{Spec}(F) = \operatorname{Spec}(- (\hat{F}^* \hat{F})^{1/2}) \cup \operatorname{Spec}((\hat{F}^* \hat{F})^{1/2}).$$

Because of (4.55) and $V_1 \sim V_2 \sim \rho$ (cf. (4.4)) the operators $\hat{F}^* \hat{F}$ and $\hat{F} \hat{F}^*$ inherit the flatness property (4.2) from $S$, i.e.,

$$\hat{F}^* \hat{F} A \sim \frac{1}{1 + \tau^2 + \eta^4} \langle A \rangle, \quad \hat{F} \hat{F}^* A \sim \frac{1}{1 + \tau^2 + \eta^4} \langle A \rangle, \quad \forall A \in \mathcal{C}_+.$$  

Thus we can apply [3, Lemma 4.8] to infer

$$\operatorname{Spec}((\hat{F}^* \hat{F})^{1/2}/\|F\|_{hs}) = \operatorname{Spec}((\hat{F} \hat{F}^*)^{1/2}/\|F\|_{hs}) \subseteq [-1 + \varepsilon, 1 - \varepsilon] \cup \{1\}. \quad (4.63)$$
where $\varepsilon \sim 1$ is a bound on the spectral gap and

$$\|\hat{F}^* F\|_{\text{hs}} = \|F\|_{\text{hs}}^2 \sim \frac{1}{1 + \tau^2 + \eta^2}.$$ 

According to the same lemma the eigenvalue 1 in (4.63) is non-degenerate with corresponding normalised eigenmatrices $F_1, F_2 \in \mathcal{C}_+$ that satisfy (4.59). In particular,

$$\hat{F}^* F_1 = \|F\|_{\text{hs}}^2 F_1, \quad \hat{F}^* F_2 = \|F\|_{\text{hs}}^2 F_2.$$ 

Therefore, $F_\pm$ are eigenvectors of $F^2$ corresponding to $\|F\|_{\text{hs}}^2$ and, consequently, $FF_\pm = \pm \|F\|_{\text{hs}} F_\pm$.

It remains to verify Properties 4 and 5. For this purpose we will use that $V V_\pm$ are approximate eigenvectors,

$$F V V_\pm = \pm V V_\pm - \eta \left( \pm \frac{C_{K_2}^{-1} V_1}{C_{K_1}^{-1} V_2} \right). \quad (4.64)$$

Indeed, (4.64) follows from using the definition of $V$ in (4.52) to identify the first summand on the right hand side of

$$F V V_\pm = \left( \pm \frac{C_{K_2}^{-1} C_{\sqrt{\tau^2} S V_2}}{C_{K_1}^{-1} C_{\sqrt{\tau^2} S^* V_1}} \right) = \left( \pm \frac{K_2^2}{K_1^2} \right) - \eta \left( \pm \frac{C_{K_2}^{-1} V_1}{C_{K_1}^{-1} V_2} \right), \quad (4.65)$$

as $\pm V V_\pm$. In (4.65) we used the definition of $V$ and $F$ for the first equality and the identities (4.54) for the second equality.

For (4.61) we choose the + in (4.64), take the scalar product with $F_+$ and use that $F$ is self-adjoint to obtain

$$\|F\|_{\text{hs}} \langle F_+, V V_+ \rangle = \langle F_+, V V_+ \rangle - \frac{\eta}{2} \left( \langle F_1, C_{K_2}^{-1} V_1 \rangle + \langle F_2, C_{K_1}^{-1} V_2 \rangle \right).$$

To establish (4.62) we apply Lemma C.3 for $S$ replaced by $\hat{F}^* F/\|F\|_{\text{hs}}^2$ and $\hat{F}^* F/\|F\|_{\text{hs}}^2$, i.e. for the diagonal entries of $(F/\|F\|_{\text{hs}}^2)^2$. Due to (4.64) the projections of $V V_+$ to the first and second component provide approximate eigenvectors for these two operators. The resolvent control from Lemma C.3 allows us to use analytic perturbation theory and the size of the error term in (4.62) is a consequence of (4.55), (4.4) and the definition of $V$ in (4.52). This finishes the proof of the lemma. \qed

**Lemma 4.12** (Spectral properties of $\mathcal{T}$). The operator $\mathcal{T}$ defined in (4.50) satisfies the following properties uniformly for $\eta > 0$ and $\tau \geq 0$:

1. It is self-adjoint, $\mathcal{T}^* = \mathcal{T}$.
2. Let \( P = \sum_{i=1}^{n} \pi_i p_i q_i^* \) with \( \pi_i \geq 0 \) and orthonormal bases \((p_i)_i\) and \((q_i)_i\) of \( \mathbb{C}^n \) be the singular value decomposition of \( P \). The eigenvectors of \( T \) are

\[
T \left[ \begin{pmatrix} p_i p_j^* \\ \pm q_i q_j^* \end{pmatrix} \right] = \frac{-1 \pm \tau \pi_i \pi_j}{\sqrt{(1 + \tau \pi_i^2)(1 + \tau \pi_j^2)}} \begin{pmatrix} p_i p_j^* \\ \pm q_i q_j^* \end{pmatrix}.
\] (4.66)

In particular, the spectrum of \( T \) is bounded away from 1 by some \( \varepsilon_1 > 0 \) satisfying

\[
\text{Spec}(T) \subseteq [-1, 1 - \varepsilon_1], \quad \varepsilon_1 \sim (1 + \tau + \eta^2)\rho^2.
\] (4.67)

3. An eigenvector of \( T \) corresponding to the eigenvalue \(-1\) is given by

\[
TV_- = -VV_-.\] (4.68)

4. On \( VV_+ \) the operator \( T \) acts contracting, i.e. there is an \( \varepsilon_2 > 0 \) such that

\[
\|TVV_+\|_{hs} \leq (1 - \varepsilon_2)\|VV_+\|_{hs}, \quad \varepsilon_2 \sim \frac{\tau}{1 + \tau + \eta^2}.
\] (4.69)

**Proof.** The self-adjointness of \( T \) follows immediately from its definition in (4.50). The form of the eigenvectors in (4.66) is a consequence of the following general fact. Let \( A = \sum_{i=1}^{n} \alpha_i a_i x_i^* \) and \( B = \sum_{i=1}^{n} \beta_i b_i y_i^* \) be singular value decompositions of matrices \( A \) and \( B \) and \( AR := ARB \) the operator that multiplies a matrix \( R \) from the left by \( A \) and from the right by \( B \). Then \( A[x_i b_k^*] = \alpha_i \beta_k a_i y_k^* \). In particular,

\[
\mathcal{C}_{f(PP^*)}[p_i p_j^*] = f(\pi_i^2) f(\pi_j^2) p_i p_j^* , \quad \mathcal{C}_{f(P^*P)}[q_i q_j^*] = f(\pi_i^2) f(\pi_j^2) q_i q_j^* ,
\]

\[
\mathcal{K}_P[p_i p_j^*] = \pi_i \pi_j q_i q_j^* , \quad \mathcal{K}_P[q_i q_j^*] = \pi_i \pi_j p_i p_j^* ,
\]

for any function \( f \) that is continuous on the positive reals. With these formulas (4.66) is easily verified using the definition of \( T \). The bound (4.67) on the spectrum of \( T \) now follows from (4.66) and (4.55).

For (4.68) and (4.69) we use the identities

\[
VV_\pm = \begin{pmatrix} (1 + \tau PP^*)^{-1/2} \\ (1 + \tau P^*P)^{-1/2} \end{pmatrix}, \quad TVV_\pm = \begin{pmatrix} \frac{-1 \pm \tau PP^*}{(1 + \tau PP^*)^{1/2}} \\ \frac{-1 \pm \tau P^*P}{(1 + \tau P^*P)^{1/2}} \end{pmatrix},
\]

that follow from the definitions of \( T \) and \( V \) in (4.50) and (4.52), respectively. To show (4.69) we also use (4.55) and (4.4). \( \Box \)

**Proof of Lemma 4.8.** We start by verifying (4.48). Indeed, owing to the representation of \( M \) in (3.7) we have
\[ \mathcal{P} \mathcal{M} \mathcal{P}^* V_\gamma = \left( V_1 S[V_2] V_1 + |\xi|^2 U S^* [V_1] U^* - V_2 S^* [V_1] V_2 - |\xi|^2 U^* S[V_2] U \right) = V_- + \eta \left( -V_1^2 - |\xi|^2 U U^* \right), \]

(4.70)

where we used the identities (4.12) for the second equality. By using the comparison relations (4.4) and (4.11) to bound the last summand on the right hand side of (4.70) we conclude (4.48a). The identity (4.48b) is verified by using the definition of \( M \) and (4.12) again.

Now we turn to the proof of the resolvent bound (4.46) for the reduced stability operator \( \mathcal{L} \). We rewrite this operator using (4.53) and apply this representation to the resolvent of \( \mathcal{P} \mathcal{M} \mathcal{P}^* \) to get

\[ \frac{1}{\mathcal{P} \mathcal{M} \mathcal{P}^* - \xi} = \mathcal{V}^{-1} \frac{1}{\mathcal{T} \mathcal{F} - \xi} \mathcal{V}. \]

(4.71)

For \( \eta \leq \delta^3 \) and \( \tau = |\xi|^2 \leq \delta \) we use \( \mathcal{T} = -1 + \mathcal{O}_{\text{hs}}(\tau) = -1 + \mathcal{O}_{\text{hs}}(\delta) \) which follows from the definition of \( \mathcal{T} \) in (4.50) and (4.55) as well as \( V_1 \sim V_2 \sim 1 \) in this regime (cf. (4.4)). From (4.56), the spectral properties of \( \mathcal{F} \), (4.60), and \( \|\mathcal{F}\|_{\text{hs}} = 1 + \mathcal{O}(\eta) = 1 + \mathcal{O}(\delta^3) \) (cf. (4.61)), as well as (4.71) we infer that there is an \( \varepsilon \in (0, 1/2) \) such that

\[
\sup \left\{ \| (\mathcal{P} \mathcal{M} \mathcal{P}^* - \xi)^{-1} \|_{\text{hs}} : \xi \in \mathbb{C}, \xi \notin (-1 + \mathbb{D}_\varepsilon) \cup \mathbb{D}_{1-2\varepsilon} \cup (1 + \mathbb{D}_\varepsilon) \right\} \lesssim \delta, 1,
\]

and that \( \varepsilon \sim \delta \). In particular, (4.46) holds true. The non-degeneracy (4.47) of the eigenvalue in \( \mathbb{D}_\varepsilon \) follows from the non-degeneracy of the eigenvalue \( \|\mathcal{F}\|_{\text{hs}} \) of \( \mathcal{F} \) as stated in Lemma 4.11. The statement \( \lambda = \mathcal{O}_\delta(\eta) \) about the non-degenerate isolated eigenvalue in (4.47) follows from \( V_- \) being an approximate eigenvector (cf. (4.48a)) and the resolvent bound (4.46).

For \( \eta \leq \delta^3 \) and \( \tau \in [\delta, 1-\delta] \) we will apply Lemma C.2 with the choices \( F := \mathcal{F}/\|\mathcal{F}\|_{\text{hs}} \) and \( T := \mathcal{T} \). We verify the assumptions of the lemma. The required upper bound \( \|T\|_{\text{hs}} \leq 1 \) follows from (4.67) and (C.10) holds true because of (4.60). Furthermore according to (4.62) and (4.69) we have

\[ \|T F_+\|_{\text{hs}} \leq 1 - \varepsilon_2 + \mathcal{O}(\eta/\rho) = 1 - \varepsilon_2 + \mathcal{O}(\delta^2), \quad \varepsilon_2 \sim \delta \geq \delta, \]

(4.72)

where \( F_+ \) is the normalized eigenvector of \( \mathcal{F} \) corresponding to the eigenvalue \( \|\mathcal{F}\|_{\text{hs}} \) and we used (4.4) to see the bounds in terms of \( \delta \). We also have

\[ \|(1 + T) F_-\|_{\text{hs}} \lesssim \frac{\eta}{\rho} \lesssim \delta^{5/2}, \]

(4.73)

by (4.62) and (4.68), where \( F_- \) is the normalised eigenvector of \( \mathcal{F} \) corresponding to the eigenvalue \(-\|\mathcal{F}\|_{\text{hs}} \). Thus Lemma C.2 is applicable because of (4.72) and (4.73) as long as \( \delta \sim 1 \) is chosen sufficiently small. Thus we find
\[
\sup \{ \|(T_\varepsilon F - |F|_{hs})^{-1}\|_{hs} : \zeta \in \mathbb{C}, \ z \notin \mathbb{D}_{1-2\varepsilon} \cup (1 + \mathbb{D}_\delta) \} \leq 1,
\]
for some \(\varepsilon \sim \delta^{5/2}\). Since \(|F|_{hs} = 1 + \mathcal{O}(\eta/\rho) = 1 + \mathcal{O}(\delta^2)\) (cf. (4.61) and (4.4)) we infer (4.46) from (4.74) by using (4.71) and (4.56). The non-degeneracy of the isolated eigenvalue \(\lambda\) in (4.47) stems from (C.12) and the resolvent bound (4.46) in combination with the approximate eigenvector equation (4.48a) for \(V_\gamma \) implies \(\lambda = \mathcal{O}(\eta)\).

Finally, note that \(\lambda \neq 0\) because the representation (4.53) shows that with \(|\mathcal{T}| \leq 1\) (cf. (4.67)) the operator \(L\) is invertible as long as \(|F|_{hs} < 1\), which is always true for \(\eta > 0\) due to the right hand side of (4.61) not vanishing. \(\square\)

**Proof of Proposition 2.4.** For \(D = 0\) the equation (4.18) is equivalent to (3.3) and thus by Theorem 4.3 for any \(|z|^2 = \tau \in [\rho, g]\) we can extend the solution \(M(z, \eta)\) analytically to \(\eta = 0\). Thus also the solution \(V_1, V_2\) of the Dyson equation (3.5) can be analytically extended to \(\eta = 0\). This proves the existence of a positive definite solution to (2.9).

For the uniqueness, note that in the proof of Theorem 4.3 and in particular for the key input, Lemma 4.4, we never used \(\eta > 0\), but only that \(M \in \mathbb{E}^\perp\) solves (3.3) and has positive definite imaginary part with lower and upper bounds depending on model parameters and \(\delta\). Thus for any positive definite solution \(V_1, V_2\) that satisfies (2.9) and (2.10) we can construct a solution \(M\) of (3.3) at \(\eta = 0\) through (3.7) and Theorem 4.3 also applies to this \(M = M(\sqrt{\tau}, 0)\) with \(c_1, c_2\) from (4.19) now depending also on the lower and upper bounds on \(V_1, V_2\). By analyticity of \(G\) in all variables \(G(z, \overline{z}, \eta, 0)\) has positive definite imaginary part for sufficiently small \(|z - \sqrt{\tau}|\) and \(\eta > 0\). We conclude \(G(z, \overline{z}, \eta, 0) = M(z, \eta)\) since it solves (4.18) with \(D = 0\) and \(M = \lim_{\eta \downarrow 0} M(\sqrt{\tau}, \eta)\), establishing uniqueness of the solution to (2.9). \(\square\)

As used in the proof of Proposition 2.4 above, the uniformity of the statement of Theorem 4.3 in \(\eta > 0\) allows for an extension of \(M\) as well as \(V_1, V_2\) to \(\eta = 0\) in the following sense.

**Corollary 4.13** (*Extension inside the spectrum*). The solution \(M\) of the MDE (3.3) has a unique continuous extension to \(\mathbb{C} \times [0, \infty)\), i.e. to \(\eta = 0\). For every \(z \notin \partial \mathbb{D}\) this extension, still denoted by \(M\), also has a continuation to a neighbourhood of \((\zeta, 0)\) that is real analytic in \(\text{Re} \zeta, \text{Im} \zeta, \eta\). The size of this neighbourhood only depends on the model parameters and on \(\text{dist}(\zeta, \partial \mathbb{D}) = |z| - 1|\).

