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Landauer vs. Nernst: What is the True Cost of Cooling a Quantum System?

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Thermodynamics connects our knowledge of the world to our capability to manipulate and thus to control it. This crucial role of control is exemplified by the third law of thermodynamics, Nernst’s unattainability principle, stating that infinite resources are required to cool a system to absolute zero temperature. But what are these resources? And how does this relate to Landauer’s principle that famously connects information and thermodynamics? We answer these questions by providing a framework for identifying the resources that enable the creation of pure quantum states. We show that perfect cooling is possible with Landauer energy cost given infinite time or control complexity. Within the context of resource theories of quantum thermodynamics, we derive a Carnot-Landauer limit, along with protocols for its saturation. This generalises Landauer’s principle to a fully thermodynamic setting, leading to a unification with the third law and emphasising the importance of control in quantum thermodynamics.

I. INTRODUCTION

What is the cost of creating a pure state? Pure states appear as ubiquitous idealisations in quantum information processing and preparing them with high fidelity is essential for quantum technologies such as reliable quantum communication [1, 2], high-precision quantum parameter estimation [3–5], and fault-tolerant quantum computation [6, 7]. Fundamentally, pure states are prerequisites for ideal measurements [8] and precise timekeeping [9]. To answer the above question, one could turn to Landauer’s principle, stating that erasing a bit of information has an energy cost of at least $k_B T \log(2)$ [10]. Alternatively, one could consult Nernst’s unattainability principle (the third law of thermodynamics) [11], stating that cooling a physical system to the ground state requires diverging resources. It may seem that these statements are at odds. However, Landauer’s protocol requires infinite time, thus identifying time as a resource according to the third law [12–16]. Does this mean infinite energy or time are needed to prepare a pure state?

The perhaps surprising answer we give here is: No. We show that finite energy and time suffice to perfectly cool any quantum system and we identify the resource—control complexity—that must diverge (in the spirit of Nernst) to do so. The ultimate limit on the energetic cost of cooling is thus still provided by the Landauer limit, but to achieve it, either time or control complexity must diverge.

At the same time, heat fluctuations and short coherence times in current quantum technologies [17] demand that both energy and time are not only finite, but minimal. Therefore, in addition to proving the necessity of diverging control complexity for perfect cooling with minimal time and energy, we conceive protocols that attain these ultimate limits. Via the introduced notion of control complexity, we demonstrate that mitigating detrimental effects comes at the cost of fine-tuned interactions that require a coherent external work source, i.e., a quantum battery [18–21]. From a thermodynamic perspective, this may seem unsatisfactory: Nonequilibrium resources imply that the joint system is not energetically closed. Therefore, we also consider an incoherent control setting restricted to global energy-conserving unitaries with a heat bath as thermodynamic energy source. This setting corresponds to minimal control, where interactions need only be switched on and off to generate transformations, i.e., a heat engine provides energy [22–26]. In this paradigm, we show that the Landauer bound is not attainable, derive a novel limit—which we dub the Carnot-Landauer bound—and construct protocols that saturate it.

Our work both generalises Landauer’s erasure principle and, at the same time, unifies it with the laws of thermodynamics. By accounting for control complexity, we emphasise a crucial resource that is oftentimes overlooked. Here, we study the asymptotic setting that allows us to connect this resource with Nernst’s unattainability principle; beyond the asymptotic case, the gained insights also open the door to a better understanding of the intricate relationship between energy, time and control complexity when all resources are finite, which will be crucial for practical applications. Lastly, our protocols saturating the Carnot-Landauer bound pave the way for thermodynamically-driven (minimal control) quan-

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II. SUMMARY OF MAIN RESULTS

There are two types of thermodynamic laws: Those, like the second law, that place bounds on the (changes of) characteristic quantities during thermodynamic processes, and those, like the third law, which state the impossibility of certain tasks. Landauer’s principle is of the former kind (indeed, it can be rephrased as a version of the second law), associating a minimal heat dissipation to any logically irreversible process, thus placing a fundamental lower bound on the energy cost of computation. The paradigmatic logically irreversible process is that of erasing information: In other words, perfectly cooling a system to the ground state, or more generally, preparing a pure state.

Nernst’s unattainability principle is a law of the latter kind, stating that perfectly cooling any system requires diverging resources. The resources typically considered are energy and time. On the one hand, perfect cooling can be achieved in finite time (i.e., in a single unitary transformation between a target system and a thermal environment) at the expense of an energy cost that diverges as the ground state is approached. However, the high levels of fine-tuned microstate control required in such protocols come at a practical cost, which motivates our consideration of the incoherent-control paradigm. There, an agent is afforded minimal control and may only switch on and off interactions between the system and thermal machines at different temperatures. Inevitably, with less control, one must deal with more heat. This is captured by the equality that generalises Landauer’s bound to the incoherent-control setting—which we dub the Carnot-Landauer limit—presented in Sec. VI. Nonetheless, such an increase in heat is likely to be more favourable than the significant costs incurred by high-level quantum control on classical computers executing coherent experimental protocols [27, 28]. We begin with a summary of our main results.

1. Perfect cooling is possible with coherent control provided either energy, time, or control complexity diverge. In particular, it is possible in finite time and at Landauer cost with diverging control complexity.

2. Perfect cooling is possible with incoherent control provided either time or control complexity diverge. It is impossible with both finite time and control complexity regardless of the energy drawn from the heat bath.

3. No process driven by a finite-temperature heat engine can (perfectly) cool a quantum system at the Landauer limit. Nonetheless, the Carnot-Landauer limit, which we introduce here, can be saturated for any initial temperature heat bath, given either diverging time or control complexity.

In the following, we discuss these results in more detail and provide a systematic study concerning the asymptotic interplay of energy, time and control complexity.

III. COOLING PROCESSES

A. Thermodynamic Framework

Consider a target system $S$ in an initial state $\rho_S$ described by a unit-trace, positive semidefinite operator with associated Hamiltonian $H_S$. An auxiliary machine $M$, initially uncorrelated with $S$ and in equilibrium with a reservoir at inverse temperature $\beta := \frac{1}{k_B T}$, is used to cool the target system. The initial state of $M$ is thus of Gibbs form,

$$\rho_M = \tau_{M}(\beta, H_M) := \frac{e^{-\beta H_M}}{Z_{M}(\beta, H_M)};$$

where $H_M$ is the machine Hamiltonian and $Z_{M}(\beta, H_M) := \text{tr} [e^{-\beta H_M}]$ its partition function. Throughout this article we only consider Hamiltonians with discrete spectra, i.e., with an associated separable Hilbert space that has a countable energy eigenbasis. Moreover, for the most part we consider finite-dimensional systems (or sequences thereof) and deal with infinite-dimensional systems separately.

Note that any full-rank state $\rho$ can be associated to some chosen temperature $\beta$, which sets the energy scale, and a Hamiltonian $H = -\frac{1}{\beta} \log (\rho)$; as we consider arbitrary Hamiltonians, we only write the state dependence on these parameters when necessary. If the state is not full rank, the rank can be used to re-define the dimension.

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FIG. 1. Framework. We consider the task of cooling a quantum system in two extremal control scenarios, with each step of both paradigms comprising two primitives. The top panel depicts the coherent-control scenario: In the control step (left), an agent can use a work source $W$ to implement any global unitary on the system $S$ and machine $M$, which both begin thermal at inverse temperature $\beta$; in cooling the target, energy and entropy is transferred to the machine. In the rethermalisation step (right), the machine rethermalises with its environment, thereby dissipating the energy it gained in the control step. The bottom panel depicts the incoherent-control scenario: The machine is bipartitioned into a cold part at inverse temperature $\beta_C$ and a hot part at inverse temperature $\beta_H < \beta$. In the control step, the agent switches on an interaction between the three systems, represented by a global energy-conserving unitary $U_{EC}$. In the rethermalisation step, the interaction is turned off and both subsystems of the machine rethermalise to their respective initial temperatures; the hot part draws energy from the heat bath while the cold part dissipates heat to its environment. Red arrows represent entropy and/or energy flows. The two steps of each cycle may be repeated throughout a cooling protocol.

As shown in Fig. 1, one step of a cooling process comprises two sub-procedures: First, a joint unitary is implemented during the control step; second, the machine rethermalises to the ambient temperature. A cooling protocol is determined by the initial conditions and any concatenation of such primitives.

We consider two extremal control paradigms corresponding to two classes of allowed global transformations. The coherent control paradigm permits arbitrary unitaries on $SM$; in general, these change the total energy but leave the global entropy invariant and thus require an external work source $W$. At the other extreme is the incoherent control paradigm, where the energy source is a heat bath. Here, the machine $M$ is bipartitioned: One part, $C$, is connected to a cold bath at inverse temperature $\beta_C$, which serves as a sink for all energy and entropy flows; the other, $H$, is connected to a hot bath at inverse temperature $\beta_H \leq \beta$, which provides energy. The composite system $SCH$ is closed and thus global unitary transformations are restricted to be energy conserving. The temperature gradient causes a natural heat flow away from the hot bath, which carries maximal entropic change with it. Cooling protocols in this setting can be run with minimal external control, i.e., they only require switching on and off interactions. We first analyse the coherent-control scenario.

B. Cooling at the Landauer Limit

Within the coherent-control scenario, a transformation $\varrho_S \to \varrho_S'$ is enacted via a unitary $U$ on $SM$ involving a thermal machine $\varrho_M = \tau_M(\beta, H_M, \beta)$, i.e.,

$$\varrho_S' := \text{tr}_S \left[ U(\varrho_S \otimes \varrho_M) U^\dagger \right].$$

For such a transformation, there are two energy costs contributing to the total energy change, which must be drawn from a work source $W$. The first is the energy change of the target $\Delta E_S := \text{tr} \left[ H_S (\varrho_S' - \varrho_S) \right]$; the second is that of the machine $\Delta E_M := \text{tr} \left[ H_M (\varrho_M' - \varrho_M) \right]$, where $\varrho_M' := \text{tr}_S \left[ U(\varrho_S \otimes \varrho_M) U^\dagger \right]$. The latter is associated with the heat dissipated into the environment and is given by [29]

$$\beta \Delta E_M = \Delta S_S + I(S:M)_{\varrho_S'} + D(\varrho_M' || \varrho_M),$$

[2] One could refer to both $M$ and the transformations applied as the machine and call the system $M$ itself the working medium inasmuch as the latter passively facilitates the process, in line with conventional parlance; however, we use the terminology established in the pertinent literature.
where $S(\rho) := -\text{tr}[\rho \log(\rho)]$ is the von Neumann entropy, $\Delta S_\text{A} := S(\rho_\text{A}) - S(\rho'_\text{A})$, $I(A : B)_{\text{EAB}} := S(\rho_\text{A}) + S(\rho'_\text{B}) - S(\rho_{\text{AB}})$ (with marginals $\rho_{\text{EAB}} := \text{tr}_{\text{\text{E}}}[\rho_{\text{AB}}]$) is the mutual information between $\text{A}$ and $\text{B}$, and $D(\rho||\sigma) := \text{tr}[\rho \log \rho] - \text{tr}[\rho \log \sigma]$ is the relative entropy of $\rho$ with respect to $\sigma$, with $D(\rho||\sigma) := \infty$ if $\text{supp} \rho \not\subseteq \text{supp} \sigma$. We derive Eq. (3) and its counterpart in the incoherent-control setting in Appendix A. The mutual information is non-negative and vanishes if $\rho_{\text{EAB}} = \rho_\text{A} \otimes \rho_\text{B}$; similarly, the relative entropy is non-negative and vanishes if $\rho = \sigma$. Dropping these terms leads to the Landauer bound [10]

$$\beta \Delta E_{\text{M}} \geq \tilde{\Delta} S_\text{E}. \quad (4)$$

The Landauer limit holds independently of the protocol implemented, i.e., it only assumes some unitary was applied to the target and thermal machine. For large machines, the dissipated heat is typically much greater than the energy change of the target; nonetheless, the contributions can be comparable at the microscopic scale. We will assume that the target begins in equilibrium with the reservoir at inverse temperature $\beta$, i.e., in the initial thermal state $\rho_\text{S} = \tau_\beta(\mathcal{H}_\text{S})$, with no loss of generality since such a relaxation can be achieved for free (by swapping the target with a suitable part of the environment). We will track all energetic and entropic quantities and refer to the asymptotic saturation of Eq. (4) with $\tilde{\rho}'_\text{S}$ pure as perfect cooling at the Landauer limit.

Equation (3) provides insight for understanding the conditions required for saturating the Landauer bound. Although for finite-dimensional machines only trivial processes of the form $U_{\text{S,M}} = U_\text{S} \otimes 1_\text{M}$ saturate the Landauer limit [29], we show how it can be asymptotically saturated with non-trivial processes by considering diverging machine and interaction properties, as we elaborate on shortly. Any such process must asymptotically exhibit no correlations such that $I(\mathcal{S} : \mathcal{M})_{\tilde{\rho}'_{\text{S,M}}} \to 0$ and effectively not disturb the machine, i.e., yield $\tilde{\rho}'_{\text{S,M}} \to \rho_{\text{S,M}}$ such that $D(\tilde{\rho}'_{\text{S,M}}||\rho_{\text{S,M}}) \to 0$. Indeed, any correlations created between initially thermal systems would come at the expense of an additional energetic cost [30–32] whose minimisation is a problem that has so far only been partially resolved [33]. However, it has been shown that for any rank non-decreasing process, there exists a thermal machine and joint unitary such that for any $\epsilon > 0$, the heat dissipated satisfies $\beta \Delta E_{\text{M}} \leq \tilde{\Delta} S_\text{E} + \epsilon$ [29], thereby saturating the Landauer limit. Here, we present protocols that asymptotically achieve both this and perfect cooling (in particular, effectively decrease the rank), and provide necessary conditions on the underlying resources required to do so.$^4$

### IV. THE ENERGY–TIME–COMPLEXITY RESOURCE TRINITY

We now describe coherently-controlled protocols that approach perfect cooling of an arbitrary finite-dimensional target system with thermal machines when any one of the relevant resources—energy, time or control complexity—diverges; moreover, the resources that are kept finite saturate protocol-independent bounds. Importantly, all of the protocols we present are constructive proofs in the sense that they prove that perfect cooling is possible under certain conditions (e.g., infinite energy, time, or control complexity), and not the inverse, i.e., that no protocol exists that achieves the same with finite resources. However, note that the existence of a finite-resource protocol that perfectly cools would violate the unattainability principle. By establishing such protocols, we show that the three resources can be traded-off among each other to approach perfect cooling as long as any one diverges. Whenever we say “achievable with diverging $X$”, we mean that one can construct a procedure that takes the initial state to within arbitrarily small distance $\epsilon$ from the ground state, i.e., $\rho_\text{S} \to \rho_{\text{S}}(0)/\rho_{\text{S}}(0)$, for any suitably-chosen distance measure, and that $X$ diverges in the limit $\epsilon \to 0$. Similarly, “at the Landauer limit” should be interpreted as the asymptotic saturation of the relevant bound. The scenarios where energy is minimised correspond to situations in which the Landauer limit is asymptotically attained as either time or control complexity diverge.

#### A. Diverging Energy

This cooling protocol is arguably the simplest of those presented. The thermal populations of any target system can be exchanged with a machine system of the same dimension, in the thermal state of $\rho_\text{S} = \omega_{\text{S}} = \sum_{n=0}^{d-1} n|n\rangle\langle n|$. As $\omega_\text{S} \to \infty$, the machine state $\tau_\beta(\beta, H_\text{M})$ approaches $|0\rangle\langle 0|$ independently of $\beta$ (as long as $\beta \neq 0$). Such a population-exchange operation is a single interaction (i.e., the protocol occurs in unit time) which is of finite complexity (in a

\[\text{terms of their “classical” counterparts, i.e., } \rho_\text{X} \to \rho_\text{X} := (p_0, \ldots, p_d) \text{ with } p_n = e^{-\beta E_n}, \text{ where } |H(\rho_\text{X})| \to |E_\text{X}| := \sum_n p_n E_n, \text{ and so on.}

Nonetheless, all of the results presented hold for the more general “quantum” properties.}
sense that we discuss below). However, the energy drawn from the resource \( W \) upon performing said swap is at least \( E = (p_s^{(1)} - p_s^{(1)}) (\omega_{s}^{k}) \), where \( p_s^{(1)} \) is the initial population of the first excited level of system \( X \) and \( \omega_{s}^{k} \) is the first energy eigenvalue of the target system. Denoting by \( \omega_{s}^{k} \) the energy eigenvalue of the \( k \)th excited level of the target system, we have above assumed that \( \omega_{s}^{(0)} = 0 \) (which we do for all Hamiltonians without loss of generality) and \( \omega_{s}^{k} > \omega_{s}^{(d-1)} \).

As such, perfect cooling will incur diverging energy cost.

In addition to being sufficient, for perfect cooling with both finite time and control complexity (i.e., using an effectively finite-dimensional machine), diverging energy is also necessary (see Appendix B).

### B. Diverging Time

We now present a protocol that uses a diverging number of operations to asymptotically attain perfect cooling at the Landauer limit \([29, 34, 35]\).

**Theorem 1.** With diverging time, any finite-dimensional quantum system can be perfectly cooled at the Landauer limit via interactions of finite complexity.

**Sketch of proof.**—We first show that any system can be cooled from \( \rho_s = \tau_s (\beta, H_s) \) to \( \tau_s (\beta^*, H_s) \), with \( \beta^* \geq \beta \), using only \( \beta^{-1} \Delta S_s \) units of energy. Our proof is constructive in the sense that we provide a protocol that achieves the Landauer energy cost as the number of operations diverges. The interactions in this protocol are of finite control complexity as they simply swap the target system with one of a sequence of thermal machines with increasing energy gaps. In this way, the final state \( \tau_s (\beta^*, H_s) \) can be made to be arbitrarily close to \( |0\rangle |0\rangle_s \) for any initial temperature.

The proof is presented in Appendix C, along with a more detailed dimension-dependent energy cost function for the special case of equally spaced Hamiltonians.

### C. Diverging Control Complexity

By reconsidering the diverging-time protocol above, a trade-off can be made between time and control complexity. As illustrated in Fig. 2, one can consider all of the operations \( \{ U_k = e^{-iH_k t_k} \}_{k=1,..,N} \) required in said protocol to make up one single joint interaction \( U_{\text{tot}} := \lim_{N \rightarrow \infty} \prod_{k=1}^{N} U_k = e^{-iH_{\text{tot}} t_{\text{tot}}} \) thus setting the time required to be unity (in terms of the number of operations). In other words, for any finite number \( N \) of unitary transformations \( U_k \), there exists a total Hamiltonian \( H_{\text{tot}}^{(N)} \) and a finite time \( t_{\text{tot}} \) that generates the overall transformation \( U^{(N)}_{\text{tot}} := \prod_{k=1}^{N} U_k \); since \( t_{\text{tot}} \) is finite, we can set it equal to one without loss of generality by rescaling the Hamiltonian as \( H_{\text{tot}}^{(N)} = t_{\text{tot}} H_{\text{tot}}^{(N)} \). Here, we refer to the limit \( N \rightarrow \infty \) as diverging control complexity. Compressing a diverging number of finite-complexity operations thus yields a protocol of diverging control complexity. The fact that there exists such an operation that minimises both the time and energy requirements follows from our constructive proof of Theorem 1. We therefore have:

**Corollary 1.** With diverging control complexity, any finite-dimensional quantum system can be perfectly cooled at the Landauer limit in unit time.

Up to this point, we have deliberately kept the notion of control complexity vague to focus on intuition, hinting that it is based on the idea that one can compress all of the operations required in the diverging-time protocol into a single interaction (whose control complexity diverges). However, one need not construct complex protocols in this way, and so the natural concern becomes understanding the generic features that enable perfect cooling at the Landauer limit in unit time. We now turn to the task of making the notion of control complexity more precise. We first present some necessary conditions on the dimensionality of the machine, before turning to analyse the key properties that diverge in our protocols. In doing so, we highlight that one requires a more complicated structure involving the full energy-level distribution than these aforementioned necessary conditions on the dimensionality in order to achieve perfect cooling at the Landauer limit in unit time.

### V. NOTIONS OF COMPLEXITY

To address this issue, we begin by provide protocol-independent structural conditions that must be fulfilled by the
machine Hamiltonian to enable (1) perfect cooling and (2) cooling at Landauer cost; combined, these independent conditions provide a necessary requirement, namely that the machine must have an unbounded spectrum (from above) and be infinite-dimensional (respectively) for the possibility of (3) perfect cooling at the Landauer limit. The properties of the machine Hamiltonian define the structural complexity, which sets the potential for how cool the target system can be made and at what energy cost. However, the extent to which a machine’s potential is utilised depends on properties of the dynamics of a given protocol, i.e., the control complexity. We show that any protocol achieving perfect cooling at the Landauer limit necessarily involves interactions between the target and infinitely many energy levels of the machine. In other words, no interaction restricted to a finite-dimensional subspace suffices. Moreover, the degrees of freedom of the machine must be individually addressed in a fine-tuned manner to permute populations optimally, demonstrating that an operationally meaningful notion of control complexity must take into account factors beyond the effective dimensionality accessed by an operation.

A. Necessary Structural Conditions

1. Perfect Cooling.—Let us consider the task of perfect cooling, independently from protocol-specific constraints, in the envisaged setting. One can lower-bound the smallest eigenvalue $\lambda_{\text{min}}$ of the final state $\rho_S^f$ (and hence how cold the system can become) after any unitary interaction with a thermal machine by [29]

$$\lambda_{\text{min}}(\rho_S^f) \geq e^{-\beta \omega_{\text{max}} M \lambda_{\text{min}}(\rho_S)},$$

where $\omega_{\text{max}} := \max_{i,j} (\omega_j - \omega_i)$ denotes the largest energy gap of the machine Hamiltonian $H_M$ with eigenvalues $\omega_i$. Without loss of generality, throughout this article we set the ground state energy of any system to be zero, i.e., $\omega_0 = 0$, such that the largest energy gap coincides with the largest energy eigenvalue. As we have made no restrictions on the size or structure of the target or machine, the above inequality pertains to cooling protocols that could, for instance, be realised via sequences of unitaries on the target and parts of the machine. It follows that perfect cooling is only possible under two conditions: either the machine begins in a pure state ($\beta \rightarrow \infty$), or $H_M$ is unbounded, i.e., $\omega_{\text{max}} \rightarrow \infty$. Requiring $\beta < \infty$, a diverging energy gap in the machine Hamiltonian is thus a necessary structural condition for perfect cooling. Indeed, the largest energy gap of the machine plays a crucial role in limiting how cool the target system can be made (see also, e.g., Refs. [36, 37]). We now detail an independent property that is required for cooling with minimal energetic cost.

2. Cooling at the Landauer Limit.—Suppose now that one wishes to cool an initial target state $\tau_{\text{c}}(\beta, H_S)$ to any thermal state $\tau_{\text{c}}(\beta^*, H_S)$ with $\beta^* > \beta$ (not necessarily close to a pure state), at an energy cost saturating the Landauer limit. In Ref. [29], it was shown that for any finite-dimensional machine, there are correction terms to the Landauer bound which imply that it cannot be saturated; these terms only vanish in the limit where the machine dimension diverges. Thus, a necessary condition for achieving cooling with energy cost at the Landauer limit is provided by the following:

**Theorem 2.** To cool a target system from $\tau_{\text{c}}(\beta, H_S)$ to $\tau_{\text{c}}(\beta^*, H_S)$, with $\beta^* > \beta$, using a machine in the initial state $\tau_{M}(\beta, H_M)$ with energy cost at the Landauer limit, the machine must be infinite-dimensional.

As we will discuss later in this section, this minimal requirement for the notion of complexity is far from sufficient to achieve cooling at Landauer cost.

3. Perfect Cooling at the Landauer Limit.—We have two independent necessary conditions on the structure of the machine that must be asymptotically achieved to enable relevant goals for cooling: the former is required to achieve perfect cooling; the latter for cooling at the Landauer limit. Together, these conditions imply that in order to achieve perfect cooling at the Landauer limit, one must have an infinite-dimensional machine with a spectrum that is unbounded (from above).

**Corollary 2.** To perfectly cool a target system with energy cost at the Landauer limit using a thermal machine $\tau_{M}(\beta, H_M)$, the machine must be infinite-dimensional and $\omega_{\text{max}}$, the maximal eigenvalue of $H_M$, must diverge.

Henceforth, we will assume that these conditions are satisfied by the machine. The question then becomes: how does one engineer an interaction between the target system and machine to achieve perfect cooling at Landauer cost?

B. Control Complexity

The unbounded structural properties of the machine support the possibility for perfect cooling at the Landauer limit; however, we now focus on the control properties of the interaction that realise said potential (see Fig. 2). This leads to the distinct notion of control complexity, which aims to differentiate between protocols that access the machine in a more or less complex manner. The structural complexity properties are protocol-independent and related to the energy spectrum and dimensionality of the machine, whereas the control complexity concerns properties of the unitary that represents a particular protocol. For instance, the diverging-time protocol previously outlined comprises a sequence of interactions, each of which is individually not very complex; at the same time, the unconstrained control complexity protocol accesses the total (overall infinite-dimensional) machine “at once”, and thus the number of (nontrivial) terms in the interaction Hamiltonian, or the effective dimensionality of the machine accessed by the unitary, becomes unbounded. Nonetheless, the net energy cost of this protocol with unconstrained control complexity remains in accordance with the Landauer limit, as the initial and final states of both the system and machine are identical to those in the diverging-time protocol.

**Effective Dimensionality.**—We begin by demonstrating that the effective dimension (nontrivially) accessed by a unitary must diverge to achieve perfect cooling at the Landauer limit.
We define the effective dimension as the dimension of the subspace of the global Hilbert space upon which the unitary acts nontrivially, which can be quantified via the minimum dimension of a subspace $A$ of the joint Hilbert space $H_{SM}$ in terms of which the unitary can be decomposed as $U_{SM} = U_A \oplus \mathbb{I}_{A^\perp}$, i.e.,

$$d_{\text{eff}} := \min \dim(A) : U_{SM} = U_A \oplus \mathbb{I}_{A^\perp}. \quad (6)$$

This quantity can be computed by considering a given cooling protocol and finite unit of time $T$ (which we can set equal to unity without loss of generality) with respect to which the target and total machine transform unitarily by decomposing the Hamiltonian in $U_{SM} = e^{-iH_{SM}T}$ in terms of local and interaction terms, i.e., $H_{SM} = H_S \otimes \mathbb{I}_M + \mathbb{I}_S \otimes H_M + H_{\text{int}}$. The effective dimension then corresponds to $\text{rank}(H_{\text{int}})$. Lastly, note that the above definition in terms of a direct sum decomposition provides an upper bound on any similar quantification of effective dimensionalities based on other tensor factorizations of the joint Hilbert space considered and makes no assumption about the underlying structure. On the other hand, knowledge of said structure would permit a more meaningful notion of complexity to be defined. For instance, the effective dimensionality of a unitary acting on a many-qubit system is better captured by considering its decomposition into a tensor product factorization rather than the direct sum. We leave the exploration of such considerations to future work.

With this definition at hand, we have the following theorem, which is proven in Appendix D.

**Theorem 3.** The unitary representing a cooling protocol that saturates the Landauer limit must act nontrivially on an infinite-dimensional subspace of $\text{supp}(H_M)$. This implies $d_{\text{eff}} \to \infty$.

Intuitively, we show that if a protocol only accesses a finite-dimensional subspace of the machine, then the machine is effectively finite-dimensional inasmuch as a suitable replacement can be made while keeping all quantities relevant for cooling invariant. Invoking the main result of Ref. [29] then implies that there are finite-dimensional correction terms such that the Landauer limit cannot be saturated. The effective dimensionality provides a minimal quantifier for a notion of control complexity, insofar as its divergence is necessary for saturating the Landauer bound, as we prove in the next section. In fact, we prove a slightly stronger statement, namely that the dimension of the machine Hilbert space to which the unitary (nontrivially) couples the target system to must diverge. However, as we will discuss below, $d_{\text{eff}} \to \infty$ is generally insufficient to achieve said goal, and fine-tuned control is required. Nonetheless, the manifestation of such control seems to be system-dependent, precluding our ability (so far) to present a universal quantifier of control complexity. Thus, even though further conditions need to be met to achieve perfect cooling at minimal energy cost in unit time (see Theorem 10), whenever we talk of an operation with finite control complexity, we mean those represented by a unitary that acts (nontrivially) only on a finite-dimensional subspace of the target system and machine. In contrast, by diverging control complexity, we mean a unitary that couples the target (nontrivially) to a full basis of the machine’s Hilbert space, whose dimension diverges.

Note that in Theorem 3 no particular structure of the machine is presupposed and the effective dimensionality relates to various notions of complexity put forth throughout the literature (see, e.g., Refs. [38, 39]). For instance, for a finite-dimensional target system with equally spaced energy levels $\omega_j$, suppose that the machine structure is decomposed as $N$ qubits with energy gaps $\omega_{M,j} \in \{\omega_j + n\epsilon\}_{n=1,...,N}$, with arbitrarily small $\epsilon > 0$ and $N \to \infty$. Then the overall unitary that approaches perfect cooling at the Landauer limit has circuit complexity equal to the diverging $N$. Taking a different perspective, in the infinite-dimensional setting, the difficulty of implementing an operation is often related to the polynomial degree of its generators. Here, we see some friction with respect to Eq. (6): a generic Gaussian unitary operation (i.e., one generated by a Hamiltonian at most quadratic in the mode operators) between a harmonic oscillator target and machine would already use an infinite effective dimensionality. For this reason, we treat the case of harmonic oscillators as a representative for infinite-dimensional systems separately in Appendix E. There, we first construct a protocol that achieves perfect cooling at the Landauer limit with diverging time using only sequences of Gaussian operations [i.e., those typically considered to be practically easily implementable, but nonetheless with infinite effective dimensionality according to Eq. (6)]. In contrast, we then present a protocol that demonstrates that perfect cooling is possible given diverging time and operations acting on only a finite effective dimensionality (i.e., using non-Gaussian operations), with a finite energy cost that is greater than the Landauer limit; whether or not a protocol that saturates the Landauer limit exists in this setting remains an open question. Lastly, we present a unit-time cooling protocol with diverging control complexity that nonetheless can be achieved using only Gaussian operations.

The above examples demonstrate some perhaps counterintuitive points regarding the difficulty of defining a universal quantifier of complexity whose divergence is both necessary and sufficient for perfect cooling at the Landauer limit. Nonetheless, as we now discuss, cooling at minimal energy cost generally requires fine-tuned control.

**Fine-tuned Control.**—Theorem 3 captures a notion of control complexity as a resource in a thermodynamically consistent manner, i.e., in line with Nernst’s unattainability principle. However, following the discussion around Theorem 2, the protocols that we present that achieve perfect cooling at Landauer cost make use of machines and interactions with a far more complicated structure than suggested by the necessary condition of infinite effective dimensionality. In particular, the interactions couple the target system to a diverging number of subspaces of the machine corresponding to distinct energy gaps in a fine-tuned manner. Moreover, there are a diverging number of energy levels of the machine both above and below the first excited level of the target.

This suggests that an operationally meaningful quantifier of control complexity must take into account the energy-level structure of the machine that is accessed throughout any given
protocol; additionally that of the target system plays a role. Indeed, both the final temperature of the target as well as the energy cost required to achieve this depends upon how the global eigenvalues are permuted via the cooling process. First, how cool the target becomes depends on the sum of the eigenvalues that are placed into the subspace spanned by the ground state. Second, for any fixed cooling amount, the energy cost depends on the constrained distribution of eigenvalues within the machine. Thus, in general, the optimal permutation of eigenvalues depends upon properties of the target and machine.

