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POINT PROCESS CONVERGENCE FOR THE OFF-DIAGONAL ENTRIES OF
SAMPLE COVARIANCE MATRICES

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We study point process convergence for sequences of i.i.d. random walks. The objective is to derive asymptotic theory for the extremes of these random walks. We show convergence of the maximum random walk to the Gumbel distribution under the existence of a $(2 + \delta)$th moment. We make heavy use of precise large deviation results for sums of i.i.d. random variables. As a consequence, we derive the joint convergence of the off-diagonal entries in sample covariance and correlation matrices of a high-dimensional sample whose dimension increases with the sample size. This generalizes known results on the asymptotic Gumbel property of the largest entry.

1. Introduction.

1.1. Motivation. An accurate probabilistic understanding of covariances and correlations is often the backbone of a thorough statistical data analysis. In many contemporary applications, one is faced with large data sets where both the dimension of the observations and the sample size are large. A major reason lies in the rapid improvement of computing power and data collection devices which has triggered the necessity to study and interpret the sometimes overwhelming amounts of data in an efficient and tractable way. Huge data sets arise naturally in genome sequence data in biology, online networks, wireless communication, large financial portfolios, and natural sciences. More applications where the dimension $p$ might be of the same or even higher magnitude than the sample size $n$ are discussed in [13, 22]. In such a high-dimensional setting, one faces new probabilistic and statistical challenges; see [23] for a review. The sample (auto)covariance matrices will typically be misleading [3, 15]. Even in the null case, that is, when the components of the time series are i.i.d., it is well known that the sample covariance matrix poorly estimates the population covariance matrix. The fluctuations of the off-diagonal entries of the sample covariance matrix aggregate, creating an estimation bias which is quantified by the famous Marčenko–Pastur theorem [29]. This paper provides insight into the joint behavior of the off-diagonal entries with a particular focus on their extremes.

Aside from the high dimension, the marginal distributions of the components present another major challenge for an accurate assessment of the dependence. In the literature, one typically assumes a finite fourth moment since otherwise the largest eigenvalue of the sample covariance matrix would tend to infinity when $n$ and $p$ increase. This moment assumption, however, excludes heavy-tailed time series from the analysis. The theory for the eigenvalues and eigenvectors of the sample autocovariance matrices stemming from such time series is quite different from the classical Marčenko–Pastur theory which applies in the light-tailed case. For detailed discussions about classical random matrix theory, we refer to the monographs [3, 40], while the developments in the heavy-tailed case can be found in [2, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40].

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19, 38, 39] and the references therein. For applications of extreme value statistics in finance and physics we refer to [6, 17].

In this paper, we study point process convergence for sequences of i.i.d. random walks. We then apply our results to derive the joint asymptotic behavior of the off-diagonal entries of sample covariance and correlation matrices. Based on this joint convergence we propose new independence tests in high dimensions.

1.2. The model. We are given \( p \)-dimensional random vectors \( \mathbf{x}_t = (X_{1t}, \ldots, X_{pt})^\top, t = 1, \ldots, n \), whose components \((X_{it})_{i,t \geq 1}\) satisfy the following standard conditions:

- \((X_{it})\) are independent and identically distributed (i.i.d.) random variables with generic element \( X \).
- \( \mathbb{E}[X] = 0 \) and \( \mathbb{E}[X^2] = 1 \).

The dimension \( p = p_n \) is some integer sequence tending to infinity as \( n \to \infty \).

We are interested in the (nonnormalized) \( p \times p \) sample covariance matrix \( \mathbf{S} \) and the sample correlation matrix \( \mathbf{R} \),

\[
\mathbf{S} = \mathbf{S}_n = \sum_{t=1}^{n} \mathbf{x}_t \mathbf{x}_t^\top \quad \text{and} \quad \mathbf{R} = \mathbf{R}_n = (\text{diag}(\mathbf{S}))^{-1/2}\mathbf{S}(\text{diag}(\mathbf{S}))^{-1/2}
\]

with entries

\[
S_{ij} = \sum_{t=1}^{n} X_{it} X_{jt} \quad \text{and} \quad R_{ij} = \frac{S_{ij}}{\sqrt{S_{ii} S_{jj}}}, \quad i, j = 1, \ldots, p,
\]

respectively. The dependence on \( n \) is often suppressed in our notation.

Our goal is to prove limit theory for the point processes of scaled and centered points \((S_{ij})\), \((R_{ij})\). Asymptotic theory for the extremes of these points can be deduced from the limit point process.

1.3. State-of-the-art. In the literature, the largest off-diagonal entry of a sample covariance or correlation matrix has been studied, but results concerning the joint behavior of the entries are currently lacking. The theoretical developments are mainly due to Jiang. In [20], he analyzed the asymptotic distributions of

\[
W_n := \frac{1}{n} \max_{1 \leq i < j \leq p} |S_{ij}| \quad \text{and} \quad L_n := \max_{1 \leq i < j \leq p} |R_{ij}|
\]

under the assumption \( p/n \to \gamma \in (0, \infty) \). If \( \mathbb{E}[|X|^{30+\delta}] < \infty \) for some \( \delta > 0 \), he proved that

\[
\lim_{n \to \infty} \mathbb{P}(nW_n^2 - 4 \log p + \log \log p \leq x) = \exp\left(-\frac{1}{\sqrt{8\pi}}e^{-x/2}\right), \quad x \in \mathbb{R},
\]

\[
\lim_{n \to \infty} \mathbb{P}(nL_n^2 - 4 \log p + \log \log p \leq x) = \exp\left(-\frac{1}{\sqrt{8\pi}}e^{-x/2}\right), \quad x \in \mathbb{R}.
\]