Similarly \(V_1, V_2\) admit a continuous extension to \([0, \infty) \times [0, \infty)\) that extends to an analytic function in a neighbourhood of \((\tau, 0)\) for any \(\tau \in [0, \infty) \backslash \{1\}\) with the size of the neighbourhood depending only on \(|\tau - 1|\) in addition to the model parameters.

5. **Self-consistent density of states**

In this section we use the information about the solution of the Dyson equation to control the self-consistent density of states \(\sigma\) corresponding to \(X\). In Subsection 5.1 we
begin with establishing upper and lower bounds on the density. These bounds rely on a novel representation of \( \sigma \) in (5.1). In Subsection 5.2 we provide a detailed description of \( V_i \) and \( \sigma \) at the edge of the spectrum. We end the subsection by summarising its results in the proof of Theorem 2.5. Subsections 5.3 and 5.4 contain the proofs of Lemma 3.5 and Proposition 2.9, respectively.

5.1. Upper and lower bounds in the bulk

In this subsection we establish lower and upper bounds on the density \( \sigma \) inside the spectrum, i.e. we show (2.12) away from the edge of the spectrum at \( |\zeta| = 1 \).

**Lemma 5.1 (Formula for density).** For any \( \zeta \in \mathbb{D} \) the density \( \sigma \) admits the formula

\[
\sigma(\zeta) = \frac{2}{\pi} \left\langle L^{-1} \left( \begin{array}{c} V_1 \frac{1}{S^* V_1} V_1 \\ V_2 \frac{1}{S^* V_2} V_2 \end{array} \right), \left( \begin{array}{c} SV_2 \\ S^* V_1 \end{array} \right) \right\rangle = \frac{1}{\pi \tau} \langle Y, (1 - T F^2 T) Y \rangle,
\]

where all expressions on the right hand side are evaluated at \( \eta = 0 \) (cf. Corollary 4.13) and \( \tau = |\zeta|^2 \) and where

\[
Y := (1 - F T)^{-1} \left| V_{[\zeta]} \right|^\perp \left( \begin{array}{c} K_2^1 \\ K_1^2 \end{array} \right),
\]

and \( K_i \) the matrices from (4.49). For \( \tau = 0 \) the very right hand side of (5.1) is interpreted as its limit \( \tau \downarrow 0 \). Here and in the following, the notation \((1 - F T)^{-1} \left| V_{[\zeta]} \right|^\perp \) on the right-hand side of (5.2) is understood as \((1 - F T) \left| V_{[\zeta]} \right|^\perp)^{-1} \).

**Proof.** By definition of \( \sigma \) in (2.11) and the identity (4.9) we have

\[
\sigma(\zeta) = \frac{1}{\pi} \partial_\tau (\tau \langle U(\tau, 0) \rangle)|_{\tau = |\zeta|^2} = -\frac{1}{2\pi} \partial_\tau (\langle V_1, SV_2 \rangle + \langle V_2, S^* V_1 \rangle)|_{\tau = |\zeta|^2, \eta = 0},
\]

for any \( \zeta \in \mathbb{D} \). By rotational symmetry it suffices to establish (5.1) at \( \zeta = \sqrt{\tau} > 0 \). Thus we denote \( \sigma = \sigma(\sqrt{\tau}) \) and \( M = M(\sqrt{\tau}, 0) \). By (5.3), the representation of \( M \) from (3.7) and the definition of \( \mathcal{I} \) in (3.4) we find

\[
\sigma = -\frac{1}{\pi} \partial_\tau \langle M, \mathcal{I} M \rangle = -\frac{2}{\pi} \langle \mathcal{P} \partial_\tau M, \mathcal{P} \mathcal{I} M \rangle,
\]

where we used the structure of \( \mathcal{I} \) and the projection \( \mathcal{P} \) from (4.42) in the second equality. We compute the derivative of \( M \) with respect to \( \tau \) by differentiating both sides of (3.3) and solving for

\[
\partial_\tau M = (\text{Id} - \mathcal{E}_M \mathcal{I})^{-1} \mathcal{E}_M \partial_\tau (\sqrt{\tau}, \sqrt{\tau}).
\]
By definition of $M$ and the identities (4.8) for $U$ we have
\[
\mathcal{P}\mathcal{C}_{M}\partial_{\tau}Z(\sqrt{\tau}, \sqrt{\tau}) = -\frac{i}{2} \left( UV_{1} + V_{1}U^{*} \right) = -i \left( \frac{V_{1}}{V_{2}} \frac{1}{\sqrt{V_{2}}} \frac{V_{1}}{V_{2}} \right).
\]  
(5.6)

Thus, inserting (5.6) into (5.5) and recalling the definition of $L$ from (4.43), shows
\[
\mathcal{P}\partial_{\tau}M = -i L^{-1} \left( \frac{V_{1}}{V_{2}} \frac{1}{\sqrt{V_{2}}} \frac{V_{1}}{V_{2}} \right).
\]  
(5.7)

We plug this into (5.4) and verify the first equality in (5.1). Note that $L^{-1}$ is applied to the orthogonal complement of $(SV_{2}, S^{*}V_{1})$ in (5.7). To check the orthogonality of the vector on the right hand side we can use (4.8) at $\eta = 0$.

For the second equality in (5.1) we recall the definitions of $T$, $F$ and $V$ from (4.50), (4.51) and (4.52), as well as the identities (4.54) that take the form
\[
1 + \tau PP^{*} = K_{2}^{-4}, \quad 1 + \tau P^{*}P = K_{1}^{-4} \quad \text{with } P = K_{2}^{-4} \sqrt{V_{1}} \sqrt{V_{2}} = \sqrt{V_{1}} \sqrt{V_{2}} K_{1}^{-4},
\] at $\eta = 0$. Then we compute
\[
(V^{*})^{-1} \left( \begin{array}{c}
SV_{2} \\
S^{*}V_{1}
\end{array} \right) = \left( \begin{array}{c}
K_{2}^{2} \\
K_{1}^{2}
\end{array} \right),
\] 
(5.8)
\[
V \left( \begin{array}{c}
V_{1} \frac{1}{\sqrt{V_{1}}} \frac{V_{1}}{V_{2}} \\
V_{2} \frac{1}{\sqrt{V_{2}}} \frac{V_{2}}{V_{2}}
\end{array} \right) = \left( \begin{array}{c}
C_{K_{2}} k_{p} K_{1}^{4} \\
C_{K_{1}} k_{p} K_{2}^{4}
\end{array} \right) = \frac{1}{\tau} \left( \begin{array}{c}
C_{K_{2}} [1 - K_{2}^{4}] \\
C_{K_{1}} [1 - K_{1}^{4}]
\end{array} \right),
\] where we used (2.9) for the last equality. Again with (2.9) we also have
\[
(1 + T) \left( \begin{array}{c}
K_{2}^{2} \\
K_{1}^{2}
\end{array} \right) = 2 \left( \begin{array}{c}
C_{K_{2}} [1 - K_{2}^{4}] \\
C_{K_{1}} [1 - K_{1}^{4}]
\end{array} \right),
\]  
(5.9)

Now we insert the representation (4.53) for the reduced stability operator into the middle formula of (5.1). Afterwards we use (5.8) and (5.9) to get
\[
\sigma = \frac{2}{\pi \tau} \left( (1 - TF)^{-1} \left( \begin{array}{c}
C_{K_{2}} [1 - K_{2}^{4}] \\
C_{K_{1}} [1 - K_{1}^{4}]
\end{array} \right) , \left( \begin{array}{c}
K_{2}^{2} \\
K_{1}^{2}
\end{array} \right) \right) = \frac{1}{\pi \tau} \left( (1 - TF)^{-1} (1 + T) \left( \begin{array}{c}
K_{2}^{2} \\
K_{1}^{2}
\end{array} \right) , \left( \begin{array}{c}
K_{2}^{2} \\
K_{1}^{2}
\end{array} \right) \right),
\] where the inverse of $1 - TF$ is restricted to $V[V]^{\perp}$. The vector in the second argument of the scalar product is a representation of the Perron-Frobenius eigenvector for $F$. Indeed, by the definitions of $F$ in (4.51) and $K_{i}$ in (5.2) we see that
Thus \( F \left( \begin{pmatrix} K_2^2 \\ K_1^2 \end{pmatrix} \right) = \left( \begin{pmatrix} K_2^2 \\ K_1^2 \end{pmatrix} \right) \). \( (5.10) \)

Because of (5.10) we also have the identity

\[
(1 - TF)^{-1}(1 + T) \begin{pmatrix} K_2^2 \\ K_1^2 \end{pmatrix} = (1 - TF)^{-1}(1 - T^2 T)(1 - T T)^{-1} \begin{pmatrix} K_2^2 \\ K_1^2 \end{pmatrix},
\]

which finishes the proof of the second equality in (5.1) and, thus, the proof of the lemma. \( \square \)

**Corollary 5.2** (Bounds on the density). For any \( \delta \in (0, 1) \), we have \( \sigma(\zeta) \sim \delta 1 \) uniformly for \( \zeta \in \mathbb{D}_{1-\delta} \).

**Proof.** We consider two separate regimes. First upper and lower bounds on \( \sigma \) follow in a neighbourhood of \( \zeta = 0 \) by continuity (cf. Corollary 4.13 and (2.11)) of \( \sigma \) and \( \sigma(0) \geq 1 \). The latter is easy to see because at \( \tau = \eta = 0 \) the Dyson equation simplifies to

\[
1 = V_1 SV_2, \quad 1 = V_2 S^* V_1, \quad (5.11)
\]

and we have

\[
U = V_1 V_2, \quad \mathcal{L} = \begin{pmatrix} 1 & C_{V_1} S \\ C_{V_2} S^* & 1 \end{pmatrix}.
\]

In particular, the reduced stability has the form \( \mathcal{L} = 1 - \mathcal{A} \), where \( \mathcal{A} \) preserves the cone of positive definite matrix pairs. Thus the first identity in (5.1) implies

\[
\sigma(0) = \frac{2}{\pi} \left\langle \mathcal{L}^{-1} \left( \begin{pmatrix} V_1 & \frac{1}{V_2} V_1 \\ V_2 & \frac{1}{V_1} V_2 \end{pmatrix} \right), \left( \begin{pmatrix} 1 \\ V_1 \\ 1 \\ V_2 \end{pmatrix} \right) \right\rangle \geq \frac{1}{\pi} \left( \left\langle V_1 V_2^{-1} \right\rangle + \left\langle V_2 V_1^{-1} \right\rangle \right).
\]

Note that we can expand \( \mathcal{L}^{-1} = (1 - \mathcal{A})^{-1} \) in a Neumann series because of the representation (4.53), \( \|T\| \leq 1 \) and \( \|\mathcal{F}\|_{\text{hs}} < 1 \).

Now we consider the regime \( 1 \leq \tau^{1/2} = |\zeta| \leq 1 - \delta \). Here, owing to the second relation in (5.1), we have the lower and upper bound

\[
\sigma(\zeta) \sim \left( K, \frac{1}{1 - \mathcal{A}(1 - \mathcal{A}^*)} \frac{1}{1 - \mathcal{A}^*} K \right),
\]

where \( K = (K_2^2, K_1^2) \in (\mathcal{V} V_\perp) \) with \( K_i \sim 1 \) and \( \mathcal{A} = T \mathcal{F} \). Thus for \( \sigma \sim 1 \) it suffices to check that \( \|\mathcal{A}\|_{(\mathcal{V} V_\perp)\perp} \leq 1 - \varepsilon \) for some \( \varepsilon \geq 1 \). We apply Lemma C.2 with \( T = \mathcal{T} \), \( F = \mathcal{F}/\|\mathcal{F}\|_{\text{hs}} \) and \( f_\pm = \mathcal{V} V_\pm/\|\mathcal{V} V_\pm\|_{\text{hs}} \) and note that the non-degenerate eigenvalue 1 of \( \mathcal{A} \) corresponds to the eigenvector \( \mathcal{V} V_\perp \) which is projected out when we take the norm. Thus we have the resolvent bound
\[ \sup_{\omega \in \mathbb{D}_{1}^{-\varepsilon}} \left\| (\mathcal{A} - \omega)^{-1} (\nu_{V_{\varepsilon}}^{-}) \right\|_{\text{hs}} \lesssim 1, \]

for some \( \varepsilon \sim 1 \), which implies the desired norm bound. \( \square \)

### 5.2. Solution close to the edge

In this subsection we explicitly determine the leading order of the solution \( V_{1}, V_{2} \) to (3.5) close to the edge \( \tau = |\zeta|^2 = 1 \) of the spectrum. We use the result to determine the jump height (2.13) of the density \( \sigma \) at the edge. Let \( S_{2} \) and \( S_{1} \) be the unique positive definite right and left eigenvectors of \( S \), respectively, i.e. \( S_{2} S_{2} = S_{2} \) and \( S^{*} S_{1} = S_{1} \), satisfying \( \langle S_{1} \rangle = \langle S_{2} \rangle = 1 \). We also write \( \rho := \rho_{\zeta} := \langle V_{1} \rangle / \pi \) for the harmonic extension of the self-consistent density of states of \( H_{\zeta} \) to the complex upper half plane and recall that \( \rho \) is comparable to the right hand side of (4.4).

**Proposition 5.3** *(Solution at the edge).* For any \( \tau, \eta \in [0, 2] \) we have the expansion

\[
V_{1} = \alpha S_{1} + \mathcal{O}(\eta + \rho^{3}), \quad V_{2} = \alpha S_{2} + \mathcal{O}(\eta + \rho^{3}), \quad \alpha := \frac{\langle S_{1}, V_{2} \rangle}{\langle S_{1}, S_{2} \rangle},
\]

where \( \alpha \) satisfies the cubic equation

\[
\alpha^{3} \langle (S_{1} S_{2})^{2} \rangle + \alpha (\tau - 1) \langle S_{1} S_{2} \rangle - \eta = \mathcal{O}(\rho^{5} + \eta \rho^{2}).
\]

**Proof.** We write \( \tau = 1 + \varepsilon \) for some small \( \varepsilon \). The case when \( \varepsilon \leq -c \) for some constant \( c \sim 1 \) is trivial since then \( \rho \sim 1 \) and the error term in (5.13) dominates. Similarly, for \( \varepsilon \geq c \) we have \( \rho \sim \alpha \sim \eta \), i.e. in both regimes the proposition does not contain any information. Solving (4.10) shows

\[
U = \frac{1}{2(1 + \varepsilon)} \left( 1 + \sqrt{1 - 4(1 + \varepsilon)V_{1}V_{2}} \right) = \frac{1}{1 + \varepsilon} - V_{1}V_{2} - (1 + \varepsilon)(V_{1}V_{2})^{2} + \mathcal{O}(\rho^{6}).
\]

We use this expansion for \( U \) in (4.9) and find

\[
0 = V_{1}(\eta + SV_{2}) + (1 + \varepsilon)U - 1
= V_{1}(\eta + SV_{2}) - (1 + \varepsilon)V_{1}V_{2} - (1 + \varepsilon)^{2}(V_{1}V_{2})^{2} + \mathcal{O}(\rho^{6}).
\]

Multiplying with \( V_{1}^{-1} \) from the left and using the decomposition \( V_{i} = \alpha_{i} S_{i} + \tilde{V}_{i} \) shows

\[
(1 + \varepsilon - S)\tilde{V}_{2} = \eta - \varepsilon \alpha_{2} S_{2} - (1 + \varepsilon)^{2}V_{2}V_{1}V_{2} + \mathcal{O}(\rho^{5}).
\]