For instance, consider an arbitrary initially thermal target qubit, whose state is given by $\rho_0 = \rho(1 - p)$ and a thermal machine of dimension $d_M$ with spectrum $\{\lambda_{n_j}^i\}_{i=0,\ldots,d_M-1}$. Now consider the decomposition of the joint Hilbert space into two orthogonal subspaces, $B_0$ and $B_1$, corresponding to the ground and excited eigenspaces of the target. The initial joint state is $\rho = \rho_0 \otimes (1 - p) \rho_1$, where we have written $\lambda_{n_j}^i$ to denote the $i$th machine eigenvalue in the subspace $B_j$. The total population in the subspaces $B_0$ and $B_1$ are $p$ and $(1 - p)$, respectively. To achieve perfect cooling one must permute the eigenvalues such that a approximately a net transfer of population $(1 - p)$ is moved from $B_1$ to $B_0$. To do this, one can take any subset $\mathcal{K}$ of $\{\lambda_{n_j}^i\}$ such that as $d_M \to \infty$, $\sum_{i \in \mathcal{K}} \lambda_{n_j}^i \to (1 - p)$ and a subset $\mathcal{K}'$ (with $|\mathcal{K}'| = |\mathcal{K}'|'$) from $\{\lambda_{n_j}^i\}$ such that $\sum_{i \in \mathcal{K}'} \lambda_{n_j}^i \to 0$ and exchange them. Although the choice of eigenvalues permuted is non-unique, the requirement must be fulfilled for some sets to perfectly cool the target. For any pair of eigenvalues exchanged between the subspaces, demanding that the exchange costs minimal energy amounts to a fine-tuning condition of the form $\lambda_{n_j}^i \to p\lambda_{n_j}^i + (1 - p)\lambda_{n_j}^j$, that must be satisfied. In general, the fine-tuned eigenvalue conditions that must be asymptotically attained depend upon target and machine eigenvalues, making it difficult to derive a closed-form expression. However, in the restricted scenario in which the target qubit begins maximally mixed (i.e., at infinite temperature), the machine begins thermal at some $\beta > 0$ and of dimension $d_M$, and that the unitary implemented is such that the target is cooled as much as possible, one can derive precise conditions in terms of the machine structure alone, as we demonstrate in Appendix D. The case for higher-dimensional target systems is similar.

This discussion highlights the importance of capturing properties beyond the effective dimensionality, e.g., those regarding the distribution of machine (and, more generally, target system) eigenvalues, in order to meaningfully quantify control complexity in thermodynamics. Our protocols display similar behaviour to that discussed above asymptotically. Moreover, the machines exhibit an energy-level structure such that every possible energy gap is present, i.e., the set of machine energy gaps $\{\omega_{ij} = \omega_i - \omega_j\}$ densely covers the interval $[\omega_{\min}, \infty)$, where $\omega_{\min}$ is the energy of the first excited level of the target. Whether or not such a condition is necessary in general remains an open question. Additionally, how any such notion of control complexity relates to what is achievable in laboratory setups remains to be characterised. This concept is even more important in the case where all resources are finite, as particular structures of machines and the types of interactions permitted play a crucial role in both how much time or energy is spent cooling a system and how cold the system can ultimately become (see, e.g., Ref. [40–42]). In the Outlook section, we discuss some of these problems in detail.

VI. INCOHERENT-CONTROL PARADIGM

Until now, we focused on the task of cooling where the only restriction has been that the machines begin in thermal states. In particular, there are no restrictions on the allowed global unitaries. In general, the operations required for cooling are not energy-conserving and require an external work source. With respect to standard considerations of thermodynamics, this may seem somewhat unsatisfactory, as the joint system is, in the coherent setting, open to the universe. When quantifying thermodynamic resources, one typically restricts the permitted transformations to be energy conserving, thereby closing the joint system and yielding a more self-contained theory. This picture is still not fully self-contained as there is an energy cost associated to switching on and off the interactions. Nonetheless, once this cost is accounted for, any such protocol can be executed without additional control.

We therefore analyse protocols using energy-conserving unitaries. With this additional restriction, it is not possible to cool a target system with machines that are thermalised at a single temperature, as considered in the coherent-control paradigm [37]. Instead, cooling can be achieved by partitioning the machine into one cold subsystem $\mathcal{C}$ that begins in equilibrium at inverse temperature $\beta$ and another hot subsystem $\mathcal{H}$ coupled to a heat bath at inverse temperature $\beta_H < \beta$ [35, 37, 40] (see Fig. 1). In other words, one uses a hot and cold bath to construct a heat engine that cools the target. As we demonstrate, perfect cooling can be achieved in this setting as pertinent resources diverge.

We now analyse the energy–time–complexity trinity within this incoherent-control context. Here, we focus on finite-dimensional systems and leave the analysis of infinite-dimensional ones to future work. As we will see, the structure of the hot bath plays a crucial role in the ability to cool perfectly, especially regarding the resource requirements. Already in the simplest cases, we see a departure from the results presented in Secs. IV and V, where no restrictions on the global unitaries were imposed.

A. Cooling at the Carnot-Landauer Limit

In the incoherent-control setting, an equality-form adaptation of Landauer’s bound on the minimum heat dissipated (or, as we phrase it here, the minimum amount of energy drawn from the hot bath), which we dub the Carnot-Landauer limit, can be derived:

**Theorem 4.** Let $F_\beta(\varrho_\beta) := \text{tr}[H_\beta \varrho_\beta] - \beta^{-1}S(\varrho_\beta)$ be the free energy of a state $\varrho_\beta$ with respect to a heat bath at inverse temperature $\beta$, $\Delta F_{\beta}^{(2)} := F_\beta(\varrho_\beta) - F_\beta(\bar{\varrho}_\beta)$, and let $\eta := 1 - \beta_H/\beta \in (0, 1)$ be the Carnot efficiency with respect to
the hot and cold baths. In the incoherent-control setting, the quantity
\[
\Delta F_s^{(\beta)} + \eta \Delta E_H
\]
\( \leq \frac{1}{\beta} [\Delta S_s + \Delta S_c + \Delta S_n + D(\rho_c || \rho_c') + D(\rho_n || \rho_n')] \) (7)
satisfies the inequality
\[
\Delta F_s^{(\beta)} + \eta \Delta E_H \leq 0. \quad (8)
\]
Equation (8) holds due to the non-negativity of the sum of local entropy decreases and the relative-entropy terms. The derivation is provided in Appendix A, where we also show that the usual Landauer bound is recovered in the limit of an infinite-temperature hot bath.

By explicitly accounting for a thermal energy source, Theorem 4 generalises Landauer’s erasure principle and, at the same time, unifies it with the laws of thermodynamics. The consideration of cooling vis-à-vis information erasure already generalises Landauer’s principle by specifying a target system Hamiltonian. By further incorporating a heat-bath energy source we connect the heat dissipated during a cooling process with the thermodynamic resource driving it. Analysing the resource costs in turn provides a unification of cooling (and, as such, erasure) with the laws of thermodynamics. For instance, the Carnot efficiency between the two baths determines the cooling efficiency, implying that Landauer-cost cooling is impossible within any resource theory of thermodynamics with finite-temperature heat baths. Moreover, considering an infinite-temperature heat bath resource (which can be identified as a work source) one recovers the relation between work and free-energy difference that embodies the second law of thermodynamics. Alternatively, if one were to consider the incoherent-control scenario with any finite-temperature hot bath and not specify an energetic structure of the target system, i.e., not attribute a Hamiltonian to the target, the Landauer information erasure procedure fuelled by a heat engine is recovered. Taking both of these reductions together yields the regular Landauer bound, i.e., perfectly efficient information erasure where the heat dissipated into the environment is exactly that drawn from the energy source.

Note that the incoherent-control setting is fundamentally distinct from the coherent-control setting in terms of what can (or cannot) be achieved with a given combination of resources. For instance, consider the case where one wishes to implement perfect cooling in unit time and with finite control complexity with diverging energy costs; in the coherent-control setting, this task is possible in principle (see Sec. IV A), whereas we show in Theorem 5 that it is impossible in the incoherent-control setting, even as the temperature of the hot bath diverges. Thus, while we have above presented a Landauer-like bound in the incoherent-control setting that posits the ultimate limitation on cooling, it is, a priori, unclear if the bound is attainable and, if so, how to attain it. This problem persists for the special case where the heat-bath temperature tends to infinity and the bound reduces to the standard Landauer limit. Indeed, the restriction to energy-conserving unitaries generally makes it difficult to tell if the ultimate bounds can be saturated in the incoherent-control setting, and which resources would be required to do so.

In the coming section, we first present a no-go theorem for perfect cooling with incoherent control, even given a diverging amount of energetic resource. We then analyse protocols that require minimum energy expenditure at the expense of diverging time or control complexity. To this end, we first present cooling protocols that saturate the Landauer bound in the regime where the heat-bath temperature goes to infinity, thereby acting as a work source of unbounded energy. We do so by fine-tuning the machine structure such that the desired cooling transitions are rendered energy conserving between the target system and the cold and hot parts of the machine. Following our analysis of the infinite-temperature heat-bath scenario, we study the more general case of finite-temperature heat baths. We detail cooling protocols that saturate the Carnot-Landauer limit for any finite-temperature heat bath, for finite-dimensional systems with arbitrary but fixed Hamiltonians. As in the coherent-control setting, these protocols use either diverging time or control complexity to asymptotically saturate the Carnot-Landauer bound. The results presented in this section therefore provide a comprehensive understanding of the resources required to perfectly cool at minimum energy cost in a setting that aligns with the resource theories of thermodynamics.

### B. The Energy-Time-Complexity Resource Trinity in the Incoherent-Control Setting

#### 1. Diverging Energy

Suppose that the hot bath is at infinite temperature, thereby providing the ultimate cooling resource possible in the incoherent scenario. Then, in the scenario where the energy drawn from the hot bath is allowed to diverge and one wishes to minimise both time and control complexity, the following no-go theorem holds (see Appendix F for proof):

**Theorem 5.** In the incoherent-control scenario, it is not possible to perfectly cool any quantum system of finite dimension in unit time and with finite control complexity, even given diverging energy drawn from the hot bath, for any non-negative inverse temperature heat bath \( \beta_H \in [0, \beta < \infty] \).

**Sketch of Proof.** In the incoherent-control setting, the target system can only interact with subspaces of the joint hot-and-cold machine with respect to which it is energy degenerate. Now, the global ground-state level of the joint hot-and-cold machine has some non-zero initial population for any finite-dimensional machine; in particular it can always be lower-bounded by \( \frac{1}{d_m} \) for any Hamiltonians and initial temperatures, which is strictly greater than zero as long as the dimensions remain finite. Fixing the control complexity of any protocol considered here to be finite thus implies a lower bound on the accessed initial ground-state population of the total machine that is larger than zero by a finite amount. To perfectly cool the target, all such populations (as well as any other non-zero populations) must be transferred to the subspace spanned
by the ground state of the system. However, in order to make such transfers in an energy-conserving manner, any $|i00\rangle_{SCN}$ must be degenerate with respect to some $|0jk\rangle_{SCM}$. We then show that the population of the latter subspace, $p_{0jk}$, is always non-vanishing for machines of finite complexity, implying that upon transferring some $p_{i00}$ into the ground-state subspace of the target, i.e., to any eigenstate of the form $|0jk\rangle_{SCM}$, one inevitably transfers some non-zero population $p_{0jk}$ away and instead into an eigenstate of the form $|i00\rangle_{SCN}$. Thus, one will always end up with some population that does not contribute to the final ground-state population of the target, implying that perfect cooling cannot be achieved.

2. Saturating the Landauer Limit with an Infinite-Temperature Heat Bath

In contrast, given an infinite-temperature hot bath and diverging time, it is possible to perfectly cool at the Landauer limit with finite control complexity interactions:

**Theorem 6.** In the incoherent-control scenario, for an infinite-temperature hot bath $\beta_H = 0$, any finite-dimensional system can be perfectly cooled at the Landauer limit with diverging time via finite control complexity interactions.

Similarly to Sec. IV, one can trade between time and control complexity:

**Corollary 3.** In the incoherent-control scenario, for an infinite-temperature hot bath $\beta_H = 0$, without restrictions on control complexity, any finite-dimensional quantum system can be perfectly cooled at the Landauer limit in unit time.

Theorem 6 and Corollary 3 are proven in Appendix F. Throughout our analysis there, we see that the situation becomes more complicated when finite-temperature hot baths are considered. The key quantities of interest in the incoherent paradigm are the populations of the subspaces in the joint hot-and-cold system (i.e., the total machine) that are energy-degenerate with respect to the target system and their corresponding effective energy gaps. We call these “effective” energy gaps because they need not correspond to physical gaps in either the hot or cold parts of the machine, rather only those in the joint Hilbert space, and furthermore because not all energy gaps of the joint hot-and-cold machine can be accessed by an energy-conserving unitary; only those that match the gaps in the target system can be. Crucially, the relevant effective energy gaps depend on the temperature of the baths in such a way that an additional energy cost is incurred when implementing the desired operations. Indeed, this fact is reflected by the Carnot efficiency factor that appears in the Carnot-Landauer limit.

3. Saturating the Carnot-Landauer Limit with a Finite-Temperature Heat Bath

We now move to analyse the resources required to perform perfect cooling in the incoherent setting at minimum energy cost, i.e., at the Carnot-Landauer limit, for any finite-temperature heat bath. By explicit construction in Appendix G, we can prove the following theorem:

**Theorem 7.** In the incoherent-control scenario, for any finite-temperature hot bath $0 < \beta_H < \beta$, any finite-dimensional quantum system can be perfectly cooled at the Carnot-Landauer limit given diverging time via finite control complexity interactions.

And:

**Corollary 4.** In the incoherent-control scenario, for any finite-temperature hot bath $0 < \beta_H < \beta$, without restrictions on control complexity, any finite-dimensional quantum system can be perfectly cooled at the Carnot-Landauer limit via a single interaction (i.e., in unit time).

These protocols thus attain the ultimate limitation for perfect cooling at minimal energetic cost within the incoherent-control setting, thereby contextualising the tasks of cooling quantum systems and erasing quantum information with respect to the resource considerations of quantum thermodynamics.

VII. CONCLUSIONS

In this work, we have presented a flexible framework that allows one to properly account for the resource costs relevant for cooling quantum systems and erasing quantum information. This framework includes two extremal paradigms of control: the coherent-control paradigm is closer in spirit to the original setting analysed by Landauer [10], whereas the incoherent-control paradigm reflects the typical setting envisaged in the context of the laws of thermodynamics [43].

In the former setting, we presented optimal protocols for asymptotically achieving perfect cooling when any one of the three identified resources—energy, time, or control complexity—diverges. In particular, we presented protocols that saturate the Landauer limit when either time or control complexity diverge. The protocol in which perfect cooling is achieved at the Landauer limit and in unit time is seemingly at odds with the unattainability principle of Nernst; however, by introducing the notion of control complexity, we resolve this apparent paradox. Importantly, we identified necessary conditions on both the machine structure and the unitary used in the corresponding protocol. Furthermore, we highlighted the importance of capturing structure beyond the effective dimensionality of the machine by exemplifying a set of fine-tuning conditions that must be satisfied to approach perfect cooling at minimal energy cost for a special case.

We then moved to analyse the task of cooling in the incoherent-control setting, which aligns our work with the typical assumptions of quantum thermodynamics, in the sense that the setting requires minimal external control. Here, we showed that there are already a number of stark differences to the coherent-control paradigm, further highlighting the important role of the level of control afforded to an agent. In particular, we derived a Carnot-Landauer bound that implies that for
any finite-temperature heat bath, the standard Landauer limit can never be attained, even asymptotically. Finally, we presented protocols that saturate the Carnot-Landauer limit for any finite-dimensional systems with arbitrary Hamiltonians.

The results of this work have wide-ranging implications. We have both generalised and unified Landauer’s bound with respect to the laws of thermodynamics. In particular, we have posed the ultimate limitations for cooling quantum systems or erasing quantum information in terms of resource costs and presented protocols that asymptotically saturate these limits. Indeed, while it is well-known that heat and time requirements must be minimised to combat the detrimental effects of fluctuation-induced errors and short decoherence times on quantum technologies [17], we have shown that this comes at a practical cost of greater control. In particular, we have demonstrated the necessity of implementing fine-tuned interventions to minimise energy and time costs, which serves to deliver a cautionary message: Control complexity must be accounted for to build meaningful resource theories of quantum thermodynamics. Our analysis of the incoherent-control setting provides pragmatic ultimate limitations for the scenario where minimal overall control is required, in the sense that all transformations are driven by thermodynamic energy and entropy flows between two heat baths, which could be viewed as a thermodynamically-driven quantum computer. Nevertheless, the intricate relationship between various resources will need to be further explored.

VIII. OUTLOOK

Looking forward, we believe it will be crucial to go beyond asymptotic limits. While Landauer erasure and the third law of thermodynamics conventionally deal with the creation of pure states, practical results would need to consider cooling to a finite temperature (i.e., creating only approximately pure states) with a finite amount of invested resources [40–42]. In this context, the trade-off between time and control complexity will gain more practical relevance, as realistic quantum technologies usually have limited coherence times and interaction Hamiltonians are practically limited to few-body terms. Here, operational measures of control complexity that fit the envisioned experimental setup present an important challenge that would need to be overcome to extend our results to realistic applications.

Furthermore, inasmuch as we have considered time to be discretised (in terms of the number of unitary operations between rethermalisations), our results can be seen as bridging the gap between thermal operations [44, 45] and collision models [46–48]. As such discrete-time transformations naturally arise out of thermodynamic master equations, we can also expect our results to translate either to continuously driven (coherently controlled) or autonomous (incoherently controlled) machines. Here, computing the rates of energy and entropy flows would complement the asymptotic picture of cooling protocols.

Our results strengthen the view that, in contrast to classical thermodynamics, the role of control is one of the most crucial issues to address before a true understanding of the limitations and potential of quantum machines is revealed. On the one hand, in classical systems, control is only ever achieved over few bulk degrees of freedom, whereas addressing and designing particular microstate control is within reach of current quantum technological platforms, offering additional routes towards operations enhanced by fine-tuned control. On the other hand, the thermodynamic cost of such control itself can quickly exceed the energy scale of the system, potentially rendering any perceived advantages a mirage. This is exacerbated by the fact that it is not possible to observe (measure) a quantum machine without incurring significant additional thermodynamic costs [8, 49] and non-negligible back-action on the operation of the machine itself [50]. A fully developed theory of quantum thermodynamics would need to take these costs into account and we hope that our study sheds light on the role of control complexity in this endeavour.

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AUTHOR CONTRIBUTIONS

P. T., F. B., A. B., R. S., N. F., F. C., and M. H. conceived the project. P. T. and M. H. supervised and organised the work. P. T., N. F., and M. H. wrote the main text. P. T., F. B., A. B., R. S., N. F., M. P. E. L., G. V., F. C., and M. H. proved the main results. E. S. made the figures. All authors read the manuscript, discussed and interpreted results, and provided feedback.


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Appendix A: Equality Forms of the (Carnot-)Landauer Limit

In this section, we present lower bounds on the energy change of the machine (or heat dissipated into its environment) in terms of the entropy change of the target system, both in the coherent and incoherent-control settings outlined in the main text. In the coherent setting, this amounts to the well-known Landauer’s principle [10], whereas the incoherent setting requires an extension of this derivation. These lower bounds are important, because they put limits on the optimal energetic performance of the machines for cooling.


The coherent setting was already studied in detail in Ref. [29], where the authors derived an equality version of Landauer’s principle. We restate the results here for convenience, since we will also use them in the incoherent paradigm. Recall that the setting we consider consists of two parts, the target system $S$ and the machine $M$. In the beginning, the joint state is $\rho_{SM} = \rho_S \otimes \tau_M(\beta, H_M)$ for some arbitrary (but fixed) Hamiltonian $H_M$ and $\beta \in \mathbb{R}^+$. We assume that both systems are finite-dimensional. Let $U$ be a global unitary on $SM$. We write $\rho'_{SM} := U[\rho_S \otimes \tau_M(\beta, H_M)]U^\dagger$ and denote by $\rho'_S$ and $\rho'_M$ the respective reduced states. The quantity $I(S : M)_{\rho'_{SM}} = S(\rho'_S) + S(\rho'_M) - S(\rho'_{SM})$ is the final mutual information between $S$
and $M$ and $D(\rho_\text{f} || \rho_M) = \text{tr} [\rho_\text{f}' \log(\rho_\text{f}')] - \text{tr} [\rho_\text{f}' \log(\rho_M)]$ is the relative entropy of the final machine state with respect to its initial state.

**Lemma 1** ([29, Lemma 2]). Let the setting be as above. Then
\[
[S(\rho_S') - S(\rho_S)] + [S(\rho_\text{f}') - S(\rho_M)] = I(S : M) \rho_\text{f}',M = 0. \tag{A1}
\]

**Proof.** We note that
\[
[S(\rho_S') - S(\rho_S)] + [S(\rho_\text{f}') - S(\rho_M)] = S(\rho_S') + S(\rho_\text{f}') - S(\rho_S) - S(\rho_M),
\]
(A2)

since the von Neumann entropy is additive for product states and invariant under unitary evolution. The assertion follows from the definition of the mutual information and the fact that it is non-negative. □

**Theorem 8** (Equality form of Landauer’s principle, [29, Theorem 3]). Let the setting be as above. Then
\[
\beta \text{tr}[H_M(\rho_\text{f}' - \rho_M)] - [S(\rho_S') - S(\rho_S)] = I(S : M) \rho_\text{f}',M + D(\rho_\text{f}' || \rho_M) \geq 0. \tag{A3}
\]

**Proof.** From Lemma 1, it follows that
\[
[S(\rho_S') - S(\rho_S)] + I(S : M) \rho_\text{f}',M = S(\rho_\text{f}') - S(\rho_M). \tag{A4}
\]

Using the fact that $\rho_M = \tau_M(\beta, H_M)$, we infer that $D(\rho_\text{f}' || \rho_M) = -S(\rho_\text{f}') + \beta \text{tr}[H_M \rho_\text{f}'] + \log [\text{tr} (e^{-\beta H_M})]$ and $S(\rho_M) = \beta \text{tr}[H_M \rho_M] + \log [\text{tr} (e^{-\beta H_M})]$. Re-expressing the first of these for $S(\rho_\text{f}')$ and inserting both yields Eq. (A4). The inequality results from non-negativity of relative entropy and mutual information. This completes the proof. □

**A2. Incoherent-Control Paradigm: the Carnot-Landauer Limit**

Landauer’s principle provides a relationship between how much heat must necessarily be dissipated into the thermal background environment upon manipulating the entropy of a given quantum system. Until now, we have assumed that the system of interest can interact arbitrarily with its environment (i.e., the machine); in other words, we have considered general joint unitary interactions between system and machine, without restriction. Furthermore, it is sensible to assume that the energy resource system can interact arbitrarily with its environment (i.e., the machine); in other words, we have considered general joint ground environment upon manipulating the entropy of a given quantum system. Until now, we have assumed that the system of interest can interact arbitrarily with its environment (i.e., the machine); in other words, we have considered general joint

To this end, note first that said resource must be out of thermal equilibrium with respect to the target and machine in order to perform any meaningful thermodynamic transformation. Furthermore, it is sensible to assume that the energy resource system is in thermal equilibrium with its own environment to begin with. The joint target-machine-resource system is then considered to be energetically closed; as such, global unitaries in this setting are restricted to be energy conserving. In order to act as a resource for cooling the target in this picture, the energy source here must begin in equilibrium with a heat bath that is hotter than the initial temperature of the machine (assuming that the machine and resource both begin in thermal states), such that a natural heat flow is induced that leads the environment of the machine to act as a final heat sink. This setting is what we call the incoherent-control scenario. In this context, Landauer’s principle translates to studying the relationship between the heat that is necessarily dissipated into the machine’s environment upon manipulating the entropy of the target system. Finally, note that the relationship between the coherent and the incoherent-control paradigms is interesting in itself: while on the one hand the incoherent setting includes an additional system and therefore increases the dimensionality of the overall joint system, on the other hand by restricting the transformations on this larger space to be energy conserving, one can explicitly include the energy resource, modelled as a quantum system itself, into the setting.

Now let us consider the incoherent-control setting. Here, we have the target system $S$ and the machine comprises of one part $C$ coupled to the cold bath and another $H$ coupled to the hot bath. We assume that all systems are finite-dimensional. Every subsystem $A$ is associated with a Hamiltonian $H_A$ and $\mathcal{C}, \mathcal{H}$ are initially in a thermal state; the cold bath has inverse temperature $\beta$ and the hot bath has inverse temperature $\beta_H < \beta$. We assume $\beta, \beta_H$. Thus, the initial joint state is $\rho_{\text{scn}} = \rho_S \otimes \tau_S(\beta, H_S) \otimes \tau_M(\beta_H, H_M)$. The global evolution on $\mathcal{SCH}$ is implemented via a unitary $U$, leading to $\rho_{\text{scn}}' = U(\rho_{\text{scn}})U^\dagger$. We further assume that the unitary evolution on the joint system is energy conserving, i.e., $[U, H_S + H_C + H_M] = 0$. We write $\Delta S_A := S(\rho_S') - S(\rho_S)$ for the entropy change on subsystem $A$ and $\Delta E_A := \text{tr}[H_A(\rho_S' - \rho_S)]$ for the average energy change. Moreover, the free energy of a state $\rho_S$ with respect to the inverse temperature $\beta$ is $F_\beta(\rho_S) = \text{tr}[H_A \rho_S] - \beta^{-1} S(\rho_S)$.

In the incoherent setting, it makes sense to look at the energy decrease in the hot bath $\mathcal{H}$, since the hot bath can be seen as the energetic resource one must to expend in order to cool the system $S$ (alternatively, as we present after the following theorem, one can consider the energy dissipated into the cold bath $C$, which serves as the heat sink).
Theorem 9. In the above setting, it holds that

\[ \Delta F^{(\beta)}_s + \eta \Delta E_n = -\frac{1}{\beta} [\Delta S_s + \Delta S_c + \Delta S_n + D(\varrho'_c||\varrho_c) + D(\varrho'_n||\varrho_n)] \leq 0, \]  

(A5)

where \( (0, 1) \ni \eta := 1 - \beta_n / \beta \) is the Carnot efficiency and \( \Delta F^{(\beta)}_s = F_\beta(\varrho'_s) - F_\beta(\varrho_s) \).

Proof. Let us consider

\[ I(S : C : H)_{\varrho_{SCN}} := S(\varrho'_s) + S(\varrho'_c) + S(\varrho'_n) - S(\varrho_{SCN}) \geq 0. \]  

(A6)

Note that the quantity \( I(S : C : H)_{\varrho_{SCN}} \), which quantifies the tripartite mutual information of the state \( \varrho_{SCN} \), is non-negative via subadditivity \( S(\varrho_s) + S(\varrho_c) \geq S(\varrho_{AB}) \) for any state \( \varrho_{AB} \). Furthermore, since the von Neumann entropy is invariant under unitary transformations and additive for tensor product states, we have

\[ I(S : C : H)_{\varrho_{SCN}} = \Delta S_s + \Delta S_c + \Delta S_n. \]  

(A7)

We also have that

\[ \Delta S_c = \beta \Delta E_c - D(\varrho'_c||\varrho_c) \]  

(A8)

and

\[ \Delta S_n = \beta_n \Delta E_n - D(\varrho'_n||\varrho_n). \]  

(A9)

Thus,

\[ I(S : C : H)_{\varrho_{SCN}} = \Delta S_s + \beta \Delta E_c - D(\varrho'_c||\varrho_c) + \beta_n \Delta E_n - D(\varrho'_n||\varrho_n). \]  

(A10)

Since the unitary is energy conserving, we infer that \( \Delta E_s + \Delta E_c + \Delta E_n = 0 \). Hence, we have

\[ \Delta S_s - \beta \Delta E_s + (\beta_n - \beta) \Delta E_n = I(S : C : H)_{\varrho_{SCN}} + D(\varrho'_c||\varrho_c) + D(\varrho'_n||\varrho_n). \]  

(A11)

Using the free energy, we can rewrite this as

\[ -\beta [F_\beta(\varrho'_s) - F_\beta(\varrho_s)] - (\beta - \beta_n) \Delta E_n = I(S : C : H)_{\varrho_{SCN}} + D(\varrho'_c||\varrho_c) + D(\varrho'_n||\varrho_n). \]  

(A12)

Dividing by \( -\beta \), we obtain the assertion, since, in particular, \( I(S : C : H)_{\varrho_{SCN}} \) and \( D(\varrho'_c||\varrho_c) + D(\varrho'_n||\varrho_n) \geq 0 \) by the non-negativity of each term.

In particular, we have shown that the energy extracted from the hot bath is lower-bounded by the increase in free energy, weighted by the inverse Carnot efficiency:

\[ \text{tr}[H_n(\varrho_n - \varrho'_n)] \geq \frac{1}{\eta} [F_\beta(\varrho'_s) - F_\beta(\varrho_s)]. \]  

(A13)

Note that if \( \varrho_s = \tau_s(\beta, H_s) \), the r.h.s. is non-negative for any nontrivial thermodynamic process, i.e., any for which the target system is heated or—of particular relevance for us—cooled. This follows by the Gibbs variational principle which states that the free energy of \( \varrho \) is minimal if and only if \( \varrho \) is the Gibbs state.

Finally, in order to make a more concrete connection to the spirit of Landauer’s original derivation, note that one can consider bounding the heat dissipated into the cold bath, rather than that drawn from the hot bath. Substituting \( \Delta E_n = - (\Delta E_s + \Delta E_c) \) into Eq. (A10) leads to

\[ -\Delta S_s - \beta_n \Delta E_s + (\beta - \beta_n) \Delta E_c \geq 0, \]  

(A14)

which recovers the standard Landauer bound for the dissipated heat in the limit of an infinitely-hot heat bath, i.e., \( \beta_n \to 0 \).
Appendix B: Necessity of Diverging Energy for Protocols with Finite Time and Control Complexity

Consider the following Hamiltonians for the target system and machine with finite but otherwise arbitrary energy levels, \( H_S = \sum_{n=0}^{d_S-1} \omega_S^n |n\rangle \langle n| \) and \( H_M = \sum_{n=0}^{d_M-1} \omega_M^n |n\rangle \langle n| \), respectively. For any finite inverse temperature \( \beta \), the initial thermal states \( \tau_S(\beta, H_S) \) and \( \tau_M(\beta, H_M) \) are of full rank. Suppose now that one can implement a single unitary transformation (i.e., a unit time protocol) of finite control complexity on the joint target and machine, yielding the joint output state \( \rho_{S,M}^\prime = \text{tr}_M \left( \tau_S(\beta, H_S) \otimes \tau_M(\beta, H_M) \right) U^\dagger \), and wishes to attain perfect cooling of the target in doing so. By invariance of the rank under unitary transformations and the fact that the system and machine begin uncorrelated, we have

\[
\text{rank}[\tau_S(\beta, H_S)] \text{rank}[\tau_M(\beta, H_M)] = \text{rank}[\rho_{S,M}'] \leq \text{rank}[\rho_S] \text{rank}[\rho_M'],
\]

(B1)

where the inequality follows from the subadditivity of the Rényi-zero entropy \([51]\), which is the logarithm of the rank. To achieve perfect cooling of the target, one must (at least asymptotically) attain \( \text{rank}[\rho_S'] < \text{rank}[\tau_S(\beta, H_S)] \), which implies that \( \text{rank}[\rho_S'] > \text{rank}[\tau_S(\beta, H_S)] \). However, if this condition is achieved, then \( D(\rho_M' \parallel \tau_M(\beta, H_M)) \) diverges, implying a diverging energy cost by Eq. (3). The above argument already appears in Ref. [29].