The limiting law is a nonstandard Gumbel distribution. Under the same assumptions Jiang [20] also derived the limits

\[
\lim_{n \to \infty} \sqrt{\frac{n}{\log p}} L_n = 2 = \lim_{n \to \infty} \sqrt{\frac{n}{\log p}} W_n \quad \text{a.s.}
\]

Several authors managed to relax Jiang’s moment condition while keeping the proportionality of \( p \) and \( n \). Zhou [41] showed that (1.4) holds if \( n^6 \mathbb{P}(|X_{11}X_{12}| > n) \to 0 \) as \( n \to \infty \). A sufficient condition is \( \mathbb{E}[X^6] < \infty \). The papers [25–27] provide refinements of Zhou’s condition.
We summarize the distributional assumptions on $X$ for the validity of (1.4) and (1.3) under proportionality of dimension $p$ and sample size $n$ as follows: $\mathbb{E}[X^6] < \infty$ is sufficient, and $\mathbb{E}[|X|^{6-\delta}] < \infty$ for any $\delta > 0$ is necessary. In that sense, finiteness of the sixth moment is the optimal moment assumption.

Interestingly, the optimal moment requirement also depends on the growth of $p$ if $p$ increases at a different rate than $n$. For the largest off-diagonal entry of the sample correlation matrix, also known as coherence of the random matrix $X = (x_1, \ldots, x_n)$, the interplay between dimension and moments was addressed in [28]. If $\mathbb{E}[|X|^s] < \infty$ for $s > 2$ and

$$c_1 n^{(s-2)/4} \leq p \leq c_2 n^{(s-2)/4}$$

with positive constants $c_1, c_2$, Theorem 1.1 in [28] shows that (1.4) still applies. Note that, for proportional $p$ and $n$, this result requires the finiteness of the sixth moment. The larger $p$ relative to $n$, the more moments of $X$ are needed. If the moment generating function of $|X|$ exists in some neighborhood of zero, (1.4) holds for $p = O(\exp(n^\beta))$ for certain $\beta \in (0, 1/3)$; see [8]. Finally, if $(\log p)/n \not\to 0$, various phase transitions appear in the limit distribution of $L_n$. These were explored in [9] under convenient assumptions on $X$ which yield an explicit formula for the density of $R_{12}$.

1.4. Objective and structure of this paper. Our main objective is to prove limit theory for the point processes of scaled and centered points $(S_{ij})$, $(R_{ij})$ in a more general framework than used for the results above. By a continuous mapping argument, the joint asymptotic distribution of functionals of a fixed number of points can easily be deduced from the limit process. In particular, we obtain the asymptotic distribution of the largest and smallest entries.

First, we establish our result for $S$. Since each $S_{ij}$ is a sum of i.i.d. random variables, we prove a useful large deviation theorem which exploits the asymptotic normal distribution of $S_{ij}$. Aside from finding suitable assumptions on $X$, the main challenge is that the $S_{ij}$ are not independent. It turns out that despite their nontrivial dependence, the maximum behaves like the maximum of i.i.d. copies. Therefore we will first solve the problem for i.i.d. random walk points $(S_n^{(i)})$ instead of $(S_{ij})$. This is done in Section 2.

We continue in Section 3 with the main results of the paper. Here we derive asymptotic theory for the point processes

$$\tilde{N}_n = \sum_{1 \leq i < j \leq p} \varepsilon_{d_p}(S_{ij}/\sqrt{n-\tilde{d}_p}) \xrightarrow{d} N,$$

for suitable constants $(\tilde{d}_p)$ and limit Poisson random measure $N$ with mean measure $\mu$ on $\mathbb{R}$ such that $\mu(x, \infty) = e^{-x}$, $x \in \mathbb{R}$. Throughout this paper $\varepsilon_x$ denotes the Dirac measure at $x$. A continuous mapping theorem implies distributional convergence of finitely many $S_{ij}$. In particular, the maximum entry $S_{ij}$ converges to the Gumbel distribution provided $X$ has suitably many finite moments. A related result holds with $S_{ij}$ replaced by the corresponding sample correlations $R_{ij} = S_{ij}/\sqrt{S_{ii}S_{jj}}$. In Section 4, we extend our results to hypercubic random matrices of the form $\sum_{t=1}^n x_t \otimes \cdots \otimes x_t$, and we briefly discuss some statistical applications. The proofs of the main results are presented in Sections 5 and 6.

2. Normal approximation to large deviation probabilities. In this section we collect some precise large deviation results for sums of independent random variables. Throughout this section, $(X_i)$ is an i.i.d. sequence of mean zero, unit variance random variables with generic element $X$, distribution $F$ and right tail $\overline{F} = 1 - F$. We define the corresponding partial sum process