Here \( \tilde{V}_{1} \) and \( \tilde{V}_{2} \) are the spectral projection of \( V_{1} \) and \( V_{2} \) corresponding to the spectrum of \( S \) and \( S^{*} \) complementary to the isolated eigenvalue 1, respectively, i.e. \( \tilde{V}_{i} = Q_{i} V_{i} \), with
\[ P_1 = \frac{\langle S_2, \cdot \rangle}{\langle S_1 S_2 \rangle} S_1, \quad Q_1 = 1 - P_1, \quad P_2 = \frac{\langle S_1, \cdot \rangle}{\langle S_1 S_2 \rangle} S_2, \quad Q_2 = 1 - P_2. \]

In particular, projecting both sides of (5.15) onto the range of \( Q_2 \) implies \( \| \tilde{V}_2 \| \lesssim \eta + \rho^3 \). Here we used that \( \|(1 + \varepsilon - \mathcal{S})^{-1} Q_2 \| \lesssim 1 \), which follows from Lemma C.3. By exchanging the roles of \( V_1 \) and \( V_2 \) we also find \( \| \tilde{V}_1 \| \lesssim \eta + \rho^3 \). Therefore, (5.15) can be expanded further as

\[ (1 + \varepsilon - \mathcal{S}) \tilde{V}_2 = \eta - \varepsilon \alpha_2 S_2 - \alpha_1 \alpha_2^2 S_2 S_1 S_2 + \mathcal{O}(\varepsilon \rho^3 + \rho^5 + \eta \rho^2). \]

Now we apply the rank one projection \( P_2 \) on both sides and get

\[ 0 = \eta - \varepsilon \alpha_2 \langle S_1 S_2 \rangle - \alpha_1 \alpha_2^2 \langle (S_1 S_2)^2 \rangle + \mathcal{O}(\varepsilon \rho^3 + \rho^5 + \eta \rho^2), \]

where we used \( \langle S_1 \rangle = 1 \) and, for the error term, \( |\varepsilon| \rho^3 \lesssim \rho^5 + \eta \rho^2 \) due to (4.4). Finally (5.13) follows from

\[ \alpha_1 = \alpha_2 + \mathcal{O}(\eta + \rho^3), \quad (5.16) \]

which is a consequence of (4.3) and \( \alpha_2 = \alpha \). Moreover, (5.16) and \( \| \tilde{V}_i \| \lesssim \eta + \rho^3 \) yield (5.12). \( \square \)

For the next corollary, we introduce \( \mathcal{M} : \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \) defined by

\[ \mathcal{M} := \frac{1}{\langle V_+ \rangle} \begin{pmatrix} \eta + S V_2 & 0 \\ 0 & \eta + S^* V_1 \end{pmatrix}. \quad (5.17) \]

Moreover, we recall that \( \mathcal{L} \) and \( E_- \) were defined in (4.43) and (4.44), respectively.

**Corollary 5.4 (Resolvent control for stability operator close to the edge).** The following holds.

1. The operator \( \mathcal{M} \mathcal{L} \) has the invariant subspace \( E_+^\perp \subset \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \), i.e. \( \mathcal{M} \mathcal{L} E_+^\perp \subset E_+^\perp \).
2. There is \( \delta \sim 1 \) such that, for any \( \eta \in (0, \delta) \) and \( \zeta \in \mathbb{D}_{1+\delta} \setminus \mathbb{D}_{1-\delta} \), the eigenvalues of \( \mathcal{M} \mathcal{L} |_{E_+^\perp} \) close to zero are isolated in the sense that

\[ \sup \left\{ \| (\mathcal{M} \mathcal{L} - \xi)^{-1} |_{E_+^\perp} \|_\# : \xi \in \mathbb{D}_{2\varepsilon} \setminus \mathbb{D}_\varepsilon \right\} \lesssim 1, \quad (5.18) \]

for some \( \varepsilon \sim 1 \) and \( \# = \text{hs}, \| \cdot \| \). In fact, \( \mathcal{M} \mathcal{L} |_{E_+^\perp} \) has only one eigenvalue in \( \mathbb{D}_\varepsilon \). This eigenvalue is simple and the spectral projection

\[ \mathcal{P} := -\frac{1}{2\pi i} \oint_{\partial \mathbb{D}_\varepsilon} d\xi (\mathcal{M} \mathcal{L} - \xi)^{-1} |_{E_+^\perp} \]
has rank one.

**Proof.** The invariance of $E_+^\perp$ under $\mathcal{ML}$ is a direct consequence of $\mathcal{L}^* \mathcal{M}^* E_\perp = \eta \langle V_+ \rangle^{-1} E_-$. The operator $\mathcal{ML}|_{E_+^\perp}$ is a small perturbation of $\mathcal{K}$ from Corollary C.4, since

$$\mathcal{ML} = \mathcal{K} + \mathcal{O}(\eta/\rho + \rho + ||\zeta| - 1||)$$

due to (5.12) and (5.14). Thus the claim follows from Corollary C.4 for $\# = \text{hs, } ||\cdot||$ by perturbation theory for sufficiently small $\delta \sim 1$. \(\square\)

**Corollary 5.5 (Density at the edge).** At the edge of the spectrum the self-consistent density of states has an expansion

$$\sigma(\zeta) = \frac{(S_1 S_2)^2}{\pi \langle S_1 S_2 \rangle^2} + \mathcal{O}(1 - |\zeta|), \quad (5.19)$$

for any $\zeta \in \mathbb{D}$.

**Proof.** We set $\eta = 0$ throughout the proof. We use the first identity in (5.1) and insert $\mathcal{M}$ from (5.17) to find

$$\sigma = \frac{2}{\pi \langle V_+ \rangle} \langle (\mathcal{ML})^{-1} \left( \begin{array}{c} (SV_2) V_1 \frac{1}{S^* V_2} V_1 \\ (S^* V_1) V_2 \frac{1}{S V_2} V_2 \end{array} \right) \rangle.$$

de (5.20)

We consider $\tau = |\zeta|^2 = 1 - \varepsilon$ for some $\varepsilon > 0$. Since $\eta = 0$, we get from (5.12) that

$$V_i = \sqrt{\kappa \varepsilon} S_i + \mathcal{O}(\varepsilon^{3/2}), \quad \kappa := \frac{\langle S_1 S_2 \rangle}{\langle S_1 S_2 \rangle^2}, \quad (5.21)$$

where we used $\rho \sim \sqrt{\varepsilon}$. From the expansion of $U$ in (5.14) this implies

$$\tau U = 1 - \varepsilon \kappa S_1 S_2 + \mathcal{O}(\varepsilon^2). \quad (5.22)$$

Plugging (5.22) and (5.21) into the definitions of $\mathcal{L}$ and $\mathcal{M}$ in (4.43) and (5.17), respectively, yields

$$\mathcal{L} = \mathcal{L}_e + \varepsilon \mathcal{D} + \mathcal{O}(\varepsilon^2), \quad \mathcal{L}_e := \left( \begin{array}{cc} 1 - S^* & 0 \\ 0 & 1 - S \end{array} \right),$$

$$\mathcal{M} = \mathcal{M}_e + \mathcal{O}(\varepsilon), \quad \mathcal{M}_e := \left( \begin{array}{cc} S_2 & 0 \\ 0 & S_1 \end{array} \right),$$

where the first order perturbation of $\mathcal{L}$ is given by
\[
\mathcal{D} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} -S^*A_1 + \kappa S_1S_2S^*A_1 + \kappa (S^*A_1)S_2S_1 + \kappa C_S S A_2 \\ \kappa C_S S S^* A_1 - S A_2 + \kappa S_2 S S A_2 + \kappa (S A_2) S A_2 \end{pmatrix}.
\]

According to Corollary 5.4 the operator \( \mathcal{M}_e = (\mathcal{M}_e + \mathcal{O}(\varepsilon))(\mathcal{L}_e + \varepsilon \mathcal{D} + \mathcal{O}(\varepsilon^2)) \) has an isolated eigenvalue \( \tilde{\lambda} \) close to 0 when restricted to \( E_\pm \). Therefore, we can use perturbation theory to determine its value to leading order

\[
\tilde{\lambda} = \varepsilon \left( \frac{\langle E_+, \mathcal{M}_e DS \rangle}{\langle E_+, S \rangle} \right) + \mathcal{O}(\varepsilon^2) = 2 \varepsilon \langle S_1 S_2 \rangle + \mathcal{O}(\varepsilon^2),
\]

(5.23)

where we used \( \mathcal{L}_e S_+ = 0 \) with \( S_+ = (S_1, S_2) \) the right eigenvector and \( E_+ = (1, 1) \) the left eigenvector of the unperturbed operator \( \mathcal{M}_e \mathcal{L}_e = \mathcal{K} \) (cf. Corollary C.4). The spectral projection corresponding to the eigenvalue 0 of \( \mathcal{M}_e \mathcal{L}_e \) is

\[
P_e = \frac{\langle E_+, \cdot \rangle}{\langle E_+, S_+ \rangle} S_+,
\]

and thus inserting (5.23) and (5.21) into (5.20) yields

\[
\sigma = \frac{2 \varepsilon \kappa}{\pi \tilde{\lambda}} \left( \mathcal{P}_e \left( \begin{array}{c} S_2 S_1 \\ S_1 S_2 \end{array} \right), \left( \begin{array}{c} S_2 \\ S_1 \end{array} \right) \right) + \mathcal{O}(\varepsilon) = \frac{(S_1 S_2)^2}{\pi (S_1 S_2)^2} + \mathcal{O}(\varepsilon).
\]

This finishes the proof. \( \Box \)

**Proof of Theorem 2.5.** The analyticity of \( V_1, V_2 \) and thus the well definedness of \( \sigma \) in (2.11) was shown in Corollary 4.13, the upper and lower bounds on \( \sigma \) from (2.12) in Corollary 5.2 away from the edge and in Corollary 5.5 close to the edge. Integrating the definition of \( \sigma(\zeta) \) over \( \mathbb{D}_1 \) and recalling \( \varrho(S) = 1 \) from (4.1) as well as \( V_1(\tau) \to 0 \) and \( V_2(\tau) \to 0 \) for \( \tau \to 1 \) due Corollary 4.13 and (4.4) imply that \( \sigma \) is a probability density on \( \mathbb{C} \). Finally the jump height (2.13) of \( \sigma \) right at the spectral edge is read off from (5.19). \( \Box \)

### 5.3. Proof of Lemma 3.5

In this subsection, we prove Lemma 3.5, the basic property of \( \sigma \) used in the proofs of the global and local inhomogeneous circular law, Theorem 2.3 and Theorem 2.7.

**Proof of Lemma 3.5.** Recall the normalization \( \varrho(S) = 1 \) from (4.1) and write \( U(\tau) = U(\tau, 0), V_1(\tau) = V_1(\tau, 0), \) and \( V_2(\tau) = V_2(\tau, 0) \).

As a first step, we now compute in the integral in the definition of \( L \) in (3.19). This will yield

\[
L(\zeta) = \frac{1}{2} \left( \langle V_1 SV_2 \rangle - \frac{1}{2} \langle \log((\tau + S^*V_1SV_2)(\tau + SV_2S^*V_1)) \rangle \right) \bigg|_{\tau = |\zeta|^2, \eta = 0}.
\]

(5.24)
To that end, let $t \mapsto A(t)$ be a differentiable map with values in the positive definite matrices. Then we have the well-known identity

$$\partial_t \langle \log A(t) \rangle = \langle A(t)^{-1} \partial_t A(t) \rangle$$  \hspace{1cm} (5.25)

(see e.g. [20, Lemma 1.1]). We apply the relation (5.25) to $A = (UU^*)^{-1}$ with $t = \eta$ and obtain

$$\frac{1}{2} \langle \partial_\eta \log(UU^*)^{-1} \rangle = \Re \langle U \partial_\eta U^{-1} \rangle = \langle V_1 \rangle + \langle V_2 \rangle + \partial_\eta \langle V_1 \gamma V_2 \rangle.$$  

Since $\langle V_1 \rangle + \langle V_2 \rangle = 2 \langle \text{Im} M \rangle$, this proves (5.24) due to (3.6), the continuity of $V_1(\tau, \eta)$ and $V_2(\tau, \eta)$ at $\eta = 0$, $\lim_{\eta \to \infty} V_1 = \lim_{\eta \to \infty} V_2 = 0$ by (4.4) as well as $\lim_{\eta \to \infty} (UU^*)^{-1}(1 + \eta)^{-4} = 1$ by (3.6).

The identity (5.24) directly shows that $L$ is rotationally symmetric on $\mathbb{C}$. Moreover, it implies that $L$ is a continuous function of $\zeta$ on $\mathbb{C}$ since $V_1(\tau)$ and $V_2(\tau)$ are continuous functions of $\tau$.

We now show that $\tau \mapsto L(\sqrt{\tau})$ is continuously differentiable on $(0, 1) \cup (1, \infty)$ with

$$\partial_\tau L(\sqrt{\tau}) = -\frac{1}{2} \begin{cases} \langle U(\tau) \rangle, & \text{if } \tau < 1, \\ \tau^{-1}, & \text{if } \tau > 1. \end{cases}$$  \hspace{1cm} (5.26)

If $\tau < 1$ then the continuous differentiability follows from the analyticity and positivity of $V_1$ and $V_2$. Moreover, from (5.25) with $A = (UU^*)^{-1}$ and $t = \tau$, we get

$$\frac{1}{2} \partial_\tau \langle \log(U(\tau)U(\tau)^*)^{-1} \rangle - \partial_\tau \langle V_1(\tau) \gamma V_2(\tau) \rangle = \langle U(\tau) \rangle,$$  

which implies the first case in (5.26) due to (5.24). If $|\zeta| \geq 1$ then $\lim_{\eta \downarrow 0} V_1 = \lim_{\eta \downarrow 0} V_2 = 0$. Hence, we get from (5.24) that $L(\zeta) = -\log|\zeta|$. Thus, the differentiability and the relation (5.26) for $\tau > 1$ follows. This completes the proof of (5.26).

Since $L$ is rotationally symmetric it suffices to show (3.19) under the same constraint on $f$. If $f \in C^2_0(\mathbb{C})$ is rotationally symmetric then a simple change of coordinates yields

$$\frac{1}{2\pi} \int_\mathbb{C} \Delta f(\zeta) L(\zeta) d^2 \zeta = 2 \int_0^\infty (\tau \partial^2_\tau f(\sqrt{\tau}) \gamma \partial_\tau f(\sqrt{\tau}) \gamma L(\sqrt{\tau}) d\tau,$$  \hspace{1cm} (5.27)

where we employed $\Delta f(\zeta) = 4(\tau \partial^2_\tau f(\sqrt{\tau}) \gamma \partial_\tau f(\sqrt{\tau}))|_{\tau = |\zeta|^2}$.

We now split up the $\tau$-integration into $(0, 1)$ and $(1, \infty)$ and use the differentiability of $L$ on both domains to integrate by parts. More precisely, integrating by parts twice, using the continuity of $L$ and $L(1) = 0$ as well as (5.26) and $\lim_{\tau \uparrow 1} \partial_\tau L(\sqrt{\tau}) = -1/2$ yield
\[
2 \int_0^1 (\tau \partial^2_\tau f(\sqrt{\tau}) + \partial_\tau f(\sqrt{\tau}))L(\sqrt{\tau})d\tau = f(1) - \int_0^1 f(\sqrt{\tau})\partial_\tau (\tau(U(\tau)))d\tau \\
= f(1) - \pi \int_0^1 f(\sqrt{\tau})\sigma(\sqrt{\tau})d\tau.
\]

Here, we used in the last step that \(\pi \sigma(\zeta) = \partial_\tau (\tau(U(\tau)))|_{\tau=|\zeta|^2}\) if \(|\zeta| < 1\) due to the definition of \(\sigma\) in (2.11) and the definition of \(U\) in (3.6).