The other situation that one must consider is the case where one attains a \( \rho_S' \) such that \( \text{rank}[\rho_S'] = \text{rank}[\tau_S(\beta, H_S)] \) but nonetheless \( \rho_S' \) is arbitrarily close to a pure state, as is the case, for instance, in the protocols that we present. Consider a sequence of machines \( \rho_M(i) \) and unitaries \( U(i) \) such that \( \rho_M(i) \rightarrow \rho_M \) and \( U(i) \rightarrow U \). Note that since we fixed the dimensions of \( S \) and \( M \), any sequence of machines has a converging subsequence by the Bolzano-Weierstrass theorem and the fact that the set of quantum states is compact. Here, \( \rho_M \) and \( U \) achieve perfect cooling. If we fix \( \rho_M \), we obtain a corresponding sequence \( \rho_M(i) \rightarrow \rho_M \). Crucially, here, since we restrict the unitary transformation to be of finite control complexity, the states \( \rho_M \) are effectively finite-dimensional, in the sense that whatever their true dimension, they can be replaced by finite-dimensional versions without changing any of the relevant quantities (see Appendix D). Since the relative entropy \( (\rho, \sigma) \rightarrow D(\rho \parallel \sigma) \) is lower semicontinuous \([52, 53]\) and since \( D(\rho_M \parallel \rho_M(i)) \rightarrow \infty \) by the arguments above, we infer that \( D(\rho_M(i) \parallel \rho_M) \rightarrow \infty \) as \( i \rightarrow \infty \). This argument holds independently of \( \text{rank}[\rho_S'] \); in particular, for the special case \( \text{rank}[\rho_S'] = \text{rank}[\tau_S(\beta, H_S)] \) that we are considering here. Thus, to approach perfect cooling in unit time and with finite control complexity, one would need a diverging energy cost.

Importantly, the above argument no longer holds if the time or control complexity is allowed to diverge. In such cases, both \( \rho_M \) and \( \rho_M' \) can be infinite-dimensional, and because of this the rank argument no longer applies and the relative entropy does not necessarily diverge in the limit of perfect cooling. On the contrary, as we show, it is even possible to saturate the Landauer bound.

Appendix C: Diverging Time Cooling Protocol for Finite-dimensional Systems

C1. Proof of Theorem 1

Proof. Consider a target system \( S \) of dimension \( d \) with associated Hamiltonian

\[
H_S = \sum_{k=0}^{d-1} \omega_S k |k\rangle \langle k|,
\]

(C1)

where we also set \( \omega_0 = 0 \) without loss of generality. Consider also the machine \( M \) to be composed of \( N \) subsystems, \( \{M_n\}_{n=1,...,N} \), each of the same dimension \( d \) as the target, whose local Hamiltonians are

\[
H_M^{(n)} = (1 + \epsilon \epsilon) H_S,
\]

(C2)

where \( \epsilon = (\beta_{\text{max}} - \beta)/(N\beta) \). We first cool the system initially at non-zero \( \beta \) to some fixed, finite \( \beta_{\text{max}} \), which we will eventually take \( \beta_{\text{max}} \rightarrow \infty \) in order to asymptotically achieve perfect cooling. We treat the case \( \beta = 0 \) as a limiting case of \( \beta \rightarrow 0 \); here, as \( \beta \rightarrow 0 \), we let \( N \rightarrow \infty \) such that \( N\beta \rightarrow \infty \), e.g., we specify a suitable function \( N(\beta) \) such that \( N(\beta) \rightarrow \infty \) “faster” than \( \beta \rightarrow 0 \).

We now show that, given the ability to perform a diverging number of operations on such a configuration, one can reach the target state \( \tau_S(\beta_{\text{max}}, H_S) \). In particular, we show that the protocol presented uses the minimal amount of energy to do so, and explicitly calculate this to be \( \beta^{-1} \Delta S \) units of energy, where \( \Delta S := S[\tau_S(\beta, H_S)] - S[\tau_S(\beta_{\text{max}}, H_S)] \). In other words, as the number of operations in the protocol diverges, we approach perfect cooling at the Landauer limit, thereby saturating the ultimate bound.
The diverging time cooling protocol is as follows. At each step, the target system interacts with a single machine labelled by \( n \) via the swap operator
\[
S_{SM_n}^d := \sum_{i,j=0}^{d-1} |i,j⟩⟨j,i|_{SM_n}.
\] (C3)

As the target and machine subsystems considered here are of the same dimension, we will drop the subscript on the states associated to each subsystem, for ease of notation. Such a transformation is, in general, not energy conserving, but one can calculate the energy change for both the target system and the machine due to the \( n^{\text{th}} \) interaction as
\[
\Delta E^{(n)}_S = \text{tr} \left[ H_S \tau(x, H^{(n)}_M) \right] - \text{tr} \left[ H_S \tau(x, H^{(n-1)}_M) \right],
\] (C4)
and so the total energy change of the system over the entire \( N \)-step protocol is given by
\[
\Delta E_S = \sum_{n=1}^{N} \Delta E^{(n)}_S = \text{tr} \left[ H_S \tau(x, H^{(N)}_M) \right] - \text{tr} \left[ H_S \tau(x, H^{(0)}_M) \right].
\] (C5)

The energy change of the machine subsystem that is swapped with the target system at each step is given by
\[
\Delta E^{(n)}_M = \text{tr} \left[ H^{(n)}_M \tau(x, H^{(n)}_M) \right] - \text{tr} \left[ H^{(n-1)}_M \tau(x, H^{(n-1)}_M) \right] = \sum_{k=0}^{d-1} (1 + n\epsilon)\omega_k \left[ p_k(x, H^{(n-1)}_M) - p_k(x, H^{(n)}_M) \right],
\] (C6)
where \( p_k(x, H^{(n)}_M) = e^{-\beta(1+n\epsilon)x}/Z_{M_n}(x, H^{(n)}_M) \) is the population in the \( k^{\text{th}} \) energy level of the thermal state of the \( n^{\text{th}} \) machine subsystem at inverse temperature \( \beta \), with \( Z_{M_n}(x, \beta, H^{(n)}_M) = \text{tr} \left[ e^{-\beta H^{(n)}_M} \right] \) being the partition function.

By summing the contributions of the energy changes in each step, one can obtain the total energy change for the overall machine throughout the entire process:
\[
\Delta E^{(N)}_M = \sum_{n=1}^{N} \Delta E^{(n)}_M = \sum_{n=0}^{d-1} \sum_{k=0}^{d-1} (1 + n\epsilon)\omega_k \left[ p_k(x, H^{(n-1)}_M) - p_k(x, H^{(n)}_M) \right],
\] (C7)

In general, it is complicated to calculate the energy cost for the protocol up until a finite time step \( N \), since this depends on the full energy structure of the target system and machine subsystems involved (we will return to resolve this problem for the special case of equally-space system and machine Hamiltonians in the coming section). Here, we focus on a special case in which \( N \to \infty \), i.e., there is a diverging number of machine subsystems that the target system interacts with throughout the protocol. This limit physically corresponds to that of requiring a diverging amount of time (in terms of the number of steps). Furthermore, we take the limit \( \epsilon \to 0 \) for any fixed \( \beta, \beta_{\text{max}} \). Considering the differentials
\[
\Delta p_k := p_k(x, H^{(n)}_M) - p_k(x, H^{(n-1)}_M),
\] (C8)
and
\[
\Delta x_n := x_n - x_{n-1} \quad \text{with} \quad x_n := 1 + n\epsilon,
\] (C9)
where \( x_n := 1 + n\epsilon \), to become infinitesimal, and noting the explicit form of the machine subsystem Hamiltonians \( H^{(n)}_M = (1 + n\epsilon)H_S \), we can make the replacement
\[
-\frac{\Delta p_k}{\Delta x_n} \Delta x_n \to -\frac{\partial p_k(x, H_S)}{\partial x} \, dx,
\] (C10)
where \( x := 1 + n\epsilon \) is a continuous parameter. This way we can express the limit \( N \to \infty \) of Eq. (C7) as a Riemann integral in the following form
\[
\lim_{N \to \infty} \Delta E^{(N)}_M = -\int_{1}^{x_{\text{max}}} \sum_{k=0}^{d-1} x\omega_k \frac{\partial p_k(x, H_S)}{\partial x} \, dx,
\] (C11)
where \( x_{\text{max}} := \beta_{\text{max}} / \beta \). Both the summation and the integral converge, so one can swap the order of their evaluation. Integrating by parts then gives

\[
\lim_{N \to \infty} \Delta E_{N,N} = \sum_{k=0}^{d-1} \left[ -x \omega_k p_k(\beta, x H_s) \right]^{x_{\text{max}}} + 1 \int_1^{x_{\text{max}}} \omega_k p_k(\beta, x H_s) \, dx
\]

\[
= \sum_{k=0}^{d-1} \left[ -x \omega_k p_k(\beta, x H_s) \right]^{x_{\text{max}}} - \int_1^{x_{\text{max}}} \frac{1}{\beta} \frac{\partial}{\partial x} \left[ \log Z(\beta, x H_s) \right] \, dx
\]

\[
= E[\tau(\beta, H_s)] - E[\tau(\beta, x_{\text{max}} H_s)] - \frac{1}{\beta} \log Z(\beta, x_{\text{max}} H_s) + \frac{1}{\beta} \log Z(\beta, H_s),
\]

where in the second line we again swapped the order of the integral and the sum to write \( \sum_{k=0}^{d-1} \omega_k p_k(\beta, x H_s) = -\frac{1}{\beta} \frac{\partial}{\partial x} [\log Z(\beta, x H_s)] \) and in the last line we invoke \( E[\tau(\beta, x H)] = \text{tr} [x H \tau(\beta, x H)] \). Finally, writing the partition function in terms of the average energy and entropy, i.e., \( \log[Z(\beta, x H)] = -\beta E[\tau(\beta, x H)] + S[\tau(\beta, x H)] \), the total energy change of the machine is given by

\[
\lim_{N \to \infty} \Delta E_{N,N} = E[\tau(\beta, H_s)] - E[\tau(\beta, x_{\text{max}} H_s)] - E[\tau(\beta, x_{\text{max}} H_s)] - \frac{1}{\beta} S[\tau(\beta, x_{\text{max}} H)] - E[\tau(\beta, H_s)] + \frac{1}{\beta} S[\tau(\beta, H_s)]
\]

\[
= \frac{1}{\beta} \{S[\tau(\beta, H_s)] - S[\tau(\beta_{\text{max}}, H_s)]\} = -\frac{1}{\beta} \Delta S_s,
\]

where we have made use of the property \( \tau_{\alpha}(\beta, x_{\text{max}} H_s) = \tau_{\alpha}(\beta_{\text{max}}, H_s) \) and the entropy decrease of the target system corresponds to that associated with the transformation \( \tau(\beta, H_s) \to \tau(\beta_{\text{max}}, H_s) \). Thus, as the number of timesteps diverges, this cooling process saturation the Landauer limit for the heat dissipated by the machine. In order to achieve perfect cooling at the Landauer limit, i.e., the final target state to approach \( |0\rangle |0\rangle \) and thus prove Theorem 1, we can now take the limit \( \beta_{\text{max}} \to \infty \).

The above proof holds for systems and machines of arbitrary (but equal) dimension, either finite or infinite, with arbitrary Hamiltonians. We now present some more detailed analysis regarding the special case where the Hamiltonians of the target system and all machine subsystems are equally spaced; this provides an opportunity both to derive a more detailed formula for the energy costs involved and to build intuition regarding some of the important differences between the finite- and infinite-dimensional settings.

### C2. Special Case: Equally Spaced Hamiltonians

Consider a finite \( d \)-dimensional target system beginning at inverse temperature \( \beta \) with an equally spaced Hamiltonian \( H_s(\omega_s) = \omega_s \sum_{n=0}^{d-1} |n\rangle \langle n| \). In this case, we can derive a more precise dimension-dependent function for the energy cost dissipated by the machines throughout the optimal cooling protocol presented above.

Consider an initial target system \( \tau_s(\beta, H_s) \) and a diverging number \( N \) of machines \( \{M_\alpha\}_{\alpha=0,\ldots,N} \) of the same dimension \( d \) as the target, which all begin in a thermal state at inverse temperature \( \beta \) with respect to an equally spaced Hamiltonian whose the gaps between neighbouring energy levels \( \omega_{M_\alpha} \) are ordered in non-decreasingly. Each machine is used once and then discarded; the particular interaction is the aforementioned swap between the target system and the \( n^\text{th} \) qudit machine, i.e., that represented by the unitary \( S^d_{s,M_\alpha} := \sum_{i,j=0}^{d-1} |i\rangle \langle j| S_{s,M_\alpha}^{i,j} \). After applying such an operation, the state of the target system is given by

\[
\tau_{s}(\beta, \omega_s) := \frac{e^{-\beta H_s(\omega_s)}}{Z_s(\beta, \omega_s)},
\]

where \( H_s(\omega_s) := \omega_s \sum_{n=0}^{d-1} |n\rangle \langle n| \) and \( Z_s(\beta, \omega_s) := \text{tr} [e^{-\beta H_s(\omega_s)}] \).

We now calculate the optimal energy cost explicitly for the diverging time cooling protocol. In order to minimise the energy cost of cooling, the target system must be cooled by the qudit system in the machines with the smallest gap between neighbouring energy levels (that permits cooling) as much as possible at each stage. In order to optimally use the given machine structure at hand, we thus order the set of energy gaps \( \omega_{\alpha} \) in non-decreasing order. In addition, the protocol to reach the Landauer erasure bound, i.e., minimal energy cost, dictates that one must infinitesimally increase \( \omega_{\alpha} \) of the machines in order to dissipate as little heat as possible throughout the interactions. Since we are here considering a diverging time limit, we have access to a diverging number of qudit machine with distinct energy gap \( \omega_{\alpha} \) at our disposal; the task is then to use these in an energy-optimal manner.
It is straightforward to see that to minimise the total energy cost, one must apply the sequence of unitaries \( S^d_{\alpha,M} \) such that \( S^d_{\alpha,M_0} \) is first applied to reach the optimally cool \( \tau_{\omega_1}(\beta, \omega_0) \), then \( S^d_{\alpha,M} \), to reach \( \tau_{\omega_1}(\beta, \omega_0) \), and so on. The heat dissipated by the reset machines in each stage of such a cooling protocol (i.e., for each value of \( \alpha \)) can thus be calculated as

\[
\Delta E_{\tau,\omega_1}(\omega_\alpha) = -\left\{ \text{tr} \left[ H^{\alpha}_{\omega_0}(\omega_\alpha) \tau^{\alpha,\omega_0}_{\omega_1}(\beta, \omega_0) \right] + \text{tr} \left[ H^{\alpha}_{\omega_1}(\omega_\alpha) \tau^{\alpha,\omega_1}_{\omega_2}(\beta, \omega_1) \right] \right\} \\
= -\text{tr} \left[ H^{\omega_0}_{\omega_1}(\omega_\alpha) \tau^{\omega_0}_{\omega_1}(\beta, \omega_0) - \tau^{\omega_0}_{\omega_2}(\beta, \omega_1) \right].
\] (C15)

In the second line, we have made use of the fact that the Hamiltonians of both the target system and each of machine are \( d \)-dimensional and equally spaced. So far, we have obtained the energy dissipated by the reset machines. To investigate the total energy cost of cooling in such a process, we also must consider the contribution of energy transferred to the target system \( S \), which is characterised via its local Hamiltonian \( H^{\omega_0}_{\omega_1} \) and calculated via

\[
\Delta E^{\omega_1}_{\omega_0}(\omega_\alpha) = \text{tr} \left[ H^{\omega_0}_{\omega_1}(\omega_\alpha) \tau^{\omega_0}_{\omega_1}(\beta, \omega_0) \right] - \text{tr} \left[ H^{\omega_0}_{\omega_1}(\omega_\alpha) \tau^{\omega_0}_{\omega_2}(\beta, \omega_1) \right],
\] (C16)

in which we set \( \omega_0 = \omega_S \). Using Eqs. (C15, C16), the total energy cost for each stage of cooling is given by

\[
\Delta E^{\omega_1}_{\omega_0}(\omega_\alpha) = \Delta E^{\omega_1}_{\omega_0}(\omega_\alpha) + \Delta E^{\omega_1}_{\omega_0}(\omega_\alpha) = \text{tr} \left\{ [H^{\omega_0}_{\omega_1}(\omega_\alpha) - H^{\omega_1}_{\omega_0}(\omega_\alpha)] \left[ \tau^{\omega_0}_{\omega_1}(\beta, \omega_0) - \tau^{\omega_1}_{\omega_2}(\beta, \omega_1) \right] \right\},
\] (C17)

which leads to the overall energy cost after \( N \) stages, where \( N \) is the number of non-zero distinct energy gaps of the reset machines, as

\[
\Delta E^{(N)}\tau_{\omega_0}(\omega_\alpha) = \sum_{\alpha=1}^{N} \Delta E_{\omega_0}(\omega_\alpha) = \sum_{\alpha=1}^{N} \text{tr} \left\{ [H^{\omega_0}_{\omega_1}(\omega_\alpha) - H^{\omega_1}_{\omega_0}(\omega_\alpha)] \left[ \tau^{\omega_0}_{\omega_1}(\beta, \omega_0) - \tau^{\omega_1}_{\omega_2}(\beta, \omega_1) \right] \right\}.
\] (C18)

Now, we can obtain the total energy cost for each stage of the protocol (i.e., each value of \( \alpha \) considered) in terms of the transformation of the target system alone. Note that in this protocol, each stage corresponding to each of the \( N \) distinct energy gaps \( \{\omega_\alpha\} \) in itself requires only one operation to perfectly reach \( \tau^{\omega_0}_{\omega_1}(\beta, \omega_0) \). The end result of this protocol is that the target system is cooled from the initial thermal state \( \tau_{\omega_1}(\beta, \omega_1) \), where \( \omega_1 \) is the energy gap between each pair of adjacent energy levels in the system, to \( \tau_{\omega_1}(\beta, \omega_{\max}) \) in the energy-optimal manner.

Starting from Eq. (C18), we have

\[
\Delta E^{(N)}\tau_{\omega_0}(\omega_\alpha) = \sum_{\alpha=1}^{N} \text{tr} \left\{ [H^{\omega_0}_{\omega_1}(\omega_\alpha) - H^{\omega_1}_{\omega_0}(\omega_\alpha)] \left[ \tau^{\omega_0}_{\omega_1}(\beta, \omega_0) - \tau^{\omega_1}_{\omega_2}(\beta, \omega_1) \right] \right\} \\
= \sum_{\alpha=1}^{N} (\omega_\alpha - \omega_0) \left[ \left( \frac{e^{-\beta\omega_0}}{1 - e^{-\beta\omega_1}} - \frac{e^{-\beta\omega_0} - e^{-\beta\omega_1}}{1 - e^{-\beta\omega_1}} \right) - \left( \frac{de^{-\beta\omega_0}}{1 - e^{-\beta\omega_1}} - \frac{de^{-\beta\omega_1}}{1 - e^{-\beta\omega_1}} \right) \right] \\
= \lim_{K \to \infty} \sum_{\alpha=1}^{N} (\omega_\alpha - \omega_0) \sum_{k=0}^{K} \left[ e^{-\beta(k+1)\omega_0} - e^{-\beta(k+1)\omega_1} - d \left( e^{-\beta(k+1)\omega_0} - e^{-\beta(k+1)\omega_1} \right) \right] \\
= \lim_{K \to \infty} \sum_{\alpha=1}^{N} (\omega_\alpha - \omega_0) \sum_{k=0}^{K} \left[ e^{-\beta(k+1)\omega_0} - e^{-\beta(k+1)\omega_1} - d \left( e^{-\beta(k+1)\omega_0} - e^{-\beta(k+1)\omega_1} \right) \right].
\] (C19)

Here, since both \( H^{\omega_0}_{\omega_1} \) and \( H^{\omega_1}_{\omega_0} \) are equally spaced Hamiltonians, the average energy can be written as

\[
E(\omega_\alpha, \omega_\beta) = \text{tr} \left[ H^{\omega_0}_{\omega_1}(\omega_\alpha) \tau^{\omega_0}_{\omega_1}(\beta, \omega_0) \right] = \frac{\sum_{n=0}^{d-1} n \omega_\alpha e^{-n\beta\omega_\beta}}{\sum_{n=0}^{d-1} e^{-n\beta\omega_\beta}} = \omega_\alpha \left( \frac{e^{-\beta\omega_\beta}}{1 - e^{-\beta\omega_\beta}} - \frac{d e^{-\beta\omega_\beta}}{1 - e^{-\beta\omega_\beta}} \right)
\] (C20)

by evaluating the geometric series

\[
Z(\beta, \omega_\beta) = \sum_{n=0}^{d-1} e^{-n\beta\omega_\beta} = \frac{1 - e^{-d\beta\omega_\beta}}{1 - e^{-\beta\omega_\beta}}
\] (C21)

and writing

\[
E(\omega_\alpha, \omega_\beta) = \sum_{n=0}^{d-1} n \omega_\alpha \frac{e^{-n\beta\omega_\beta}}{Z(\beta, \omega_\beta)} = \omega_\alpha \left\{ -\frac{\partial}{\partial \omega_\beta} \log \left[ Z(\beta, \omega_\beta) \right] \right\} = -\omega_\alpha \frac{\partial}{\partial \omega_\beta} \left[ \log \left( 1 - e^{-d\beta\omega_\beta} \right) - \log \left( 1 - e^{-\beta\omega_\beta} \right) \right]
\] (C22)
as we do in the second line of Eq. (C19) and then using the infinite series expression \((1-x)^{-1} = \lim_{K \to \infty} \sum_{k=0}^{K} x^{k}\) for any \(|x| < 1\) as per the third line.

As we will see in Appendix E2a, the energy cost for cooling an infinite-dimensional system when both target and machines have equally spaced Hamiltonians (i.e., harmonic oscillators) is similar to the form of Eq. (C19). Importantly, the second term in square parenthesis vanishes as \(d \to \infty\), simplifying the expression even further.

We now assume that the energy gaps of the machine are given by \(\omega_{\alpha} = \omega_{\beta} + \epsilon\alpha\) and so the total energy cost can be written as follows:

\[
\Delta E_{SM}^{(N)} = -\lim_{K \to \infty} \sum_{\alpha=1}^{N} \sum_{k=0}^{K} e^{-\beta k(\omega_{\alpha} + \epsilon \alpha)} (1 - e^{\beta k\epsilon}) + \lim_{K \to \infty} \sum_{\alpha=1}^{N} \sum_{k=0}^{K} e^{-\beta k(\omega_{\beta} + \epsilon \alpha)} (1 - e^{\beta k\epsilon})
\]

\[
= \lim_{K \to \infty} \sum_{k=0}^{K} \left[ e^{-\beta k\omega_{\beta} (1 - e^{\beta k\epsilon})} \right] - \lim_{K \to \infty} \sum_{k=0}^{K} \left[ e^{-\beta k\omega_{\beta} (1 - e^{\beta k\epsilon})} \right] \] \quad \text{(C23)}

where we can swap the order of summation since both sums converge and the summands are non-positive. This can be seen from the first line above, using the fact that \(e^{-\alpha x} (1 - e^{x}) \in [-1, 0]\) for all \(\alpha \geq 1\) and \(x \geq 0\). We now calculate the sum over \(\alpha\).

\[
\sum_{\alpha=1}^{N} \alpha e^{-\beta \alpha \epsilon} = -\frac{\partial}{\partial \beta} \sum_{\alpha=0}^{\infty} e^{-\beta \alpha \epsilon} = -\frac{\partial}{\partial \beta} \left( \frac{1 - e^{-\beta(N+1)\epsilon}}{1 - e^{-\beta \epsilon}} \right)
\]

\[
= \frac{e^{-\beta \epsilon}}{(1 - e^{-\beta \epsilon})^2} \left( (N + 1)e^{-\beta(N+1)\epsilon} - (N + 1)e^{-\beta(N+2)\epsilon} - e^{-\beta \epsilon} + e^{-\beta(N+2)\epsilon} \right)
\]

\[
= \frac{e^{-\beta \epsilon}}{(1 - e^{-\beta \epsilon})^2} \left( 1 - N e^{-\beta \epsilon} \right)
\]

\[
\quad \text{(C24)}
\]

Combining Eqs. (C23) and (C24), we arrive at

\[
\Delta E_{SM}^{(N)} = -\lim_{K \to \infty} \sum_{k=0}^{K} \left[ e^{-\beta k\omega_{\beta} (1 - e^{\beta k\epsilon})} \right] - \lim_{K \to \infty} \sum_{k=0}^{K} \left[ e^{-\beta k\omega_{\beta} (1 - e^{\beta k\epsilon})} \right]
\]

\[
\quad \text{(C25)}
\]

In order to optimise the energy cost, we now assume that the energy gaps of the machine can be chosen to be smoothly increasing in such way that \(\epsilon = \Delta \omega / N := (\omega_{\max} - \omega_{\beta}) / N\). Substituting this expression for \(\epsilon\) into the above equation yields

\[
\Delta E_{SM}^{(N)} = \lim_{K \to \infty} \sum_{k=0}^{K} \left[ \frac{e^{-\beta k\omega_{\beta} (1 - e^{\beta k\epsilon})}}{k} \right] - \lim_{K \to \infty} \sum_{k=0}^{K} \left[ \frac{e^{-\beta k\omega_{\beta} (1 - e^{\beta k\epsilon})}}{k} \right]
\]

\[
\quad \text{(C26)}
\]

We now wish to take the limit of \(N \gg K \to \infty\). This assumption means that energy change of the system is approximately equal to its free energy change; in other words, the process occurs quasi-adiabatically. The ability to switch the order of taking the limits of \(K \text{ and } N\) going to \(\infty\) follows from the monotonic convergence of the sum over \(k\). In particular, note that the term inside square parenthesis in each summand converges and the first term in each summation (which is the only part that depends on \(N\)) is positive and bounded.

Under this assumption, we can use the approximation \(\lim_{x \to 0} \frac{x}{e^x - 1} = \frac{1}{e^x - 1}\). Since \(0 < e^{-\beta x} < 1\) for any positive \(x\), the sum over \(k\) converges to a finite value. In general, this approximation introduces a correction term for the energy change, however under said assumption the error incurred becomes negligible. Then, the total energy change \(\Delta E_{SM}^{\text{tot}}\) for the transformation \(\tau_{\beta}(\beta, \omega_{\beta}) \to \tau_{\beta}(\beta, \omega_{\max})\) throughout the overall process is

\[
\Delta E_{SM}^{\text{tot}} = \lim_{K \to \infty} \sum_{k=0}^{K} \left[ \frac{e^{-\beta k\omega_{\beta}} - e^{-\beta k\omega_{\max}}}{\beta k} \right]
\]

\[
\quad \text{(21)}
\]
we immediately see that the heat dissipated by the resetting of machines in such a cooling process, i.e.,

\[
- \lim_{K \to \infty} \sum_{k=0}^{K} \left[ \frac{e^{-\beta k \omega_{\max}}}{\beta k} - \frac{e^{-\beta k \omega_{\max}}}{\beta k} - d(\omega_{\max} - \omega) e^{-\beta k \omega_{\max}} \right]. 
\]  

(C27)

As a side remark, note that here one can see that in the special case of equally spaced Hamiltonians, one indeed requires a diverging number of machine subsystems to attain perfect cooling at the Landauer limit, as this is the only way to fulfill the condition of Theorem 3. This follows from the fact that the approximation \( \frac{\omega}{e^{-\beta \omega}} \approx \frac{1}{e^{\beta \omega}} \) only holds for small \( \beta \omega \) and in general one would need to include higher-order terms that lead to an increase in energy cost.

We then have, using the expression for \( E(\omega_x, \omega_y) \) derived earlier:

\[
\Delta E_{s,M}^{\text{tot}} = -\frac{1}{\beta} \log(1 - e^{-\beta \omega_s}) + \frac{1}{\beta} \log(1 - e^{-\beta \omega_{\max}}) - \left( \frac{\omega_{\max} - \omega_s e^{-\beta \omega_{\max}}}{1 - e^{-\beta \omega_{\max}}} \right) + \frac{1}{\beta} \log(1 - e^{-\beta \omega_{\max}}) - \left( \frac{\omega_{\max} - \omega_s e^{-\beta \omega_{\max}}}{1 - e^{-\beta \omega_{\max}}} \right) 
\]

\[
= \frac{1}{\beta} \log \left[ Z_\beta(\beta, \omega_s) \right] - \frac{1}{\beta} \log \left[ Z_\beta(\beta, \omega_{\max}) \right] - \text{tr} \left[ H_\beta(\omega_{\max}) \tau_\beta(\beta, \omega_{\max}) \right] + \text{tr} \left[ H_\beta(\omega_s) \tau_\beta(\beta, \omega_{\max}) \right] 
\]

\[
= \frac{1}{\beta} \log \left[ Z_\beta(\beta, \omega_s) \right] - \frac{1}{\beta} \log \left[ Z_\beta(\beta, \omega_{\max}) \right] - \text{tr} \left[ H_\beta(\omega_{\max}) \tau_\beta(\beta, \omega_{\max}) \right] + \text{tr} \left[ H_\beta(\omega_s) \tau_\beta(\beta, \omega_{\max}) \right] 
\]

\[
= \frac{1}{\beta} \Delta S_\beta + \Delta E_g, \quad \quad (C28)
\]

where we have explicitly written the von Neumann entropy \( S(\rho) = -\text{tr} \left[ \rho \log(\rho) \right] \) of a thermal state at inverse temperature \( \beta \) as \( S[\tau_\beta(\beta, \omega)] = \log[Z_\beta(\beta, \omega)] + \beta E[\tau_\beta(\beta, \omega)] \). Since the energy change of the target system only concerns its local Hamiltonian, we immediately see that the heat dissipated by the resetting of machines in such a cooling process, i.e., \( \Delta E_M \), saturates the Landauer bound as it is equal to \( \beta^{-1} \Delta S_\beta \). The process described is thus energy-optimal.

Appendix D: Conditions for Structural and Control Complexity

D1. Necessary Conditions: Proof of Theorem 2, Corollary 2, and Theorem 3

Here we prove Theorem 3, which implies Theorem 2 and leads to Corollary 2.

Proof. Let \( \mathcal{H}_X \) be a separable Hilbert space associated with the system \( X \). Consider

\[
H_M = \sum_{n=0}^{\infty} \omega_n |n\rangle \langle n| \quad \text{and} \quad H_{M'} = \text{span}_{n \leq m} \{|n\rangle \}, \quad (D1)
\]

for some finite \( m \). In other words, \( H_{M'} \) is a finite-dimensional restriction of \( H_M \). We will show that any unitary that (nontrivially) interacts the target system with only a subspace spanned by finitely many eigenstates of \( H_M \) cannot attain Landauer’s bound. Consider a general unitary \( U \). Suppose that \( U \) only couples \( \mathcal{H}_\beta \) with \( \mathcal{H}_{M'} \); whenever we talk of an operation with finite control complexity in this article, we mean specifically such a \( U \), and by diverging control complexity we mean a unitary that couples the target to any subspace of \( H_M \) whose dimension diverges. Since

\[
\mathcal{H}_\beta \otimes \mathcal{H}_M = \mathcal{H}_\beta \otimes (\mathcal{H}_{M'} \oplus \mathcal{H}_{M'}^\perp) \simeq (\mathcal{H}_\beta \otimes \mathcal{H}_{M'}) \oplus (\mathcal{H}_\beta \otimes \mathcal{H}_{M'}^\perp), \quad (D2)
\]

we can associate the subspace \( \mathcal{H}_\beta \otimes \mathcal{H}_{M'} \) with the label \( A \) and \( \mathcal{H}_\beta \otimes \mathcal{H}_{M'}^\perp \) with \( B \) and write \( U = U_A \oplus I_B \). Then the initial configuration can be expressed as:

\[
\theta_\beta \otimes \tau_M(\beta, H_M) = \begin{bmatrix} \theta_\beta \otimes \theta_{M'} & 0 \\ 0 & \theta_\beta \otimes \theta_{M'}^\perp \end{bmatrix}, \quad (D3)
\]

where

\[
\theta_{M'} := \frac{1}{Z_{M'}(\beta, H_M)} \sum_{n \leq m} e^{-\beta \omega_n} |n\rangle \langle n| \quad \text{and} \quad \theta_{M'}^\perp := \frac{1}{Z_M(\beta, H_M)} \sum_{n > m} e^{-\beta \omega_n} |n\rangle \langle n|. \quad (D4)
\]
add up to a (normalised) thermal state. Now consider the state
\[
\tilde{\varrho}_M = \begin{bmatrix} \varrho_{M'} & 0 \\ 0 & \text{tr} [\varrho_{M'}^+] \end{bmatrix}.
\] (D5)

It is straightforward to check that is indeed a quantum state; moreover, it is the Gibbs state (at inverse temperature $\beta$) associated with the Hamiltonian
\[
\tilde{H}_M = \sum_{n \leq m} \omega_n |n\rangle\langle n| - \frac{1}{\beta} \log \left( \sum_{n > m} e^{-\beta \omega_n} \right) |m+1\rangle\langle m+1|.
\] (D6)

To see this, note that $Z_M(\beta, H_M) = Z_M(\beta, \tilde{H}_M)$ and that
\[
\exp \left\{ -\beta \left[ -\frac{1}{\beta} \log \left( \sum_{n > m} e^{-\beta \omega_n} \right) \right] \right\} = \sum_{n > m} e^{-\beta \omega_n}.
\] (D7)

Thus $\tilde{\varrho}_M = \tau_M(\beta, \tilde{H}_M)$. To ease notation in what follows, we will write $\tilde{\omega}_{m+1} := -\frac{1}{\beta} \log \left( \sum_{n > m} e^{-\beta \omega_n} \right)$. In the rest of the proof, we will show that the unitary $U$ and the Hamiltonian $H_M$ can be replaced by finite-dimensional versions without changing the quantities relevant for Landauer’s principle.