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad n \geq 1.$$
Consider i.i.d. copies \((S^{(i)}_n)_{i \geq 1}\) of \(S_n\). We also introduce an integer sequence \((p_n)\) such that \(p = p_n \to \infty\) as \(n \to \infty\). We are interested in the limit behavior of the \(k\) largest values among \((S^{(i)}_n)_{i = 1, \ldots, p}\), in particular in the possible limit laws of the maximum \(\max_{i = 1, \ldots, p} S^{(i)}_n\). More generally, we are interested in the limit behavior of the point processes \(N_n\),

\[
N_n = \sum_{i=1}^{p} \varepsilon_{dp(S^{(i)}_n/\sqrt{n-d_p})} \Rightarrow N, \quad n \to \infty,
\]
toward a Poisson random measure \(N\) on \(\mathbb{R}\) with mean measure \(\mu\) given by \(\mu(x, \infty) = e^{-x}\), \(x \in \mathbb{R}\). The sequence \((dp)\) is chosen such that \(p \Phi(dp) = p(1 - \Phi(dp)) \to 1\) as \(p \to \infty\) where \(\Phi\) is the standard normal distribution function. In this paper, we work with

\[
dp = \sqrt{2\log p - \log \log p + \log 4\pi (2\log p)^{1/2}}.
\]

A motivation for this choice is that for an i.i.d. sequence \((X_i)\) with distribution function \(\Phi\) we have

\[
\lim_{n \to \infty} \mathbb{P}\left(\max_{i = 1, \ldots, p} X_i - dp \leq x\right) = \exp(-e^{-x}) = \Lambda(x), \quad x \in \mathbb{R}.
\]

The limit distribution function is the standard Gumbel \(\Lambda\); see Embrechts et al. [16], Example 3.3.29.

By [35], Theorem 5.3, relation (2.1) is equivalent to the following limit relation for the tails:

\[
p \mathbb{P}(dp(S_n/\sqrt{n-d_p}) > x) \to e^{-x}, \quad n \to \infty, x \in \mathbb{R},
\]

and also to convergence of the maximum of the random walks \((S^{(i)}_n)_{i = 1, \ldots, p}\) to the Gumbel distribution:

\[
\lim_{n \to \infty} \mathbb{P}\left(\max_{i = 1, \ldots, p} d_p(S^{(i)}_n/\sqrt{n-d_p}) \leq x\right) = \Lambda(x), \quad x \in \mathbb{R}.
\]

Equations (2.3) and (2.4) involve precise large deviation probabilities for the random walk \((S_n)\). To state some results which are relevant in this context, we assume one of the following three moment conditions:

(C1) There exists \(s > 2\) such that \(\mathbb{E}[|X|^s] < \infty\).

(C2) There exists an increasing differentiable function \(g\) on \((0, \infty)\) such that \(\mathbb{E}[\exp(g(|X|))] < \infty\), \(g'(x) \leq \tau g(x)/x\) for sufficiently large \(x\) and some \(\tau < 1\), and \(\lim_{x \to \infty} g(x)/\log x = \infty\).

(C3) There exists a constant \(h > 0\) such that \(\mathbb{E}[\exp(h|X|)] < \infty\).

Note that the conditions (C1)–(C3) are increasing in strength. One has the implications (C3) \(\Rightarrow\) (C2) \(\Rightarrow\) (C1). The following result explains the connection between the rate of \(p_n \to \infty\) in (2.3) and the conditions (C1)–(C3) on the distribution of \(X\).

**THEOREM 2.1.** Assume the standard conditions on \((X_i)\) and that \(p = p_n \to \infty\) satisfies:

- \(p = O(n^{(s-2)/2})\) if (C1) holds.
- \(p = \exp(o(g_n^2 \land n^{1/3}))\) where \(g_n\) is the solution of the equation \(g_n^2 = g(g_n \sqrt{n})\), if (C2) holds.
- \(p = \exp(o(n^{1/3}))\) if (C3) holds.

Then we have

\[
p \mathbb{P}(S_n/\sqrt{n} > dp + x/d_p) \sim p \Phi(dp + x/d_p) \to e^{-x}, \quad n \to \infty, x \in \mathbb{R}.
\]
REMARK 2.2. The proofs of these results follow from the definition of $d_p$ and precise large deviation bounds of the type

$$\sup_{0 \leq y \leq \gamma_n} \left| \frac{\mathbb{P}(S_n/\sqrt{n} > y)}{\Phi(y)} - 1 \right| \to 0, \quad n \to \infty,$$

where $\gamma_n \to \infty$ are sequences depending on the conditions (C1)–(C3). If (C3) holds, one can choose $\gamma_n = o(n^{1/6})$ implying the growth rate $p = \exp(o(n^{1/3}))$. This follows from Petrov’s large deviation result [33], Theorem VIII.2. Under (C2) one can choose $\gamma_n = o(n^{1/6} \wedge g_n)$ implying the growth rate $p = \exp(o(g_n^2 \wedge n^{1/3}))$. This follows from S.V. Nagaev’s [31], Theorem 3. Under (C1) he also derived $\gamma_n = \sqrt{(s/2 - 1) \log n}$ in [31], Theorem 4. The best possible range under (C1) is $\gamma_n = \sqrt{s - 2} \log n$; see Michel [30], Theorem 4.

The aforementioned large deviation results cannot be improved in general unless additional conditions are assumed. For example, under (C3) if the cumulants of $X$ of order $k = 3, \ldots, r + 2$ vanish then (2.5) holds for $p = \exp(o(n^{(r+1)/(r+3)}))$. This follows from the fact that one can choose $\gamma_n = o(n^{(r+1)/(2(r+3))})$; see [33], Theorem VIII.7. In Section VIII.3 of [33] one also finds necessary and sufficient conditions for (2.6) to hold in certain intervals. As a matter of fact, such conditions are not of moment-type. Therefore one cannot expect that necessary and sufficient conditions for (2.5) for general sequences $(p_n)$ can be expressed in terms of moments. There is, however, a clear relationship between possible rates of $(p_n)$ and the existence of moments: the higher moments exist the larger we can choose $(p_n)$, but the growth cannot be arbitrarily fast.