Secondly, an integration by parts, (5.26), the continuity of \(L\) and \(L(1) = 0\) imply

\[
2 \int_1^\infty (\tau \partial^2_\tau f(\sqrt{\tau}) + \partial_\tau f(\sqrt{\tau}))L(\sqrt{\tau})d\tau = -f(1).
\]

By plugging these identities into (5.27), we obtain

\[
-\frac{1}{2\pi} \int_\mathbb{C} \Delta f(\zeta)L(\zeta)d^2\zeta = \int_\mathbb{D} f(\zeta)\sigma(\zeta)d^2\zeta = \int_\mathbb{C} f(\zeta)\sigma(\zeta)d^2\zeta,
\]

where the last step follows from \(\sigma(\zeta) = 0\) if \(|\zeta| \geq 1\) by definition (see (2.11)). This proves (3.19). \(\square\)

5.4. Proof of Proposition 2.9

In this section, we establish Proposition 2.9.

**Proof of Proposition 2.9.** In the light of Theorem 2.5, it suffices to show that (2.16) holds when \(\mu_X(d\zeta)\) is replaced by \(\sigma(\zeta)d^2\zeta\). To that end, let \(L(\zeta)\) be defined as in (3.19). We first show that

\[
-L(\zeta) = \log D(X - \zeta) \tag{5.28}
\]

for all \(\zeta \in \mathbb{C}\). Using [39, Theorem 11 and Proposition 13 in Chapter 9] it is easy to see that \(M(\zeta, \eta) := E[(H_\zeta - i\eta)^{-1}]\) satisfies (3.3), where \(E := \text{id} \otimes \tau : \mathcal{M}^{2n \times 2n} \to \mathbb{C}^{2n \times 2n}\), \(H_\zeta \in \mathcal{M}^{2n \times 2n}\) is defined analogously to (3.1) with \(X\) from (2.15) and \(\mathcal{S}\) is defined as in (3.4) with \(S\) and \(S^\ast\) from (2.18). We introduce the tracial state \(\varphi := \langle \cdot \rangle \otimes \tau\) on \(\mathcal{M}^{n \times n}\) and the matrix \(E_{22} \in \mathbb{C}^{2n \times 2n}\) which has the identity matrix in its lower-right \(n \times n\)-block and vanishes otherwise. Thus, the definitions of \(\varphi\) and \(M\) as well as (C.27) imply

\[
\varphi \left( \frac{\eta}{(X - \zeta)^\ast(X - \zeta) + \eta^2} \right) = -2i\langle E_{22} M(\zeta, \eta) E_{22} \rangle = \text{Im}(M(\zeta, \eta)). \tag{5.29}
\]

We set \(f_\epsilon(\zeta) := \varphi (\log((X - \zeta)^\ast(X - \zeta) + \epsilon^2)^{1/2}) - \log(1 + \epsilon)\) for \(\epsilon > 0\) and \(\zeta \in \mathbb{C}\) and compute
\[ f_\varepsilon(\zeta) = -\int_{\varepsilon}^{\infty} \frac{\partial}{\partial \eta} \left( \frac{1}{2} \varphi \left( \log((X - \zeta)^*(X - \zeta) + \eta^2) - \log(1 + \eta) \right) \right) d\eta \]

\[ = -\int_{\varepsilon}^{\infty} \varphi \frac{\eta}{(X - \zeta)^*(X - \zeta) + \eta^2} - \frac{1}{1 + \eta} d\eta \]

\[ = -\int_{\varepsilon}^{\infty} \operatorname{Im} \langle M(\zeta, \eta) \rangle - \frac{1}{1 + \eta} d\eta. \]

We remark that the integrals exist due to (3.18). In (5.30), we used (5.25) for \( \varphi \) instead of \( \langle \cdot \rangle \) in the second step and (5.29) in the third step. Sending \( \varepsilon \downarrow 0 \) this shows (5.28) by (2.17) and (3.18).

By Lemma 3.5 and standard results from potential theory (see e.g. [11, Chapter 4.3]), we know that

\[ \int_{\mathbb{C}} \log|\lambda - \zeta| \sigma(\zeta) d^2 \zeta = -L(\lambda) + h(\lambda) \]  

(5.31)

for all \( \lambda \in \mathbb{C} \) and some harmonic function \( h: \mathbb{C} \to \mathbb{C} \). In the proof of Lemma 3.5, we saw that \( L(\lambda) = -\log|\lambda| \) if \(|\lambda|\) is sufficiently large. Hence, \( h(\lambda) \to 0 \) if \(|\lambda| \to \infty \), which implies \( h \equiv 0 \). Therefore, (5.28) and (5.31) with \( h \equiv 0 \) prove (2.20) and, thus, Proposition 2.9. \( \square \)

6. Local inhomogeneous circular law

This section is devoted to the proof of Theorem 2.7 which is based on the next theorem. Its formulation and the notation in the next arguments is simplified by the use of the following notion of high probability estimate first introduced in [27].

**Definition 6.1 (Stochastic domination).** Let \( X = X^{(n)} \) and \( Y = Y^{(n)} \) be two sequences of two non-negative random variables. We say that \( X \) is stochastically dominated by \( Y \), denoted by \( X \prec Y \), if, for any \( \varepsilon > 0 \) and \( \nu > 0 \), there is \( C \equiv C_{\varepsilon, \nu} \) such that

\[ \mathbb{P}(X > n^\varepsilon Y) \leq C_{\varepsilon, \nu} n^{-\nu} \]  

(6.1)

for all \( n \in \mathbb{N} \).

We remark that stochastic domination is compatible with basic arithmetic operation (see e.g. [27, Lemma 4.4]). The constants \( C_{\varepsilon, \nu} \) in (6.1) will typically depend on the model parameters.

To simplify the formulation of the next result, we fix \( \tau_* \in (0, g(S)) \) and define the spectral domains
\[ \mathbb{D}_< = \{ \zeta \in \mathbb{C} : |\zeta|^2 \leq g(S) - \tau_s \}, \quad \mathbb{D}_> = \{ \zeta \in \mathbb{C} : \tau_s \leq |\zeta|^2 - g(S) \leq 1/\tau_s \}. \]

**Theorem 6.2** (Local law for \( H_\zeta \)). Let \( \varepsilon \in (0, 1) \), \( X \) satisfy A1 – A3 and \( M \) be the solution of (3.3). Then we have the isotropic local law:

\[
|\langle x, (G(\zeta, \eta) - M(\zeta, \eta))y \rangle| < \|x\| \|y\| \quad \left\{ \begin{array}{ll}
\frac{1}{\sqrt{n\eta}}, & \text{if } \zeta \in \mathbb{D}_<, \eta \in [n^{-1+\varepsilon}, 1], \\
\frac{1}{\sqrt{\varepsilon}}, & \text{if } \zeta \in \mathbb{D}_>, \eta \in [n^{-1+\varepsilon}, 1], \\
\frac{1}{\eta^3}, & \text{if } \zeta \in \mathbb{D}_< \cup \mathbb{D}_>, \eta \in [1, n^{100}],
\end{array} \right.
\]

uniformly for all deterministic vectors \( x, y \in \mathbb{C}^{2n} \). Moreover, the averaged local law

\[
|\langle R(G(\zeta, \eta) - M(\zeta, \eta)) \rangle| < \|R\| \quad \left\{ \begin{array}{ll}
\frac{1}{n\eta}, & \text{if } \zeta \in \mathbb{D}_<, \eta \in [n^{-1+\varepsilon}, 1], \\
\frac{1}{n}, & \text{if } \zeta \in \mathbb{D}_>, \eta \in [n^{-1+\varepsilon}, 1], \\
\frac{1}{\eta^3}, & \text{if } \zeta \in \mathbb{D}_< \cup \mathbb{D}_>, \eta \in [1, n^{100}]
\end{array} \right.
\]

holds uniformly for all deterministic matrices \( R \in \mathbb{C}^{2n \times 2n} \).

We will prove Theorem 6.2 in Section 6.1 below. The next lemma is an application of Theorem 6.2 and estimates the number of small, in modulus, eigenvalues of \( H_\zeta \). It will be used in the proof of Theorem 2.7 to control the integral in (3.16) for small \( \eta \).

**Lemma 6.3** (Number of small singular values of \( X - \zeta \)). Let \( X \) satisfy A1 and A2. Then, for each \( \varepsilon > 0 \), we have

\[ \# \{ i \in [2n] : |\lambda_i(\zeta)| \leq \eta \} < n\eta \quad (6.4) \]

uniformly for all \( \eta \in [n^{-1+\varepsilon}, n^{100}] \) and \( \zeta \in \mathbb{D}_< \).

**Proof.** We follow the proof of Lemma 3.8 and use \( |\text{Tr} G| \prec n \) for all \( \eta \geq n^{-1+\varepsilon} \) due to (6.3) instead of \( |\text{Tr} G| \lesssim n \). This proves Lemma 6.3. \( \square \)

**Proof of Theorem 2.7.** We first remark that the condition \( \|\Delta f\|_{L_1^{1+\beta}} \leq n^D \|\Delta f\|_{L_1} \) is not needed in Theorem 2.7 if the stronger Assumption A4' holds. This can be seen by following the proof of [5, Theorem 2.5] and using, in the proof of [5, Lemma 5.8], Proposition 7.1 below instead of [5, Proposition 5.7], (6.4) instead of [5, Eq. (5.22)] and (6.3) instead of [5, Eq. (5.4)].

We now prove Theorem 2.7 assuming A1–A4. In fact, the proof is a simple refinement of the proof of Theorem 2.3 and we solely describe the necessary modifications. We replace \( f \) by \( f_{\zeta_0, \alpha} \) and choose \( \Omega = \mathbb{D}_{\sqrt{g(S)} - \tau_s/2} \). We remark that \( \text{supp} f_{\zeta_0, \alpha} \subseteq \Omega \) for all sufficiently large \( n \) as \( \alpha > 0 \). The functions \( F \) and \( h \) as well as the measure \( \mu \) are defined analogously according to the new choices of \( f \) and \( \Omega \).
In contrast to the proof of Theorem 2.3, we formulate all estimates in the proof of Theorem 2.7 with respect to stochastic domination $\prec$. In particular, analogously to (3.25), we obtain

$$\left| \int F d\mu - \frac{1}{m} \sum_{i=1}^{m} F(\xi_i) \right| \prec n^{-A} \| \Delta f \|_{L^{1+\beta}}$$  \hspace{1cm} (6.5)

for all $A > 0$, where $m$ was chosen sufficiently large and $\xi_1, \ldots, \xi_m$ are independent random variables distributed according to $\mu$.

The next step is proving that, for $T = n^{100}$ and for each $\varepsilon > 0$, we have

$$|F(\zeta)| \prec n^{-1+\varepsilon} |\Delta f_{\zeta_0,\alpha}(\zeta)|$$ \hspace{1cm} (6.6)

uniformly for all $\zeta \in \Omega$. This is the analogue of (3.26) and shown by decomposing $h = h_1 + \ldots + h_4$, where $h_1, \ldots, h_4$ are defined as before but with the choice $\eta_* = n^{-1+\varepsilon}$. As in the proof of Theorem 2.3, we see that $|h_3| \prec n^{-10}$ and $|h_4| \prec n^{-1+\varepsilon}$ uniformly for $\zeta \in \Omega$. To establish $|h_1(\zeta)| \prec n^{-1+\varepsilon}$, we distinguish the regimes $\eta \in [n^{-1+\varepsilon}, 1]$ and $\eta \in [1, T]$ in the integral as well as apply a union bound and a continuity argument in $\eta$ to (6.3) with $R = 1$. For the bound $|h_2| \prec n^{-1+\varepsilon}$, we decompose the sum into three regimes, $|\lambda_j| \prec n^{-1+\varepsilon}$, $|\lambda_j| \in [n^{-1+\varepsilon}, n^{-1/2}]$ and $|\lambda_j| > n^{-1/2}$, (instead of two regimes in the proof of Theorem 2.3) and estimate each regime separately. The first and the third regime are treated as (3.27) and (3.28), respectively, using Lemma 6.3 instead of Lemma 3.8. In the second regime, we restrict to the nonnegative eigenvalues of $H_\zeta$ due to $\text{Spec } H_\zeta = - \text{Spec } H_\zeta$. We decompose $[n^{-1+\varepsilon}, n^{-1/2}]$ dyadically into intervals $[\eta_k, \eta_{k+1}]$ with $\eta_k := 2^k n^{-1+\varepsilon}$ and obtain

$$\frac{1}{4n} \sum_{|\lambda_j| \in [n^{-1+\varepsilon}, n^{-1/2}]} \log \left( 1 + \frac{n^{-2+2\varepsilon}}{\lambda_j^2} \right) \leq \frac{1}{2n} \sum_{k=0}^{K} \sum_{\lambda_j \in [\eta_k, \eta_{k+1}]} \log \left( 1 + \frac{n^{-2+2\varepsilon}}{\lambda_j^2} \right) \prec \frac{n^\varepsilon}{n},$$

where $K = O(\log n)$. In the last step, we used the monotonicity of the logarithm, $\log(1 + x) \leq x$ and $\# \{ j : \lambda_j \in [\eta_k, \eta_{k+1}] \} \leq \# \{ j : |\lambda_j| \leq \eta_{k+1} \} \prec n^{2k+1}$ due to (6.4). This completes the proof of $|h_2| \prec n^{-1+\varepsilon}$ and, thus, the one of (6.6).

Therefore, following the remaining steps in the proof of Theorem 2.3 yields

$$\left| \frac{1}{n} \sum_{\xi \in \text{Spec } X} f_{\zeta_0,\alpha}(\xi) - \int f_{\zeta_0,\alpha}(\zeta) \sigma(\zeta) d\zeta \right| \prec n^{-1+2\alpha} \| \Delta f \|_{L^1} + n^{-A} \| \Delta f \|_{L^{1+\beta}}$$ \hspace{1cm} (6.7)

for all $A > 0$. Using the condition $\| \Delta f \|_{L^{1+\beta}} \leq n^D \| \Delta f \|_{L^1}$ in (6.7) as well as choosing $A$ appropriately complete the proof. €
6.1. Local law for $H_\zeta$ – proof of Theorem 6.2

This section is devoted to the proof of Theorem 6.2. The local law for Hermitian random matrices with decaying correlations was established in [3,26].

In order to get the isotropic version stated in Theorem 6.2 we will follow the strategy from [26]. Its main result, [26, Theorem 2.2] is not directly applicable to our current situation since Assumption (E) from [26] is violated for $H_\zeta$. The reason why Assumption (E) is needed in the proof of [26, Theorem 2.2] is to ensure the invertibility of $\mathcal{L}$ in the stability result [26, Theorem 5.2] for the MDE. The purpose of this section is to show how the proof is adjusted by using our new stability results, Theorem 4.3 and Corollary 4.7, instead.