Let $\tilde{U} = U_A \otimes (1_A \otimes |m+1\rangle\langle m+1|)$. We then have
\[
\tilde{U} (\varrho_s \otimes \tilde{\varrho}_M) \tilde{U}^\dagger = \begin{bmatrix} U_A(\varrho_s \otimes \varrho_{M'}) U_A^\dagger & 0 \\ 0 & \frac{e^{-\beta \omega_{m+1}}}{Z_M(\beta, H_M)} \varrho_s \end{bmatrix}
\] (D8)

and
\[
\text{tr}_M \left[ \tilde{U} (\varrho_s \otimes \tilde{\varrho}_M) \tilde{U}^\dagger \right] = \text{tr}_{M'} \left[ U_A(\varrho_s \otimes \varrho_{M'}) U_A^\dagger \right] + \frac{e^{-\beta \omega_{m+1}}}{Z_M(\beta, H_M)} \varrho_s.
\] (D9)

Compare this to the expression
\[
\text{tr}_M \left[ U(\varrho_s \otimes \varrho_{M'}) U^\dagger \right] = \text{tr}_{M'} \left[ U_A(\varrho_s \otimes \varrho_{M'}) U_A^\dagger \right] + \frac{e^{-\beta \omega_{m+1}}}{Z_M(\beta, H_M)} \varrho_s,
\] (D10)

since $\text{tr} [\varrho_{M'}^+] = \frac{1}{Z_M(\beta, H_M)} \sum_{n \geq m} e^{-\beta \omega_n}$. Thus, the final system state is the same as it would be if we replaced the full initial machine state with $\tilde{\varrho}_M$; in particular, the entropy decrease of the system for any unitary that cools it is also unchanged.

The last thing we need to check is that the energy change of the machine similarly remains invariant. To that end, we have that
\[
\tilde{\varrho}_M = \text{tr}_S \left[ \tilde{U} (\varrho_s \otimes \tilde{\varrho}_M) \tilde{U}^\dagger \right] = \text{tr}_S \left[ U_A(\varrho_s \otimes \varrho_{M'}) U_A^\dagger \right] + \frac{e^{-\beta \omega_{m+1}}}{Z_M(\beta, H_M)} |m+1\rangle\langle m+1|,
\] (D11)

Thus, we have
\[
\text{tr} \left[ \tilde{H}_M (\tilde{\varrho}_M - \varrho_M) \right] = \text{tr} \left[ \tilde{H}_M \left\{ \text{tr}_S \left[ U_A(\varrho_s \otimes \varrho_{M'}) U_A^\dagger \right] - \varrho_{M'} \right\} \right],
\] (D12)

since $U_S$ only acts on $\mathcal{H}_S \otimes \mathcal{H}_{M'}$ and $\tilde{H}_{M|M'} = H_{M|M'}$. In the same way, we have
\[
\text{tr}_S \left[ U(\varrho_s \otimes \varrho_{M'}) U^\dagger \right] = \text{tr}_S \left[ U_A(\varrho_s \otimes \varrho_{M'}) U_A^\dagger \right] + \varrho_{M'}^+ \varrho_{M'} = \varrho_{M'} + \varrho_{M'}^+.\] (D13)
Thus, the energy difference is also
\[
\text{tr} \left\{ H_M \left[ \text{tr}_S \left[ U_M (\varrho_S \otimes \varrho_{M'}) U_M^\dagger \right] - \varrho_{M'} \right] \right\}.
\] (D14)

Hence, we have shown that we can replace $M$ by some $m + 1$ dimensional machine $\tilde{M}$ if the joint unitary $U$ only acts on $m$ levels of $H_M$. By Theorem 6 of Ref. [29], there are finite-dimensional corrections to the Landauer bound, which then imply that it cannot be reached for finite $m$. Thus, the effective machine dimension, i.e., that which is actually (nontrivially) accessed throughout the interaction, must diverge in order for cooling to be possible at the Landauer limit. This proves Theorem 3, which implies Theorem 2.

D2. Sufficient Conditions for Complexity

Having shown the necessary requirements for cooling at Landauer cost, namely a control interaction that acts nontrivially on an infinite-dimensional (sub)space of the machine’s Hilbert space, let us now return to emphasise the properties of the machine and cooling protocol that are sufficient to achieve perfect cooling at Landauer cost. For simplicity, we consider the case of a qubit, which exemplifies the discussion of finite-dimensional systems. The case of infinite-dimensional systems shall be treated independently in the next appendix.

We first consider the structural properties of the machine. The diverging-time protocol discussed in Appendix C makes use of a diverging number $N$ of machines. Thus, the machine begins in the thermal state $\rho_0$, which is set at zero energy. More generally, the energy eigenvalue corresponding to an eigenstate $|i_{\epsilon}, \ldots, i_N\rangle$ is given by $\omega_1$ multiplied by the number of indices $i_k$ that are equal to $1$, plus a sum of terms $k \epsilon$ where $k$ is the label of each index equal to $1$. Thus, the energy eigenvalue of the eigenstate $|1, \ldots, 1\rangle$ diverges as the number of subsystems diverges. At the same time, letting the factor $\epsilon$ go to zero renders all eigenstates with the same (constant) number of indices such that $i_k = 1$ approach the same energy. Thus, in the limit $\epsilon \to 0$, one obtains subspaces of energy $E_M(k) = k \omega_1$ with degeneracy given by $D_k = \binom{N}{k}$, which also diverges for each constant $k$ and diverging $N$. Therefore, in addition to satisfying the structural conditions that are necessary for perfect cooling, as stated in Theorem 2, the machine used here features additional properties, which are crucially important for this particular protocol, in particular because they are sufficient for perfect cooling at Landauer cost. As a remark, we also emphasise that for fixed (large) $N$ and (small) $\epsilon$, the machine is finite-dimensional and has a non-degenerate Hamiltonian without any energy levels formally at infinity.

Concerning the control complexity properties of the unitary that achieves perfect cooling in unit time, note that it is a cyclic shift operator, which can be written as
\[
U_{SM} = \Pi_{n=0}^N \sigma_{SM,n} = \Pi_{n=0}^N \left( \sum_{i,j_1,\ldots,j_N=0}^{1} |i,j_1,\ldots,j_N\rangle\langle j_N,j_1,\ldots,i\rangle \right)_{SM}.
\] (D16)

As it is evident from its form, this unitary acts nontrivially on all of the (divergingly many) energy levels of the machine. The only basis vectors of the system-plus-machine Hilbert space that are left invariant are $|i = 0, j_1 = 0, \ldots, j_N = 0\rangle$ and $|i = 1, j_1 = 1, \ldots, j_N = 1\rangle$.

D3. Fine-tuned Control Conditions

Here we analyse the fine-tuned control conditions that are asymptotically required for cooling at the Landauer limit. We begin with some general considerations before focusing on a special case for which an analytic expression can be derived. Furthermore, we demand that the unitary implemented is such that the target is cooled as much as possible: this does not preclude the possibility for cooling the target system less (albeit still close to a pure state) at a cost closer to the Landauer bound.
without satisfying all of the fine-tuning conditions. Nonetheless, in general there are a number of such conditions to be satisfied, and the special case serves as a pertinent example that demonstrates how the particular set of fine-tuning conditions for any considered scenario can be derived.

Consider an arbitrary thermal target system and machine of finite dimensions, with respective spectra \( \lambda_\mathcal{S} := \{\lambda_0^\mathcal{S}, \ldots, \lambda_{d_{\mathcal{S}}}^\mathcal{S}\} \) and \( \lambda_\mathcal{M} := \{\lambda_0^\mathcal{M}, \ldots, \lambda_{d_{\mathcal{M}}}^\mathcal{M}\} \). The states begin uncorrelated, so the global spectrum of the initial joint state is \( \lambda_{\mathcal{S}\mathcal{M}} := \{\lambda_0^\mathcal{S} \cdot \lambda_0^\mathcal{M}, \ldots, \lambda_{d_{\mathcal{S}}}^\mathcal{S} \cdot \lambda_{d_{\mathcal{M}}}^\mathcal{M}\} \). Consider now a global unitary transformation; such a transformation cannot change the values of the spectrum, but merely permute them. In other words, the spectrum of the final global state after any such unitary is invariant and we have equivalence of the (unordered) sets \( \lambda_{\mathcal{S}\mathcal{M}} \) and \( \lambda_{\mathcal{S}\mathcal{M}} \).

The transformation that cools the target system as much as possible is the one that places the \( d_\mathcal{M} \) largest of the global eigenvalues into the subspace spanned by the ground state of the target, the second \( d_\mathcal{M} \) largest into that spanned by the first excited state of the target, and so forth, with the smallest \( d_\mathcal{M} \) global eigenvalues placed in the subspace corresponding to the highest energy eigenstate of the target system (we prove this statement shortly). More precisely, we denote by \( \lambda^1 \) the non-increasing ordering of the set \( \lambda \). Since the target and machine begin thermal, the local spectra \( \lambda_\mathcal{S} \) and \( \lambda_\mathcal{M} \) are already ordered in this way with respect to their energy eigenbases, which we consider to be labelled in non-decreasing order. Cooling the target system as much as possible amounts to achieving the final reduced state of the target

\[
d'_{\mathcal{S}} = \sum_{i=0}^{d_{\mathcal{S}}-1} \left( \sum_{j=0}^{d_{\mathcal{M}}-1} \lambda_{\mathcal{S}\mathcal{M}}^{i \cdot d_{\mathcal{M}}+j} \right) |i\rangle \langle i|_{\mathcal{S}}. \tag{D17}
\]

As a side remark, note that since each of the global eigenvalues are a product of the initial local eigenvalues (due to the initial tensor product structure), which are in turn related to the energy-level structure of the target system and machine (as they begin as thermal states), one can already see here that in order to approach perfect cooling, the machine must have some diverging energy gaps, such that the (finite) sum of the global eigenvalues contributing to the ground-state population of the target approaches 1.

Of course, there is an equivalence class of unitaries that can achieve the same amount of cooling; in particular, any permutation of the set of the \( d_\mathcal{M} \) global eigenvalues within each energy eigenspace of the target system achieves the same amount of cooling, since it is the sum of these values that contribute to the total population in each subspace. Importantly, although such unitaries cool the target system to the same extent, their effect on the machine differs, and therefore so too does the energy cost. However, demanding that such cooling is achieved at minimal energy cost amounts to a unique constraint on the global post-transformation state, namely that it must be

\[
d'_{\mathcal{S}\mathcal{M}} = \sum_{i=0}^{d_{\mathcal{S}}-1} \sum_{j=0}^{d_{\mathcal{M}}-1} \lambda_{\mathcal{S}\mathcal{M}}^{i \cdot d_{\mathcal{M}}+j} |ij\rangle \langle ij|_{\mathcal{S}\mathcal{M}}. \tag{D18}
\]

We can derive the above form of the final joint state as follows. Consider the following ordering for the energy eigenbasis of \( \mathcal{S}\mathcal{M} \) chosen to match the above form

\[
\{ |00\rangle_{\mathcal{S}\mathcal{M}}, |01\rangle_{\mathcal{S}\mathcal{M}}, \ldots, |0, d_{\mathcal{M}}-1\rangle_{\mathcal{S}\mathcal{M}}, |10\rangle_{\mathcal{S}\mathcal{M}}, \ldots, |1, d_{\mathcal{M}}-1\rangle_{\mathcal{S}\mathcal{M}}, \ldots, |d_{\mathcal{S}}-1, 0\rangle_{\mathcal{S}\mathcal{M}}, \ldots, |d_{\mathcal{S}}-1, d_{\mathcal{M}}-1\rangle_{\mathcal{S}\mathcal{M}} \}. \tag{D19}
\]

This ordering is monotonically non-decreasing primarily with respect to the energy of \( \mathcal{S} \), and secondarily w.r.t. \( \mathcal{M} \). We take the final state \( \rho'_{\mathcal{S}\mathcal{M}} \) to be expressed in this basis. To maximise the cooling in a single unitary operation, we maximise the sum of the first \( k \cdot d_{\mathcal{M}} \) diagonal elements, for each \( k \in \{1, 2, \ldots, d_{\mathcal{S}}\} \), as each sum corresponds to the total population in the \( k \)th lowest energy eigenstate of \( \mathcal{S} \). The initial state \( \rho_{\mathcal{S}\mathcal{M}} \) is diagonal in this basis, so the vector of initial diagonal elements, which we label \( \theta := \text{diag}(\rho_{\mathcal{S}\mathcal{M}}) \), is also the vector of eigenvalues, \( \lambda_{\mathcal{S}\mathcal{M}} \), i.e., \( \theta = \lambda_{\mathcal{S}\mathcal{M}} \). Furthermore, since the unitary operation leaves the set of eigenvalues invariant, we have via the Schur-Horn lemma [54] that the vector of final diagonal elements, which we label \( \theta' := \text{diag}(\rho'_{\mathcal{S}\mathcal{M}}) \), is majorised by the vector of initial ones, i.e., \( \theta' \preceq \theta \). It follows that the partial sums we wish to maximise are upper bounded by the corresponding partial sums of the \( k \cdot d_{\mathcal{M}} \) largest diagonal elements of the initial state. We claim that the unitary that cools this maximal cooling amount at minimum energy cost is the one that permutes the diagonal elements to be ordered w.r.t. the basis ordering in Eq. (D19).

More precisely, via the Schur-Horn lemma, one can always write \( \theta' = D \theta \), with \( D \) a doubly stochastic matrix. The partial sums of the \( k \cdot d_{\mathcal{M}} \) first elements are linear functions of the elements of \( \theta \). Thus the maximum values are obtained at the extremal points of the convex set of doubly stochastic matrices, which are the permutation matrices, via the Birkhoff-von Neumann theorem [54]. One can see by inspection that the optimal permutation matrices are the ones that place the largest \( d_{\mathcal{M}} \) diagonal

\footnote{We take majorisation among passive states to be the measure of cooling; this implies the highest possible ground state population and purity, and lowest possible entropy and average energy via Schur convexity.}
elements in the first block (i.e., the ground state eigenspace of $S$), the next largest $d_{S,i}$ elements in the second block (i.e., the first excited state eigenspace of $S$), and so on. Within each block, the ordering does not affect the cooling of the target, so there is an equivalence class of permutations that satisfy the maximal cooling criterion. However, adding the optimisation over the energy cost eliminates this freedom. We may consider the reduced set of stochastic matrices that satisfy maximal cooling, generated by the permutations described above. Since the average energy of the final state is again a linear function of the diagonal elements, here too the minimum corresponds to a permutation matrix. Clearly the permutation that minimises the average energy is the one that orders the elements within each block to be decreasing w.r.t. the energies of the permutations described above. Since the average energy of the final state is again a linear function of the diagonal elements, the action of the stochastic matrix which leads to maximal cooling at the least energy cost possible is the one that permutes the energy eigenvalues to be ordered decreasing primarily w.r.t. the system energies, and secondarily w.r.t. the machine energies. The action of the stochastic matrix on diagonal elements of the state is related to the unitary operation on the entire quantum state by
\[
|U_{ij}|^2 = D_{ij},
\]
so that the unitary operation is also a permutation (up to an energy dependent phase, which is irrelevant since the initial and final states are diagonal).

We may understand this optimal operation through the notion of passivity, by noting that it cools at minimal energy cost by rendering the machine into the most passive reduced state in the joint unitary orbit with respect to the cooling constraint on the target. Intuitively, one has cooled the target system maximally at the expense of heating the machine as little as possible. The final reduced state of the machine corresponding to this energetically-optimal cooling transformation is
\[
d'_{SM} = \sum_{j=0}^{d_{SM}-1} \left( \sum_{i=0}^{d_{S,1}} S_{SM}^{i,j} + j \right) |j\rangle\langle j|_M.
\]
(D20)

In general, any unitary that achieves these desired conditions simultaneously depends upon the energy-level structure of both the target system and machine, precluding a closed-form set of conditions that can be expressed only in terms of the machine. However, for the special case of a maximally-mixed initial target state (i.e., cooling a thermal state at infinite temperature or erasing quantum information from its most entropic state), one can deduce this ordering precisely and moreover relate it directly to properties of the machine Hamiltonian, as we now demonstrate. In the following, we assume that $d_{SM}$ is even; the case for odd $d_{SM}$ can be derived similarly.

**Theorem 10.** Consider the target system to begin in the maximally mixed state and a thermal machine at temperature $\beta > 0$, whose eigenvalues are labelled in non-increasing order, $\{\lambda_M^{i}\} = (0, \ldots, d_{SM}-1)$. In order to cool the target perfectly, with the restriction that the target must be cooled as much as possible, at an energy cost that saturates the Landauer limit, the machine eigenvalues must satisfy
\[
\sum_{i=0}^{d_{SM}-1} \lambda_M^{i} \rightarrow 1, \quad \sum_{i=d_{SM}}^{d_{SM}-1} \lambda_M^{i} \rightarrow 0,
\]
and
\[
\frac{1}{2} \left( \lambda_M^{-\frac{d_{SM}}{2}} + \lambda_M^{-\frac{d_{SM}}{2}+1} \right) \rightarrow 1
\]
(D22)
for all $i \in \{0, \ldots, d_{SM}-1\}$, where $\lfloor \cdot \rfloor$ denotes the floor function and $\rightarrow$ denotes that the condition is satisfied asymptotically, i.e., as $d_{SM} \rightarrow \infty$.

**Proof.** We consider a qubit for simplicity, but the generalisation to cooling an arbitrary-dimensional maximally-mixed state is straightforward. The initial joint spectrum of the system and machine is
\[
\lambda_{SM} = \frac{1}{2} \{ \lambda_M^{1}, \lambda_M^{1}, \ldots, \lambda_M^{d_{SM}-1}, \lambda_M^{1}, \ldots, \lambda_M^{d_{SM}-1} \}.
\]
(D23)
As each $\lambda_M^{i} = \frac{1}{\sum_{i}(d_{SM})} e^{-\beta \omega_i}$ for any thermal state with Hamiltonian $H_M = \sum_{i} \omega_i |i\rangle\langle i|_M$ written with respect to non-decreasing energy eigenvalues, it follows that the globally ordered spectrum is
\[
\lambda_{SM}^{\downarrow} = \frac{1}{2} \{ \lambda_M^{1}, \lambda_M^{1}, \ldots, \lambda_M^{d_{SM}-1}, \lambda_M^{1}, \ldots, \lambda_M^{d_{SM}-1} \}.
\]
(D24)

\[^6\] Note that degeneracies in energy eigenvalues would lead to sets of equal diagonal elements, and prevent one from choosing a unique permutation. However, as the state in such degenerate subspaces is proportional to the identity matrix, we may take any unitary that is block diagonal w.r.t. the degeneracies without affecting the state, and hence the final cooling or average energy change.

\[^7\] Strictly speaking, in the limit $d_{SM} \rightarrow \infty$ the conditions in Eq. (D22) must only be satisfied for almost all $i$, i.e., for all but a small subset that contributes negligibly to the relative entropy, as we discuss below.
Expressing the global states with respect to the product of local energy eigenbases, we have that the initial joint state is \( |\psi_0\rangle = \frac{1}{\sqrt{d}} \otimes \tau_M(\beta, H_M) = \text{diag}(\lambda_M^{ij}) \) [see Eq. (D23)] and the unitary that cools the target as much as possible at minimum energy cost is the one achieving the globally passive final joint state \( g'_{\beta, M} = \text{diag}(\lambda_M^{ij}) \). This leads to the following reduced states

\[
g'_M = \left( \sum_{i=0}^{d_M-1} \lambda_M^{i\downarrow} \right) |0\rangle\langle 0| + \left( \sum_{i=0}^{d_M-1} \lambda_M^{i\uparrow} \right) |1\rangle\langle 1|,
\]

\[
g'_M = \frac{1}{2} \left( \lambda_M^{0\downarrow} + \lambda_M^{2d_M}\downarrow \right) |0\rangle\langle 0| + \frac{1}{2} \left( \lambda_M^{0\uparrow} + \lambda_M^{2d_M}\uparrow \right) |1\rangle\langle 1| + \frac{1}{2} \left( \lambda_M^{1\downarrow} + \lambda_M^{2d_M+1}\downarrow \right) |2\rangle\langle 2| + \frac{1}{2} \left( \lambda_M^{1\uparrow} + \lambda_M^{2d_M+1}\uparrow \right) |3\rangle\langle 3| + \ldots
\]

Intuitively, the reduced target state has the larger half of the initial machine eigenvalues in the ground state and the smaller half in the excited state; the reduced machine state has the sum of the largest elements from each of these halves in its ground state, the next largest element from each half (which, in this case, is equal to the first) in its first excited state, and so forth.

Let us denote the spectrum of the final state of the machine by \( \lambda_M^{ij} := \{ \lambda_M^{ij_0}, \lambda_M^{ij_1}, \ldots, \lambda_M^{ij_{d_M-1}} \} = \frac{1}{2} \{ \lambda_M^{ij_0} + \lambda_M^{ij_{d_M-1}} \}, \lambda_M^{ij_0} + \lambda_M^{ij_{d_M-1}} \}, \ldots, \lambda_M^{ij_{d_M-1}} \}. Importantly, by construction, the reduced state of the final machine has its local eigenvalues in non-increasing order, i.e., it is energetically passive.

We therefore have the final reduced states of the protocol that cools the initially maximally-mixed target as much as possible at minimal energy cost, in particular with minimal heat dissipation by the machine, given the structural resources at hand. We can now analyse the properties that are required to saturate the Landauer limit by considering the terms on the r.h.s. of Eq. (3) for any fixed initial inverse temperature of the machine \( \beta \geq 0 \).

First note that cooling the target system by any amount fixes the change in entropy of the target system, so the first term is irrelevant. The second term concerns the mutual information built up between the target system and machine. In general, there are processes that asymptotically display the behaviour \( I \rightarrow \infty \) for all \( \omega_i \) for \( i \geq d_M \), diverge (since the summation contains only non-negative summands).

The final term that must be minimised to saturate the Landauer limit is the relative entropy of the final with respect to the initial state, the next largest element from each half (which, in this case, is equal to the first) in its first excited state, and so forth.

In order to achieve perfect cooling at the Landauer limit, one thus must simultaneously satisfy the conditions outlined in Theorem 10. In other words, to minimise the relative entropy term with the additional constraints \( \sum_{i=0}^{d_M-1} \lambda_M^{i\downarrow} \rightarrow 1 \) and \( \sum_{i=0}^{d_M-1} \lambda_M^{i\uparrow} \rightarrow 0 \). The first thing to note is that since the eigenvalues \( \lambda_M^{ij} \) contribute to different sums depending on whether \( i \) is in the larger half \( \{0, \ldots, \frac{d_M}{2} - 1\} \) or smaller half \( \{ \frac{d_M}{2}, \ldots, d_M \} \), one cannot have \( \lambda_M^{i\downarrow} + \lambda_M^{i\uparrow} \rightarrow 1 \) \( \forall i \) (i.e., a completely degenerate machine), since then both summations would be over identical values and there is no way for them to converge to distinct values. This precludes the trivial solution that satisfies the constraints of Eq. (D22) alone, namely the maximally mixed machine state which cannot be used to perform any cooling [as, in particular, it does not satisfy the constraints of Eq. (D21)]. For the conditions to be simultaneously satisfied, we intuitively require that, although they must be distinct, for each \( i \) both \( \lambda_M^{i\downarrow} \) and
and \((d_{\mathcal{M}} + \frac{1}{2})\) become “close” to each other, but with a difference that decays rapidly as \(d_{\mathcal{M}} \to \infty\), such that in the infinite-dimensional limit the larger “half” of the eigenvalues sum to one and the smaller “half” sum to zero. A subtle point to note is that because the relative entropy involves the ratio of final to initial eigenvalues it is not enough that the absolute difference \(|\lambda_{M}^{i} - \lambda_{M}^{j}|\) goes to zero, as in the infinite \(d_{\mathcal{M}}\) limit, it is possible for this to happen for all of the eigenvalues approaching zero without the ratios of final to initial eigenvalues approaching unity (and hence the relative entropy not vanishing). One manner of satisfying such a constraint, as evidenced by the construction we proceed with next, is for the ratios of final to initial eigenvalues to go to unity for all but a small number energy levels, with the population in this exceptional subspace going to zero in the infinite \(d_{\mathcal{M}}\) limit (along with the ratios not diverging within said subspace).

The natural question that arises here is whether or not it is possible to satisfy these constraints concurrently. (Note that none of the cooling protocols provided throughout this article use the max-cooling operation, so do not serve as examples.) To this end, we now construct a family of machine Hamiltonians \(H_{M}\) of increasing dimension that in the limit \(d_{\mathcal{M}} \to \infty\) manages to attain both perfect cooling of a maximally-mixed qubit and the Landauer limit for the energy cost using the maximal cooling operation discussed above. The form of the Hamiltonian is instructive regarding the complexity requirements for perfect cooling at the Landauer limit. The construction is inspired by the infinite-dimensional Hamiltonian found in Ref. [29] (Appendix D), therein used to perfectly cool a qubit with energy cost arbitrarily close to the Landauer limit. Their construction already begins with infinitely many machine eigenvalues, as well as infinitely many of them corresponding to diverging energy levels. In the following, we demonstrate that one can arbitrarily closely attain perfect cooling and the Landauer limit with finite-dimensional Hamiltonians, and by taking the limit \(d_{\mathcal{M}} \to \infty\), recover the result of Ref. [29].

The Hamiltonian of the machine is \(d_{\mathcal{M}} := 2^{N+1}\) dimensional,

\[
H_{M} = \sum_{n=0}^{N} \sum_{j=1}^{2^{n}} \left( n \Delta |n;j\rangle\langle n;j|_{M} \right) + N \Delta |N;2^{N+1}\rangle\langle N;2^{N+1}|_{M} \tag{D28}
\]

Here, each energy eigenvalue labelled by \(n\) is \(2^n\)-fold degenerate. Thus the ground state is unique, the first excited state is two-fold degenerate, the second excited state four-fold degenerate, and so on, with the degeneracy doubling every energy level. In order to make the Hamiltonian of even dimensionality for convenience, we have added an extra degenerate state to the final level (which makes this level \((2^{N+1})\)-fold degenerate). Also note that the Hamiltonian is equally spaced with energy gap \(\Delta\).

In the following, we use the index \(n\) to denote any one of the degenerate states in the \(n\)th energy level from \(n = 0\) to \(n = N\), and the index \(i\) to denote individual energy eigenstates from \(i = 1\) to \(i = 2^{N+1}\) (note that in contrast to the previous section, we are here beginning with \(i = 1\) in order to simplify some future notation). With these indices, the eigenvalues are related by

\[
\lambda_{M}^{M} = e^{-\beta \Delta} \lambda_{M}^{\frac{1}{2}} \quad \forall \, i \in \{2, \ldots, d_{\mathcal{M}} - 1\},
\]

\[
\lambda_{M}^{L} = e^{-\beta \Delta} \lambda_{M}^{n-1} \quad \forall \, n \in \{1, \ldots, N\}. \tag{D29}
\]

We introduce a parameter \(\epsilon\) to express the Gibbs ratio as

\[
e^{-\beta \Delta} = \frac{1 - \epsilon}{2}, \tag{D31}
\]

where \(0 < \epsilon < 1\), and we will eventually take the limit \(\epsilon \to 0\) appropriately as the dimension diverges. Note that this constrains the Gibbs ratio to be smaller than \(\frac{1}{2}\), which in turn ensures that the total population over all of the degenerate eigenstates in the \(n\)th level is smaller than that in the \((n-1)\)th level (as it has twice the number of eigenstates, but less than half the population in each). If this constraint failed to hold, then in the asymptotic limit, all of the population would lie in energy levels that diverge.

We now consider using this machine to cool a maximally-mixed qubit target. The final ground-state population of the qubit under the maximal cooling operation is the sum over the larger half of the eigenvalues of the machine, corresponding to the eigenvalues from \(i = 1\) to \(i = 2^N\) (equivalently, from \(n = 0\) to \(n = N - 1\) plus a single eigenvalue from the \(n = N\) energy level), and is thus given by

\[
p_{0} = \frac{1}{Z_{M}} \sum_{n=0}^{N-1} 2^{n} \left( \frac{1 - \epsilon}{2} \right)^{n} + \left( \frac{1 - \epsilon}{2} \right)^{N}, \tag{D32}
\]

where

\[
Z_{M} = \sum_{n=0}^{N} 2^{n} \left( \frac{1 - \epsilon}{2} \right)^{n} + \left( \frac{1 - \epsilon}{2} \right)^{N}, \tag{D33}
\]

is the partition function of the machine. The geometric series above evaluates to

\[
p_{0} = \left( 1 + \frac{\epsilon (1 - \epsilon)^N}{1 - (1 - \epsilon)^N + \epsilon (1 - \epsilon)^N 2^{-N}} \right)^{-1}. \tag{D34}
\]
As an ansatz, supposing that $\epsilon$ scales inversely with $N$ as $\epsilon := \frac{\theta}{N}$ leads to the simplification $(1 - \epsilon)^N \to e^{-\theta}$ as $d_M$ (and hence $N$) diverges. The asymptotic behaviour of the ground-state population is thus

$$p'_0 = 1 - \frac{1}{N} \left( \frac{\theta}{e^\theta - 1} \right) + O \left( \frac{1}{N^2} \right), \quad \text{(D35)}$$

and so $p'_0 \to 1$ in the $N \to \infty$ limit.

We now move to calculate the energy cost. Rather than considering the optimal max-cooling operation described above, we consider a slight modification in order to make the connection to the construction in Ref. [29] clear as well as to simplify notation. Nonetheless, the energy cost of this modified protocol upper-bounds that of the max-cooling operation (for the same achieved ground-state population), and so showing that the Landauer limit is attained for the modified protocol implies that it would be too for the max-cooling protocol. The modification is simply to relabel the smallest eigenvalue of the machine $\lambda_{M}^{X_{\mu+1}}$ as $\lambda_{M}^{X_{\mu}}$, and treat it as the ground-state eigenvalue in the max-cooling operation. For general machine states, such a switch would lead to less cooling (if the same unitary were applied), but in this case it does not because the sum of the first half of the eigenvalues, from $i = 0$ to $i = 2^N - 1$, is the same as the original sum from $i = 1$ to $i = 2^N$, due to the relabelling $\lambda_0 = \lambda_{2N}$, since they are both eigenvalues of states corresponding the maximum excited energy level of the machine spectrum.