In passing we mention a sharp large deviation result for a sequence of i.i.d. regularly varying random variables $(X_i)$ with tail index $\alpha > 2$, that is, a generic element $X$ has tails

$$\mathbb{P}(\pm X > x) \sim p_\pm \frac{L(x)}{x^\alpha}, \quad x \to \infty,$$

where $p_+ + p_- = 1$ and $L$ is slowly varying. Then, due to S.V. Nagaev’s results in [32], one has (2.6) with $\gamma_n = \sqrt{c_1 \log n}$ for $c_1 < \alpha - 2$, while for $\xi_n = \sqrt{c_2 \log n}$ and any $c_2 > \alpha - 2$,

$$\sup_{y > \xi_n} \left| \frac{\mathbb{P}(\pm S_n/\sqrt{n} > y)}{n \Phi(y/\sqrt{n})} - p_\pm \right| \to 0, \quad n \to \infty.$$

There exists a small but increasing literature on precise large deviation results; we refer to [12, 37] and the references therein.

Now consider i.i.d. copies $(S^{(i)}_n)_{i \geq 1}$ of $(S_n)$. The following result is an immediate consequence of Theorem 2.1.

THEOREM 2.3. Assume the conditions of Theorem 2.1. Relation (2.5) is equivalent to either of the following two limit relations:

$$\mathbb{P} \left( d_p \max_{i=1, \ldots, p} \left( S^{(i)}_n/\sqrt{n} - d_p \right) \leq x \right) \to \Lambda(x), \quad x \in \mathbb{R}, n \to \infty$$

and

$$N_n = \sum_{i=1}^p \varepsilon \cdots \varepsilon d_p (S^{(i)}_n/\sqrt{n} - d_p) \overset{d}{\to} N = \sum_{i=1}^\infty \varepsilon \log \Gamma_i, \quad n \to \infty,$$

where $\Gamma_i = E_1 + \cdots + E_i$, $i \geq 1$, and $(E_i)$ is i.i.d. standard exponential, that is, $N$ is a Poisson random measure with mean measure $\mu(x, \infty) = e^{-x}, x \in \mathbb{R}$. 
THE OFF-DIAGONAL POINT PROCESS

PROOF. Following Resnick [35], Theorem 5.3, (2.10) and (2.5) are equivalent. Moreover, a continuous mapping argument implies that, if \( N_n \xrightarrow{d} N \), then

\[
\mathbb{P}(N_n(x, \infty) = 0) = \mathbb{P} \left( d_p \max_{i=1,...,p} (S_n^{(i)}/\sqrt{n} - d_p) \leq x \right)
\]

(2.11)

\[
\rightarrow \mathbb{P}(N(x, \infty) = 0) = \mathbb{P}(- \log \Gamma_1 \leq x) = \exp(-e^{-x}).
\]

Moreover, if (2.9) holds a Taylor expansion argument shows that

\[
\mathbb{P} \left( d_p \max_{i=1,...,p} (S_n^{(i)}/\sqrt{n} - d_p) \leq x \right) = \left( 1 - p \mathbb{P}(S_n/\sqrt{n} > d_p + x/d_p) \right)^p
\]

\[
\rightarrow \exp(-e^{-x}), \quad n \to \infty,
\]

holds if and only if (2.5) does. □

This means that in case of i.i.d. points \((S^{(i)}_n)\) the convergence of the maximum is equivalent to the convergence of the point processes \((N_n)\). In general, the latter is a stronger statement. If \((N_n)\) converges, the distribution of the maximum can always be recovered using (2.11).

3. Main results.

3.1. Point process convergence of a sample covariance matrix. We consider the sample covariance matrix \( S = (S_{ij})_{i,j=1,...,p} \) introduced in Section 1.2. The problem of showing limit theory for the associated point process is similar to Theorem 2.3 for i.i.d. random walks \((S^{(i)}_n)\). In contrast to the i.i.d. copies \((S^{(i)}_n)\) in Section 2 here we deal with \(p(p-1)/2\) dependent off-diagonal entries of \( S \). Nevertheless, Theorem 2.1 will again be a main tool for proving these results.

Since the summands of \( S_{ij} \) are i.i.d. products \( X_{it} X_{jt} \) we need to adjust the conditions (C2) and (C3) to this situation while (C1) remains unchanged.

(C2') There exists an increasing differentiable function \( g \) on \((0, \infty)\) such that \( \mathbb{E}[\exp(g(|X_{11} X_{12}|))] < \infty \), \( g'(x) \leq \tau g(x)/x \) for sufficiently large \( x \) and some \( \tau < 1 \), and \( \lim_{x \to \infty} g(x)/\log x = \infty \).

(C3') There exists a constant \( h > 0 \) such that \( \mathbb{E}[\exp(h|X_{11} X_{12}|)] < \infty \).

REMARK 3.1. By Lemma 6.3, (C3') implies (C3). The reverse implication is not true. For example, if \( X \) is standard exponential, which satisfies (C3), then \( X_{11} X_{21} \) has Weibull-type tail with parameter \( 1/2 \); see [1]; which does not satisfy (C3'). By Lemma 6.3, (C2') implies \( \mathbb{E}[\exp(g(|X|))] < \infty \).

THEOREM 3.2. Assume the standard conditions on \((X_{it})\) and that \( p = p_n \to \infty \) satisfies:

- \( p = O(n^{(s-2)/4}) \) if (C1) holds.
- \( p = \exp(o(s_n^2 \land n^{1/3})) \), where \( g_n \) is the solution of the equation \( g_n^2 = g(g_n \sqrt{n}) \), if (C2') holds.
- \( p = \exp(o(n^{1/3})) \) if (C3') holds.