The resolvent $G = (H_\zeta - i\eta)^{-1}$ satisfies the perturbed MDE

$$1 + (i\eta + Z + \mathcal{L}G)G = D, \quad D := (H_\zeta + Z + \mathcal{L}G)G.$$  \hfill (6.8)

The main input for the local law for $H_\zeta$, Theorem 6.2, is the following estimate on the error term $D$ in terms of the $p$-norms for random variables $Y$ and random matrices $A \in \mathbb{C}^{2n \times 2n}$ defined through

$$\|Y\|_p := (\mathbb{E}|Y|^p)^{1/p}, \quad \|A\|_p := \sup_{\|x\|,\|y\| \leq 1} (\mathbb{E}|\langle x, Ay \rangle|^p)^{1/p}.$$  \hfill (6.9)

**Proposition 6.4 (Bound on error matrix).** There is a constant $C > 0$, depending only on model parameters, such that for any $\eta \in [n^{-1}, n^{100}]$, $p \in \mathbb{N}$, $\varepsilon > 0$, $R \in \mathbb{C}^{2n \times 2n}$ and $x, y \in \mathbb{C}^{2n}$ with $q := Cp^4 / \varepsilon$ the following holds true:

$$\|\langle x, Dy \rangle\|_p \lesssim_{\varepsilon, p} \|x\|\|y\|^{n^\varepsilon} \sqrt{\frac{\|\text{Im} G\|_q}{n\eta}} (1 + \|G\|_q)^C \left(1 + \frac{\|G\|_q}{n^{1/2-\varepsilon}}\right)^{Cp},$$  \hfill (6.10)

$$\|\langle R, D \rangle\|_p \lesssim_{\varepsilon, p} \|R\|^{n^\varepsilon} (1 + \eta) \frac{\|\text{Im} G\|_q}{n\eta} (1 + \|G\|_q)^C \left(1 + \frac{\|G\|_q}{n^{1/2-\varepsilon}}\right)^{Cp}. $$  \hfill (6.11)

Before deriving Proposition 6.4 from [26, Theorem 4.1], we now explain the definition of the self-energy operator in [26] which differs from the self-energy operator $\mathcal{L}$ used in the present work and defined in (3.4). Instead of $\mathcal{L}$, the self-energy operator considered in [26] (and denoted by $S$ in [26]) is

$$\tilde{\mathcal{L}} := \mathbb{E}(H_\zeta + Z)R(H_\zeta + Z) = \begin{pmatrix} SR_{22} & RR_{21} \\ R^* R_{12} & S^* R_{11} \end{pmatrix},$$  \hfill (6.12)

with $Z = Z(\zeta, \overline{\zeta})$ and the operators $R, R^*: \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ defined through

$$RR := \mathbb{E}XX, \quad R^* R := \mathbb{E}X^* X^*.$$  \hfill (6.13)
Moreover, [26] works with the solution $\widetilde{M}$ (denoted by $M$ in [26]) of the MDE, (4.36), with the self-energy $\widetilde{\mathcal{S}}$ instead of $\mathcal{S}$, i.e. $\widetilde{M} = \widetilde{M}(\zeta, z)$ satisfies

$$-\widetilde{M}^{-1} = z1 + Z + \widetilde{\mathcal{S}}\widetilde{M}$$

(6.13)

for all $z \in \mathbb{H}$ and all $\zeta \in \mathbb{C}$, where $Z = Z(\zeta, \zeta)$.

**Proof.** The bounds (6.9) and (6.10) are an immediate consequence of [26, equations (23a) and (23b)], respectively, with the choice $\mu = 1/2 - \varepsilon$. We will use the following lemma.

**Lemma 6.5.** Let $X$ satisfy Assumptions A1 and A2, then $H_\zeta$ defined in (1.1) satisfies [26, Assumption (C)] and the following modification of [26, Assumption (D)].

**Modification of [26, Assumption (D)]:** With the notation from the formulation of [26, Assumption (D)] the matrix $H_\zeta$ satisfies

$$\kappa(f, g_1, \ldots, g_q) \leq R, q, \mu \cdot n^{-3q} ||f||_{2q} \prod_{j=1}^{q} ||g_j||_{2q},$$

(6.14)

i.e. the $||\cdot||_{q+1}$-norms on the right hand side of [26, Assumption (D)] are replaced by $||\cdot||_{2q}$-norms. This change does not affect any of the proofs in [26].

The proof of Lemma 6.5 is given in Appendix C below. The matrix $H_\zeta$ satisfies [26, Assumptions (A), (B), (C)] and the modified version (6.14) of [26, Assumption (D)] according to Assumption A1, A2 and Lemma 6.5. Since the modification (6.14) does not affect any of the proofs in [26] we can apply [26, Theorem 4.1] to $H_\zeta$. Owing to the different self-energy operator in [26] as explained above, the bounds from [26, Theorem 4.1] are for

$$\bar{D} := D + (\widetilde{\mathcal{S}}G - \mathcal{S}G)G.$$  

(6.15)

Thus to prove the proposition it suffices to show the following bounds on the additional error term

$$\|\langle x, ((\widetilde{\mathcal{S}} - \mathcal{S})G)y\rangle\|_p \lesssim \|x\|\|y\|n^\varepsilon\|G\|_q \left(\|\text{Im } G\|_q / m\right)^{1/2},$$

(6.16)

$$\|\langle R, ((\widetilde{\mathcal{S}} - \mathcal{S})G)G\rangle\| \lesssim \|R\|\|\text{Im } G\| / m\right).$$

(6.17)

To see (6.16) we use that for any unit vectors $x, y \in \mathbb{C}^n$ and $R, Q \in \mathbb{C}^{n \times n}$ we have

---

5 Note that there is a typo in the statement of [26, equation (23b)]. Compared to (6.10) the first $(1 + \eta)$-factor on the right hand side was missing. Indeed, the right hand side of [26, equation (23b)] should have been multiplied with a factor $\langle z \rangle := (1 + |z|)$. In the arXiv version of [26] this typo was corrected.
\[
\| \langle x, (RR)Qy \rangle \|_p \leq \| \sum_{j,k} (Rv_{jk})_j (Qy)_k \|_p \leq n^{2\varepsilon} \| R \|_{1/\varepsilon} \| \sum_{j,k} v_{jk} \| \| (Qy)_k \|_{2p},
\]
where \( v_{jk} := \left( \sum_i T_i X_{ij} X_{ik} \right)_t \in \mathbb{C}^n \) and we employed the generalized inequality for random variables \( (X_t, Y_t)^n_{t=1} \) and \( \varepsilon \in (0, 1/2p) \) of the form \( \| \sum_i X_i Y_i \|_p \leq n^{2\varepsilon} \operatorname{sup}_t \| X_i \|_{1/\varepsilon} \| \sum_i Y_i \|_{2p} \). Since the diagonal contributions of \( \mathcal{F} \) and \( \mathcal{S} \) coincide, we conclude that
\[
\| \langle x, ((\mathcal{F} - \mathcal{S})G)Gy \rangle \|_p \lesssim n^{-1/2+2\varepsilon} \| G \|_{1/\varepsilon} \| G^* G \|_{2p},
\]
where the decay \( \| v_{jk} \| \lesssim n^{-1}(1 + d(j, k))^{-\nu} \) of arbitrarily high order \( \nu \in \mathbb{N} \) was used. The Ward identity \( \eta \mathcal{G}^* \mathcal{G} = \operatorname{Im} \mathcal{G} \) now implies (6.16).

The remaining inequality, (6.17), follows from the Ward identity and \( \| \mathcal{F} - \mathcal{S} \|_{\text{hs}} \lesssim \| R \|_{\text{hs}} \lesssim \frac{1}{n} \) (cf. (A.2) for the bound on \( R \)). This finishes the proof of Proposition 6.4. \( \square \)

**Proof of Theorem 6.2.** To prove the theorem we follow the arguments from the proof of [26, Theorems 2.1 and 2.2] in [26, Sections 5.3 and 5.4] line by line. The spectral parameter \( \zeta \in \mathcal{D}_< \cup \mathcal{D}_< \) associated to \( X \) (cf. (1.1)) is fixed throughout the proof. The different definition of the self-energy in [26] as explained after Proposition 6.4 necessitates replacing a few objects in the arguments in [26] by their counterparts in the present setup. Indeed, \( \mathcal{S}, M \) and \( D \) in [26] are replaced by \( \mathcal{S}, \mathcal{M} \) and \( \mathcal{D} \) from (1.3), (3.7) and (6.8), respectively. The role of the spectral parameter \( z \) in [26] is played here by \( i\eta \) which is associated to \( \mathcal{H}_\zeta \). Correspondingly the domains \( \mathcal{D}^\delta_{\text{out}} \) and \( \mathcal{D}^\delta_{\gamma} \) in \( \mathbb{C} \) from [26] are replaced by
\[
\mathcal{D}^\delta_{\text{out}} := \{ \eta : \eta \in (0, n^{100}], \eta + |\zeta|^{-1} \} \quad \text{and} \quad \mathcal{D}^\delta_{\gamma} := \{ \eta : \eta \in [n^{-1+\gamma}, n^{100}], \eta + |\zeta|^{-1} \} \geq \delta \}
\]
respectively. Here, \((\xi)_+ := \max\{0, \xi\}\) denotes the positive part.

Furthermore, whenever [26, Theorem 4.1] is used in [26] we will use Proposition 6.4 instead. The now missing Assumption (E) from [26] was used along the argument solely for the purpose of establishing stability of the MDE, i.e. to show that the inverse of \( \mathcal{L} \) defined in (4.24) is bounded (Note that \( \mathcal{L} \) is the analogue of \( 1 - C_M \mathcal{S} \) from [26]). We will now point out where the boundedness of \( \mathcal{L}^{-1} \) and the resulting stability in the form of [26, equation (74)] has to be replaced by the use of Corollary 4.7.

Any direct use of [26, equation (74)] is simply replaced by (4.35), using that \( \mathcal{G} \in \mathcal{E}_+^{\perp} \) by Lemma C.5 below. Otherwise the boundedness of \( \mathcal{L}^{-1} \) is only used to establish the averaged bound [26, equation (84)]. To establish this bound in the current setting we start from the quadratic equation
\[
\mathcal{L} \Delta = \hat{D}, \quad \hat{D} := -MD + M(\mathcal{S} \Delta) \Delta \quad (6.18)
\]
for the difference \( \Delta := \mathcal{G} - \mathcal{M} \) right after [26, equation (83)].
Away from the self-consistent spectrum $\text{supp} \rho_\zeta$ with $\rho_\zeta$ defined in (4.37), i.e. for $i\eta \in \mathbb{D}_\text{out}$ we can invert $\mathcal{L}$ and follow the argument from [26] exactly since $\|\mathcal{L}^{-1}\|_{\text{hs}} \lesssim \delta$. This bounded invertibility of $\mathcal{L}$ follows from [8, Lemma 3.7] because $\text{dist}(i\eta, \text{supp} \rho_\zeta) \gtrsim \delta$ by Corollary A.1. In the regime $|\zeta| \leq 1 - \delta$ and $\eta \leq \delta$ the operator $\mathcal{L}$ does not have a bounded inverse. Thus, we have to proceed more cautiously from (6.18) and use the operator $\mathcal{M}$ defined in (4.24). Since $\Delta \in \mathbb{E}^\perp$ and $\mathcal{M} \mathcal{L}$ preserve the subspace $\mathbb{E}^\perp$, we see that $\mathcal{M} \tilde{\mathcal{D}} \in \mathbb{E}^\perp$ by acting with $\mathcal{M}$ on both sides of (6.18). Therefore we can use (4.25) to invert $\mathcal{L}$ on $\mathcal{M}^{-1} \mathbb{E}^\perp$ and after that follow [26] again until the end of [26, Step 3 in Section 5.4]. This proves Theorem 6.2 in the regime $\mathbb{D}^\delta$ for any $\gamma > 0$ without the $\eta^{-2}$-decay of the bound in the regime $\eta \geq 1$ on the right hand side of (6.3) and (6.2).

For the $\eta^{-2}$-decay we replace [26, Step 4 in Section 5.4] by Lemma 3.3, the analogue of [26, Corollary 2.3], to see that there are no eigenvalues in a $\varepsilon$-neighbourhood of the origin for $\zeta \in \mathbb{D}_>$ and follow [26, Step 5 in Section 5.4], again using Proposition 6.4 instead of [26, Theorem 4.1] and (4.35) instead of [26, equation (74)]. This finishes the proof of Theorem 6.2. \hfill \Box

6.2. Eigenvector delocalisation for $X$

In this subsection we prove Corollary 2.8 which is a consequence of the local law for $H_\zeta$, Theorem 6.2.

**Proof of Corollary 2.8.** Take $v \in \mathbb{C}^n$ and $\varepsilon > 0$. Let $u \in \mathcal{U}_{\tau_*}$. Then there is $\zeta \in \mathbb{D}_<$ such that $Xu = \zeta u$. With $u := (0, u)^t \in \mathbb{C}^{2n}$, we obtain $H_\zeta u = 0$. Extending $u/\|u\|$ to an orthonormal basis $u/\|u\|, u_2, \ldots, u_{2n} \in \mathbb{C}^{2n}$ consisting of eigenvectors of $H_\zeta$ associated to the eigenvalues $\lambda_1(\zeta) = 0, \lambda_2(\zeta), \ldots, \lambda_{2n}(\zeta)$ and using the spectral theorem for any $v \in \mathbb{C}^{2n}$ and $\eta > 0$ yields

$$\text{Im} \langle v, G(\zeta, \eta)v \rangle = \frac{|\langle v, u \rangle|^2}{\eta \|u\|^2} + \sum_{i=2}^{2n} \frac{\eta|\langle v, u_i \rangle|^2}{\lambda_i(\zeta)^2 + \eta^2} \geq \frac{1}{\eta} \frac{|\langle v, u \rangle|^2}{\|u\|^2}, \quad (6.19)$$

where, for the last step, we chose $v := (0, v)^t$. Thus, for any $\eta > 0$, the bound (6.19) implies

$$\{ \exists u \in \mathcal{U}_{\tau_*} : |\langle v, u \rangle| \geq n^{-1/2+\varepsilon} \|v\|\|u\| \} \subset \{ \exists \zeta \in \mathbb{D}_< : \eta|\langle v, G(\zeta, \eta)v \rangle| \geq n^{-1+2\varepsilon}\|v\|^2 \} \quad (6.20)$$

with $v := (0, v)^t$.

From (6.2) in Theorem 6.2 and $\|\mathcal{M}\| \lesssim 1$ due to (3.17), we conclude that, for each $\varepsilon \in (0, 1)$, the bound $|\langle v, G(\zeta, \eta)v \rangle| \lesssim \|v\|^2$ holds with very high probability uniformly for all $\zeta \in \mathbb{D}_<$ with $\eta = n^{-1+\varepsilon}$. Therefore, a grid- and continuity argument in $\zeta$ shows that $\sup_{\zeta \in \mathbb{D}_<} |\langle v, G(\zeta, \eta)v \rangle| \lesssim \|v\|^2$ with very high probability for $\eta = n^{-1+\varepsilon}$. We conclude that (6.20) with $\eta = n^{-1+\varepsilon}$ and sufficiently small $\varepsilon > 0$ proves Corollary 2.8. \hfill \Box
7. Bound on the smallest singular value

In this section we bound the smallest singular value of $X + A$ if $X$ satisfies Assumption $\mathbf{A4}'$ and $A$ is deterministic. This is done in Proposition 7.1 below, which, in particular, implies Proposition 2.10. Moreover, we prove Lemma 2.13 in the next subsection.

We recall that $s_{\text{min}}(R)$ denotes the smallest singular value of a matrix $R \in \mathbb{C}^{n \times n}$.

**Proposition 7.1** (Smallest singular value). Let $X = (x_{ij})_{i,j} \in \mathbb{C}^{n \times n}$ be a correlated random matrix satisfying $\mathbb{E}X = 0$ and $\mathbf{A4}'$. Then, for any deterministic matrix $A \in \mathbb{C}^{n \times n}$, we have

$$\mathbb{P}(s_{\text{min}}(X + A) \leq u) \leq \pi n^{\kappa + 5/3} u^{1/4}$$

for all $u \in (0, 1]$.

**Proof.** The following proof mimics the one of [16, Lemma 4.12] that is valid for independent entries.

Going back to [44], the smallest singular value is often estimated by the inequality

$$s_{\text{min}}(X + A) \geq n^{-1/2} \min_{i \in [n]} \text{dist}(R_i, R_{-i}),$$

where $R_1, \ldots, R_n$ are the rows of $X + A$ and $R_{-i} := \text{span}\{R_j : j \neq i\}$ (see also [16, Lemma 4.16]).