The spectrum of the final state of the machine is then given by

$$\lambda^{i,i}_{M} = \frac{1}{2} \left( \lambda^{i,i}_{M} + \lambda^{i,i}_{M} + \frac{\epsilon}{\lambda_{M}^{X_{\mu}}} \right) \quad \forall \ i \in \{0, \ldots, d_M - 1\}, \quad \text{(D36)}$$

which leads to

$$\begin{align*}
\lambda^{i,0}_{M} &= \frac{1}{2} \left( \lambda^{i,0}_{M} + \lambda^{i,2N}_{M} \right) = \lambda^{i,0}_{M}, \\
\lambda^{i,1}_{M} &= \frac{1}{2} \left( \lambda^{i,1}_{M} + \lambda^{i,1}_{M} + \frac{\epsilon}{\lambda_{M}^{X_{\mu}}} \right) \quad \forall \ i \in \{2, \ldots, d_M - 1\} \\
&= \frac{1}{2} \left( \frac{2}{1 - \epsilon} \lambda^{i,1}_{M} + \lambda^{n=N}_{M} \right) = \frac{1}{2} \frac{1}{Z_M} \left[ \left( 1 - \epsilon \right) \lambda^{i,0}_{M} + \left( 1 - \epsilon \right) \lambda^{i,1}_{M} \right],
\end{align*}\quad \text{(D37)}$$

where we observe that the index $i \frac{1}{2} + \frac{\epsilon}{\lambda_{M}^{X_{\mu}}}$ corresponds to the largest energy level of the machine for all $i$, and we have used Eq. (D29) for the spectrum of initial eigenvalues. Using the index $n$ instead to denote a generic eigenvalue of the $n^{th}$ energy level, we have the simpler expression

$$\lambda^{i,n}_{M} = \frac{1}{2} \left( \lambda^{i,n-1}_{M} + \lambda^{i,n}_{M} \right), \quad \forall \ n \in \{1, 2, \ldots, N\}. \quad \text{(D38)}$$

The energy cost can now be simply calculated from the difference in the average energy of the machine state,

$$\Delta E_M = \sum_{i=0}^{d_M - 1} (\lambda^{i,i}_{M} - \lambda^{i,i}_{M}) \omega_i, \quad \text{(D39)}$$

where we denote the $\beta^{th}$ energy eigenvalue by $\omega_i$. $\lambda^{i,0}_{M}$ is unchanged, and although $\lambda^{i,1}_{M}$ does change, $\omega_1 = 0$ corresponds to the ground state and thus this eigenvalue change does not affect the energy cost. We can thus express the energy cost in terms of the index $n$ instead, starting from $n = 1$ (corresponding to $i = 2$ onward), as

$$\Delta E_M = \sum_{n=1}^{N} (\lambda^{i,n}_{M} - \lambda^{i,n}_{M}) \omega_n = \frac{1}{\beta} \left[ 1 - \frac{2(1 - \epsilon)^N}{1 - (1 - \epsilon)^N + (1 - 2^{-N})(1 - \epsilon)^N} \right] \log \left( \frac{2}{1 - \epsilon} \right). \quad \text{(D40)}$$

As we did above, we parameterise $\epsilon = \frac{\theta}{N}$. The asymptotic behaviour of the energy cost is then

$$\beta \Delta E_M = \log(2) + \frac{1}{N} \left( 1 - \frac{2 \log(2)}{e^\theta - 1} \right) \theta + O \left( \frac{1}{N^2} \right), \quad \text{(D41)}$$

or in terms of the decrease in entropy of the system,

$$\beta \Delta E_M = \Delta S_M + \frac{\log N}{N} \left( \frac{\theta}{e^\theta - 1} \right) + O \left( \frac{1}{N} \right). \quad \text{(D42)}$$
Combining (D35) and (D41), we have that in the limit $N \to \infty$, which is also $d_M \to \infty$, the ground state population approaches 1—corresponding to perfect cooling—and the energy cost approaches $\beta^{-1} \log(2)$, which is the Landauer limit for the perfect erasure of a maximally-mixed qubit. To connect this construction to the constraints of Eq. (D22), note that in the limit $N \to \infty$ (recalling that $\epsilon = \frac{\theta}{B}$),

$$\frac{\lambda_i^{\text{in}}}{\lambda_i^{\text{in}}} = \lim_{N \to \infty} \frac{\left(\frac{1}{2} - \epsilon\right)^{n-1} + \left(\frac{1}{2} + \epsilon\right)^N}{\left(\frac{1}{2} + \epsilon\right)^N} = \lim_{N \to \infty} \left[\frac{1}{1 - \epsilon} + \frac{1}{2N-n+1} \left(\frac{e^{-\theta}}{(1 - \epsilon)^n}\right)\right] = 1,$$

(D43)

for all $n \geq 1$, leaving only the ground state eigenvalue (corresponding to $n = 0$ and $i = 1$) not satisfying the condition. However this term is actually a negative contribution to the relative entropy as this eigenvalue decreases, and in any case can be verified independently to approach zero.

To see this, note that a necessary condition that ensures the contribution of any set of eigenvalues that do not satisfy Eq. (D22) to the relative entropy to be negligible is that the total population of the relevant subspace is vanishingly small. Writing the relative entropy between two states in terms of their eigenvalues, we have $D(\rho'\|\rho) = \sum_n \lambda_n' \log\left(\frac{\lambda_n}{\lambda_n'}\right)$, which we split up into two sets: $S_0$ containing all $n$ for which Eq. (D22) is satisfied and $S_\pm$ containing all $n$ for which Eq. (D22) is not satisfied. The contribution of the first term to the relative entropy is asymptotically zero, so we are left with $D(\rho'\|\rho) = \sum_{n \in S_\pm} \lambda_n' \log\left(\frac{\lambda_n}{\lambda_n'}\right)$. For each term in the sum here, one can write $\lambda_n = \lambda_n'(1 + \Delta_n)$ with the condition $|\Delta_n| \geq \theta > 0$ for some $\theta$, i.e., the ratio of eigenvalues is bounded away from unity (on either side) by at least $\theta$. This leads to the expression

$$D(\rho'\|\rho) = -\sum_{n \in S_\pm} \lambda_n' \log(1 + \Delta_n) = -N_\pm \sum_{n \in S_\pm} p_n \log(1 + \Delta_n),$$

(D44)

where we have renormalised the eigenvalues (which here correspond to a subnormalised probability distribution) by writing $\lambda_n' = N_\pm p_n$, with $N_\pm := \sum_{n \in S_\pm} \lambda_n'$ being the total population of the subspace $S_\pm$ and $\{p_n\}$ here forming a probability distribution. Note that the ratio of eigenvalues going to unity in the $S_\pm$ subspace implies that the total populations of initial and final eigenvalues in this subspace are equal, i.e., $\sum_{n \in S_\pm} \lambda_n = \sum_{n \in S_0} \lambda_n'$, which in turn implies that the same is true for the $S_\pm$ subspace, leading to $\sum_{n \in S_\pm} p_n \Delta_n = 0$.

We argue from the concavity of the logarithm function that

$$\frac{1}{2} \log(1 + \theta) + \frac{1}{2} \log(1 - \theta) \geq \sum_{n \in S_\pm} p_n \log(1 + \Delta_n).$$

(D45)

Visualising the graph of the function $y = \log(1 + x)$, the latter expression above must evaluate to a point that lies within the intersection of the convex hull of $(\Delta_n, \log(1 + \Delta_n))$ and the linear equality $\sum_{n \in S_\pm} p_n \Delta_n = 0$, the latter of which is the line $x = 0$. By the concavity of the logarithm, the aforementioned convex hull lies entirely below the line segment connecting $(1 - \theta, \log(1 - \theta))$ to $(1 + \theta, \log(1 + \theta))$, and thus the expression is upper bounded by the intersection of this line segment with $x = 0$, which is precisely the l.h.s. of the inequality above. Thus we have the inequality

$$D(\rho'\|\rho) \geq -N_\pm \left[\frac{1}{2} \log(1 + \theta) + \frac{1}{2} \log(1 - \theta)\right] = -\frac{N_\pm}{2} \log(1 - \theta^2) \geq \frac{N_\pm}{2} \theta^2,$$

(D46)

where we used $\log(1 - \theta^2) \leq -\theta^2$ for all $\theta \in [-1, 1]$. As $\theta > 0$, the only way that this contribution to the relative entropy by the eigenvalues that do not satisfy Eq. (D22) can be asymptotically negligible is if the total population of their associated subspace $N_\pm$ goes to zero.

Finally note that, as mentioned in the main text, the above result pertains to the restricted setting where the target system is cooled as much as possible. However, this is not the only way to approach perfect cooling at the Landauer cost: instead of the largest half of global eigenvalues being placed into the ground state subspace of the target system, any amount of them such that their sum is sufficiently close to one would suffice. Although it is complicated to derive an exact set of conditions that would need to be satisfied in such cases (since it depends upon exactly which eigenvalues are permuted to which subspaces), the fact that fine-tuned control over particular degrees of freedom is required remains. Lastly, note that even in the restricted setting of cooling the target as much as possible, the situation becomes even more complicated when considering target systems that begin at a finite temperature. Here, the choice of which global eigenvalues should be permuted to which subspaces, the fact that fine-tuned control over particular degrees of freedom is required remains. Nonetheless, given that cooling at minimal energy cost requires fine-tuned control to access precisely distributed populations still holds true. We leave the further exploration of such scenarios, for instance constructing optimal machines for particular initial target systems, to future work.
Appendix E: Diverging-time / Control Complexity Cooling Protocols for Harmonic Oscillators

We now analyse the case of cooling infinite-dimensional quantum systems in detail. More specifically, we consider ensembles of harmonic oscillators. For the sake of completeness, we first briefly present some key concepts that will become relevant throughout this analysis. Following this, in Appendix E2a, we construct a protocol that achieves perfect cooling at the Landauer limit using a diverging number of Gaussian operations. Although such operations are typically considered to be relatively “simple” both when it comes to experimental implementation and theoretical description, according to the effective dimension notion of control complexity that we have shown must necessarily diverge to cool at the Landauer limit [see Eq. (6)], such Gaussian operations have infinite control complexity. Subsequently, in Appendix E2b, we consider the task of perfect cooling with diverging time but restricting the individual operations to be of finite control complexity. In particular, note that such operations are non-Gaussian in general. Here, we present a protocol that approaches perfect cooling of the target system as the number of operations diverges, with finite energy cost—albeit not at the Landauer limit. Whether or not a similar protocol exists that also saturates the Landauer bound remains an open question. Finally, in Appendix E3, we reconsider the protocol from Appendix E2b in terms of a single transformation, i.e., unit time. By explicitly constructing the joint unitary transformation that is applied throughout the entire protocol, we show this to be a multi-mode Gaussian operation acting on a diverging number of harmonic oscillators. The key message to be taken away from these protocols is that, while the distinction between Gaussian and non-Gaussian operations is a significant one in terms of experimental feasibility, and it certainly plays a role regarding the task of cooling—in particular, the energy cost incurred—these concepts alone cannot be used to characterise a notion of control complexity that we have shown must necessarily diverge to cool at the Landauer limit [see Eq. (6)], such “simple” both when it comes to experimental implementation and theoretical description, according to the effective dimension notion of control complexity that we have shown must necessarily diverge to cool at the Landauer limit [see Eq. (6)], such Gaussian operations have infinite control complexity. Subsequently, in Appendix E2b, we consider the task of perfect cooling with diverging time but restricting the individual operations to be of finite control complexity. In particular, note that such operations are non-Gaussian in general. Here, we present a protocol that approaches perfect cooling of the target system as the number of operations diverges, with finite energy cost—albeit not at the Landauer limit. Whether or not a similar protocol exists that also saturates the Landauer bound remains an open question. Finally, in Appendix E3, we reconsider the protocol from Appendix E2b in terms of a single transformation, i.e., unit time. By explicitly constructing the joint unitary transformation that is applied throughout the entire protocol, we show this to be a multi-mode Gaussian operation acting on a diverging number of harmonic oscillators. The key message to be taken away from these protocols is that, while the distinction between Gaussian and non-Gaussian operations is a significant one in terms of experimental feasibility, and it certainly plays a role regarding the task of cooling—in particular, the energy cost incurred—these concepts alone cannot be used to characterise a notion of control complexity that we have shown must necessarily diverge to cool at the Landauer limit [see Eq. (6)], such Gaussian operations have infinite control complexity.

E1. Preliminaries

We consider ensembles of $N$ harmonic oscillators (i.e., infinite-dimensional systems consisting of $N$ bosonic modes), which are associated to a tensor product Hilbert space $H_{\text{tot}} = \bigotimes_{j=1}^{N} H_j$ and (resp: lowering, raising) mode operators $\{a_k, a_k^\dagger\}$ satisfying the bosonic commutation relations:

\[ [a_k, a_{k'}^\dagger] = \delta_{kk'}, \quad [a_k, a_{k'}] = 0, \quad \forall k, k' = 1, 2, \ldots, N. \tag{E1} \]

The free Hamiltonian of any such system can be written as $H_{\text{tot}} = \sum_{k=1}^{N} \omega_k a_k a_k^\dagger$, where $\omega_k$ represents the energy gap of the $k$-th mode (in units where $\hbar = 1$). Position- and momentum-like operators for each mode can be defined as follows (for simplicity, we use the rescaled version below where the $\omega_k$ are omitted from the pre-factors)

\[ q_k := \frac{1}{\sqrt{2}} (a_k + a_k^\dagger), \quad p_k := \frac{1}{i\sqrt{2}} (a_k - a_k^\dagger). \tag{E2} \]

As a consequence of the commutation relations in Eq. (E1), the generalised position and momentum operators satisfy the canonical commutation relations

\[ [q_k, p_{k'}] = i\delta_{kk'}. \tag{E3} \]

To simplify notation, one may further introduce the vector of quadrature operators $\mathbf{X} := (q_1, p_1, \ldots, q_N, p_N)$; then, the commutation relations can be expressed succinctly as

\[ [X_k, X_{k'}] = i\Omega_{kk'}, \tag{E4} \]

where the $\Omega_{kl}$ are the components of the symplectic form

\[ \Omega = \bigoplus_{j=1}^{N} \Omega_j, \quad \Omega_j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{E5} \]

The density operator associated to $N$ harmonic oscillators can be written in the so-called phase-space representation as

\[ \varrho = \frac{1}{(2\pi)^N} \int \chi(\Omega \xi) \mathcal{W}(-\Omega \xi) d^{2N} \xi, \tag{E6} \]
where $\mathcal{W}(\xi) := e^{i\xi^T X}$ is the Weyl operator and $\chi(\xi) := \text{tr}[\rho \mathcal{W}(\xi)]$ is called the characteristic function.

Throughout our analysis, we will see that a particular class of states and operations, namely those that are known as *Gaussian* states, are of particular importance. A Gaussian state is one for which the characteristic function is Gaussian

$$\chi(\xi) = e^{-i\xi^T \Gamma \xi + \xi^T \xi}. \quad \text{(E7)}$$

Here, $\vec{X} := \langle X \rangle_\rho$ is the displacement vector or vector of first moments, and $\Gamma$ is a real symmetric matrix that collects the second statistical moments of the quadratures, which is known as the covariance matrix. Its entries are given by

$$\Gamma_{mn} := \langle X_m X_n + X_n X_m \rangle_\rho - 2 \langle X_m \rangle_\rho \langle X_n \rangle_\rho. \quad \text{(E8)}$$

We see that any Gaussian state is thus uniquely determined by its first and second moments. As an example of specific interest here, we recall that any thermal state $\tau$ of a harmonic oscillator with frequency $\omega$ is a Gaussian state and has vanishing first moments, $\vec{X} = 0$. Here and throughout this article, we are assuming that the infinite-dimensional thermal state is well-defined (see, e.g., Ref. [55] for discussion). The covariance matrix of a thermal state is proportional to the $2 \times 2$ identity, and given by

$$\Gamma[\tau(\beta, H)] = \coth \left( \frac{\beta \omega}{2} \right) \mathbb{1}_2. \quad \text{(E9)}$$

Gaussian operations are transformations that map the set of Gaussian states onto itself. Such operations, which include, e.g., beam-splitting and phase-space displacement, are generally considered to be relatively easily implementable in the laboratory. Although non-unitary Gaussian operations exist as well, all of the examples mentioned above are Gaussian unitaries. Such Gaussian unitaries are generated by Hamiltonians that are at most quadratic in the raising and lowering operators. Conversely, any Hamiltonian that can be expressed as a polynomial of at most second order in the mode operators generates a Gaussian unitary. Any unitary Gaussian transformation can be represented by an affine map $(M, \kappa)$,

$$\vec{X} \mapsto M \vec{X} + \kappa, \quad \text{(E9)}$$

where $\kappa \in \mathbb{R}^{2N}$ is a displacement vector in the phase-space representation and $M$ is a symplectic $2N \times 2N$ matrix that leaves the symplectic form $\Omega$ invariant, i.e.,

$$M \Omega M^T = \Omega. \quad \text{(E10)}$$

Under such a mapping, the first and second moments transform according to

$$\vec{X} \mapsto M \vec{X} + \kappa, \quad \Gamma \mapsto M \Gamma M^T. \quad \text{(E11)}$$

Lastly, note that the energy of a Gaussian state $\rho_G$ with respect to its free Hamiltonian $H = \sum_k \omega_k a_k^\dagger a_k$ can be calculated in terms of the first and second moments as follows [20]

$$E(\rho_G) = \sum_k \omega_k \left( \frac{1}{4} \text{tr} \left[ \Gamma^{(k)} - 2 \right] + \frac{1}{2} \| \vec{X}^{(k)} \|^2 \right), \quad \text{(E12)}$$

where $\| \cdot \|$ denotes the Euclidean norm. Here, $\Gamma^{(k)}$ indicates the $(2 \times 2)$-submatrix of the full covariance matrix $\Gamma$ corresponding to the reduced state of the $k^{th}$ mode. Similarly $\vec{X}^{(k)}$ denotes the two-component subvector of first moments for the $k^{th}$ mode of the displacement vector $\vec{X}$.

### E2. Diverging-time Cooling Protocol for Harmonic Oscillators

#### E2a. Diverging-time Protocol using Gaussian Operations (with Diverging Control Complexity)

We now consider a simple protocol for lowering the temperature of a single-mode system within the coherent-control paradigm using a single harmonic oscillator machine. This protocol will form the basic step of a protocol for achieving perfect cooling at the Landauer limit using diverging time, which we subsequently present.

In the situation we consider here, the target system $S$ to be cooled is a harmonic oscillator with frequency $\omega_s$ interacting with a harmonic oscillator machine $M$ at frequency $\omega_M \geq \omega_s$ via a (non-energy-conserving) unitary acting on the joint system $S \otimes M$ initialised as a tensor product of thermal states $\tau_s(\beta, H_s) \otimes \tau_M(\beta, H_M)$ at inverse temperature $\beta$ with respect to their local Hamiltonians $H_s$ and $H_M$, respectively. The joint covariance matrix of the system and machine modes is block-diagonal since the initial state is of product form, i.e.,

$$\Gamma[\tau_s(\beta, H_s) \otimes \tau_M(\beta, H_M)] = \Gamma[\tau_s(\beta, H_s)] \otimes \Gamma[\tau_M(\beta, H_M)]. \quad \text{(E13)}$$
and the $2 \times 2$ blocks of the individual modes are also diagonal, with the explicit expression $\Gamma[\tau_S(\beta, H_s)] = \coth \left( \frac{\beta \omega_s}{2} \right) \mathbb{1}_2$.

In this setting, it has been shown that the minimum reachable temperature of the target system is given by $T_{\text{min}} = \frac{\omega_s}{\omega_M} T$ (for the case $\omega_M \geq \omega_s$) [40]. The non-energy-conserving unitary transformation that achieves this is of the form

$$U = e^{-\frac{i}{2} \left(a^\dagger b + a b^\dagger\right)}, \quad (E14)$$

where the operators $a$ ($a^\dagger$) and $b$ ($b^\dagger$) denote the annihilation (creation) operators of the target system and machine, respectively. This beam-splitter-like unitary acts as a swap with a relative phase factor imparted on the resultant state; nonetheless, this phase is irrelevant at the level of the covariance matrix, which fully characterises the (Gaussian) thermal states considered, and transforms it according to a standard swapping of the systems. After acting with such a swap operator, which is a Gaussian operation, the first moment remains vanishing and the covariance matrix transforms as [see Eq. (E11)]

$$\begin{bmatrix}
\coth \left( \frac{\beta \omega_s}{2} \right) & 0 \\
0 & \coth \left( \frac{\beta \omega_M}{2} \right)
\end{bmatrix}
\xrightarrow{\text{SWAP}}
\begin{bmatrix}
\coth \left( \frac{\beta \omega_M}{2} \right) & 0 \\
0 & \coth \left( \frac{\beta \omega_s}{2} \right)
\end{bmatrix}. \quad (E15)$$

This means that both the output target system and machine are thermal states at different temperatures $T_s' = \frac{\omega_M}{\omega_s} T$ and $T_M' = \frac{\omega_M}{\omega_s} T$. Making use of Eq. (E12), we can calculate the energy change for the system and machine as

$$\Delta E_s = E \left[ \tau_S \left( \frac{\omega_M}{\omega_s}, \beta, H_s \right) \right] - E \left[ \tau_S(\beta, H_s) \right] = \frac{\omega_s}{2} \left[ \coth \left( \frac{\beta \omega_M}{2} \right) - \coth \left( \frac{\beta \omega_s}{2} \right) \right],$$

$$\Delta E_M = E \left[ \tau_M \left( \frac{\omega_M}{\omega_s}, \beta, H_M \right) \right] - E \left[ \tau_M(\beta, H_M) \right] = \frac{\omega_M}{2} \left[ \coth \left( \frac{\beta \omega_M}{2} \right) - \coth \left( \frac{\beta \omega_s}{2} \right) \right]. \quad (E16)$$

The total energy cost associated to such a swap operation is thus

$$\Delta E_{s,M} = \Delta E_s + \Delta E_M = \frac{(\omega_M - \omega_s)}{2} \left[ \coth \left( \frac{\beta \omega_M}{2} \right) - \coth \left( \frac{\beta \omega_s}{2} \right) \right] = (\omega_M - \omega_s) e^{-\beta \omega_s} \frac{(1 - e^{-\beta (\omega_M - \omega_s)})}{(1 - e^{-\beta \omega_s})(1 - e^{-\beta \omega_M})}. \quad (E17)$$

Note that this form is similar to that for finite-dimensional systems with equally spaced Hamiltonian [cf., Eq. (C19)]; the dimension-dependent term vanishes as $d \to \infty$, simplifying the expression in the infinite-dimensional case.

With this simple protocol for lowering the temperature of a harmonic oscillator target using a single harmonic oscillator machine at hand, we are now in a position to describe an energy-optimal (in the sense of saturating the Landauer bound) cooling protocol for cooling a single harmonic oscillator to the ground state and one of the harmonic oscillators in the machine. Here, we argue that in the case of a non-Gaussian diverging-time protocol for cooling a single harmonic oscillator to the ground state using finite control complexity in each of the infinitely many steps, and at an overall finite (albeit not minimal, i.e., not at the Landauer limit) energy cost. Before presenting this non-Gaussian protocol, let us now discuss the details of the Gaussian diverging-time protocol for cooling at the Landauer limit.

We consider a harmonic oscillator with the frequency $\omega_s$ as the target system and the machine to comprise $N$ harmonic oscillators, where the $n^{\text{th}}$ oscillator has frequency $\omega_M^n = \omega_s + n \epsilon$. In addition, we assume that all modes are initially uncorrelated and in thermal states at the same inverse temperature $\beta$ with respect to their free Hamiltonians, i.e., the target system is $\tau_S(\beta, H_s)$ and the multi-mode thermal machine is $\tau_M(\beta, H_M) = \otimes_{n=1}^{N} \tau_{M_n}(\beta, H_{M_n})$.

In this case, the cooling process is divided into $N$ time steps. During each step, there is an interaction between the target system and one of the harmonic oscillators in the machine. Here, we assume that at the $n^{\text{th}}$ time step, the target system interacts only with the $n^{\text{th}}$ harmonic oscillator, which has frequency $\omega_s + n \epsilon$. To obtain the minimum temperature for the target system, we perform the previously outlined cooling process at each step, which is given by swapping the corresponding two modes. Using Eq. (E15), the covariance matrix transformation of the two-mode process at the first time step takes the form

$$\Gamma^{(1)}(\tau_S(\beta) \otimes \tau_{M_1}(\beta)) = \begin{bmatrix}
\coth \left( \frac{\beta \omega_s}{2} \right) & 0 \\
0 & \coth \left( \frac{\beta (\omega_s + \epsilon)}{2} \right)
\end{bmatrix}
\xrightarrow{\text{SWAP}}
\begin{bmatrix}
\coth \left( \frac{\beta (\omega_s + \epsilon)}{2} \right) & 0 \\
0 & \coth \left( \frac{\beta \omega_s}{2} \right)
\end{bmatrix}. \quad (E18)$$
By repeating this process on each of the harmonic oscillators in the machine, after the \((n-1)\)th step, the \(2\times2\) block corresponding to the target system \(S\) in the covariance matrix is given by \(\coth \left( \frac{\beta (\omega - (n-1)\epsilon)}{2} \right) \mathbb{I}_2\). Therefore, one can show inductively that the covariance matrix transformation associated to the \(n\)th interaction is given by

\[
\Gamma^{(n)}(\tau_S (\beta) \otimes \tau_{M_n}(\beta)) = \begin{bmatrix}
\coth \left( \frac{\beta (\omega - (n-1)\epsilon)}{2} \right) \mathbb{I}_2 & 0 \\
0 & \coth \left( \frac{\beta (\omega + n\epsilon)}{2} \right) \mathbb{I}_2
\end{bmatrix}.
\]

(E19)

Based on this process, after \(N\) steps (i.e., after the system has interacted with all \(N\) harmonic oscillators), the minimal achievable temperature of the target system is \(T_{\text{min}}^{(N)} = \frac{\omega}{\omega + N\epsilon T}\). Moreover, by using Eq. (E16), one can calculate the energy changes of the target system and the machine at each time step as

\[
\Delta E_S^{(n)} = \frac{\omega_S}{2} \left[ \coth \left( \frac{\beta (\omega_S + n\epsilon)}{2} \right) - \coth \left( \frac{\beta (\omega_S + (n-1)\epsilon)}{2} \right) \right],
\]

\[
\Delta E_M^{(n)} = \frac{(\omega_S + n\epsilon)}{2} \left[ \coth \left( \frac{\beta (\omega_S + (n-1)\epsilon)}{2} \right) - \coth \left( \frac{\beta (\omega_S + n\epsilon)}{2} \right) \right].
\]

(E20)

The total energy change for the target system during the overall process (i.e., throughout the \(N\) steps) is thus given by

\[
\Delta E_S = \sum_{n=1}^{N} \Delta E_S^{(n)} = \sum_{n=1}^{N} \frac{\omega_S}{2} \left[ \coth \left( \frac{\beta (\omega_S + n\epsilon)}{2} \right) - \coth \left( \frac{\beta (\omega_S + (n-1)\epsilon)}{2} \right) \right]
= \frac{\omega_S}{2} \left[ \coth \left( \frac{\beta (\omega_S + N\epsilon)}{2} \right) - \coth \left( \frac{\beta (\omega_S)}{2} \right) \right] = \omega_S \left[ \frac{e^{-\beta (\omega_S + N\epsilon)}}{1 - e^{-\beta (\omega_S + N\epsilon)}} - \frac{e^{-\beta (\omega_S)}}{1 - e^{-\beta (\omega_S)}} \right].
\]

(E21)

Here, we have written \(\coth (x) = 1 + (2e^{-2x})/(1 - e^{-2x})\). Similarly, one can obtain the total energy change of the overall machine

\[
\Delta E_M = \sum_{n=1}^{N} \Delta E_M^{(n)} = \sum_{n=1}^{N} \frac{\omega_S + n\epsilon}{2} \left[ \coth \left( \frac{\beta (\omega_S + (n-1)\epsilon)}{2} \right) - \coth \left( \frac{\beta (\omega_S + n\epsilon)}{2} \right) \right]
= \frac{N}{\omega_S + n\epsilon} \left[ \frac{e^{-\beta (\omega_S + (n-1)\epsilon)}}{1 - e^{-\beta (\omega_S + (n-1)\epsilon)}} - \frac{e^{-\beta (\omega_S + n\epsilon)}}{1 - e^{-\beta (\omega_S + n\epsilon)}} \right].
\]

(E22)

It is straightforward to check that the total energy change, i.e., the sum of Eqs. (E21) and (E22), is equal to the energy cost obtained in Eq. (C19) with \(d \to \infty\). In particular, this can be seen by considering the second line of Eq. (C19), where the second term in round parenthesis vanishes as \(d \to \infty\) for any value of \(N\). Thus, when the number of operations diverges \(N \to \infty\) and \(\epsilon = (\omega_{\text{max}} - \omega_S)/N \to 0\), where \(\omega_{\text{max}} := \frac{\beta}{\omega_S}\) is the maximum frequency of the machines, the heat dissipated by the machines throughout the process saturates the Landauer bound and is therefore energetically optimal. Moreover, by taking \(\omega_{\text{max}} \to \infty\) one approaches perfect cooling.

At this point, a comment on the notion of control complexity is in order. According to Eq. (6), the effective dimension of the machine in the protocol we consider here diverges in addition to time. Indeed, the notion of control complexity thus defined diverges for any Gaussian operation acting on the machine, in particular, it diverges for any single one of the infinitely many steps of the protocol, as each operation is a two-mode Gaussian operation. At first glance, this appears to be in contrast to the common conception that Gaussian operations are typically easily implementable (cf. Refs. [20, 56]). However, an alternative way of interpreting this protocol is that, exactly because of the simple structure of Gaussian operations, reaching the ground state at finite energy cost requires a diverging number of two-mode Gaussian unitaries, and thus divergingly many modes on which to act (see also Appendix E3). In fact, if non-Gaussian unitaries are employed, then the ground state can be approached at finite energy cost using just a single harmonic oscillator machine, as we now show.

\textit{E2b. Diverging-time Protocol using Non-Gaussian Operations (with Finite Control Complexity)}

We now consider a protocol for cooling a single harmonic oscillator at frequency \(\omega_S\) to the ground state using a diverging amount of time, but requiring only a finite overall energy input as well as finite control complexity in each of the diverging
number of steps of the protocol. In this protocol, the machine $\mathcal{M}$ is also represented by a single harmonic oscillator whose frequency matches that of the target oscillator that is to be cooled, $\omega_{\mathcal{M}} = \omega =: \omega$. The initial states of both the target system $S$ and machine $\mathcal{M}$ are assumed to be thermal at the same inverse temperature $\beta$, and are hence both described by thermal states of the form

$$\tau(\beta) = \frac{e^{-\beta H}}{\text{tr}[e^{-\beta H}]} = \sum_{n=0}^\infty e^{-\beta \omega n} (1 - e^{-\beta \omega}) |n\rangle\langle n| = \sum_{n=0}^\infty p_n |n\rangle\langle n|_{S,M},$$  \hfill (E23)

where the Hamiltonian $H$ is given by $H = \sum_{n=0}^\infty \omega_n |n\rangle\langle n|$ and the $p_n = e^{-\beta \omega n} (1 - e^{-\beta \omega})$ are the eigenvalues of $\tau$. The joint initial state is a product state that we can then write as

$$\tau_S(\beta) \otimes \tau_M(\beta) = \sum_{m,n=0}^\infty p_m p_n |m\rangle\langle m| \otimes |n\rangle\langle n| = \sum_{m,n=0}^\infty \tilde{p}_{m+n} |m\rangle\langle m| \otimes |n\rangle\langle n|,$$  \hfill (E24)

where we have defined $\tilde{p}_k := e^{-\beta \omega k} (1 - e^{-\beta \omega})^2$. We then note that the eigenvalues $\tilde{p}_k$ of the joint initial state have degeneracy $k + 1$. For instance, the largest value $\tilde{p}_0 = p_0 p_0$, corresponding to both the system and machine being in the ground state, is the single largest eigenvalue, but there are two eigenstates, $|0, 1\rangle$ and $|1, 0\rangle$, corresponding to the second largest eigenvalue $\tilde{p}_1$, three states, $|0, 2\rangle$, $|1, 1\rangle$, and $|2, 0\rangle$ for the third largest eigenvalue $\tilde{p}_2$, and so forth. Obviously, not all of these eigenvalues correspond to eigenstates for which the target system is in the ground state.