Define \( \tilde{d}_p = d_p(p-1)/2 \). Then the following point process convergence holds:

\[
N_n^S := \sum_{1 \leq i < j \leq p} \varepsilon_{\tilde{d}_p(S_{ij}/\sqrt{n} - \tilde{d}_p)} \xrightarrow{d} N,
\]

where \( N \) is the Poisson random measure defined in (2.10).

The proof is given in Section 6.1.
Some comments:

- The point process convergence in Theorem 3.2 remains valid if the standard conditions on \((X_{it})\) are relaxed to the following two conditions:
  - The columns \(x_1, \ldots, x_n\) of the matrix \((X_{it})_{i=1,\ldots,p; t=1,\ldots,n}\) are i.i.d.
  - The random variables \(X_{11}, \ldots, X_{p1}\) are independent, with mean zero and unit variance, but they are not necessarily identically distributed.

The proof is the same as that of Theorem 3.2. All results in Section 3 hold under these relaxed conditions. For clarity of presentation and proof, all statements are presented under the standard conditions.

- Theorem 3.2 can be extended by introducing additional time stamps:
  \[
  \sum_{1 \leq i < j \leq p} \frac{\varepsilon_{(i,j)}}{p} \Delta_p(S_{ij}/\sqrt{n} - \tilde{d}_p) \overset{d}{\to} \tilde{N}, \quad n \to \infty,
  \]
on \{(x_1, x_2) : 0 \leq x_1 \leq 1, x_1 \leq x_2\} \times \mathbb{R} where \(\tilde{N}\) is a Poisson random measure with mean measure \(\text{LEB} \times \mu\). This follows for example, by using the techniques of [36], Proposition 3.21.

- Under any of the moment conditions (C2'),(C3') one can choose \(p \sim \gamma n\) for \(\gamma > 0\) in Theorem 3.2. Under (C1), one needs the condition \(\mathbb{E}[|X|^\delta] < \infty\) in order to guarantee \(p = O(n)\). This is in agreement with the minimal moment requirement for the results on \(W_n\) (see (1.3)).

Next we consider the order statistics of \(S_{ij}\), \(1 \leq i < j \leq p\):

\[
\min_{1 \leq i < j \leq p} S_{ij} =: S_{(p(p-1)/2)} \leq \cdots \leq S_{(1)} := \max_{1 \leq i < j \leq p} S_{ij}.
\]

Theorem 3.2 implies the convergence of the largest and smallest off-diagonal entries of \(S\).

**Corollary 3.3.** Under the conditions of Theorem 3.2 we have joint convergence of the upper and lower order statistics: for any \(k \geq 1\),

\[
(3.1) \quad \tilde{d}_p(S_{(i)}/\sqrt{n} - \tilde{d}_p)_{i=1,\ldots,k} \overset{d}{\to} (-\log \Gamma_i)_{i=1,\ldots,k},
\]

\[
(3.2) \quad \tilde{d}_p(S_{(i)}/\sqrt{n} + \tilde{d}_p)_{i=p(p-1)/2,\ldots,p(p-1)/2-k+1} \overset{d}{\to} (\log \Gamma_i)_{i=1,\ldots,k}.
\]

Moreover, the properly normalized maxima and minima are asymptotically independent, that is for any \(x, y \in \mathbb{R}\) we have as \(n \to \infty\),

\[
(3.3) \quad \mathbb{P}(\tilde{d}_p(S_{(1)}/\sqrt{n} - \tilde{d}_p) \leq x, \tilde{d}_p(S_{(p(p-1)/2)}/\sqrt{n} + \tilde{d}_p) \leq y) \to \Lambda(x)(1 - \Lambda(-y)).
\]

**Proof.** Relation (3.1) is immediate from \(N_n^S \overset{d}{\to} N\) and the continuous mapping theorem. The same argument works for (3.2) if one observes that

\[
\tilde{d}_p(S_{(i)}/\sqrt{n} + \tilde{d}_p)_{i=p(p-1)/2,\ldots,p(p-1)/2-k+1} = -\tilde{d}_p((-S_{(i)}/\sqrt{n} - \tilde{d}_p)_{i=1,\ldots,k},
\]

where \((-S)_{(i,j)}\) is the ordered sample of \((-S_{ij})\). An application of (3.1) with \((S_{ij})\) replaced by \((-S_{ij})\) then yields (3.2).

Now we consider joint convergence of the maximum and the minimum: for \(x, y \in \mathbb{R}\),

\[
G_n(x, y)
\]

\[
= \mathbb{P}(\tilde{d}_p(S_{(1)}/\sqrt{n} - \tilde{d}_p) \leq x, \tilde{d}_p(S_{(p(p-1)/2)}/\sqrt{n} + \tilde{d}_p) > y)
\]

\[
= \mathbb{P}(-\tilde{d}_p + y/\tilde{d}_p < S_{ij}/\sqrt{n} \leq \tilde{d}_p + x/\tilde{d}_p \text{ for all } 1 \leq i < j \leq p)
\]