Owing to this inequality and a union, we obtain

$$\mathbb{P}(s_{\text{min}}(X + A) \leq u) \leq n \max_{i \in [n]} \mathbb{P}(n^{-1/2} \text{dist}(R_i, R_{-i}) \leq u).$$

We fix $i \in [n]$. Let $y$ be a unit vector that is orthogonal to $R_{-i}$ and measurable with respect to $\{R_j : j \neq i\}$. The Cauchy-Schwarz inequality implies

$$|\langle R_i, y \rangle| \leq \|\pi_i(R_i)\| \|y\| = \text{dist}(R_i, R_{-i}),$$

where $\pi_i$ is the orthogonal projection onto the orthogonal complement of $R_{-i}$. Therefore, we obtain

$$\mathbb{P}(\text{dist}(R_i, R_{-i}) \leq un^{1/2}) \leq \mathbb{P}(|\langle R_i, y \rangle| \leq un^{1/2}).$$

Since $y$ is normalised, we find $j \in [n]$ such that $|y_j| \geq n^{-1/2}$. This yields

$$\mathbb{P}(|\langle R_i, y \rangle| \leq un^{1/2}) = \mathbb{E}\left[\mathbb{E}\left[\sum_{j \in [n]} 1(j = \min\{k : |y_k| \geq n^{-1/2}\}) \mathbb{P}(|\langle R_i, y \rangle| \leq un^{1/2} | R_{ij} = y_j)\right]\right], \quad (7.1)$$

$$\mathbb{P}(|\langle R_i, y \rangle| \leq un^{1/2}) = \mathbb{E}\left[\mathbb{E}\left[\sum_{j \in [n]} 1(j = \min\{k : |y_k| \geq n^{-1/2}\}) \mathbb{P}(|\langle R_i, y \rangle| \leq un^{1/2} | R_{ij} = y_j)\right]\right], \quad (7.1)$$
where we denote by \( X_{ij} \) the family of random variables \( X_{ij} := \{ x_{kl} : (k, l) \in [n]^2 \setminus \{(i, j)\} \} \).

We now estimate the conditional probability with respect to \( X_{ij} \) for any \( j \in [n] \) such that \( |y_j| \geq n^{-1/2} \). We only consider the case that \( \psi_{ij} \) is a density on \( \mathbb{C} \). (If \( \psi \) is a density on \( \mathbb{R} \) then we proceed completely analogously.) The condition \( |y_j| \geq n^{-1/2} \), the identity (2.7) in A4' and Hölder’s inequality imply

\[
\mathbb{P}\left( |\langle R_i, y \rangle| \leq un^{1/2} \mid X_{ij} \right) = \int_{\mathbb{C}} \mathbf{1}\left( |a_{ij} + z| \leq \frac{un}{|y_j|} \right) \psi_{ij}(z) d^2z \leq \pi^{(q-1)/2}n^{3(q-1)/2}u^{2(q-1)/q} \|\psi_{ij}\|_q,
\]

for some \( C \)-valued random variable \( a_{ij} \), which is measurable with respect to \( X_{ij} \). Thus, estimating the sum in (7.1) by \( n \) and using the bound on \( \mathbb{E}\|\psi_{ij}\|_q \) from A4' complete the proof of Proposition 2.10. \( \Box \)

### 7.1. Proof of Lemma 2.13

**Proof of Lemma 2.13.** For all \( i, j \in [N] \) and \( \alpha, \beta \in [K] \), we set \( X_{i,j,\alpha,\beta} := \{ (x_{kl})_{\gamma,\delta} : (k, l, \gamma, \delta) \neq (i, j, \alpha, \beta) \} \) and

\[
\psi_{i,j,\alpha,\beta}(z) := \frac{f_{ij}((z_{\gamma,\delta})_{\gamma,\delta} \in [K])}{\int_{\mathbb{C}} f_{ij}((z_{\gamma,\delta})_{\gamma,\delta} \in [K]) d^2z_{\alpha,\beta}} \bigg| \begin{array}{l} z_{\alpha,\beta} = z, \\ z_{\gamma,\delta} = x_{ij} \text{ if } (\gamma, \delta) \neq (\alpha, \beta), \end{array}
\]

with the convention \( \psi_{i,j,\alpha,\beta}(z) = 0 \) if the denominator vanishes. A simple computation shows that

\[
\mathbb{P}\left( \langle e_\alpha, x_{ij} e_\beta \rangle \sqrt{NK} \in B \mid X_{i,j,\alpha,\beta} \right) = \int_B \psi_{i,j,\alpha,\beta}(z) d^2z
\]

for all measurable \( B \subset \mathbb{C} \), where \( e_1, \ldots, e_K \) denote the standard basis vectors of \( \mathbb{C}^K \). Hence, for each entry \( \langle e_\alpha, x_{ij} e_\beta \rangle \) of \( X \) as defined in (2.22), we have determined the density in (2.7).

From the definition of \( \psi_{i,j,\alpha,\beta} \), it is easy to conclude that

\[
\mathbb{E}\|\psi_{i,j,\alpha,\beta}\|_q = \int_{\mathbb{C}^K \times K - 1} \left( \int_{\mathbb{C}} f_{ij}(z) d^2z_{\alpha,\beta} \right)^{1/q} d^2z_{11} d^2z_{12} \ldots d^2z_{\alpha,\beta} \ldots d^2z_{KK}.
\]

(7.2)

Finally, applying (2.21) to (7.2) implies Assumption A4' for \( X \). \( \Box \)
Appendix A. Exclusion of eigenvalues outside disk and global law for $H_\zeta$

In this appendix we show how Lemma 3.3 and Proposition 3.7 can be derived from existing results. We recall that the self-consistent density of states $\rho_\zeta$ was defined in (4.37) and the self-consistent spectrum is supp $\rho_\zeta$.

The following corollary to Lemma 3.1 states that the self-consistent spectrum supp $\rho_\zeta$ is bounded away from zero for any spectral parameter $\zeta$ outside the disk of radius $\sqrt{g(S)}$.

**Corollary A.1.** Let $\zeta \in \mathbb{C}$ with $|\zeta|^2 > g(S) + \delta$ for some $\delta > 0$. Assuming A1, A2 and A5, the self-consistent spectrum supp $\rho_\zeta$ is bounded away from zero, i.e. dist$(0, \text{supp } \rho_\zeta) \geq \delta$.

**Proof.** The corollary follows from Lemma 3.1 and the implication [6, (i) implies (v) in Lemma D.1]. □

Lemma 3.3 and Proposition 3.7 will follow from [26, Corollary 2.3] and [26, Theorem 2.1], respectively. As explained after Proposition 6.4, the self-energy operator $\tilde{\mathcal{F}}$ used in [26] (cf. (6.11)) differs slightly from $\mathcal{F}$ defined in (3.4) and used in the present work. Therefore, applying results from [26] requires controlling the difference between $M(\zeta, z)$ and $\tilde{M}(\zeta, z)$, the solutions of the MDE’s (4.36) and (6.13), respectively. This is done in the next lemma. In analogy to $\rho_\zeta$, we define $\tilde{\rho}_\zeta$ as the unique probability measure on $\mathbb{R}$ with Stieltjes transform $z \mapsto \langle \tilde{M}(\zeta, z) \rangle$.

**Lemma A.2 (Properties of $\tilde{M}$).** Assume A1 and A2. Let $M = M(\zeta, \eta)$ and $\tilde{M} = \tilde{M}(\zeta, \eta)$ for some $\zeta \in \mathbb{C}$ and $\eta > 0$. If $\eta \geq n^{-\varepsilon}$ for some small enough $\varepsilon > 0$ or $|\zeta|^2 \geq g(S) + \delta$ for some $\delta > 0$ then the following holds.

(i) The solutions are close in operator norm: $\|\tilde{M} - M\| \leq \frac{n^{C_\varepsilon}}{(1+\eta^{-2})^2 n}$ for some universal constant $C > 0$.

(ii) The solutions are close in hs-norm: $\|\tilde{M} - M\|_{hs} \leq \frac{n^{C_\varepsilon}}{(1+\eta^{-2})^2 n}$ for some universal constant $C > 0$.

(iii) If $|\zeta|^2 \geq g(S) + \delta$ and we also assume A5, then 0 is outside the self-consistent spectrum associated to $\tilde{M}$, i.e. dist$(0, \text{supp } \tilde{\rho}_\zeta) \geq 1$.

Before establishing Lemma A.2 we use it to show Lemma 3.3.

**Proof of Lemma 3.3.** Given Lemma A.2 (iii), Lemma 3.3 is a direct consequence of [26, Corollary 2.3]. □

For the reader’s convenience we record the auxiliary result proven in [8, Lemma 3.4(i)] and [8, Lemma 3.7(ii), (iii)].

**Lemma A.3.** Assume A1 and A2. Let $\mathcal{L} = \mathcal{L}(\zeta, \eta)$ be defined as in (4.24). The following holds.
(i) For all $\zeta \in \mathbb{C}$ and $\eta > 0$, we have

$$\|M(\zeta, \eta)\| \leq \frac{1}{\text{dist}(\eta, \text{supp } \rho_\zeta)}.$$ 

(ii) There is a universal constant $K > 0$ such that, for all $\eta > 0$ and $\zeta \in \mathbb{C}$ with $|\zeta| \lesssim 1$, we have

$$\|\mathcal{L}^{-1}\|_{\text{hs}} + \|\mathcal{L}^{-1}\| + \|\mathcal{L}^{-1}\|^* \lesssim 1 + \frac{1}{\text{dist}(\eta, \text{supp } \rho_\zeta)}.$$ 

Proof of Lemma A.2. We start by showing that the operators $\mathcal{R}$ and $\mathcal{R}^*$ from (6.12) that constitute the off-diagonal entries of $\tilde{T} - \mathcal{T}$ can be considered small perturbations. Indeed, we will prove that

$$\|\mathcal{R}R\| + \|\mathcal{R}^* R\| \lesssim \frac{1}{\sqrt{n}} \|R\|_{\text{hs}} \leq \frac{1}{\sqrt{n}} \|R\| \quad (A.1)$$

for every $R \in \mathbb{C}^{n \times n}$. To check (A.1) we simply use that

$$|\langle x, (\mathcal{R}R)y \rangle| \leq \frac{1}{\sqrt{n}} \left( \sum_{i,j,k,l} |x_i y_j x_l y_k| |K_{ij,kl}| \right)^{1/2} \|R\|_{\text{hs}}.$$ 

Here, we introduce the coefficients $K_{ij,kl}$ given by

$$K_{ij,kl} := \sum_{u,v} \left| \text{Cov}(X_{iu} \sqrt{n}, X_{vj} \sqrt{n}) \text{Cov}(X_{ju} \sqrt{n}, X_{vk} \sqrt{n}) \right| \lesssim \frac{1}{(1 + d(i, l) + d(j, k))^\nu},$$

where we used Assumptions A1 and A2 as well as Young’s inequality to see that they still have a polynomial decay of arbitrarily high order $\nu \in \mathbb{N}$. Thus, the volume growth condition (2.3) for the metric $d$ implies (A.1).

Since the difference $\Delta = \tilde{M} - M$ satisfies the quadratic equation (6.18) with the error matrix $D := (\tilde{\mathcal{T}}M - \mathcal{T}M)\tilde{M}$ that satisfies the bound $\|D\| \lesssim \|\mathcal{R}\|\|\tilde{M}\|^2 \lesssim n^{-1/2+2\varepsilon}/(1 + \eta^2)$ due to (A.1) and the trivial bound $\|M\| \lesssim \frac{1}{\eta}$, we use the invertibility of the stability operator $\mathcal{L}$ from (ii) of Lemma A.3 to conclude (i) of the lemma in case $\eta \geq n^{-\varepsilon}$. In case $|\zeta|^2 \geq g(S) + \delta$ and $\eta \leq 1$ the invertibility of $\mathcal{L}$ is still guaranteed by (ii) of Lemma A.3 and we have

$$\|\Delta\| \lesssim \|\mathcal{R}\|\|\tilde{M}\|^2 + \|\Delta\|^2 \lesssim n^{-1/2} + \|\Delta\|^2,$$

where we used $\|\mathcal{R}\| \lesssim n^{-1/2}$ by (A.1) and $\|\tilde{M}\| \lesssim \|M\| + \|\Delta\|$. Thus we can bootstrap the bound $\|\Delta\| \lesssim n^{-1/2}$ from the regime $\eta \geq 1$.

For the proof of (ii), we show the improved norm bound on $\mathcal{R}$ in the hs-sense

$$\|\mathcal{R}\|_{\text{hs}} \lesssim \frac{1}{n}. \quad (A.2)$$
To show (A.2), for each $R \in \mathbb{C}^{n \times n}$, we estimate the hs-norm through

$$
\|R \|_{\text{hs}} \leq \frac{1}{n^3} \sum_{u,v,u',v'} \hat{K}_{uv,u'v'} |R_{uv}R_{u'v'}| \lesssim \frac{1}{n^2} \|R\|_{\text{hs}}^2,
$$

where the second bound holds because for any $\nu \in \mathbb{N}$ the coefficients $\hat{K}_{uv,u'v'}$ satisfy

$$
\hat{K}_{uv,u'v'} := \sum_{i,j} |\text{Cov}(X_{iu}\sqrt{n}, X_{v'j}\sqrt{n})\text{Cov}(X_{iuv}\sqrt{n}, X_{v'j}\sqrt{n})|
\lesssim \frac{1}{(1 + d(v,v') + d(u,u'))^\nu}.
$$

As above, from (A.2), we get $\|D\|_{\text{hs}} \lesssim \|R\|_{\text{hs}} \|\bar{M}\|^2 \lesssim n^{-1+2\varepsilon}/(1 + \eta^2)$ and infer (ii) of the lemma from (ii) of Lemma A.3.

Now we verify (iii). First we have $\text{dist}(i\eta, \text{supp } \rho_\xi) \gtrsim 1$ by Corollary A.1. We use the implication [6, (v) implies (ii) in Lemma D.1]. By (i) of Lemma A.2 the property [6, (ii) in Lemma D.1] is satisfied for $\bar{M}$ whenever it is satisfied for $M$ due to their closeness. Finally, by the implication [6, (ii) implies (v) in Lemma D.1] we see that property [6, (v) in Lemma D.1] holds for $\bar{M}$, i.e. $\text{dist}(i\eta, \text{supp } \tilde{\rho}_\xi) \gtrsim 1$. \hfill \Box

The next proposition is a generalization of Proposition 3.7.

**Proposition A.4** (Global law for $H_\xi$, general version). Let $X$ satisfy $\textbf{A1}$ and $\textbf{A2}$. Then there is $C > 0$ such that for all $\varphi > 0$ and all sufficiently small $\delta > 0$ we have

$$
\langle x, (G - M)y \rangle \leq \|x\|\|y\| \frac{\eta^{C\delta}}{(1 + \eta^2)\sqrt{n}},
$$

(A.3a)

$$
\langle R(G - M) \rangle \leq \|R\| \frac{\eta^{C\delta}}{(1 + \eta^2)n},
$$

(A.3b)

with very high probability uniformly for all $n \in \mathbb{N}$, $\xi \in \mathbb{D}_\varphi$ and $\eta \in [n^{-\delta}, n^{100}]$ as well as deterministic vectors $x, y \in \mathbb{C}^{2n}$ and deterministic matrices $R \in \mathbb{C}^{2n \times 2n}$. Here $K$ is some absolute constant and the constant $C_\varphi$ implicit in Definition 3.2 of ‘very high probability’ depends only on $\delta$ and $\varphi$ as well as the constants from $\textbf{A1}$ and $\textbf{A2}$, in addition to $\nu$.