In order to increase the ground-state population of the target system oscillator, we can now apply a sequence of ‘two-level’ unitaries, i.e., unitaries that act only on a subspace spanned by two particular eigenstates and exchange their respective populations. The two-dimensional subspaces are chosen such that one of the two eigenstates corresponds to the system $S$ being in the ground state, $|0, k\rangle$, while the other eigenstate corresponds to $S$ being in an excited state, $|i, j\rangle$ for $i \neq j$. In addition, these pairs of levels are selected such that, at the time the unitary operation is to be performed, the population of $|0, k\rangle$ is smaller than that of $|j, i\rangle$, such that the two-level exchange increases the ground-state population of $S$ at each step.

More specifically, at the $k$th step of this sequence, the joint system $S\mathcal{M}$ is in the state $\rho_{S\mathcal{M}}^{(k)}$, and one determines the set $\Omega_k$ of index pairs $(i \neq 0, j)$ such that $\tilde{p}_k < \langle i, j | \rho_{S\mathcal{M}}^{(k)} | i, j \rangle$, i.e., the set of eigenstates for which $S$ is not in the ground state and which have a larger associated population (at the beginning of the $k$th step) than $|0, k\rangle$. One then determines an index pair $(m_k, n_k)$ for which this population is maximal, i.e., $\langle m_k, n_k | \rho_{S\mathcal{M}}^{(k)} | m_k, n_k \rangle = \max \{ \langle i, j | \rho_{S\mathcal{M}}^{(k)}(i, j) | (i, j) \in \Omega_k \}$, and performs the unitary

$$U_{S\mathcal{M}}^{(k)} = \mathbb{I}_{S\mathcal{M}} - |0, k\rangle\langle 0, k| - |m_k, n_k\rangle\langle m_k, n_k| + (|0, k\rangle\langle m_k, n_k| + |m_k, n_k\rangle\langle 0, k|).$$  \hfill (E25)

If there is no larger population that is not already in the subspace of the ground state of the target system, i.e., when $\Omega_k = \emptyset$, which is only the case for the first step ($k = 1$), then no unitary is performed. After the $k$th step, the joint state $\rho_{S\mathcal{M}}^{(k+1)}$ is still diagonal in the energy eigenbasis, and the subspace of the joint Hilbert spaces for which $S$ is in the ground state is populated with the $k + 1$ largest eigenvalues $\tilde{p}_k$ in non-increasing order with respect to non-decreasing energy eigenvalues of the subspace’s basis vectors $|0, i\rangle$. That is, for all $i \in \{0, 1, 2, \ldots, k\}$ and for all $j \in \mathbb{N}$ with $j > i$, we have $\langle 0, i | \rho_{S\mathcal{M}}^{(k+1)} | 0, i \rangle \geq \langle 0, j | \rho_{S\mathcal{M}}^{(k+1)} | 0, j \rangle$.

Since the Hilbert spaces of both $S$ and $\mathcal{M}$ are infinite-dimensional, we can continue with such a sequence of two-level exchanges indefinitely, starting with $k = 1$ and continuing step-by-step as $k \to \infty$. Here we note that the choice of $(m_k, n_k)$ is generally not unique at the $k$-th step, but as $k \to \infty$, the resulting final state is independent of the particular choices of $(m_k, n_k)$ made along the way. In particular, in a fashion that is reminiscent of the famed Hilbert hotel paradox (see, e.g., Ref. [57, p. 17]), this sequence places all of the infinitely many eigenvalues $\tilde{p}_k$ of the joint state of $S\mathcal{M}$ (which must hence sum to one) into the subspace where $S$ is in the ground state. In other words, in the limit of infinitely many steps, the population of the ground-state subspace evaluates to

$$\sum_{k=0}^\infty (k+1) \tilde{p}_k = \sum_{k=0}^\infty (k+1) e^{-\beta \omega k} (1 - e^{-\beta \omega})^2 = 1,$$  \hfill (E26)

where we have taken into account the $(k+1)$-fold degeneracy of the $k$th eigenvalue $\tilde{p}_k$. We thus have $\lim_{k \to \infty} \text{tr}_M [\rho_{S\mathcal{M}}^{(k)}] = |0\rangle\langle 0|_S$, the reduced state of the system is asymptotically the pure state $|0\rangle_S$. As per our requirement on the structural complexity (see Sec. V), the Hilbert space of the machine required to achieve this is infinite-dimensional, and since each step of the protocol is assumed to take a finite amount of time, the overall time for reaching the ground state diverges. At the same time, the control complexity for each individual step is finite, since each $U_k$ acts nontrivially only on a two-dimensional subspace. To see that also the energy cost for this protocol is finite, we first note that the protocol results in a final state of the machine that is diagonal in the energy eigenbasis $|n\rangle_{M,M}$ with probability weights $\tilde{p}_k$ decreasing (but not strictly) with increasing energy. Due to the degeneracy of the eigenvalues $\tilde{p}_k$, each one appears $(k+1)$ times.
on the diagonal (w.r.t. the energy eigenbasis) of the resulting machine state, populating adjacent energy levels. The label \( n(k) \) of the lowest energy level that is populated by a particular value \( \tilde{p}_k \) can be calculated as

\[
\tilde{n}(k) := \sum_{n=0}^{k-1} (n+1) = \frac{1}{2} k(k + 1),
\]

(E27)

while the largest energy populated by \( \tilde{p}_k \) is given by \( \tilde{n}(k + 1) - 1 \). With this, we calculate the energy of the machine after the protocol, which evaluates to

\[
\frac{E_{\text{total}}}{\omega} = \sum_{k=1}^{\infty} e^{-\beta \omega k} (1 - e^{-\beta \omega})^2 \sum_{n=\tilde{n}(k)}^{\tilde{n}(k+1)-1} n = \sum_{k=1}^{\infty} e^{-\beta \omega k} (1 - e^{-\beta \omega})^2 \frac{1}{2} k(k + 1)(k + 2) = \frac{3}{2} \coth^2 \left( \frac{\beta \omega}{2} \right).
\]

(E28)

Since the energy of the initial thermal state is given by

\[
\frac{E[\beta]}{\omega} = \sum_{n=0}^{\infty} n e^{-\beta \omega n} (1 - e^{-\beta \omega}) = \frac{e^{-\beta \omega}}{1 - e^{-\beta \omega}},
\]

we thus arrive at the energy cost

\[
\frac{\Delta E_{\text{tot}}}{\omega} = \frac{\frac{E_{\text{total}}}{\omega} - E[\beta]}{\omega} = \frac{e^{-\beta \omega}(2 + e^{-\beta \omega})}{(1 - e^{-\beta \omega})^2}.
\]

(E30)

We thus see that this energy cost is finite for all finite initial temperatures (although note that the energy cost diverges when \( \beta \to 0 \)).

However, as we shall show next, the energy cost for attaining the ground state is not minimal, i.e., the protocol achieves perfect cooling (with finite energy and control complexity, but infinite time) but not at the Landauer limit. To see this, we first observe that the entropy of the final pure state of the system \( S \) vanishes, such that \( \Delta S_S = S[\tau(\beta)] \). Evaluating this entropy, one obtains

\[
S[\tau(\beta)] = -\text{tr} [\tau \log(\tau)] = -\sum_{n=0}^{\infty} e^{-\beta \omega n} (1 - e^{-\beta \omega}) \log [e^{-\beta \omega n} (1 - e^{-\beta \omega})] = -\sum_{n=0}^{\infty} e^{-\beta \omega n} (1 - e^{-\beta \omega}) \left[ -\beta \omega n + \log(1 - e^{-\beta \omega}) \right]
\]

\[
= \beta \omega e^{-\beta \omega} + \beta \omega + \log \left( \frac{e^{-\beta \omega}}{1 - e^{-\beta \omega}} \right) = \frac{\beta \omega}{1 - e^{-\beta \omega}} + \log \left( \frac{e^{-\beta \omega}}{1 - e^{-\beta \omega}} \right).
\]

(E31)

Using the results from Eqs. (E30) and (E31), we can thus compare the expressions for \( \beta \Delta E_{\text{tot}} \) and \( \Delta S_S \), and we find that \( \beta \Delta E_{\text{tot}} = \Delta S_S > 0 \) for all nonzero initial temperatures. The origin of this difference is easily identified: Although the protocol results in an uncorrelated final state because the system is left in a pure state, that is, \( I(S : M) \parallel \rho_{\omega}^{\text{fin}} = 0 \), the last term \( D(\rho_{\omega}^{\text{fin}} \parallel \tau_M) \) in Eq. (3) is nonvanishing for nonzero temperatures because the protocol does not result in a thermal state of the machine.

With this, we have thus shown that perfect cooling is indeed possible using a finite energy cost and a finite control complexity in every one of infinitely many steps (thus using diverging time). As we have seen, the structural requirement of an infinite-dimensional effective machine Hilbert space can be met by realising \( M \) as a single harmonic oscillator. Although the presented protocol does not minimise the energy cost to saturate the Landauer bound, we cannot at this point conclusively say that it is not possible to do so in this setting. However, we suspect that a more complicated energy-level structure of the machine is necessary.

Finally, let us comment again on the notion of control complexity in terms of effective machine dimension as opposed to the notion of complexity that is often (loosely) associated with the distinction between Gaussian and non-Gaussian operations. As we see from the protocols presented here, the concept of control complexity based on the nontrivially accessed Hilbert-space dimension of the machine indeed captures the resource that must diverge in order to reach the ground state, while the intuition of complexity associated with (non)-Gaussian operations, albeit valid as a characterisation of a certain practical difficulty in realising such operations, seems to be irrelevant for determining if the ground state can be reached. In the protocol presented in this section, non-Gaussian operations with finite control complexity are used in each step to reach the ground state. Infinitely many steps (i.e., diverging time) could then be traded for a single (also non-Gaussian) operation of infinite control complexity, performed in unit time. In the previous protocol based on Gaussian operations (Appendix E2a), the control complexity diverges in every single step of the cooling protocol, but only when there are infinitely many such steps (diverging time) or one operation in unit time on infinitely many modes (see below), can we reach the ground state. However, in the latter case, the operation, though acting on a diverging number of harmonic oscillators, remains Gaussian, as we now show explicitly.
E3. Diverging Control Complexity Cooling Protocol for Harmonic Oscillators

Here we give a protocol for perfectly cooling a harmonic oscillator in unit time and with the minimum energy cost, but with diverging control complexity. In accordance with Theorem 3, the machines used to cool the target system will likewise be harmonic oscillators. Let the operators $a$ ($a^\dagger$) and $b_k$ ($b_k^\dagger$) respectively denote the annihilation (creation) operators of the target system and a machine subsystem labelled $k$. We then consider the the unitary transformation in Eq. (E14), namely

$$U_k := e^{i \vec{\xi} (a^\dagger b_k + a b_k^\dagger)}.$$  \hspace{1cm} (E32)

One can then apply the diverging-time cooling protocol from Appendix E2a to cool the system to the ground state at the Landauer limit via the total unitary transformation

$$U_{tot} := \lim_{N \to \infty} U_{(N)}, \quad \text{with} \quad U_{(N)} := \prod_{k=1}^N U_k.$$  \hspace{1cm} (E33)

We now seek the Hamiltonian that generates $U_{tot}$. First note that $U_{(N)}a U_{(N)}^\dagger = ib_1$ and

$$U_{(N)} b_k U_{(N)}^\dagger = \begin{cases} -b_{k+1}, & \text{for } k < N \\ i a, & \text{for } k = N \\ b_k, & \text{for } k > N \end{cases}$$  \hspace{1cm} (E34)

which can be proven by induction. In contrast with Appendix E2a, here we use the complex representation of the symplectic group to describe the transformation, i.e., the set of matrices $S$ satisfying $SKS^T = K$, where $K := 1_N \oplus (-1_N)$. Gathering the raising and lowering operators of the target system and the first $N$ machines into the vector $\xi := (a \ b_1 \ b_2 \ldots b_N \ a^\dagger \ b_1^\dagger \ b_2^\dagger \ldots b_N^\dagger)^T$, we can write the transformation above as $U_{(N)} \xi U_{(N)}^\dagger = S^T \xi$ [58], where

$$S = \begin{pmatrix} \alpha_{(N)} & 0 \\ 0 & \alpha_{(N)} \end{pmatrix}, \quad \text{with} \quad \alpha_{(N)} := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & i \\ i & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (E35)

Now, defining the matrix of Hamiltonian coefficients $h_{(N)}$ implicitly by $U_{(N)} := \exp(-iS^T \cdot h_{(N)} \cdot \xi)$, we have that $S = \exp(-iK h_{(N)})$ [58], i.e., $h_{(N)} = iK \log(S^T) = iK \log(S)^T$, where we take the principal logarithm. To calculate this, we must diagonalise the matrix $\alpha_{(N)}$ in Eq. (E35). The eigenvalues of $\alpha_{(N)}$ are

$$\lambda_k := -e^{-i \frac{2\pi k-1}{N+1}}, \quad \text{with} \quad k \in \{1, 2, \ldots, N+1\},$$  \hspace{1cm} (E36)

i.e., the negative of the $(N + 1)^{th}$ roots of $-1$, and it is diagonalised by the unitary matrix $V$ constructed from the eigenvectors $\vec{v}_k$:

$$V := \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \ldots & \vec{v}_{N+1} \end{pmatrix} \quad \text{with} \quad \vec{v}_k := \frac{1}{\sqrt{N+1}} \begin{pmatrix} i(-\lambda_k)^{-1} \\ (-\lambda_k)^{-2} \\ (-\lambda_k)^{-3} \\ \vdots \\ (-\lambda_k)^{-(N+1)} \end{pmatrix}.$$  \hspace{1cm} (E37)

Specifically, $\alpha_{(N)} = VDV^\dagger$, where $D := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{N+1})$, and thus

$$h_{(N)}^T = iK \log \begin{pmatrix} VDV^\dagger & 0 \\ 0 & VDV^\dagger \end{pmatrix} = iK \begin{pmatrix} V & 0 \end{pmatrix} \begin{pmatrix} \log(D) & 0 \\ 0 & \log(D) \end{pmatrix} \begin{pmatrix} V^\dagger & 0 \end{pmatrix} =: \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$$  \hspace{1cm} (E38)
for some matrix $A$. By direct calculation, one finds that

$$A_{jk} = \epsilon^{i\theta_{j,l} k_{l+1}} \frac{\pi}{(N+1)^2} \sum_{p=1}^{N+1} (2p - 2 - N) e^{-i\frac{\pi}{N+1} (j-k)}. \quad (E39)$$

Now, considering the identity

$$\sum_{p=1}^{N+1} e^{i\theta p} = \frac{e^{i\theta(N+1)} - 1}{1 - e^{i\theta}} \quad (E40)$$

for $\theta \in \mathbb{R}$, as well as its derivative with respect to $\theta$, one can calculate the sum in Eq. (E39). We then have

$$\lim_{N \to \infty} A_{jk} = \begin{cases} 0, & \text{for } j = k \smallskip \vspace{0.5em} \\ \frac{i}{j-k} \frac{1}{\tau}, & \text{for } j \neq k \end{cases} \quad (E41)$$

Then, finally, we have that $U_{\text{tot}} = e^{-iH_{\text{tot}}}$, where $H_{\text{tot}} = \lim_{N \to \infty} (\mathbf{v}^\dagger \cdot h_{\langle N \rangle} \cdot \mathbf{v})$, i.e.,

$$H_{\text{tot}} = -\sum_{j=2}^{\infty} \left( \frac{1}{j-1} h_j^\dagger a + \text{H.c.} \right) + \sum_{j,k=1; j \neq k}^{\infty} \frac{i}{j-k} h_j^\dagger b_k. \quad (E42)$$

Thus, the system is cooled to the ground state at an energy cost saturating the Landauer bound, and in unit time, but via a procedure that implements a multi-mode Gaussian unitary on a diverging number of modes.

**Appendix F: Cooling Protocols in the Incoherent-control Paradigm**

In this section, we investigate the required resources to cool the target system within the incoherent-control paradigm. For simplicity, we consider only the finite-dimensional setting. Here, we have a qudit target system $S$ interacting resonantly (i.e., in an energy-conserving manner) with a qudit machine $M$, which is partitioned into one part, $C$, in thermal contact with the ambient environment at inverse temperature $\beta$ and another part, $H$, in contact with a hot bath at inverse temperature $\beta_H < \beta$. The Hamiltonians for each subsystem are $H_x = \sum_{n=0}^{d_x-1} n \omega_x |n\rangle \langle n|_x$; the energy resonance condition enforces that $\omega_x = \omega_c - \omega_d$.

For the most part in this section, we focus on equally spaced Hamiltonians for simplicity; we comment specifically whenever we consider otherwise.

In order to cool the target system, we aim to compress as much population as possible into the its lowest energy eigenstates via interactions that are restricted to the energy-degenerate subspaces of the joint $SCH$ system. Thus we are restricted to global energy-conserving unitaries $U_{\text{ec}}$ that satisfy

$$[H_x + H_c + H_{\text{tot}}, U_{\text{ec}}] = 0. \quad (F1)$$

In Ref. [40], it was shown that for the case where all three subsystems are qubits, the optimal global unitary in this setting (inasmuch as they render the target system in the coldest state possible given the restrictions) is

$$U_{\text{ec}} = |0, 1, 0\rangle \langle 1, 0, 1|_{\text{ecN}} + |1, 0, 1\rangle \langle 0, 1, 0|_{\text{ecN}} + \mathbb{I}, \quad (F2)$$

where $\mathbb{I}$ denotes the identity matrix on all subspaces that are not energy-degenerate. Considering the generalisation to qudit subsystems, it is straightforward to see that, for equally spaced Hamiltonians, the optimal global unitaries must be of the form

$$U_{\text{ec}} = \sum_{m,n,l=0}^{d-2} \left[ |m, n + 1, l\rangle \langle m + 1, n, l|_{\text{ecN}} + |m + 1, n, l + 1\rangle \langle m, n + 1, l|_{\text{ecN}} \right] + \mathbb{I}. \quad (F3)$$

For the most general case where the Hamiltonians of each subsystem are arbitrary, it is not possible to write down a generic form of the optimal unitary, since the energy-resonant transitions that lead to cooling the target now depend on the microscopic details of the energetic structure. Nonetheless, in Appendix G, we provide a protocol (i.e., not the unitary per se, but a sequence of steps) in this setting that attains perfect cooling and saturates the Carnot-Landauer limit.

Intuitively, the above types of unitaries simply reshuffle populations that are accessible through resonant transitions. For the purpose of cooling, one wishes to do this in such a way that the largest population is placed in the lowest energy eigenstate of the target system, the second largest in the second lowest energy eigenstate, and so on (in line with the optimal unitaries in the
coherent-control setting); indeed, on the energy-degenerate subspaces accessible, such unitaries act precisely in this way. It is straightforward to show that interactions of this form satisfy Eq. (F1).

For the sake of simplicity, we now focus on the case where all systems are qubits, although the results generalise to the qudit setting. Consider the initial joint state \( \rho_{SCN} = \sum_{m,n,l=0}^{d \times d \times d} p_{mnl} |m,n,l\rangle\langle m,n,l|_{SCN} \). By applying a unitary \( U_{SC} \) of the form given in Eq. (F3), the post-transformation joint state is

\[
\rho'_{SCN} = U_{SC} \rho_{SCN} U_{SC}^\dagger = \rho_{SCN} + \Delta \rho |0,1,0\rangle\langle 0,1,0|_{SCN} - \Delta \rho |1,0,1\rangle\langle 1,0,1|_{SCN},
\]

where \( \Delta \rho := p_{010} - p_{001} \) indicates the amount of population that has been transferred from the excited state of the target system to the ground state throughout the interaction. Naturally, in order to cool the target system, \( \Delta \rho \geq 0 \), i.e., the initial population \( p_{010} \) must be at least as large as \( p_{001} \).

Due to the energy-conserving nature of the global interaction, the energy exchanged between the subsystems throughout a single such interaction, \( \Delta E = \text{tr} \left[ H_X (\rho'_X - \rho_X) \right] \), can be calculated via

\[
\Delta E_g = -\omega_g \Delta \rho, \quad \Delta E_C = \omega_C \Delta \rho, \quad \Delta E_H = -\omega_H \Delta \rho.
\]

Thus, for a fixed energy-level structure of all subsystems (i.e., given the local Hamiltonians), one only requires knowledge of the pre- and post-transformation state of any one of the subsystems to calculate the energy change for all of them.

**F1. Diverging Energy: Proof of Theorem 5**

The first thing to note is that in the incoherent-control paradigm, even when one allows for the energy cost, i.e., the heat drawn from the hot bath, to be diverging, it is not possible to perfectly cool the target system, as presented in Theorem 5. The intuition behind this result is that the target system can only interact with energy-degenerate subspaces of the hot and cold machine subsystems. The optimal transformation that one can do here to achieve cooling is to transfer the highest populations of any such subspace to the lowest energy eigenstate of the target system; however, any such subspace has population strictly less than one for any \( 0 \leq \beta_H \leq \beta < \infty \) independently of the energy structure. Moreover, the difference from one can be bounded by a finite amount that does not vanish independent of the energy level structure of any machine of finite dimension. This makes it impossible to attain a subspace population of one even as the energy cost diverges for any fixed and finite control complexity. It follows that the ground-state population of the target system can never reach unity in a single operation of finite control complexity and hence perfect cooling cannot be achieved.

Precisely, we show the following. Let \( S \) be a finite-dimensional system of dimension \( d_S \) with associated Hamiltonian with finite but otherwise arbitrary energy gaps \( H_S = \sum_{i=0}^{d_S-1} \omega_i |i\rangle\langle i|_S \), and let \( d_C \) and \( d_H \) be integers denoting the dimensions of the cold and hot parts of the machine respectively. Then it is impossible to cool the system \( S \) in the incoherent-control paradigm, i.e., using energy-conserving unitaries involving \( C \) and \( H \) at some initial inverse temperatures \( \beta, \beta_H \) respectively, arbitrarily close to the ground state. Note that, in particular, this result holds irrespective of the energy level structure of \( C \) and \( H \) and no matter how much energy is drawn from the hot bath as a resource.

In order to set notation for the following, we assume \( \omega_i \geq \omega_j \) for \( i \leq j \) and \( \omega_0 = 0 \), where \( \omega_i \) denotes the \( i \)th energy eigenvalue of system \( X \) with \( X \in \{ S, C, H \} \). We also assume the initial states on \( S \) and \( C \) to be thermal at inverse temperature \( \beta \) and \( H \) is assumed to be initially in a thermal state at inverse temperature \( \beta_H \leq \beta \). We denote by \( p_i \) the \( i \)th population of system \( X \) in a given state, i.e., \( p_i := \langle i | \rho_X | i \rangle \), where \( | i \rangle \) denotes the \( i \)th energy eigenstate of \( \rho_X \). We will also write \( p_{ijk} := p_j p_k \).

The intuition behind the proof is as follows. The global ground state level of the joint hot-and-cold machine has some non-zero initial population for any finite-dimensional machine; in particular it can always be lower-bounded by \( \frac{1}{\pi^2 d_S} \) for any Hamiltonians and initial temperatures, which is strictly greater than zero as long as the dimensions remain finite. Fixing the control complexity of any protocol considered here to be finite in value thus implies a lower bound on the initial ground state population of the total machine that is larger than zero by a finite amount. Depending on the energy level structure of the hot and cold parts of the machine, there may be other non-zero initial populations, but in order to cool the target system \( S \) perfectly, at least all of the previously mentioned populations must be transferred into spaces spanned by energy eigenstates of the form \( |0,j,k\rangle_{SCN} \). This intuition is formalised via Lemma 2, where we show that independent of the energy structure of \( C \) and \( H \), one must be able to make such transfers of population in order to perfectly cool \( S \). However, in order to make such transfers in an energy conserving manner, all energy eigenstates of the form \( |0,j,k\rangle_{SCN} \) must be degenerate with some of the form \( |0,j,k\rangle_{SCN} \). This degeneracy condition, in turn, also implies that every energy eigenstate of the form \( |0,j,k\rangle_{SCN} \) has an associated initial population \( p_{0jk} \) that is non-vanishing for all machines of finite dimension (i.e., for all protocols with finite control complexity). Thus, upon transferring some population \( p_{000} \) into the subspace spanned by \( |0,j,k\rangle_{SCN} \), i.e., one of a relevant form for the population to contribute to the final ground state population of the target, one inevitably transfers some finite amount of population away from the relevant space and into \( |000\rangle_{SCN} \), which does not contribute to the final ground state population of the target. We formalise this intuition in the discussion following Lemma 2. In this way, no matter what one does, there is always a finite amount of population, which
is lower-bounded by some strictly positive number due to the constraint on control complexity, that does not contribute to the final ground state population of the target, implying that perfect cooling is not possible.

The formal proof occurs in two steps. We first show that some specific degeneracies in the joint $\mathcal{SCH}$ system must be present in order to be able to even potentially cool $\mathcal{S}$ arbitrarily close to the ground state. We then prove that, given the above degeneracies, one cannot cool the system $\mathcal{S}$ beyond a fixed ground-state population that is independent of the energy structure of $\mathcal{C}$ and $\mathcal{H}$; in particular, one can draw as much energy from the hot bath as they like and still do no better. We begin with the following lemma.

**Lemma 2.** Given $\mathcal{S}$, $d_c$ and $d_H$ as above, one can reach a final ground-state population of the system $\mathcal{S}$ arbitrarily close to one in the incoherent-control paradigm only if each $|0\nu\rangle_{\mathcal{SCH}}$, where $\nu \in \{1, \ldots, d_{S} - 1\}$, energy eigenstate is degenerate with at least one $|0jk\rangle_{\mathcal{SCH}}$ energy eigenstate, where $j \in \{0, \ldots, d_{S} - 1\}, k \in \{0, \ldots, d_{H} - 1\}$.

**Proof.** Suppose that there exists an $i^* \in \{1, \ldots, d_{S} - 1\}$ such that $|i^*0\rangle_{\mathcal{SCH}}$ is not degenerate with any $|0jk\rangle_{\mathcal{SCH}}$, where $j \in \{0, \ldots, d_{S} - 1\}, k \in \{0, \ldots, d_{H} - 1\}$. We show that, then, one cannot cool $\mathcal{S}$ arbitrarily close to zero.

Let $B_i$ denote the degenerate subspace of the total Hamiltonian $H_{S} + H_{C} + H_{H}$, where $H_{H}$ denotes the Hamiltonian of system $X \in \{\mathcal{C}, \mathcal{H}\}$, that contains the eigenvector $|i0\rangle_{\mathcal{SCH}}$. Then, any energy conserving unitary $U_{EC}$ used to cool the system in the incoherent-control paradigm must act within such $B_{i}$ subspaces, i.e., $U_{EC} = \bigoplus_{i} U_{B_{i}}$ (this is a direct consequence of $[U_{EC}, H_{S} + H_{C} + H_{H}] = 0$, see, e.g., Lemma 5 of Ref. [59]). This means in particular that the initial population of $|i0\rangle_{\mathcal{SCH}}$ can only be distributed within $B_{i}$, and as no eigenvector of the form $|0jk\rangle_{\mathcal{SCH}}$ is contained in $B_{i}$ by assumption, it can never contribute to the final ground state population of $\mathcal{S}$, which we denote $p_{f0}$. So we have

$$p_{f0} \leq 1 - p_{i*00}. \quad (F6)$$

Now, for $X \in \{\mathcal{C}, \mathcal{H}\}$, with any $\{\omega_{x}^{i}\} \in \{\omega_{x}^{i}\}$ such that each $\omega_{x}^{i} \geq 0$ with $\omega_{x}^{0} = 0$ and any inverse temperature $\beta \geq 0$, we have for the partition function $Z_{x}$ that

$$Z_{x} = 1 + e^{-\beta\omega_{x}^{i}} + \ldots + e^{-\beta\omega_{x}^{i-1}} \leq d_{x}, \quad (F7)$$

and so we have the following bound on the initial populations associated to each eigenvector $|00\rangle_{\mathcal{SCH}}$

$$p_{00} = \frac{e^{-\beta\omega_{x}^{i}}}{Z_{x}Z_{C}Z_{H}} \geq \frac{e^{-\beta\omega_{x}^{i}}}{Z_{x}d_{C}d_{H}} > 0 \quad \forall i \in \{1, \ldots, d_{S} - 1\}. \quad (F8)$$

Combining the above, we have that

$$p_{f0} \leq 1 - \frac{e^{-\beta\omega_{x}^{i}}}{Z_{x}d_{C}d_{H}} < 1. \quad (F9)$$

So as desired, we have shown that one cannot cool beyond $1 - \frac{e^{-\beta\omega_{x}^{i}}}{Z_{x}d_{C}d_{H}}$, a bound strictly smaller than 1 for any finite-dimensional machine (i.e., for any protocol using only finite control complexity) and independent of the energies of $\mathcal{C}$ and $\mathcal{H}$.

We can now proceed to the second step of the proof of Theorem 5.

**Proof.** To this end, consider any $i^* \in \{1, \ldots, d_{S} - 1\}$. If $|i^*0\rangle_{\mathcal{SCH}}$ is not degenerate with any $|0jk\rangle_{\mathcal{SCH}}$, our assertion is proven by Lemma 2. On the other hand, if there is a $j^* \in \{0, \ldots, d_{S} - 1\}$ and a $k^* \in \{0, \ldots, d_{H} - 1\}$ for which $|i^*0\rangle_{\mathcal{SCH}}$ and $|0j^*k^*\rangle_{\mathcal{SCH}}$ are degenerate, then $B_{i^*}$, the degenerate subspace containing $|i^*0\rangle_{\mathcal{SCH}}$, also contains $|0j^*k^*\rangle_{\mathcal{SCH}}$. Now $B_{i^*}$ may also contain other eigenvectors of the form $|0k\rangle_{\mathcal{SCH}}$, i.e., some other $|0j^*k\rangle_{\mathcal{SCH}}$ with $j^* \in \{0, \ldots, d_{S} - 1\}, k \in \{0, \ldots, d_{H} - 1\}$. Crucially, each such eigenvector in $B_{i^*}$ must have an associated minimal amount of initial population as long as the machine is finite-dimensional. Indeed, for any such $|0j^*k\rangle_{\mathcal{SCH}}$ in $B_{i^*}$, we have the condition $\omega_{c}^{j^*} + \omega_{H}^{k^*} = \omega_{c}^{i^*}$ and so $\omega_{c}^{j^*} \leq \omega_{c}^{i^*}$, implying that $\beta\omega_{c}^{j^*} \leq \beta\omega_{c}^{i^*}$ and $\beta\omega_{H}^{k^*} \leq \beta\omega_{H}^{i^*}$. Thus we have the bound

$$p_{0j^*k^*} = \frac{e^{-\beta\omega_{c}^{j^*}}e^{-\beta\omega_{H}^{k^*}}}{Z_{x}Z_{C}Z_{H}} \geq \frac{e^{-\beta\omega_{c}^{j^*}}e^{-\beta\omega_{H}^{i^*}}}{Z_{x}d_{C}d_{H}} =: q_{i^*}. \quad (F10)$$

Now, take any particular $i^* \in \{1, \ldots, d_{S} - 1\}$ and let $\pi_{i^*}$ be the dimension of $B_{i^*}$, $\mu$ the number of energy eigenstates of the form $|0j\rangle_{\mathcal{SCH}}$ that $B_{i^*}$ contains and $\nu = \pi - \mu$ the number of energy eigenstates of the form $|i\rangle_{\mathcal{SCH}}$, where $i \neq 0$, that $B_{i^*}$ contains. So

$$B_{i^*} = \text{span}\{|0jk_1\rangle,|0jk_2\rangle,\ldots,|0j_k\rangle_{\mu},|i0\rangle_{\nu},|i2\rangle_{\mu_2},\ldots,|i\nu\rangle_{\mu_\nu}\}. \quad (F11)$$
As we now claim that $\sum_{\alpha=1}^{\nu} v_{\alpha}^{*} \geq q_{i^{*}}$ from Eq. (F10). Indeed, as $v$ has at most $\nu - 1$ elements that do not belong to the set $\mathcal{A} := \{ p_{0j}, p_{0j_{k}}, \ldots, p_{0j_{k_{\mu}}, p_{0j_{00}}}, \}$, at least one element of $\mathcal{A}$ must contribute to the sum $\sum_{\alpha=1}^{\nu} v_{\alpha}^{*}$. Let $x$ be that element. As $v_{\alpha}^{*} \geq 0$ for all $\alpha = 1, \ldots, \pi = \mu + \nu$, we have
\begin{equation}
\sum_{\alpha=1}^{\nu} v_{\alpha}^{*} \geq x.
\end{equation}

Now as $p_{0j_{\gamma}} \geq q_{i^{*}}$ for all $\gamma = 2, \ldots, \mu$, we have
\begin{equation}
x \geq \min(q_{i^{*}}, p_{0j_{00}}) = q_{i^{*}}, \tag{F14}
\end{equation}
where $p_{0j_{00}} \geq q_{i^{*}}$ can be seen from Eq. (F10), as claimed.