\[
= 1 - \mathbb{P}\left( \bigcup_{1 \leq i < j \leq p} \{S_{ij}/\sqrt{n} > \tilde{d}_p + x/\tilde{d}_p\} \cup \{-S_{ij}/\sqrt{n} \geq \tilde{d}_p - y/\tilde{d}_p\} \right).
\]
Writing
\[ A_{ij} = \{ S_{ij}/\sqrt{n} > \tilde{d}_p + x/\tilde{d}_p \} \cup \{ -S_{ij}/\sqrt{n} \geq \tilde{d}_p - y/\tilde{d}_p \}, \]
one can use the same arguments used for establishing \( \mathbb{P}(N_n^S(B) = 0) \to \mathbb{P}(N(B) = 0) \) in the proof of Theorem 3.2 to show that
\[ G_n(x, y) \to \exp(- (e^x + e^{-x})) = \Lambda(x) \Lambda(-y), \quad n \to \infty. \]
Hence
\[
\mathbb{P}(\tilde{d}_p(S(1)/\sqrt{n} - \tilde{d}_p) \leq x, \tilde{d}_p(S_{(p(p-1)/2)}/\sqrt{n} + \tilde{d}_p) \leq y)
= \mathbb{P}(\tilde{d}_p(S(1)/\sqrt{n} - \tilde{d}_p) \leq x) - G_n(x, y)
\to \Lambda(x) - \Lambda(x) \Lambda(-y) = \Lambda(x)(1 - \Lambda(-y)), \quad n \to \infty.
\]

**Remark 3.4.** An immediate consequence is
\[
\frac{S_{(1)}}{\sqrt{n} \log p} \xrightarrow{\mathbb{P}} 2 \quad \text{and} \quad \frac{S_{(p(p-1)/2)}}{\sqrt{n} \log p} \xrightarrow{\mathbb{P}} -2.
\]

**Remark 3.5.** If \( \mathbb{E}[|X|^s] < \infty \) for some \( s > 4 \) and \( \text{var}(X^2) > 0 \), we conclude from Theorem 2.3 that for \( p = O(n^{(s-4)/4}) \),
\[
\sum_{i=1}^{p} \epsilon_{d_p((S_{ii} - n)/\sqrt{n \text{var}(X^2)} - d_p)} \xrightarrow{d} N.
\]

In particular, \( (\max_{i=1, \ldots, p} d_p((S_{ii} - n)/\sqrt{n \text{var}(X^2)} - d_p)) \) converges to a Gumbel distribution. We notice that \( d_p \sim \sqrt{2 \log p} \) while the normalizing and centering constants for \( (S_{ij}/\sqrt{n})_{i \neq j} \) in (3.3) are \( \tilde{d}_p \sim 2\sqrt{\log p} \).

Moreover, while we still have Gumbel convergence for the maxima of the off-diagonal elements \( S_{ij} \) for suitable \( (p_n) \) if \( \mathbb{E}[|X|^s] < \infty \) for some \( s \in (2, 4) \), the point process convergence in (3.4) cannot hold. Indeed, then an appeal to Nagaev’s large deviation result (2.8) shows that, under the regular variation condition (2.7) on \( X \) with \( \alpha \in (2, 4) \),
\[
\sum_{i=1}^{p} \epsilon_{a_{np}^{-2}(S_{ii} - n)} \xrightarrow{d} N,
\]
where \( N \) is Poisson random measure on the state space \((0, \infty)\) with mean measure \( \mu_{\alpha}(x, \infty) = x^{-\alpha/2}, \quad x > 0 \), and \( a_k \) satisfies \( k\mathbb{P}(|X| > a_k) \to 1 \) as \( k \to \infty \). In particular, the maxima of \( (S_{ii}) \) converge toward a standard Fréchet distribution:
\[
\mathbb{P}(\max_{i=1, \ldots, p} (S_{ii} - n) \leq x) \to \Phi_{\alpha/2}(x) = \exp(-x^{-\alpha/2}), \quad x > 0.
\]

Assume (2.7) on \( X \) with \( \alpha \in (2, 4) \). If we construct a point process by choosing the normalization \( a_{np}^2 \) for the diagonal and off-diagonal entries, the contribution of the \( (S_{ij}) \) vanishes in the limit:
\[
\sum_{i=1}^{p} \epsilon_{a_{np}^{-2}(S_{ii} - n)} + \sum_{1 \leq i < j \leq p} \epsilon_{a_{np}^{-2}S_{ij}} \xrightarrow{d} N.
\]
It is also proved in Heiny and Mikosch [19] that the diagonal entries \( (S_{ii}) \) of the sample covariance matrix dominate the off-diagonal terms in operator norm, that is \( \|S - \text{diag}(S)\|/\|\text{diag}(S)\| \xrightarrow{\mathbb{P}} 0 \) as \( n \to \infty \). In turn, the asymptotic behavior of the largest eigenvalues of the sample covariance matrix are determined by the corresponding largest eigenvalues of \( (S_{ii}) \).
The techniques in this paper straightforwardly extend to other transformations of the points \((S_{ij})\). As an example, we provide one such result for the squares \((S_{ij}^2)\).

**Corollary 3.6.** Assume the conditions of Theorem 3.2. Then

\[ N_n^{S^2} = \sum_{1 \leq i < j \leq p} \varepsilon_{0.5S_{ij}^2/n - 0.5\tilde{d}^2_p - \log 2} \]

converges to the Poisson random measure \(N\) described in Theorem 3.2.