**Proof.** The proposition is an immediate consequence of [26, Theorem 2.1] since $\eta \geq n^{-\delta}$ means that the spectral parameter in the MDE is sufficiently far away from the self-consistent spectrum associated to $H_\xi$. As alluded to after Proposition 6.4, the self-energy in [26] is $\bar{F}$ instead of $F$. Consequently, the resolvent $G$ is compared to $\bar{M}$, the solution of (6.13), instead of $M$. Thus (A.3a) and (A.3b) follow from the closeness of $\bar{M}$ to $M$ from (i) and (ii) in Lemma A.2, respectively. \hfill \Box
Appendix B. Quantitative law of large numbers

In this section, we state a law of large numbers with an explicit rate of convergence for random variables with only \( a \)-moments for some \( a > 1 \).

**Proposition B.1** (Quantitative law of large numbers). Let \( m \in \mathbb{N} \). Let \( (X_i)_{i=1}^m \) be centred i.i.d. random variables with \( \mathbb{E}|X_1|^a < \infty \) for some \( a > 1 \). Then, for any \( \delta \in (0, 1] \), we have

\[
P\left( \left| \frac{1}{m} \sum_{i=1}^m X_i \right| \leq \left( \frac{10\mathbb{E}|X_1|^a}{m^{a-1}\delta} \right)^{1/a} \right) \geq 1 - \delta.
\]

For the convenience of the reader, we provide a short proof of Proposition B.1, which is a quantitative variant of the standard proof of the law of large numbers.

**Proof.** We set \( \mu_a := \mathbb{E}|X_1|^a \) and \( \varepsilon^a := \frac{10\mu_a}{m^{a-1}\delta} \). We split into different terms and estimate

\[
P\left( \left| \frac{1}{m} \sum_{i=1}^m X_i \right| > \varepsilon \right) \leq P(|E_1| > \varepsilon/3) + P(|E_2| > \varepsilon/3) + 1(|E_3| > \varepsilon/3),
\]

where we introduced the random variables

\[
E_1 := \frac{1}{m} \sum_{i=1}^m (X_i - Y_i), \quad E_2 := \frac{1}{m} \sum_{i=1}^m (Y_i - \mathbb{E}Y_i), \quad E_3 := \frac{1}{m} \sum_{i=1}^m \mathbb{E}Y_i,
\]

\[
Y_i := X_i \mathbb{1}_{|X_i| \leq \varepsilon m}.
\]

We now estimate the different terms in (B.1) separately. As a preparation, we conclude from Markov’s inequality that

\[
P(|X_1| \geq t) \leq \frac{\mu_a}{t^a}
\]

for any \( t > 0 \). Hence, a simple union bound for the first term in (B.1) and \( X_i = Y_i \) if \( |X_i| \leq \varepsilon n \) by definition of \( Y_i \) yield

\[
P(|E_1| > \varepsilon/3) \leq \sum_{i=1}^m P(|X_i - Y_i| > \varepsilon/3) \leq mP(|X_1| > \varepsilon m) \leq \frac{\mu_a}{\varepsilon^a m^{a-1}}.
\]

The second term in (B.1) is bounded by Chebyshev’s inequality using independence, i.e. by

\[
P(|E_2| > \varepsilon/3) \leq \frac{9\text{Var}(Y_1)}{\varepsilon^2 m} \leq \frac{9\mu_a}{\varepsilon^a m^{a-1}},
\]

where in the last step we used that \( \text{Var}(Y_1) \leq \mathbb{E}[|X_1|^{2-a}|X_1|^a \mathbb{1}_{X_1| \leq \varepsilon m}] \leq (\varepsilon m)^{2-a} \mu_a. \)
Finally, since $\mathbb{E}X_1 = 0$, Hölder’s inequality and (B.2) imply

$$|E_3| \leq |EY_1| = |\mathbb{E}(Y_1 - X_1)| = |\mathbb{E}X_1 \mathbb{1}_{|X_1| > \varepsilon m}| \leq \frac{\mu_a}{\varepsilon a - 1} m^{a-1}.$$  

 Altogether we conclude

$$\mathbb{P}\left(\frac{1}{m} \sum_{i=1}^m X_i > \varepsilon \right) \leq \frac{10\mu_a}{\varepsilon a - 1} m^{a-1} + \mathbb{1}\left(\frac{3\mu_a}{m^{a-1}} > \varepsilon^a\right),$$

which completes the proof as the indicator function vanishes due to the definition of $\varepsilon^a$ and $\delta \leq 1$. \qed

Appendix C. Auxiliary results

Proof of Lemma 6.5. We start by verifying [26, Assumption (C)]. For the definition of the norms used inside this proof we refer to [26]. To show $\|\kappa\|_{B_0} \lesssim 1$ we split the covariances $\kappa(\alpha, \beta) := \mathbb{E}w_\alpha w_\beta$ with double indices $\alpha = (a_1, a_2), \beta = (b_1, b_2) \in [2n]^2$ and

$$W := (w_\alpha)_{\alpha \in [2n]^2} := H_\zeta + Z = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix},$$

into two summands $\kappa = \kappa_d + \kappa_c$ through

$$\kappa_c(a_1 a_2, b_1 b_2) := \kappa(a_1 a_2, b_1 b_2)\mathbb{1}((a_1, b_2) \in [n]^2 \cup (n + [n])^2).$$

We remark that with the definition of $\kappa_{\#}$ from (4.15) and $\widetilde{\kappa}$ from (6.11) we get $\kappa = \kappa_{\widetilde{\kappa}}, \kappa_d = \kappa_{\widetilde{\kappa} - \kappa}$ and $\kappa_c = \kappa_{\kappa}$. Now we verify that $\|\kappa_{\#}\|_{\#} \lesssim 1$ for $\# = c, d$. If $\# = d$ then we estimate

$$\|\kappa_d\|_d = \sup_{\|x\| \leq 1} \left\| \left( \sum_{b_1} \sum_{a_1} a_1 \kappa_d(a_1 a_2, b_1 b_2) \right)^{1/2} \right\|_{a_2, b_2} \lesssim \sup_{\|x\| \leq 1} \left\| \left( \frac{1}{1 + d(a_2, b_2)} \right)^{1/2} \right\|_{a_2, b_2} \lesssim 1,$$

where the norm on the right side of the equality refers to the standard operator norm of the matrix indexed by $a_2, b_2$ and where we used the decay of correlation from Assumption A2 via

$$|\kappa_d(a_1 a_2, b_1 b_2)| \lesssim \frac{1}{(1 + d(a_1, b_1))(1 + d(a_2, b_2))^\nu}, \quad (C.1)$$

The case $\# = c$ is seen by interchanging the roles of $b_1$ and $b_2$ and using
\[ |\kappa_c(a_1a_2,b_1b_2)| \lesssim \frac{1}{(1 + d(a_1, b_2))^{\nu}(1 + d(a_2, b_1))^{\nu}}. \quad \text{(C.2)} \]

The bound \( \|\kappa\|_2^\alpha \leq \|\kappa_d\|_2^\alpha + \|\kappa_c\|_2^\alpha \lesssim 1 \) also follows from (C.1) and (C.2). This implies \( \|\kappa\|_2 \lesssim 1 \).

The proof of \( \|\kappa\|_k \lesssim n^\varepsilon \) for \( k \geq 3 \) relies on [26, Lemma A.1]. We demonstrate the strategy for these bounds for \( \|\kappa\|_3^\alpha \) and leave the other simpler cases to the reader. Writing the third order cumulant of three centred matrices \( R_1, R_2, R_3 \) with 
\[
R_i = (r_i^{(j)})_{\alpha \in [2n]^2} \text{ as } \kappa_{R_1R_2R_3}(\alpha, \beta, \gamma) := \mathbb{E}r_1^{(1)}r_2^{(2)}r_3^{(3)} \text{ we split } \kappa := \kappa_{WWW} \text{ into four summands } \kappa = \kappa_{dd} + \kappa_{cc} + \kappa_{dc} + \kappa_{cd}. \text{ This split is performed by plugging in }
\]
\[
W = X + X^*, \quad X = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}
\]
for each of the three \( W \)-factors in the definition of \( \kappa \), multiplying out and then grouping the summands according to
\[
\kappa_{dd} := \kappa_{XXX} + \kappa_{XX} + \kappa_{X*}, \quad \kappa_{cc} := \kappa_{XXX} + \kappa_{XX} + \kappa_{X*},
\]
\[
\kappa_{dc} := \kappa_{XXX} + \kappa_{XX} + \kappa_{X*}, \quad \kappa_{cd} := \kappa_{XXX} + \kappa_{XX} + \kappa_{X*}.
\]

Since all cases \( \|\kappa_{#1#2}\|_{#1#2} \lesssim n^\varepsilon \) with \( #_i = d, c \) are proven similarly by simply interchanging the role of certain indices, we only show the case \( #_1 = #_2 = d \). Due to [26, Lemma A.1] and Assumption A2 we have for any fixed \( \nu \in \mathbb{N} \) and \( \varepsilon > 0 \) that
\[
\kappa_{dd}(\alpha, \beta, \gamma) \lesssim n^{-\nu} \text{ whenever } d \times d(\alpha, \beta) + d \times d(\alpha, \gamma) + d \times d(\beta, \gamma) \geq n^\varepsilon.
\]

Thus using (2.3) we conclude
\[
\|\kappa_{dd}\|_3^\alpha \lesssim \frac{1}{n^3} \sum_{b_2,c_1}(\sum_{b_1,c_2} \sum_{a_1,a_2} \kappa_{dd}(a_1a_2,b_1b_2,c_1c_2))^2 \\
\lesssim \max_{b_2,c_1}\{ (a_1, a_2, b_1, c_2) : d(c_1, a_1) + d(c_1, b_1) + d(b_2, a_2) + d(b_2, c_2) \leq n^\varepsilon \}^2 \\
+ n^{-\nu} \lesssim n^C \varepsilon.
\]

We proceed by verifying the modification of [26, Assumption (D)] described in the lemma, where the constant \( \mu > 0 \) from the formulation of the assumption can be chosen arbitrarily. With the choice of nested neighbourhoods \( \mathcal{N}_k(\alpha) := \{ \beta : d \times d(\alpha, \beta) \leq kn^{(1-3\mu)/4p} \} \), where \( p \in \mathbb{N} \) is from (2.3), [26, Assumption (D)] is satisfied. Indeed, with the functions \( f, g_1, \ldots, g_q \) from the formulation of the assumptions we have
\[
\kappa(f, g_1, \ldots, g_q) \lesssim n^{-\nu}\|f\|_2\|g_1\|_2 \cdots \|g_q\|_2/q, \quad \text{(C.3)}
\]
for any \( \nu \in \mathbb{N} \). To see (C.3) we follow the proof of [26, Lemma A.1] with the choice \( w_A = (f) \) and \( w_B = (g_1, \ldots, g_q) \). The covariance term in the last equation of the
proof we estimate using (2.4) with \( f_1 = \Pi \omega P_{\cap A} = f \) and \( f_2 = \Pi \omega P_{\cap B} \). Since \( d \times d(\text{supp } f_1, \text{supp } f_2) \geq n^{(1-3\mu)/4p} \) we get (C.3) after applying Hölder inequality to \( \| f_2 \|_2 \) on the right hand side of (2.4). \( \Box \)

**Proof of Lemma 4.6.** Let \( w \in \mathbb{C}^d \) with \( w \perp b \). To prove (4.32) we use the spectral projection \( P \) from (4.29) and its complementary projection \( Q := 1 - P \) as well as \( \| a \|_\# = 1 \) to estimate

\[
\| Aw \|_\# \geq \| AQw \|_\# - \| APw \|_\# \geq \| Qw \|_\# - |\alpha| \| \langle p, w \rangle \|. \tag{C.4}
\]

Since \( w \) is orthogonal to \( b \) we have the identity

\[
0 = \langle b, w \rangle = \langle b, a \rangle \langle p, w \rangle + \langle b, Qw \rangle. \tag{C.5}
\]

In particular, we find an upper bound on \( |\langle p, w \rangle| \) in terms of \( \| Qw \|_\# \), namely

\[
|\langle p, w \rangle| \leq \frac{|\langle b, Qw \rangle|}{|\langle b, a \rangle|} \leq \frac{1}{2\varepsilon} \| Qw \|_\#, \tag{C.6}
\]

where we used the assumption from (4.30). Continuing from (C.4) we see that

\[
\| Aw \|_\# \geq \left( 1 - \frac{\varepsilon}{2} \right) \| Qw \|_\# \geq \frac{1}{2} \| Qw \|_\#, \tag{C.7}
\]

because \( |\alpha| \leq \varepsilon \) by assumption.

To finish the proof of (4.32) we use

\[
\| Qw \|_\# \geq \| w \|_\# - \| Pw \|_\# \geq \| w \|_\# - |\langle p, w \rangle|. \tag{C.8}
\]

Combining the two lower bounds (C.6) and (C.8) on \( \| Qw \|_\# \) and optimizing over the values of \( |\langle p, w \rangle| \) while using that \( \varepsilon \leq 1 \) yields

\[
\| Qw \|_\# \geq \frac{2\varepsilon}{3} \| w \|_\#.
\]

Together with (C.7) this finishes the proof of Lemma 4.6. \( \Box \)

**Lemma C.1** *(Quantitative implicit function theorem).* Let \( T : \mathbb{C}^A \times \mathbb{C}^D \to \mathbb{C}^A \) be a continuously differentiable function with invertible derivative \( \nabla^{(1)} T(0,0) \) at the origin with respect to the first argument and \( T(0,0) = 0 \). Suppose \( \mathbb{C}^A \) and \( \mathbb{C}^D \) are equipped with norms that we both denote by \( \| \cdot \| \) and let the linear operators on these spaces be equipped with the corresponding induced operator norms. Let \( \delta > 0 \) such that

\[
\sup_{(a,d) \in B^A_\delta \times B^D_\delta} \| \text{Id}_{\mathbb{C}^A} - (\nabla^{(1)} T(0,0))^{-1} \nabla^{(1)} T(a,d) \| \leq \frac{1}{2}, \tag{C.9}
\]
where $B_\delta^\#$ is the $\delta$-ball around 0 with respect to $\| \cdot \|$ in $\mathbb{C}^\#$. Suppose

$$\|(\nabla^{(1)} T(0,0))^{-1}\| \leq C_1, \quad \sup_{(a,d) \in B_\delta^A \times B_\delta^B} \|\nabla^{(2)} T(a,d)\| \leq C_2,$$

for some positive constants $C_1, C_2$, where $\nabla^{(2)}$ is the derivative with respect to the second variable. Then there is a constant $\varepsilon > 0$, depending only on $\delta$, $C_1$ and $C_2$, and a unique function $f : B_\varepsilon^D \to B_\varepsilon^A$ such that $T(f(d),d) = 0$ for all $d \in B_\varepsilon^D$. The function $f$ is continuously differentiable. If $T$ is analytic, then so is $f$.