As the l.h.s. of Eq. (F12) represents the amount of population in the subspace $\mathcal{B}_{i^{*}}$ that does not contribute to the final ground-state population of the target system, we have
\begin{equation}
\tilde{p}_{s}^{0} \leq 1 - \left( \tilde{p}_{i^{*}00} + \sum_{\alpha=2}^{\nu} \tilde{p}_{i_{0\alpha}m_{\alpha}} \right) \leq 1 - q_{i^{*}} = 1 - \frac{e^{-2\beta}\omega_{S}^{*}}{Z_{d_{s}\cdot d_{u}}^{*}}. \tag{F15}
\end{equation}
So, for any finite-dimensional machine, one cannot cool the system $\mathcal{S}$ beyond $1 - \frac{e^{-2\beta}\omega_{S}^{*}}{Z_{d_{s}\cdot d_{u}}^{*}}$, a bound strictly smaller than 1 and independent of the energy structure of $\mathcal{C}$ and $\mathcal{H}$, as desired.

As a concrete example, consider the case where all systems are qubits. The initial joint state is
\begin{equation}
\rho_{SCN}^{(0)} = \frac{(|0\rangle\langle0| + e^{-\beta}\omega_{S}|1\rangle\langle1|)_{S} \otimes (|0\rangle\langle0| + e^{-\beta}\omega_{C}|1\rangle\langle1|)_{C} \otimes (|0\rangle\langle0| + e^{-\beta}\omega_{\mathcal{H}}|1\rangle\langle1|)_{\mathcal{H}}}{Z_{S}(\beta, \omega_{S})Z_{C}(\beta, \omega_{C})Z_{\mathcal{H}}(\beta_{H}, \omega_{\mathcal{H}})}. \tag{F16}
\end{equation}
The only energy-conserving unitary interaction that is relevant for cooling is the one that exchanges the populations in the levels spanned by $|0\rangle$ and $|1\rangle$, which have initial populations $e^{-\beta\omega_{S}}e^{-\beta\omega_{\mathcal{H}}} \geq e^{-\beta\omega_{C}}$; now, performing the optimal cooling unitary leads to the final ground-state population of the target system
\begin{equation}
p_{s}^{(0)} = \langle 0 | tr_{CN} \left[U_{SCN}^{(0)} U_{SCN}^{(0)} \right] | 0 \rangle_{S} = \frac{1 + e^{-\beta}\omega_{\mathcal{H}} + e^{-\beta}\omega_{S} + e^{-\beta}\omega_{C}}{Z_{S}(\beta, \omega_{S})Z_{C}(\beta, \omega_{C})Z_{\mathcal{H}}(\beta_{H}, \omega_{\mathcal{H}})} < 1. \tag{F17}
\end{equation}
Indeed, using $e^{-\beta}\omega_{S}e^{-\beta}\omega_{\mathcal{H}} \geq e^{-\beta}\omega_{C}$,
\begin{equation}
p_{s}^{(0)} \leq \frac{1 + e^{-\beta}\omega_{\mathcal{H}} + e^{-\beta}\omega_{C}}{Z_{S}(\beta, \omega_{S})Z_{C}(\beta, \omega_{C})} \leq \frac{1}{Z_{C}(\beta, \omega_{C})} \leq 1. \tag{F18}
\end{equation}
The second inequality is strict unless $\beta_{H} = 0$ or $\omega_{\mathcal{H}} = 0$. In the both cases, for equality in the first inequality, we need $\beta\omega_{S} = \beta\omega_{\mathcal{H}}$. If $\beta = 0$, then $Z_{C}(\beta, \omega_{C}) = 2$ and the last inequality is strict. If $\omega_{S} = \omega_{C}$, no cooling is possible; hence $p_{s}^{(0)} = p_{s}^{(0)} < 1$.

F2. Diverging Time / Control Complexity

We now move to analyse the case where diverging time is allowed, where we wish to minimise the energy cost and control complexity throughout the protocol over a diverging number of energy-conserving interactions between the target system and the hot and cold subsystems of the machine. We again consider all three systems to be qubits, but the results generalise to arbitrary
(finite) dimensions. Here, the machines and ancillas begin as thermal states with initial inverse temperatures $\beta$ and $\beta_n \leq \beta$ respectively. Just as in the diverging time cooling protocol in the coherent-control setting presented in Appendix C, we will consider a diverging number of machines, with slightly increasing energy gaps, in a configuration such that the target system interacts with the $n^{th}$ machine at time step $n$. Suppose that after $n$ steps of the protocol, the target qubit has been cooled to some inverse temperature $\beta_n > \beta$; equivalently, this can be expressed as a thermal state with corresponding energy gap $\omega_n = \frac{\beta}{\beta_n}$.

We now wish to interact the target system $\tau_n(\beta_n, \omega_n)$ with a machine $M_{n+1}$ with slightly increased energy gaps with respect to the most recent one $M_n$, i.e., we increase the energy gaps of the cold subsystem $C$ from $\omega_n$ to $\omega_{n+1} = \omega_n + \epsilon_n$; the resonance condition enforces the energy gap of the hot subsystem $H$ to be similarly increased to $\omega_n + \epsilon_n - \omega_S$. Thus, the next step of the protocol is the usual acting on the global state

$$g^{(n)}_{\beta,\omega} = \tau_n(\beta, \omega_n) \otimes \tau_n(\beta, \omega_n + \epsilon_n) \otimes \tau_n(\beta_n, \omega_n + \epsilon_n - \omega_S).$$  \hspace{1cm} \text{(F19)}$$

In order to cool the target system via said unitary, we must have that $p_{101} \geq p_{010}$ for the state in Eq. (F19), which implies that $\epsilon_n$ must satisfy the following condition:

$$e^{-\beta_0 - \beta_H (\omega_n + \epsilon_n - \omega_S)} \geq e^{-\beta(\omega_n + \epsilon_n)} \Rightarrow \epsilon_n \geq \gamma(\omega_n - \omega_S) \quad \text{where} \quad \gamma := \frac{\beta}{\beta - \beta_n}. \hspace{1cm} \text{(F20)}$$

This condition is crucial. It means that if the hot subsystem $H$ is coupled to a heat bath at any finite temperature, i.e., $\beta_H > 0$, $\epsilon_n$ depends linearly on the inverse temperature of the target system at the previous step $\beta_n$, and can thus not be taken to be arbitrarily small. As we will now show, this condition prohibits the ability to perfectly cool the target system at the Landauer limit for the energy cost whenever the heat bath is at finite temperature.

On the other hand, for infinite-temperature heat baths, perfect cooling at the Landauer limit is seemingly achievable; here, $\beta_H \to 0$ and so $\gamma \to 0$, leading to the trivial constraint $\epsilon_n \geq 0$ which allows it to be arbitrarily small, as is required. Nonetheless, the explicit construction of any protocol doing so in the incoherent-control setting is \textit{a priori} unclear, as the restriction of energy conservation makes for a fundamentally different setting from the coherent-control paradigm. We now explicitly derive the optimal diverging-time protocol to perfectly cool at the Landauer limit for an infinite-temperature heat bath, thereby proving Theorem 6; Corollary 3 follows via a straightforward trade-off between time and control complexity, as previously discussed.

F3. Saturating the Landauer Limit with an Infinite-temperature Heat Bath

Before calculating the energy cost, we briefly discuss the attainability of the optimally cool target state. We begin with all subsystems as qubits, for the sake of simplicity, but the logic generalises to higher dimensions. In the incoherent paradigm, the target system $S$ interacts with a virtual qubit of the total machine $M = CH$ that consists of the energy eigenstates $|0,1\rangle_{CH}$ and $|1,0\rangle_{CH}$, with populations $p_{01,10}$ and $p_{10,01}$ respectively. Suppose that at step $n+1$ the cold subsystem involved in the interaction has energy gap $\omega_n + \epsilon_n$. In Ref. [40], it is shown that by repeating the incoherent cooling process (i.e., implementing the unitary in Eq. (F3)) and taking the limit of infinite cycles, this scenario equivalently corresponds to the general (coherent) setting where arbitrary unitaries are permitted and the target system interacts with a virtual qubit machine with effective energy gap $\omega^{eff}_n$ given by

$$e^{-\beta_0 - \beta_H (\omega_n + \epsilon_n - \omega_S)} = e^{-\beta(\omega_n + \epsilon_n)} \Rightarrow \omega^{eff}_n = \omega_n + \epsilon_n - \frac{\beta}{\beta - \beta_n}(\omega_n + \epsilon_n - \omega_S). \hspace{1cm} \text{(F21)}$$

It is clear that for finite-temperature heat baths, i.e., $\beta_H > 0$, the effective energy gap $\omega^{eff}_n$ is always smaller than the energy gap of the machine at any given step, i.e., $\omega^{eff}_n \leq \omega_n + \epsilon_n$; on the other hand, equality holds if the heat bath is at infinite temperature, i.e., $\beta_H \to 0$. Thus, in the infinite-temperature case, given a target system beginning at some step of the protocol in the state $g^{*}_n(\beta, \omega_n)$, it is possible to get close to the asymptotic state $g^{*}_n(\beta, \omega_n + \epsilon_n)$; if the temperature is finite, however, this state is not attainable (even asymptotically). Following the arguments in Appendix C, i.e., considering a diverging number of machines, each of which having the part connected to the cold bath with energy gap $\omega^{CH}_n = \omega_n + \epsilon_n$ and taking the limit of $\epsilon_n \to 0$, which one can only do if the hot bath temperature is infinite, allows one to cool perfectly in diverging time in the incoherent paradigm at the Landauer limit.

We now calculate the energy cost explicitly for the infinite-temperature heat bath case, precisely demonstrating attainability of the Landauer limit. We use a similar approach to that described in Appendix C: we have a diverging number of cold machines for each energy gap $\omega_n$, with which the target system at the $n - 1^{th}$ time step interacts; for an infinite-temperature heat bath, i.e., $H$ is in the maximally mixed state independent of its energy structure, the state of the target system at each step $g^{*}_n(\beta, \omega_n - 1)$ is achievable. From Eq. (F5), the energy change between all subsystems for a given step of the protocol, i.e., taking $g^{*}_n(\beta, \omega_n - 1) \rightarrow g^{*}_n(\beta, \omega_n)$, can be calculated as

$$\Delta E^{(n)} = \text{tr} \left[H_S(\omega_S)(g^{*}_n(\beta, \omega_n) - g^{*}_n(\beta, \omega_n - 1))\right]$$
\[ \Delta E_c^{(n)} = -\text{tr} \left[ H_c(\omega_n) (g^*_\beta(\omega_n) - g^*_\beta(\omega_{n-1})) \right] \]
\[ \Delta E_s^{(n)} = \text{tr} \left[ H_s(\omega_n - \omega_s) (g^*_\beta(\omega_n) - g^*_\beta(\omega_{n-1})) \right] \]

In general, i.e., for finite-temperature heat baths, we would have \( \omega_n = \omega_{n-1} + \epsilon_{n-1} \), with a lower bound on \( \epsilon_{n-1} \) for cooling to be possible [in accordance with Eq. (F20)]. However, for infinite-temperature heat baths, this lower bound trivialises since the energy structure of the hot machine subsystem plays no role in its state; thus we can choose the energy gap structure for the target system to be possible [in accordance with Eq. (F20)]. However, for infinite-temperature heat baths, this lower bound trivialises since the energy gap structure for the target system to be possible [in accordance with Eq. (F20)]. Nonetheless, for any \( \omega_N = \omega_{\text{max}} \) for the maximum energy gap of the cold machine subsystems, the energy exchanged throughout the entire cooling protocol here is given by

\[ \Delta E_s = \lim_{N \to \infty} \sum_{n=1}^{N} \Delta E_s^{(n)} = \text{tr} \left[ H_s(\omega) (g^*_\beta(\omega) - g^*_\beta(\omega_s)) \right] \]
\[ \Delta E_c = \lim_{N \to \infty} \sum_{n=1}^{N} \Delta E_c^{(n)} = \frac{1}{\beta} \left\{ S(g^*_\beta(\omega_s)) - S(g^*_\beta(\omega_{\text{max}})) \right\} = \frac{1}{\beta} \Delta S_s \]
\[ \Delta E_{n} = \lim_{N \to \infty} \sum_{n=1}^{N} \Delta E_{n}^{(n)} = -\Delta E_s - \Delta E_c. \]

Here, the expression for \( \Delta E_c \) can be derived using the same arguments as presented in Appendix C1. In particular, the heat dissipated by the cold part of the machine, which is naturally connected to the heat sink in the incoherent setting as an infinite-temperature heat-bath can be considered a work source since any energy drawn comes with no entropy change, is in accordance with the Landauer limit. It is straightforward to obtain the same result for qudit systems. Lastly, in a similar way to the other protocols we have presented, one could compress all of the diverging number of operations into a single one whose control complexity diverges, thereby trading off between time and control complexity.

**F4. An Analysis of Finite-temperature Heat Baths**

We now return to the more general consideration of finite-temperature heat baths, i.e., \( 0 < \beta \leq \beta \). In the case where \( \beta = \beta \), from Eq. (F21), it is straightforward to see that for any machine energy gap \( \omega_n \), the effective gap \( \omega^\text{eff} \) is equal to the gap of the target system, which means that no cooling can be achieved in the incoherent paradigm. Nonetheless, for any \( \beta \) subsystem coupled to a heat bath of inverse temperature \( \beta < \beta \), cooling is possible. We first provide more detail regarding why cooling at the Landauer limit is not possible in this setting, before deriving the minimal energy cost in accordance with the Carnot-Landauer limit presented in Theorem 4; in Appendix G, we provide explicit protocols that saturate this bound for any finite-temperature heat bath and arbitrary finite-dimensional systems and machines.

Suppose that at some step \( n \) one has the initial joint state of Eq. (F19), where \( \epsilon_n = \gamma (\omega_n - \omega_s) + \epsilon \) and \( \omega_n = \omega_s + n \epsilon \). Here, \( \gamma \) is as in Eq. (F20). We now wish to cool the target system to \( g^*_\beta(\omega, \omega_n + \epsilon) \). For cooling to be possible in the incoherent setting here, we need the cold machine subsystem to have an energy gap of at least \( \omega_n + \epsilon \); moreover, with a finite-temperature heat bath, this energy gap is insufficient to achieve the desired transformation [see Eq. (F20)]. Based on Eq. (F5), we can see that nonetheless, if we calculate the hypothetical energy change in this scenario if it were possible, we can derive a lower bound for the actual energy cost incurred. Employing Eq. (F22), we have

\[ \Delta E_c^{(n+1)} \geq -\text{tr} \left\{ H_c(\omega_n + \epsilon_n) [g^*_\beta(\omega, \omega_n + \epsilon) - g^*_\beta(\omega, \omega_n)] \right\} \]
\[ = -\text{tr} \left\{ H_c(\omega_n + \epsilon_n) [g^*_\beta(\omega_n + \epsilon) - g^*_\beta(\omega_n)] \right\} \]
\[ = -\text{tr} \left\{ H_c(\omega_n + \epsilon_n) [g^*_\beta(\omega_n + \epsilon) - g^*_\beta(\omega_n)] \right\} + \gamma \text{tr} \left\{ H_c(\omega_n + \epsilon_n) [g^*_\beta(\omega_n + \epsilon) - g^*_\beta(\omega_n)] \right\} \]
\[ = (\gamma + 1) \Delta E_c^{(n+1)} + \gamma \Delta E_s^{(n+1)} + \gamma \text{tr} \left\{ H_c(\omega_n + \epsilon_n) [g^*_\beta(\omega_n + \epsilon) - g^*_\beta(\omega_n)] \right\}, \]

where we have made use of the fact that for equally spaced Hamiltonians, the structure of the Hamiltonians on each subsystem take the same form [i.e., we can write, with slight abuse of notation, \( H_c(\omega + \omega_s) = H_c(\omega) + H_s(\omega_s) \)]. We use the star in \( \Delta E_s^\star \) to denote the idealised energy cost [i.e., that corresponding to what would be achievable in the infinite-temperature setting; see Eq. (F23)] and the energy costs without the star to represent those for when the temperature of the heat bath is finite. The additional term \( \text{tr} \left\{ H(\gamma) [g^*_\beta(\omega_n + \epsilon) - g^*_\beta(\omega_n)] \right\} \) vanishes for \( \epsilon \to 0 \).

Summing up these contributions for a diverging number of steps gives the lower bound for the heat dissipated throughout the
The entire protocol for cooling an initial state $\tau_\beta(\omega_s)$ to some final $\tau_\beta(\beta_{\text{max}},\omega_s)$ is given by

$$\Delta E_C = \lim_{N \to \infty} \sum_{n=1}^{N} \Delta E^{(n+1)}_C \geq (\gamma + 1) \frac{1}{\beta} \Delta S_S + \gamma \Delta E_S = \frac{1}{\beta} \Delta S_S + \gamma \left( \Delta E_S + \frac{1}{\beta} \Delta S_S \right).$$

\hspace{1cm} (F25)

Note that for infinite-temperature heat baths, $\gamma \to 0$ and the usual Landauer limit is recovered; nonetheless, for finite-temperature heat baths, $\gamma > 0$ and there is an additional energy contribution, implying that the Landauer limit cannot be achieved. Moreover, note that the expression inside the parenthesis in the second term above is always non-negative, as it is the free energy difference of the system during the cooling process. Lastly, it is straightforward to show that this lower bound is equivalent to the Carnot-Landauer limit in Eq. (A14), which was derived in a protocol-independent manner as the ultimate limit in the incoherent-control setting. We now present explicit protocols that saturate this bound.

**Appendix G: Perfect Cooling at the Carnot-Landauer Limit in the Incoherent-control Paradigm**

The precise statement that we wish to prove regarding saturation of the Carnot-Landauer limit is the following:

**Lemma 3.** For any $\beta^* \geq \beta > \beta_H$ and $\epsilon_{1,2} > 0$, there exists a cooling protocol in the incoherent-control setting comprising a number of unitaries of finite control complexity which, when the number of operations diverges, cools to some final temperature $\beta'$ that is arbitrarily close to the ideal temperature value $\beta^*$, i.e.,

$$|\beta' - \beta^*| < \epsilon_1,$$

\hspace{1cm} (G1)

with an energy cost, measured as heat drawn from the hot bath, that is arbitrarily close to the ideal Carnot-Landauer limit, i.e.,

$$\left| \Delta E_H - \eta^{-1} \Delta F_S(\beta) \right| < \epsilon_2,$$

\hspace{1cm} (G2)

where $\eta = 1 - \beta_H / \beta$ and $\Delta F_S(\beta) = F_S^{(\beta)}(\varrho_s') - F_S^{(\beta)}(\varrho_s)$ is the free energy difference between the initial $\varrho_s = \tau_S(\beta, H_S)$ and final $\varrho_s' = \tau_S(\beta^*, H_S)$ system states (w.r.t. inverse temperature $\beta$).

We begin by presenting the diverging-time protocol that saturates the Carnot-Landauer limit when all three subsystems $S, C, H$ are qubits. The simplicity of this special case allows us to calculate precisely bounds on the number of operations required to reach any chosen error threshold. Building on this intuition, we then present the generalisation to the case where all systems are qudits. The protocols with diverging control complexity follow directly via the same line of reasoning presented in the main text.

**G1. Qubit Case**

We begin with setting some notation and intuition for the proof, before expanding on mathematical details.

**Sketch of Protocol.**—The protocol consists of the following. There are $N$ stages, each labelled by $n \in \{1, 2, ..., N\}$. Each stage proceeds as follows:

- A qubit with energy gap $\omega_s + n\theta$ is taken from the cold part of the machine, and a qubit with energy gap $n\theta$ is taken from the hot part (see below). The initial state of the machine at the beginning of the $n$th stage is thus $\tau_c(\beta, \omega_s + n\theta) \otimes \tau_H(\beta_H, n\theta)$.

- The energy-preserving three qubit unitary cycle in the $\{010, 101\}_{SCH}$ subspace is performed [see Eq. (F3)], after which the cold and hot qubits are rethermalised to their respective initial temperatures.

- The above steps are repeated $m_n$ times.

The energy increment $\theta$ is defined as

$$\theta := \frac{\omega_s}{N} \left( \frac{\beta^* - \beta}{\beta - \beta_H} \right),$$

\hspace{1cm} (G3)
while the number of repetitions within each stage is given by
\[ m_n = \left\lfloor \frac{\log(\delta)}{\log(1 - N_v^{(n)})} \right\rfloor. \]  
\text{(G4)}

\[ \lfloor \cdot \rfloor \] is the ceiling function, and \( N_v^{(n)} \) is the sum of the initial thermal populations in the \( \{01, 10\} \) subspace of the machine, i.e.,
\[ N_v^{(n)} := \langle 01 | \tau_c(\beta, \omega_s + n\theta) \otimes \tau_n(\beta_n, n\theta)|01 \rangle + \langle 10 | \tau_c(\beta, \omega_s + n\theta) \otimes \tau_n(\beta_n, n\theta)|10 \rangle. \]  
\text{(G5)}

The parameter \( \delta \) will be chosen appropriately to complete the proof (\( \delta = 1/N^2 \) works).

The intuition for the proof is as follows. We first consider how the populations of the target system changes in the idealised protocol where \( m_n \to \infty \), so that in each stage, the system reaches the virtual temperature determined by the \( CH \) qubits. We can use this ideal setting to find expressions for the final temperature and energy cost, which serves as a baseline that we wish to attain without arbitrary precision. We then consider the protocol as constructed above with a finite number of repetitions \( m_n \) in each stage, and show that its expressions for temperature and work cost are close (w.r.t. \( 1/N \)) to the original expressions, and by taking \( N \) to be sufficiently large but still finite (i.e., in the diverging time limit), we prove that the protocol can be arbitrarily close in temperature and energy cost to the ideal values.

\textbf{Proof.} We label the population in the excited state of the target system at the end of stage \( n \) as \( p_n \). Thus \( p_0 \) is the initial population and \( p_N \) is the final population in the excited level of the target system qubit, i.e., that spanned by \( |1\rangle_1 \). We also label by \( q_n \) what the corresponding population \( p_n \) would hypothetically be in the limit \( m_n \to \infty \). This value can be calculated by matching the temperature of the target system qubit to the temperature of the \( \{01, 10\} \) virtual qubit within the machine (see Appendix G in Ref. [40]). Thus \( q_n \) is defined via the Gibbs ratio
\[ \frac{q_n}{1 - q_n} = e^{-\beta(\omega_s + n\theta)}e^{+\beta_n n\theta} = e^{-\beta_0 e^{-n\theta}(\beta - \beta_n) n\theta}. \]  
\text{(G6)}

Note that
1. \( \{p_n\}, \{q_n\} \) are both monotonically decreasing sequences, as each stage cools the target qubit further.
2. \( p_n > q_n \) for all \( n \), as more repetitions within each stage keep cooling the target qubit further.

To keep track of the energetic resource cost, which we will take here to be the total heat drawn from the hot bath, we must sum the energetic contribution from each time the hot qubit is rethermalised to \( \beta \)
\[ \beta \text{ in Eq. (G3)}. \] We can thus identify \( q_N = q^* \), since it is the population associated with the ideal final temperature \( \beta^* \).

We also have the following expression for the total energetic cost of the ideal protocol after \( N \) stages
\[ \tilde{\Delta} E^*_n = \sum_{n=1}^{N} n\theta (q_{n-1} - q_n), \]  
\text{(G9)}

which can alternatively be expressed as
\[ \tilde{\Delta} E^*_n = \sum_{n=1}^{N} [(n - 1)\theta (q_{n-1} - q_n)] + \theta (q_0 - q_N). \]  
\text{(G10)}
The sums appearing in the two alternate expressions are the left and right Riemann sums of the integral of the variable \( y = n\theta \) integrated with respect to the variable \( q \), i.e.,

\[
I := -\int_{q_0}^{q^*} y \, dq.
\]

where \( \frac{q(y)}{1 - q(y)} = e^{-\beta \omega_{y} e^{-(\beta - \beta_n)y}} \),

from Eq. (G6). For \( y > 0 \), \( q(y) \) is monotonically decreasing and so the converse is also true, i.e., \( y \) is monotonically decreasing w.r.t. \( q(y) \). This implies that the integral is bounded by the left and right Riemann sums, so we have

\[
\sum_{n=1}^{N} (n - 1)\theta(q_{n-1} - q_n) \leq \int_{q_0}^{q^*} y \, dq \leq \sum_{n=1}^{N} n\theta(q_{n-1} - q_n),
\]

from which we can deduce that the value of \( \Delta E^*_n \) is itself is bounded both ways from Eqs. (G9) and (G10):

\[
\int_{q_n}^{q^*} y \, dq \leq \Delta E^*_n \leq \int_{q_0}^{q^*} y \, dq + \theta(q_0 - q^*).
\]

The integral itself can be expressed in terms of the free energy of the qubit target system with respect to the environment inverse temperature \( \beta \). Expressing the free energy as a function of the excited state population \( q \) and differentiating w.r.t. \( q \) gives

\[
F(q) = \langle E \rangle(q) - \frac{S(q)}{\beta} = q \omega_{y} + \frac{1}{\beta} \left[ g \log(q) + (1 - q) \log(1 - q) \right].
\]

\[
\frac{\partial F}{\partial q} = \omega_{y} + \frac{1}{\beta} \log \left( \frac{q}{1 - q} \right) = \left( \omega_{y} + \frac{1}{\beta} \left( -\beta \omega_{y} - (\beta - \beta_n)y \right) \right) = -\beta \frac{\beta_n}{\beta} y.
\]

Using the above expression, the definite integral in Eq. (G11) amounts to

\[
I = \frac{1}{\eta} \left[ F(q^*) - F(q_0) \right] =: \frac{1}{\eta} (F^* - F_0),
\]

where we have identified the Carnot efficiency \( \eta = 1 - \beta_n/\beta \) and for ease of notation written \( F^* := F(q^*) \) and \( F_0 := F(q_0) \). Thus we can bound \( \Delta E^*_n \) on both sides

\[
\frac{1}{\eta} (F^* - F_0) \leq \Delta E^*_n \leq \frac{1}{\eta} (F^* - F_0) + \theta(q_0 - q^*) \leq \frac{1}{\eta} (F^* - F_0) + \frac{\omega_{y}}{N} \left( \frac{\beta^* - \beta}{\beta - \beta_n} \right),
\]

where the inequality in the second line follows from the fact that \( \{q_n\} \) forms a decreasing sequence.