**Proof.** One can follow the arguments in the proof of Theorem 3.2. In order to show condition (i), observe that for \(x \in \mathbb{R}\),

\[
E[N_n^{S^2}(x, \infty)] = \frac{p(p-1)}{2} \mathbb{P}\left(\frac{S_{12}^2}{2n} - \frac{\tilde{d}^2_p}{2} - \log 2 > x\right)
\]

\[
= \frac{p(p-1)}{2} \mathbb{P}\left(\left|\frac{S_{12}}{\sqrt{n}}\right| > \sqrt{2(x + \log 2 + \tilde{d}^2_p)}\right)
\]

\[ \sim p^2 \Phi\left(\sqrt{2(x + \log 2 + \tilde{d}^2_p)}\right) \to e^{-x}, \quad n \to \infty. \]

\[ \square \]

3.2. **Point process convergence of a sample correlation matrix.** Based on Theorem 3.2 we can also derive point process convergence for the sample correlation matrix \(R = (R_{ij})_{i,j=1,...,p}\) defined in (1.1) and (1.2).

**Theorem 3.7.** Assume the standard conditions on \((X_{it})\) and that \(p = p_n \to \infty\) satisfies:

- \(p = O(n^{(s-2)/4})\) if (C1) holds.
- \(p = \exp(o(g_n^2 \wedge n^{1/3}))\) where \(g_n\) is the solution of the equation \(g_n^2 = g(g_n\sqrt{n})\) if (C2’) holds.
- \(p = \exp(o(n^{1/3}))\), if (C3’) holds.

Then the following point process convergence holds:

\[ N_n^R := \sum_{1 \leq i < j \leq p} \varepsilon_{\tilde{d}_p(\sqrt{n}R_{ij} - \tilde{d}_p)} \xrightarrow{d} N, \]

where \(N\) is the Poisson random measure defined in (2.10).

The proof is given in Section 6.3.

The results for the order statistics of \(R_{ij}, 1 \leq i < j \leq p:\)

\[ \min_{1 \leq i < j \leq p} R_{ij} =: R_{(p(p-1)/2)} \leq \cdots \leq R_{(1)} := \max_{1 \leq i < j \leq p} R_{ij}, \]

carry over from those for the order statistics of \((S_{ij})\).

**Corollary 3.8.** Under the conditions of Theorem 3.7 we have joint convergence of the upper and lower order statistics: for any \(k \geq 1\),

\[ \tilde{d}_p(\sqrt{n}R_{(i)}) - \tilde{d}_p \xrightarrow{d} (-\log \Gamma_i)_{i=1,...,k}, \]

\[ \tilde{d}_p(\sqrt{n}R_{(i)} + \tilde{d}_p)_{i=p(p-1)/2,...,p(p-1)/2-k+1} \xrightarrow{d} (\log \Gamma_i)_{i=1,...,k}. \]

Moreover, for any \(x, y \in \mathbb{R}\),

\[ \lim_{n \to \infty} \mathbb{P}(\tilde{d}_p(\sqrt{n}R_{(1)}) \leq x, \tilde{d}_p(\sqrt{n}R_{(p(p-1)/2)} + \tilde{d}_p) \leq y) = \Lambda(x)(1 - \Lambda(-y)), \]
and
\[ \sqrt{\frac{n}{\log p}} R_1 \overset{\mathbb{P}}{\to} 2 \quad \text{and} \quad \sqrt{\frac{n}{\log p}} R_{p(p-1)/2} \overset{\mathbb{P}}{\to} -2. \]

4. Extensions and applications.

4.1. Extensions. In this section, we extend our results for the point processes constructed from the off-diagonal entries of the sample covariance matrices \( S_n = \sum_{t=1}^n x_t x_t^\top \), where \( x_t \) are the \( p \)-dimensional columns of the data matrix \( X \). We introduce the hypercubic random matrices (or tensors) of order \( m \):

\[ (4.1) \quad S^{(m)} = S^{(m)}_n = \sum_{t=1}^n x_t \otimes \cdots \otimes x_t, \quad m \in \mathbb{N}, n \geq 1, \]

with entries
\[ S^{(m)}_{i_1, \ldots, i_m} = \left( \sum_{t=1}^n x_{i_1t} x_{i_2t} \cdots x_{i_mt} \right), \quad 1 \leq i_1, \ldots, i_m \leq p. \]

It is easy to see that \( S^{(2)} = S \) arises as a special case.

Next, we generalize the moment conditions (C2’) and (C3’) to the \( m \)-fold product \( X_{i_1} \cdots X_{i_m} \).

(C2(m)) There exists an increasing differentiable function \( g \) on \((0, \infty)\) such that \( \mathbb{E}[\exp(g(|X_{i_1} \cdots X_{i_m}|))] < \infty \), \( g'(x) \leq \tau g(x)/x \) for sufficiently large \( x \) and some \( \tau < 1 \), and \( \lim_{x \to \infty} g(x)/\log x = \infty \).

(C3(m)) There exists a constant \( h > 0 \) such that \( \mathbb{E}[\exp(h|X_{i_1} \cdots X_{i_m}|)] < \infty \).

The following result extends Theorem 3.2 to hypercubic matrices of order \( m \).

**THEOREM 4.1.** Let \( m \in \mathbb{N} \) and define \( d_{p,m} = d_{p,m} \). Assume the standard conditions on \((X_{i_t})\) and that \( p = p_n \to \infty \) satisfies:

- \( p = O(n^{(s-2)/4}) \) if (C1) holds.
- \( p = \exp(o(g_n^2 \land n^{1/3})) \), where \( g_n \) is the solution of the equation \( g_n^2 = g(g_n \sqrt{n}) \), if (C2(m)) holds.
- \( p = \exp(o(n^{1/3})) \) if (C3(m)) holds.