**Lemma C.2.** Let $\| a \|$ denote the Euclidean norm of a vector $a \in \mathbb{C}^d$ and $\| A \|$ the induced operator norm for a matrix $A \in \mathbb{C}^{d \times d}$. Fix $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0,1)$ with $100 \varepsilon_3 \leq \varepsilon_1 \varepsilon_2^2$. Let $F,T \in \mathbb{C}^{d \times d}$ be self-adjoint matrices such that $\| T \| \leq 1$ and

$$\text{Spec}(F) \subseteq \{-1\} \cup \{-1 + \varepsilon_1, 1 - \varepsilon_1\} \cup \{1\}, \quad (C.10)$$

where $\pm 1$ are non-degenerate eigenvalues of $F$ with corresponding normalized eigenvectors $f_\pm$, i.e. $F f_\pm = \pm f_\pm$. Suppose that $\| T f_+ \| \leq 1 - \varepsilon_2$ and $\| (1 + T) f_- \| \leq \varepsilon_3$. Then the resolvent of $TF$ satisfies

$$\sup \left\{ \| (TF - \zeta)^{-1} \| : \zeta \in \mathbb{C}, \zeta \notin D_{1-\varepsilon_3} \cup (1 + D_{3\varepsilon_3}) \right\} \leq \frac{4}{\varepsilon_3}, \quad (C.11)$$

Furthermore, there is a single eigenvalue $\zeta_0$ close to 1 and this eigenvalue is non-degenerate, more precisely,

$$\text{Spec}(TF) \cap (1 + D_{3\varepsilon_3}) = \{ \zeta_0 \}, \quad \text{dim ker}(TF - \zeta_0)^2 = 1. \quad (C.12)$$

**Proof.** First we realize that $f_-$ satisfies approximate eigenvalue equations for both $TF$ and $FT$, namely

$$\|(1 - TF) f_- \| \leq \varepsilon_3, \quad \|(1 - FT) f_- \| = \| F(1 + T) f_- \| \leq \varepsilon_3. \quad (C.13)$$

We now prove that when restricted to the orthogonal complement of $f_-$, the matrix $TF$ is strictly smaller than 1. More precisely, we will establish that

$$\| TF a \| \leq \left( 1 - \frac{\varepsilon_1 \varepsilon_2^2}{8} \right) \| a \|, \quad a \perp f_- . \quad (C.14)$$

To show (C.14) we fix a unit vector $a \in f_\perp$, $\| a \| = 1$, and decompose it according to $f_+$ and its orthogonal complement,

$$a = \sqrt{1 - \alpha^2} f_+ + \alpha \tilde{a}, \quad \tilde{a} \perp f_+, \quad \| \tilde{a} \| = 1,$$
for some \( \alpha \in [0, 1] \). Because \( \|T\| \leq 1 \) and \( F \) has a spectral gap (cf. (C.10)) we see that \( \|TFa\| \) is bounded from above by

\[
\|TFa\| \leq \|Fa\| \leq \sqrt{1 - \alpha^2 + \alpha^2(1 - \varepsilon_1)^2} \leq 1 - \frac{\varepsilon_1 \alpha^2}{2}.
\]

(C.15)

On the other hand, by using the assumption \( \|Tf_+\| \leq 1 - \varepsilon_2 \) we also get a second bound,

\[
\|TFa\| \leq \sqrt{1 - \alpha^2 \|Tf_+\| + \alpha \|Fa\|} \leq (1 - \varepsilon_2) \sqrt{1 - \alpha^2 + (1 - \varepsilon_1)\alpha}
\]

(C.16)

For \( \alpha \leq \varepsilon_2/2 \) we use (C.16) while for \( \alpha \geq \varepsilon_2/2 \) we use (C.15) to infer (C.14).

With the help of (C.13) and (C.14) we represent \( TF \) with respect to \( f_- \) and an orthonormal basis of \( f_\perp \). Thus we see that there is a unitary matrix \( U \in \mathbb{C}^{d \times d} \) as well as \( \alpha \in \mathbb{C} \), \( b, a \in \mathbb{C}^{d-1} \) and \( B \in \mathbb{C}^{(d-1) \times (d-1)} \) such that \( A := U^*TFU \) has the structure

\[
A = \begin{pmatrix} 1 + \varepsilon_3 \alpha & \varepsilon_3 b^* \\ \varepsilon_3 a & B \end{pmatrix}, \quad |\alpha| \leq 1, \quad \|b\| \leq 1, \quad \|a\| \leq 1, \quad \|B\| \leq 1 - 2\varepsilon_4, \quad \varepsilon_4 := \frac{\varepsilon_1 \varepsilon_2}{16}.
\]

(C.17)

Therefore it suffices to prove the resolvent bound (C.11) for any matrix \( A \) with the structure (C.17) in place of \( TF \). For this purpose we fix a spectral parameter \( \zeta \) with

\[
|\zeta| \geq 1 - \varepsilon_4, \quad |1 - \zeta| \geq 2\varepsilon_5, \quad \varepsilon_5 := \varepsilon_3 + \frac{\varepsilon_3^2}{\varepsilon_4}, \quad (C.18)
\]

and use the Schur complement formula for \( A - \zeta \) with respect to the block structure (C.17), i.e. we write

\[
(A - \zeta)^{-1} = \begin{pmatrix} ((A - \zeta)^{-1})_{11} & ((A - \zeta)^{-1})_{1\perp} \\ ((A - \zeta)^{-1})_{\perp1} & ((A - \zeta)^{-1})_{\perp\perp} \end{pmatrix},
\]

where \( \perp \) refers to the component in the orthogonal complement of the first canonical basis vector \( e_1 \) of \( \mathbb{C}^d \). The Schur complement itself is

\[
s_A(\zeta) := 1 + \varepsilon_3 \alpha - \zeta - \varepsilon_3^2 b^* (B - \zeta)^{-1} a,
\]

(C.19)

and because of (C.18) and the bound on \( B \) from (C.17), we find

\[
|1 - \zeta - s_A(\zeta)| \leq \varepsilon_5.
\]

We conclude that \( |((A - \zeta)^{-1})_{11}| \leq \varepsilon_5^{-1} \leq \varepsilon_3^{-1} \) and also
\[
\|((A - \zeta)^{-1})_{1\perp}\| \leq \frac{\varepsilon_3}{\varepsilon_4\varepsilon_5} \leq \frac{1}{\varepsilon_3}, \quad \|((A - \zeta)^{-1})_{\perp 1}\| \leq \frac{\varepsilon_3}{\varepsilon_4\varepsilon_5} \leq \frac{1}{\varepsilon_3},
\]

which implies (C.11), since \(2\varepsilon_4^{-1} \leq \varepsilon_3^{-1}\) and \(\zeta \notin \mathbb{D}_{1-\varepsilon_4} \cup \mathbb{D}_{2\varepsilon_5} \subseteq \mathbb{D}_{1-6\varepsilon_3} \cup \mathbb{D}_{3\varepsilon_5}\) due to \(\varepsilon_4 \geq 6\varepsilon_3 \geq 4\varepsilon_5\).

To show (C.12) we use a simple interpolation argument. Consider the family of matrices
\[
A_{\omega} := e_1e_1^* + \omega(A - e_1e_1^*), \quad \omega \in [0,1],
\]
interpolating between \(A_0 = e_1e_1^*\) and \(A_1 = A\). Since every element of this family has the same block structure (C.17) as \(A\), we conclude that (C.11) holds with \(TF\) replaced by \(A_{\omega}\). Since the eigenvalues of \(A_{\omega}\) (as the \(d\) zeros of the characteristic polynomial counted with multiplicity) depend continuously on \(\omega\) and they cannot enter the regime in which the resolvent of \(A_{\omega}\) is bounded, we conclude that the number of eigenvalues for \(A_1 = A\) within \(1 + \mathbb{D}_{3\varepsilon_3}\) is that same as for \(A_0\). Thus (C.12) is proven. \(\square\)

Lemma C.3 (Resolvent control for \(S\)). Let \(S : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}\) be a positivity preserving operator such that \(g(S) = 1\) and \(c\langle A\rangle \leq SA \leq C\langle A\rangle\) for any \(A \in \mathbb{C}_+\). Then \(S\) satisfies the resolvent control
\[
\sup\left\{\| (S - \xi)^{-1} \| : \xi \in \mathbb{C}, \xi \notin \mathbb{D}_{1-2\varepsilon} \cup (1 + \mathbb{D}_{\varepsilon}) \right\} \lesssim 1, \quad (C.20)
\]
for any sufficiently small \(\varepsilon > 0\) (depending on the constants \(c\) and \(C\)) and \# = hs, \(\|\cdot\|\).
The algebraic multiplicity of the eigenvalue \(g(S) = 1\) is one and the corresponding left and right Perron Frobenius eigenvectors, \(S_1\) and \(S_2\), satisfy
\[
S_1 \sim 1, \quad S_2 \sim 1, \quad (C.21)
\]
where \(S^*S_1 = S_1\) and \(SS_2 = S_2\) as well as \(\langle S_1\rangle = \langle S_2\rangle = 1\).

Proof. We start the proof for \# = hs. We denote by \(S_1\) and \(S_2\) the positive definite left and right Perron Frobenius eigenvectors of \(S\) with normalisation \(\langle S_1\rangle = \langle S_2\rangle = 1\). The assumption \(SA \sim \langle A\rangle\) immediately implies (C.21) and also the statement about the multiplicity. Instead of studying \(S\) we study \(\Sigma : \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}\) defined as
\[
\Sigma := \begin{pmatrix} S^* & 0 \\ 0 & S \end{pmatrix} = V^{-1}\mathcal{F}\mathcal{V}, \quad (C.22)
\]
where the representation on the very right is in terms of the invertible operator...
\[ V := \begin{pmatrix} C_{K_2}^{-1} C_{\sqrt{S_1}}^{-1} & 0 \\ 0 & C_{K_1}^{-1} C_{\sqrt{S_2}}^{-1} \end{pmatrix}, \]

and the two self-adjoint operators

\[ T := \begin{pmatrix} 0 & C_{K_2} K_{\overline{P}} C_{K_2} \\ C_{K_1} K_{\overline{P}} C_{K_2} & 0 \end{pmatrix}, \]
\[ F := \begin{pmatrix} 0 & C_{K_2}^{-1} C_{\sqrt{S_2}}^{-1} S C_{\sqrt{S_1}} C_{K_2}^{-1} \\ C_{K_1}^{-1} C_{\sqrt{S_2}}^{-1} S^* C_{\sqrt{S_1}}^{-1} C_{K_2} & 0 \end{pmatrix}. \]

(C.23)

Here we introduced a short hand notation for the matrices

\[ \widetilde{K}_1 := (\sqrt{S_2} S_1 \sqrt{S_2})^{1/4}, \quad \widetilde{K}_2 = (\sqrt{S_1} S_2 \sqrt{S_1})^{1/4}, \quad \widetilde{P} := \frac{1}{\sqrt{S_2} \sqrt{S_1}}. \]

Note that the definitions of \( V, F \) and \( T \) above are compatible with (4.52), (4.51) and (4.50) in the limit \( \tau \to 1, \eta \downarrow 0 \), while with the same limit we have \( \widetilde{K}_i := \lim K_i/\sqrt{V_i} \) and \( \widetilde{P} := \lim P(V_i) \).

Since \( \Sigma \) from (C.22) is a direct sum of \( S \) and \( S^* \), the claim (C.20) is equivalent to the same statement with \( S \) replaced by \( \Sigma \). Owing to (C.21) we have \( \|V\|_{\text{hs}} \|V^{-1}\|_{\text{hs}} \sim \|V\| \|V^{-1}\| \sim 1 \). Therefore, (C.20) for \( \Sigma \) now follows from the following facts about \( T \) and \( F \):

\[ T V S_\pm = \pm V S_\pm, \quad F V S_\pm = \pm V S_\pm, \quad \|T\|_{\text{hs}} \leq 1, \]
\[ \|F\|_{\text{hs}(V S_+ \cap (V S_-)')} \leq 1 - 2\varepsilon, \]

for some \( \varepsilon \sim 1 \), where \( S_\pm = (S_1, \pm S_2) \). Here, the last bound is obtained from [3, Lemma 4.8] in the same way as (4.60) in Lemma 4.11 was obtained. Since via Lemma 4.5 we can lift the resolvent control to the other norm \( \# = \|\cdot\| \), this finishes the proof of the lemma. \( \square \)

**Corollary C.4 (Resolvent control for edge stability operator).** Let \( S \) and \( S_1, S_2 \) be as in Lemma C.3 and define the operator \( K : \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \) via

\[ K := \begin{pmatrix} S_2 (1 - S^*) & 0 \\ 0 & S_1 (1 - S) \end{pmatrix}. \]

Then this operator satisfies the resolvent control

\[ \sup\left\{ \|(K - \xi)^{-1}\|_{\#} : \xi \in \mathbb{D}_{2\varepsilon} \setminus \mathbb{D}_\varepsilon \right\} \lesssim \varepsilon 1, \quad (C.24) \]

for any sufficiently small \( \varepsilon > 0 \) and \( \# = \|\cdot\|_{\text{hs}},\|\cdot\| \). Furthermore, the eigenvalue 0 has algebraic and geometric multiplicity equal to 2 and corresponding right and left eigenvectors
\begin{equation}
K S_\pm = 0, \quad K^* E_\pm = 0, \quad S_\pm := \left( \begin{array}{c} S_1 \\ \pm S_2 \end{array} \right), \tag{C.25}
\end{equation}

where \( E_\pm \) are defined in (4.44).

**Proof.** Since the operator \( K \) separately acts on the first and second component of a pair of matrices, the assertions about the multiplicity of 0 and (C.25) follow from Lemma C.3 and a simple computation. Similarly, it suffices to prove the resolvent control (C.24) for each component, i.e. to show it for \( S_1(1 - S) \) and \( S_2(1 - S^*) \). We will only consider the first since the latter is treated similarly with the roles on \( S \) and \( S^* \) interchanged. We define the projections

\[
P A := \frac{\langle S_1, A \rangle}{\langle S_1, S_2 \rangle} S_2, \quad \tilde{P} A := \frac{\langle 1, A \rangle}{\langle 1, S_2 \rangle} S_2, \quad P^\perp A := \frac{\langle S_2, A \rangle}{\langle S_2, S_2 \rangle} S_2,
\]

and their complements

\[
Q := 1 - P, \quad \tilde{Q} := 1 - \tilde{P}, \quad Q^\perp := 1 - P^\perp.
\]

Due to Lemma C.3 the rank one projections \( P \) and \( \tilde{P} \) are the spectral projections associated to the non-degenerate eigenvalue 0 of \( 1 - S \) and \( S_1(1 - S) \), respectively. The claim follows now because the operator \( S_1(1 - S) \) has a bounded inverse on the image of \( \tilde{Q} \), i.e.

\[
\| S_1(1 - S) A \|_{\text{hs}} = \| S_1(1 - S) Q A \|_{\text{hs}} \gtrsim \| Q A \|_{\text{hs}} \sim \| \tilde{Q} A \|_{\text{hs}}, \tag{C.26}
\]

where for the inequality we used \( S_1 \sim 1 \) and Lemma C.3 and for the last relation

\[
\| Q A \|_{\text{hs}}^2 = \| Q \hat{A} \|_{\text{hs}}^2 = \| \hat{A} \|_{\text{hs}}^2 + \frac{\| S_1(1 - S) \|_{\text{hs}}^2}{\langle S_1, S_2 \rangle^2} \sim \| \hat{A} \|_{\text{hs}}^2 \sim \| \tilde{Q} A \|_{\text{hs}}^2 = \| \tilde{Q} A \|_{\text{hs}}^2.
\]

Here we used the short hand \( \hat{A} = Q^\perp A, Q A = Q \hat{A}, \tilde{Q} \hat{A} = \tilde{Q} A \) and the second comparison relation holds for the same reason as the first. This finishes the proof of (C.24) for \( \# = \text{hs} \). For \( \# = \| \cdot \| \) we use Lemma 4.5. \( \square \)

**Lemma C.5.** Let \( X \in \mathbb{C}^{n \times n} \) be an arbitrary matrix. Then, for all \( z \in \mathbb{C} \setminus \mathbb{R} \), we have

\[
\begin{pmatrix} z & X \\ X^* & z \end{pmatrix}^{-1} \in E_\perp.
\]

**Proof.** Schur’s complement formula directly implies that

\[
\begin{pmatrix} z & X \\ X^* & z \end{pmatrix}^{-1} = \begin{pmatrix} z(z^2 - XX^*)^{-1} & -(z^2 - XX^*)^{-1}X \\ -X^*(z^2 - XX^*)^{-1} & z(z^2 - XX^*)^{-1} \end{pmatrix}. \tag{C.27}
\]
As $XX^*$ and $X^*X$ have the same eigenvalues and their multiplicities coincide, this proves the lemma. □

References