We now proceed to consider the cooling protocol with a finite number of repetitions \( m_n \) within each stage. We first bound the difference between \( p_n \) and \( q_n \). Using the properties of the exchange unitary under repetitions [40, 60] (in particular, see Appendix G in Ref. [40]), we have that in each stage

\[
\frac{p_n - q_n}{p_{n-1} - q_n} = \left( 1 - N_v^{(n)} \right)^{m_n}.
\]

Thus, the population difference to the asymptotically achievable population given by the virtual temperature shrinks as a power law w.r.t. the number of repetitions. Since \( 0 < N_v^{(n)} < 1 \) (all strict inequalities), three points follow: First, the population \( q_n \) can never be attained with a finite number of steps within the stage \( n \); Second, that every repetition cools the system further by some finite amount; Third, that one can get arbitrarily close to \( q_n \) by taking \( m_n \) sufficiently large. In fact, by our definition of \( m_n \), we have that

\[
\frac{p_n - q_n}{p_{n-1} - q_n} \leq \delta.
\]

From this, we can prove that

\[
p_n - q_n \leq \delta^n q_0 - \delta q_n + (1 - \delta) \sum_{j=1}^{n-1} \delta^{n-j-1} q_j.
\]
The proof is by induction. For \( n = 0, p_0 = q_0 \) (initial state), and for \( n = 1 \), using Eq. (G19)

\[
p_1 - q_1 \leq \delta(p_0 - q_1) = \delta(q_0 - q_1).
\]  

(G21)

Suppose that the above statement holds true for \( p_k \). Then from Eq. (G19)

\[
p_{k+1} - q_{k+1} \leq \delta(p_k - q_{k+1}) = \delta(p_k - q_k + q_k - q_{k+1})
\]

\[
\leq \delta^{k+1}q_0 - \delta q_{k+1}(1 - \delta) + \sum_{j=1}^{(k+1)-1} \delta^{(k+1)-j-1}q_j.
\]

(G22)

With this result, we can now bound the difference between the energy cost of this finite-repetition protocol and that of the idealised one. We now proceed to prove that

\[
\tilde{E}_n - \tilde{E}^*_n = \sum_{n=1}^{N} n\theta(p_{n-1} - p_n) - \sum_{n=1}^{N} n\theta(q_{n-1} - q_n) \leq \theta \left( q_0 \sum_{j=1}^{N-1} \delta^{N-j} - \sum_{j=1}^{N-1} \delta^{N-j} q_j \right),
\]

(G23)

We again use proof by induction. First note that we can rewrite

\[
\sum_{n=1}^{N} n\theta(f_{n-1} - f_n) = \theta \left( \sum_{n=1}^{N} f_{n-1} \right) - N\theta f_N,
\]

(G24)

for \( f_n \in \{p_n, q_n\} \). Therefore, we can rewrite the difference

\[
\tilde{E}_n - \tilde{E}^*_n = \theta \sum_{n=1}^{N} (p_{n-1} - q_{n-1}) - N\theta(p_N - q_N) \leq \theta \left( \sum_{n=1}^{N} (p_{n-1} - q_{n-1}) \right),
\]

(G25)

since the last subtracted term is always strictly positive. Consider now the partial sum

\[
\mathcal{E}_k = \sum_{n=1}^{k} (p_{n-1} - q_{n-1}).
\]

(G26)

For \( k = 1, \mathcal{E}_1 = 0 \), since \( p_0 = q_0 \). For \( k = 2 \), we have

\[
\mathcal{E}_1 = (p_1 - q_1) \leq \delta(q_0 - q_1) = \left( q_0 \sum_{j=1}^{k} \delta^{2-j} - \sum_{j=1}^{k} \delta^{2-j} q_j \right),
\]

(G27)

which matches the hypothesis of Eq. (G23). Assuming that the same holds true for \( \mathcal{E}_k \), then for \( \mathcal{E}_{k+1} \), we have

\[
\mathcal{E}_{k+1} = \mathcal{E}_k + (p_k - q_k)
\]

\[
\leq \left( q_0 \sum_{j=1}^{k} \delta^{k-j} - \sum_{j=1}^{k} \delta^{k-j} q_j \right) + \left( \delta^k q_0 + (1 - \delta) \delta \sum_{j=1}^{k} \delta^{k-j-1} q_j - \delta q_k \right)
\]

\[
\leq q_0 \sum_{j=1}^{k} \delta^{k+1-j} - \sum_{j=1}^{k} \delta^{k+1-j} q_j.
\]

(G28)

Then, by dropping the second sum, which is a strictly positive quantity, the difference in Eq. (G23) can be further simplified to

\[
\tilde{E}_n - \tilde{E}^*_n \leq \theta q_0 \sum_{j=1}^{N-1} \delta^{N-j} = \theta q_0 \delta \sum_{k=0}^{N-2} \delta^k < \theta q_0 \delta(N - 1) < \theta q_0 \delta N < \omega_s \left( \frac{\beta^* - \beta}{\beta - \beta_N} \right) \delta,
\]

(G29)
In summary, we have the following bounds on the protocol in which each stage consists of a finite number of steps.

\[ N^{(n)}_v = \langle 01 | \tau_c(\beta, \omega_s + n\theta) \otimes \tau_H(\beta, n\theta) | 01 \rangle + \langle 10 | \tau_c(\beta, \omega_s + n\theta) \otimes \tau_H(\beta, n\theta) | 10 \rangle \]

\[ = \frac{e^{-\beta_H n\theta} + e^{-\beta(\omega_s + n\theta)}}{(1 + e^{-\beta_H n\theta})(1 + e^{-\beta(\omega_s + n\theta)})} \]

\[ > \frac{4}{e^{-\beta(\omega_s + n\theta)}}. \]

\[ \Rightarrow \log \left[ 1 - N^{(n)}_v \right] < \log \left[ 1 - \frac{e^{-\beta(\omega_s + n\theta)}}{4} \right] \]

\[ < -\frac{e^{-\beta(\omega_s + n\theta)}}{4} \text{ if } x \in (0, 1) \Rightarrow \log(1 - x) < -x. \]

\[ \Rightarrow -\frac{1}{\log \left[ 1 - N^{(n)}_v \right]} < 4e^{\beta(\omega_s + n\theta)} \]

(G30)

Thus we can bound the number of repetitions in each stage from Eq. (G4). Noting that \( \log(\delta) < 0 \), we have

\[ m_n < 4 \log \left( \frac{1}{\delta} \right) e^{\beta(\omega_s + n\theta)} + 1. \]

(G31)

For a crude bound, we can replace \( n \) by its maximum value \( N \), and sum over all the stages to find an upper bound on the total number of three-qubit exchange unitaries implemented throughout the entire protocol, which gives

\[ M = \sum_{n=1}^{N} m_n < N \left[ 4 \log \left( \frac{1}{\delta} \right) e^{\beta(\omega_s + N\theta)} + 1 \right] = N \left[ 4 \log \left( \frac{1}{\delta} \right) e^{\omega_s(\beta - \beta_H)/\eta} + 1 \right]. \]

(G32)

Also, note that \( \lim_{\delta \to 0} p_N = q_N = q^* \). More precisely, using Eq. (G20), we have

\[ p_N - q^* < \delta \left( \delta^{N-1} q_0 + (1 - \delta) \sum_{j=1}^{N-1} \delta^{n-j-1} q_j - q_N \right) \]

\[ < \delta (1 + (1 - \delta)(N - 1)) < \delta N. \]

(G33)

In summary, we have the following bounds on the protocol in which each stage consists of a finite number of steps

\[ p_N - q^* < \delta N \]

\[ \tilde{\Delta} E_N < \frac{1}{\eta} (F^* - F_0) + \omega_s \left( \frac{\beta^* - \beta}{\beta - \beta_H} \right) \left( \frac{1}{N} + \delta \right), \]

(G34)

where we have combined Eqs. (G17) and (G29) for the second expression. For simplicity, we choose \( \delta = 1/N^2 \), so that

\[ p_N - q^* < \frac{1}{N} \]

\[ \tilde{\Delta} E_N < \frac{1}{\eta} (F^* - F_0) + \omega_s \left( \frac{\beta^* - \beta}{\beta - \beta_H} \right) \left( \frac{2}{N} \right). \]

(G35)

Thus, given any final temperature (encoded by the population \( q^* \)), and allowed errors \( \epsilon_1 \) and \( \epsilon_2 \) for the final population and energy cost respectively, one can always choose \( N \) large enough so that both quantities are within the error threshold. Specifically, choosing \( N \) as

\[ N = \left\lceil \max \left\{ \epsilon_1, 2\omega_s \left( \frac{\beta^* - \beta}{\beta - \beta_H} \right) \epsilon_2 \right\} \right\rceil, \]

(G36)

we automatically have that \( p_N - q^* < \epsilon_1 \) and \( \Delta E_N < (F^* - F_0)/\eta + \epsilon_2 \). The total number of unitary operations (each of which is followed by rethermalisation of the machine) is then bounded by Eq. (G32)

\[ M < N \left( 8 \log[N] e^{\omega_s(\beta^* - \beta_H)/\eta} + 1 \right). \]

(G37)

We can see from Theorem 9 that the protocol is asymptotically optimal with respect to the energy extracted from the hot bath. \( \square \)
G2. Qudit Case

The extension of the proof above to the case of qudits is nontrivial. This is because, while for qubits there is only one energy-resonant subspace that leads to cooling and hence a unique protocol [see Eq. (F3)] that asymptotically attains perfect cooling at the Carnot-Landauer bound, this is no longer the case for higher-dimensional systems; here, there can be a number of energy-resonant subspaces that cool the target and the question of optimality hinges crucially on the complex energy-level structure of all systems involved. Hence, it is not possible to provide a unique unitary that generates the optimal protocol independently of the subsystem Hamiltonians. Nonetheless, we slightly modify the protocol for the qubit case above to be implemented on a number of particular three-qubit subspaces of the three-qudit global state such that, at the end of each stage, the state of the target system is arbitrarily close to the (known) state which would be achieved in an abstract protocol in the diverging-time limit. This asymptotically-attainable state is precisely that which would be achieved in the coherent-control paradigm with a machine the same dimension as the joint hot-cold qudits. Thus, we will first begin by presenting the necessary steps for the proof in the coherent-control setting, which we will then adapt as appropriate for the incoherent setting.

An idealised sequence of temperatures and system states. We construct the incoherent protocol in the following manner. We will seek to take the system through a sequence of thermal states starting at inverse temperature $\beta^*$ with $N$ equally spaced intermediary steps, i.e.,

$$\beta_n = \beta + n\theta (\beta - \beta^*),$$  \hspace{1cm} (G38)

$$\theta = \frac{1}{N} \left( \frac{\beta^* - \beta}{\beta - \beta^*} \right),$$ \hspace{1cm} (G39)

so that $\beta_N = \beta^*$ by construction. This corresponds to taking the system through the following sequence of thermal states

$$\varrho_s^{(n)} = \frac{e^{-\beta_n H_s}}{Z_s(H_s, \beta_n)}.$$ \hspace{1cm} (G40)

Note that, in contrast to the coherent protocol where such a sequence can be traversed by simply swapping the target system with a sequence of appropriate machines, in the incoherent setting such a protocol is generally not possible as such swaps are not energy conserving. Nonetheless, we will develop a modified protocol that is energy conserving and mimics this idealised one.

Corresponding to each step in the sequence, we define the following quantity, which we will eventually show to be related to the heat drawn from the hot bath:

$$G^{(n)} = -n\theta \Delta E_s^{(n)} = -n\theta \text{tr} \left[ H_s \left( \varrho_s^{(n)} - \varrho_s^{(n-1)} \right) \right].$$ \hspace{1cm} (G41)

We proceed to show that the total $\sum_n G^{(n)}$ that we label the idealised heat cost $\Delta E_s^*$ is close to the free energy difference over the entire sequence. We have

$$\Delta E_s^* = \sum_{n=1}^{N} G^{(n)}$$

$$= \sum_{n=1}^{N} n\theta \text{tr} \left[ H_s \left( \varrho_s^{(n-1)} - \varrho_s^{(n)} \right) \right]$$

$$= \left\{ \sum_{n=1}^{N} (n-1)\theta \text{tr} \left[ H_s \left( \varrho_s^{(n-1)} - \varrho_s^{(n)} \right) \right] \right\} + \theta \text{tr} \left[ H_s \left( \varrho_s^{(0)} - \varrho_s^{(N)} \right) \right].$$ \hspace{1cm} (G42)

The sums on the second and third lines above, (G42) and (G43) respectively, are the right and left Riemann sums corresponding to the following integral:

$$I = \int_{q_l}^{q_u} q \left( -dx \right) = \int_{q_l}^{q_u} q \, dx,$$

where $n\theta \rightarrow q$,

$$x = \text{tr} \left[ H_s \varrho_s(q) \right],$$

$$\varrho_s(q) = \frac{e^{-[\beta + q(\beta - \beta_u)]H_s}}{\text{tr} \left[ e^{-[\beta + q(\beta - \beta_u)]H_s} \right]}.$$ \hspace{1cm} (G44)
We observe that \( x \) is the average energy of the thermal state of temperature \( \beta + q(\beta - \beta_n) \), and thus \( x \) and \( q \) are strictly monotonically decreasing w.r.t. each other (which explains why the left and right sums are switched). It follows that the Riemann sums bound the integral

\[
\sum_{n=1}^{N} (n-1) \theta \left( q \int_{q_{c}}^{q_{f}} q \, dq \right) \leq \sum_{n=1}^{N} n \theta \left( H_{s} \left( q_{c}^{(n-1)} - q_{f}^{(n)} \right) \right) . \tag{G45}
\]

We can thus bound the idealised heat cost in both directions via

\[
I \leq \tilde{\Delta} E_{n}^{*} \leq I + \theta \left( H_{s} \left( q_{c}^{(0)} - q_{f}^{(N)} \right) \right) . \tag{G46}
\]

The integral in Eq. (G44) can be shown to be equal to the change in free energy of the target system (w.r.t. inverse temperature \( \beta \))

\[
F_{\beta}[q_{s}(q)] = \text{tr} \left( H_{s} q_{s}(q) \right) + \frac{1}{\beta} \text{tr} \left[ q_{s}(q) \log q_{s}(q) \right] .
\]

\[
\frac{d}{dq} F_{\beta}[q_{s}(q)] = \left( H_{s} + \frac{\log q_{s}(q)}{\beta} \right) \frac{d q_{s}(q)}{dq} . \tag{G47}
\]

Note that \( q_{s}(q) \) and \( d q_{s}(q) \) are both always diagonal in \( H_{s} \) and full rank for all \( q \in \mathbb{R} \), so we have no problems with \( \log q_{s}(q) \), and all of the operators in the expression are well-defined and commute. Proceeding, we repeatedly use \( \text{tr} \left[ d q_{s}(q) \right] = \text{d} \text{tr} \left[ q_{s}(q) \right] = 0 \) and label the partition function \( Z(q) := \text{tr} \left[ e^{-\beta H_{s}} \right] \) to obtain

\[
\frac{d}{dq} F_{\beta}[q_{s}(q)] = \left( H_{s} + \frac{\log q_{s}(q)}{\beta} \right) \frac{d q_{s}(q)}{dq} = \left( H_{s} - \beta + q(\beta - \beta_n) \right) \frac{\log Z(q)}{\beta} \frac{d q_{s}(q)}{dq} = -q \left( 1 - \frac{\beta}{\beta} \right) \frac{d}{dq} \text{tr} \left( H_{s} q_{s}(q) \right) = -q \eta \frac{dx}{dq}, \tag{G48}
\]

where we have identified the Carnot efficiency \( \eta \) for an engine operating between \( \beta \) and \( \beta_n \). The integral thus simplifies to

\[
I = \eta^{-1} \left( F_{\beta}[q_{s}(q)] - F_{\beta}[q_{s}(q)] \right) =: \eta^{-1} \Delta F_{\beta}^{(\beta)}. \tag{G49}
\]

The idealised heat cost is thus bounded by

\[
\eta^{-1} \Delta F_{\beta}^{(\beta)} \leq \tilde{\Delta} E_{n}^{*} \leq \eta^{-1} \Delta F_{\beta}^{(\beta)} + \theta \left( H_{s} \left( q_{c}^{(0)} - q_{f}^{(N)} \right) \right) . \tag{G50}
\]

The left inequality is Landauer’s bound applied to cooling a target system with Hamiltonian \( H_{s} \) (see Theorem 4), and the error term on the right can be bounded quite easily; for instance, for \( \beta > 0 \), we have

\[
\text{tr} \left( H_{s} \left( q_{c}^{(0)} - q_{f}^{(N)} \right) \right) = \text{tr} \left( H_{s} - E_{s}^{\min} \frac{\beta}{\beta} \right) \left( q_{c}^{(0)} - q_{f}^{(N)} \right) \leq \text{tr} \left( H_{s} - E_{s}^{\min} \frac{\beta}{\beta} \right) q_{c}^{(0)} \quad \text{since } H_{s} - E_{s}^{\min} \frac{\beta}{\beta} \text{ is a positive operator},
\]

\[
\leq \text{tr} \left( H_{s} - E_{s}^{\min} \frac{\beta}{\beta} \right) \frac{\beta}{\beta} \frac{d q_{s}(q)}{dq} \leq \frac{\omega_{s}^{\max}}{d s_{s}}. \tag{G51}
\]

where \( \omega_{s}^{\max} := E_{s}^{\max} - E_{s}^{\min} \) is the largest energy gap in the target system Hamiltonian and \( d s_{s} \) is the system dimension. We have used the fact that since \( q_{c}^{(0)} \) is a thermal state of positive temperature, its average energy is less than that of the infinite temperature thermal state, \( \frac{\beta}{\beta} / d s_{s} \). Since \( \beta \propto 1/N \), it follows that one can always find an \( N \) large enough such that the error is smaller than a given value, thereby saturating the Landauer bound.

**A sequence of machine Hamiltonians to mimic the idealised sequence.** Next we construct a protocol that mimics the above sequence and obeys the global energy conservation condition imposed in the incoherent-control setting. The protocol is split into \( N \) stages (like above). In each stage, the Hamiltonian of the machine is fixed. The machine here comprises to two parts: the “cold” part and the “hot” part. The cold part is chosen to begin in a thermal state at temperature \( \beta \) of the Hamiltonian

\[
H_{c} = (1 + n \theta) H_{s} \tag{G52}
\]
At this point we note that this sequence of cold machine states is exactly the same as in the coherent protocol, which would proceed by simply swapping the full state of target system and machine in each stage. However, that is not possible here since this is not an energy preserving operation. To allow for energy preserving operations, the hot part of the machine consists of $d_q(d_q - 1)/2$ qubits, each corresponding to a pair of levels $(i,j)$ of the target system (henceforth we take $i < j$ to avoid double-counting), whose energy gap is equal to the difference in energies of the target and cold qubit subspaces (hence rendering the desired exchange energy-resonant)

$$H_{n}^{(ij)} = [\omega_i + (1 + n\theta)\omega_j - (\omega_j + (1 + n\theta)\omega_i)] |1\rangle\langle 1|_{n}^{(ij)} = n\theta (\omega_j - \omega_i) |1\rangle\langle 1|_{n}^{(ij)},$$

where we have labelled the energy eigenvalues of $H_n$ by $\{\omega_i\}$. Each of these hot qubits begins at inverse temperature $\beta_n$. After every unitary operation, the cold and hot parts of the machine are rethermalised to their respective initial temperatures.

To understand the choice of machine Hamiltonians, consider the following two energy eigenstates of the machine: $|i\rangle_c \otimes |1\rangle_n^{(i)}$ and $|j\rangle_c \otimes |0\rangle_n^{(j)}$. The energy difference is

$$\Delta^{(ij)} = \omega_j (1 + n\theta) - \omega_i (1 + n\theta) - n\theta(\omega_j - \omega_i) = \omega_j - \omega_i,$$

matching the energy difference between the corresponding pair of energy eigenstates of the target system. Furthermore, calculating the ratio of populations of the two levels we find

$$g^{(ij)} = \frac{e^{-\beta \omega_j(1 + n\theta)} - e^{-\beta \omega_i(1 + n\theta)}}{e^{-\beta \omega_i(1 + n\theta)} + e^{-\beta \omega_j(1 + n\theta)}} = \frac{e^{-n\theta(\omega_j - \omega_i)(\beta + n\theta(\beta - \beta_n))}}{e^{-n\theta(\omega_j - \omega_i)(\beta + n\theta(\beta - \beta_n))} + 1},$$

This corresponds to the Gibbs ratio of a qubit at the temperature $\beta + n\theta(\beta - \beta_n)$, which is the temperature that defines stage $n$ [see Eq. (G38)]. In summary, we have constructed a machine featuring $d_q(d_q - 1)/2$ qubit subspaces (or virtual qubits), each of the same energy gap as one pair of energy eigenstates of the system, and all of which have a Gibbs ratio (or virtual temperature) corresponding the $n$th temperature of our desired sequence.

**A single step of the protocol: the max-exchange.** Within each stage of the protocol, a single step consists of a unitary operation on $\mathcal{S}\mathcal{C}\mathcal{H}$, followed by the rethermalisation of the machine parts to their respective initial temperatures. We construct the unitary operation as follows: for every pair $(i,j)$ of system energy levels, one can calculate the absolute value of the difference in populations of the following two degenerate eigenstates $|i\rangle_s |j\rangle_c |0\rangle_n^{(ij)}$ and $|j\rangle_s |i\rangle_c |1\rangle_n^{(ij)}$. This value corresponds to the amount of population that would move under an exchange $|i\rangle_s |j\rangle_c |0\rangle_n^{(ij)} \leftrightarrow |j\rangle_s |i\rangle_c |1\rangle_n^{(ij)}$. We then choose the pair with the largest absolute value of this difference and perform that exchange, with an identity operation applied to all other subspaces.

We call this unitary operation the max-exchange. We proceed to prove two statements about the max-exchange operation. First, that the heat extracted from the hot bath is proportional to the change in average energy of the system; and second, that system state under repetition of said operation converges to the thermal state of the temperature that defines the stage $n$.

Consider the change in average energy of the target system under the exchange unitary. The only two populations that change are those of the $|i\rangle_s$ and $|j\rangle_s$. We label the increase in the population of $|i\rangle_s$ as $\delta p$. Then, we have

$$\Delta E_s = \text{tr} \left[ H_s (g'_s - g_s) \right] = -\delta p (\omega_j - \omega_i).$$

On the other hand, the populations of the corresponding hot qubit (i.e., tracing out the target system and cold machine) change by the same amount, i.e., there is a move of $\delta p$ from $|1\rangle_n^{(ij)}$ to $|0\rangle_n^{(ij)}$. In order to rethermalise the hot qubit, the heat drawn from the hot bath is thus

$$\tilde{\Delta} E_n = \delta p n\theta(\omega_j - \omega_i) = -n\theta \Delta E_s.$$

This is an expression conveniently independent of the pair $(i,j)$ that applies after an arbitrary number of repetitions of the max-exchange operation (which will use different pairs in general).

**Convergence of the max-exchange protocol to the virtual temperature.** To show that the max-exchange protocol indeed converges to the desired system state in each stage of the protocol, we first prove a rather general statement: given a state $\varrho$ diagonal in the energy eigenbasis, if we exchange any qubit subspace within this system with a virtual qubit of a particular virtual temperature, then the relative entropy of the target system w.r.t. the thermal state of that (virtual) temperature decreases.

To this end, consider the relative entropy of a state $\varrho$ that is diagonal in the energy eigenbasis to a thermal state $\tau$. Labelling the populations of $\varrho$ as $p_i$ and those of $\tau$ as $q_i$, this can be expressed as

$$D(\varrho||\tau) = \sum_k p_k \log \left( \frac{p_k}{q_k} \right),$$

where we have labelled the energy eigenvalues of $H_n$ by $\{\omega_i\}$. Each of these hot qubits begins at inverse temperature $\beta_n$. After every unitary operation, the cold and hot parts of the machine are rethermalised to their respective initial temperatures.

To understand the choice of machine Hamiltonians, consider the following two energy eigenstates of the machine: $|i\rangle_c \otimes |1\rangle_n^{(i)}$ and $|j\rangle_c \otimes |0\rangle_n^{(j)}$. The energy difference is

$$\Delta^{(ij)} = \omega_j (1 + n\theta) - \omega_i (1 + n\theta) - n\theta(\omega_j - \omega_i) = \omega_j - \omega_i,$$

matching the energy difference between the corresponding pair of energy eigenstates of the target system. Furthermore, calculating the ratio of populations of the two levels we find

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This corresponds to the Gibbs ratio of a qubit at the temperature $\beta + n\theta(\beta - \beta_n)$, which is the temperature that defines stage $n$ [see Eq. (G38)]. In summary, we have constructed a machine featuring $d_q(d_q - 1)/2$ qubit subspaces (or virtual qubits), each of the same energy gap as one pair of energy eigenstates of the system, and all of which have a Gibbs ratio (or virtual temperature) corresponding the $n$th temperature of our desired sequence.

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Consider the change in average energy of the target system under the exchange unitary. The only two populations that change are those of the $|i\rangle_s$ and $|j\rangle_s$. We label the increase in the population of $|i\rangle_s$ as $\delta p$. Then, we have

$$\Delta E_s = \text{tr} \left[ H_s (g'_s - g_s) \right] = -\delta p (\omega_j - \omega_i).$$

On the other hand, the populations of the corresponding hot qubit (i.e., tracing out the target system and cold machine) change by the same amount, i.e., there is a move of $\delta p$ from $|1\rangle_n^{(ij)}$ to $|0\rangle_n^{(ij)}$. In order to rethermalise the hot qubit, the heat drawn from the hot bath is thus

$$\tilde{\Delta} E_n = \delta p n\theta(\omega_j - \omega_i) = -n\theta \Delta E_s.$$

This is an expression conveniently independent of the pair $(i,j)$ that applies after an arbitrary number of repetitions of the max-exchange operation (which will use different pairs in general).

**Convergence of the max-exchange protocol to the virtual temperature.** To show that the max-exchange protocol indeed converges to the desired system state in each stage of the protocol, we first prove a rather general statement: given a state $\varrho$ diagonal in the energy eigenbasis, if we exchange any qubit subspace within this system with a virtual qubit of a particular virtual temperature, then the relative entropy of the target system w.r.t. the thermal state of that (virtual) temperature decreases.

To this end, consider the relative entropy of a state $\varrho$ that is diagonal in the energy eigenbasis to a thermal state $\tau$. Labelling the populations of $\varrho$ as $p_i$ and those of $\tau$ as $q_i$, this can be expressed as

$$D(\varrho||\tau) = \sum_k p_k \log \left( \frac{p_k}{q_k} \right),$$
We now focus on a single qubit subspace labelled by \( \{i, j\} \), which leads to

\[
D(\rho|\tau) = p_i \log \left( \frac{p_i}{q_i} \right) + p_j \log \left( \frac{p_j}{q_j} \right) + \sum_{k \notin \{i, j\}} p_k \log \left( \frac{p_k}{q_k} \right)
\]

\[
= (p_i + p_j) \left[ \frac{p_i}{p_i + p_j} \log \left( \frac{p_i + p_j}{q_i + q_j} \right) + \frac{p_j}{p_i + p_j} \log \left( \frac{p_i + p_j}{q_i + q_j} \right) \right] + \sum_{k \notin \{i, j\}} p_k \log \left( \frac{p_k}{q_k} \right)
\]

\[
= N \left( \frac{p_i}{p_i + p_j} \log \left( \frac{p_i + p_j}{q_i} \right) + \frac{p_j}{p_i + p_j} \log \left( \frac{p_i + p_j}{q_j} \right) + \log \left( \frac{N}{N_V} \right) \right) + \sum_{k \notin \{i, j\}} p_k \log \left( \frac{p_k}{q_k} \right),
\]

\[\text{(G59)}\]

In the last line we have renormalised the populations within the qubit subspace and labelled the total populations of the system and thermal state qubit subspaces of interest by \( N \) and \( N_V \), respectively. Labelling the normalised states within these subspaces as \( \rho_V \) and \( \tau_V \) respectively, we have

\[
D(\rho|\tau) = N \left[ D(\rho_V||\tau_V) + \log \left( \frac{N}{N_V} \right) \right] + \sum_{k \notin \{i, j\}} p_k \log \left( \frac{p_k}{q_k} \right).
\]

\[\text{(G60)}\]

Suppose now that this qubit subspace of the target system is exchanged with a qubit subspace of any machine that has the same temperature as the thermal state above. The only object that changes in the the above expression is \( \rho_V \), since the norm \( N \) remains the same. In addition, \( \rho_V \) always gets closer to \( \tau_V \) under such an exchange \([40, 60]\), implying that the relative entropy always strictly decreases under such an operation.

Returning to the max-exchange protocol, note that by construction, every virtual qubit in the machine that is exchanged with the qubit subspace \( \{i, j\} \) of the target system in a given stage \( n \) has the same virtual temperature, \( \beta_n = \beta + n\theta (\beta - \beta_n) \). Thus the relative entropy of the system to the thermal state at this temperature always decreases under this operation, unless the operation does not shift any population, which only happens at the unique fixed point where every qubit subspace of the system is already at the virtual temperature \( \beta_n \). By monotone convergence, the relative entropy must converge, and moreover converge to the value that it has at the fixed point of the operation, which is the thermal state at inverse temperature \( \beta_n \). Note that rather than choosing the qubit subspace with maximum population difference to exchange we could also have picked at random from among the pairs \( \{i, j\} \) and convergence would still hold; the max-exchange protocol simply ensured the fastest rate of convergence among these choices.

**Choosing a large enough number of repetitions in each stage so that the overall heat cost is close to the idealised heat cost.** Given that the max-exchange protocol in stage \( n \) converges to the thermal state that we label \( \hat{\rho}_s^{(n)} \), given any error \( \delta_E \), we choose a number of repetitions \( m_n \) that is large enough so that the difference between the average energy of the actual final state of this stage, which we label \( \tilde{\rho}_s^{(n)} \), and that of the ideal state \( \hat{\rho}_s^{(n)} \) is less than \( \delta_E \). In this case, the total heat cost over all stages is close to the idealised heat cost

\[
|\Delta E_N - \Delta E^{*}_N| = \sum_{n=1}^{N} \left\{ -n\theta \text{ tr} \left[ H_s \left( \hat{\rho}_s^{(n)} - \hat{\rho}_s^{(n-1)} \right) \right] \right\} - \sum_{n=1}^{N} \left\{ -n\theta \text{ tr} \left[ H_s \left( \hat{\rho}_s^{(n)} - \hat{\rho}_s^{(n-1)} \right) \right] \right\}
\]

\[
= \sum_{n=0}^{N-1} \theta \text{ tr} \left[ H_s \left( \tilde{\rho}_s^{(n)} - \hat{\rho}_s^{(n)} \right) \right] - N\theta \left( \hat{\rho}_s^{(N)} - \hat{\rho}_s^{(N)} \right)
\]

\[
\leq 2N\theta\delta_E = 2 \left( \frac{\beta^* - \beta}{\beta - \beta_n} \right) \delta_E.
\]

\[\text{(G61)}\]

The number of repetitions in each stage \( m_n \) required depends only upon the initial choice of \( \beta^* \) and \( N \).

**Completing the proof.** Finally, suppose that one is given any target temperature \( \beta^* \) and two arbitrarily small errors, \( \epsilon_{\beta} \) for the cooling and \( \epsilon_E \) for the heat cost, and asked to cool incoherently in such a way that achieves

\[
|\beta' - \beta^*| \leq \epsilon_{\beta},
\]

\[\text{(G62)}\]

\[
|\Delta E_N - \eta^{-1} \Delta E^{(\beta)}_s| \leq \epsilon_E.
\]

\[\text{(G63)}\]

We proceed by first choosing a number of stages \( N \) so that the idealised heat cost \( \Delta E^{*}_N \) is within \( \frac{\epsilon_E}{\eta} \) to the Carnot-Landauer bound above. The idealised sequence of temperatures satisfies \( \beta_N = \beta^* \) by construction. Once \( N \) is fixed, for each stage from
\( n = 1 \) to \( N - 1 \) we choose a number of repetitions for each stage \( m_n \) such that the actual heat cost is within \( \frac{\epsilon}{2} \) of the idealised heat cost, as discussed above. This ensures that the total heat cost is within \( \epsilon E \) of the bound. Finally, we check that the number of repetitions of the last stage \( m_N \) is large enough for us to be within \( \epsilon \beta \) of \( \beta^* \). If not, we increase the number of repetitions (this can only decrease the error in the heat cost anyway) until we are close enough, as required.

Appendix H: Summary Tables of Cooling Protocols

To aid in navigability, we present a set of tables summarising the key protocols that we have detailed. Landauer energy cost refers to saturation of Eq. (4); Carnot-Landauer energy cost refers to saturation of Eq. (8). Here, we use the proxy of effective dimensionality to represent the control complexity, see Eq. (6). The control complexity of a complete swap between a target system and a machine of the same dimension \( d = d_s = d_m \) is \( d_{\text{eff}} = \frac{1}{2}d(d - 1) \) and that of any exchange of two energy levels (e.g., \( |i, j\rangle \leftrightarrow |j, i\rangle \) or \( |i, j + 1, k\rangle \leftrightarrow |i + 1, j, k + 1\rangle \) as considered in the incoherent-control protocols) is \( d_{\text{eff}} = 1 \). We quantify the time duration of any given protocol by the number of required consecutive unitary operations with fixed complexity. While the physical time passing is also a function of the interaction strength and multipartite nature of the interactions giving rise to the respective unitaries [61, 62], it is clear that each operation takes finite time, and a diverging number of unitaries thus implies diverging physical time. In the incoherent setting, \( \beta \) refers to the initial inverse temperature of the target system and cold part of the machine, whereas \( \beta_H \) is the initial temperature of the hot part of the machine. In the special case of \( \beta_H = 0 \) in the incoherent-control setting, the Carnot-Landauer limit reduces to the Landauer limit.

<table>
<thead>
<tr>
<th>Energy</th>
<th>Time</th>
<th>Control Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sec. IV A</td>
<td>( \rightarrow \infty )</td>
<td>1</td>
</tr>
<tr>
<td>Sec. IV B: Thm. 1 (App. C)</td>
<td>Landauer</td>
<td>( \rightarrow \infty )</td>
</tr>
<tr>
<td>Sec. IV C: Cor. 1 (App. C)</td>
<td>Landauer</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE I. Cooling protocols for finite-dimensional systems in the coherent-control paradigm.

<table>
<thead>
<tr>
<th>Energy</th>
<th>Time</th>
<th>Control Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sec. IV A</td>
<td>( \rightarrow \infty )</td>
<td>1</td>
</tr>
<tr>
<td>App. E2a</td>
<td>Landauer</td>
<td>( \rightarrow \infty )</td>
</tr>
<tr>
<td>App. E2b</td>
<td>Finite (( \succ ) Landauer)</td>
<td>( \rightarrow \infty )</td>
</tr>
<tr>
<td>App. E3</td>
<td>Landauer</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE II. Cooling protocols for infinite-dimensional systems in the coherent-control paradigm.

<table>
<thead>
<tr>
<th>Heat Bath</th>
<th>Energy</th>
<th>Time</th>
<th>Control Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sec. VI B 1: Thm. 5 (App. F1)</td>
<td>( \beta_H \in [0, \beta] )</td>
<td>( \rightarrow \infty )</td>
<td>( \times )</td>
</tr>
<tr>
<td>Sec. VI B 2: Thm. 6 (App. F3)</td>
<td>( \beta_H = 0 )</td>
<td>Landauer</td>
<td>( \rightarrow \infty )</td>
</tr>
<tr>
<td>Sec. VI B 2: Cor. 3 (App. F3)</td>
<td>( \beta_H = 0 )</td>
<td>Landauer</td>
<td>1</td>
</tr>
<tr>
<td>Sec. VI B 3: Thm. 7 (App. G)</td>
<td>( \beta_H \in [0, \beta] )</td>
<td>Carnot-Landauer</td>
<td>( \rightarrow \infty )</td>
</tr>
<tr>
<td>Sec. VI B 3: Cor. 4 (App. G)</td>
<td>( \beta_H \in [0, \beta] )</td>
<td>Carnot-Landauer</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE III. Cooling protocols for finite-dimensional systems in the incoherent-control paradigm.