Then the following point process convergence holds:

\[ N^{(m)}_n = \sum_{1 \leq i_1 < \cdots < i_m \leq p} \mathbb{E}_{d_{p,m}(S^{(m)}_{i_1, \ldots, i_m}/\sqrt{n-d_{p,m}})} \overset{d}{\to} N, \]

where \( N \) is the Poisson random measure defined in (2.10).

The proof is given in Section 6.2. Since \( d_{p,m} \sim \sqrt{2m \log p} \), Theorem 4.1 implies the convergence of the largest and smallest off-diagonal entries of \( S^{(m)} \).

**COROLLARY 4.2.** Under the assumptions of Theorem 4.1, we have, as \( n \to \infty \),

\[ \max_{1 \leq i_1 < \cdots < i_m \leq p} \frac{S^{(m)}_{i_1, \ldots, i_m}}{\sqrt{n \log p}} \overset{\mathbb{P}}{\to} \sqrt{2m} \quad \text{and} \quad \min_{1 \leq i_1 < \cdots < i_m \leq p} \frac{S^{(m)}_{i_1, \ldots, i_m}}{\sqrt{n \log p}} \overset{\mathbb{P}}{\to} -\sqrt{2m}. \]

Analogously to Corollary 3.3, Theorem 4.1 yields the joint weak convergence of the off-diagonal entries of \( S^{(m)} \), thus extending Theorems 1 and 2 in [21] on the asymptotic Gumbel property of the largest off-diagonal entry of \( S^{(m)} \).
4.2. An application to threshold based estimators. A fundamental task in statistics is the estimation of the population covariance or correlation matrix of a multivariate distribution. If the dimension $p$ becomes large, the sample versions $n^{-1}S$ and $R$ cease to be suitable estimators. Even for our simple model in Section 1.2, that is, when the population covariance and correlation matrices are the $p$-dimensional identity matrix $I_p$, the estimators $n^{-1}S$ and $R$ are not asymptotically consistent for $I_p$. This phenomenon was explored in [18] among many other papers. Assuming $E[X^4] < \infty$ and $p/n \to \gamma \in [0, \infty)$, [18] shows that, as $n \to \infty$,

$$\sqrt{n/p} \|n^{-1}S - I_p\| \to P 2 + \sqrt{\gamma} \quad \text{and} \quad \sqrt{n/p} \|R - I_p\| \to P 2 + \sqrt{\gamma}.$$  

Note that $p$ is allowed to grow at a slower rate than $n$. It was also observed in [18] that

$$\sqrt{n/p} \|n^{-1} \text{diag}(S) - I_p\| \to 0.$$  

We would like to construct estimators $\hat{S}$, $\hat{R}$ based on $S$ and $R$, respectively, such that as $n \to \infty$,

$$\sqrt{n/p} \|n^{-1}\hat{S} - I_p\| \to 0 \quad \text{and} \quad \sqrt{n/p} \|\hat{R} - I_p\| \to 0.$$  

In view of (4.2), we know that we are able to deal with the diagonal. A natural approach is to eliminate the smallest off-diagonal entries by thresholding. Bickel and Levina [4, 5] considered estimators of the form

$$\hat{S} = (S_{ij} \mathbb{1}(\{|S_{ij}| > n \tau_n\})) \quad \text{and} \quad \hat{R} = (R_{ij} \mathbb{1}(\{|R_{ij}| > \tau_n\})),$$  

for some threshold sequence $\tau_n \to 0$. Choosing $\tau_n = \tau_n(C) = C \sqrt{\log p}/n$ with a sufficiently large constant $C$, [4], Theorem 1, shows (4.3) for standard normal $X$. In view of Remark 3.4, the order of the threshold perfectly matches the order of the largest off-diagonal entries. Based on our results, we provide a simple proof of (4.3) for a more general class of distributions.

**Corollary 4.3.** Assume $p/n \to \gamma \in [0, \infty)$ and the conditions of Theorem 3.7. Then the estimators $\hat{R}$, $\hat{S}$ in (4.4) specified for $\tau_n(C)$, $C > 2$, satisfy relation (4.3).

**Proof.** The diagonal part is taken care of by (4.2) and the fact that $\text{diag}(R) = I_p$. The off-diagonal entries of $\hat{R}$ and $\hat{S}$ asymptotically vanish in view of Remark 3.4 and Corollary 3.8, respectively. □

Corollary 4.3 shows that the order of the threshold $\tau_n(C)$ is not affected by the distributional assumption. Under (C1) we thus allow for $p = O(n^{(s-2)/4})$ provided $E[|X|^s] < \infty$. For comparison, Bickel and Levina [4], p. 2585, showed the first limit relation in (4.3) for the bigger threshold $\tau_n(C) = C p^{3/s} / \sqrt{n}$ and dimension $p = o(n^{s/8})$.

4.3. An independence test. If the data $(X_{1i})$ is centered Gaussian with identical distribution the null hypothesis of independence is equivalent to $H_0 : n^{-1}E[S] = I_p$. Based on (1.3), Jiang [20] proposed the following test of $H_0$ with significance level $\alpha \in (0, 1)$:

$$\Psi_\alpha = \mathbb{1}(n W_n^2 - 4 \log p + \log \log p \geq q_\alpha),$$

where

$$q_\alpha = - \log(8\pi) - 2 \log \log(1 - \alpha)^{-1}$$

is the $(1 - \alpha)$-quantile of the limiting nonstandard Gumbel distribution. If $\Psi_\alpha = 1$, we reject $H_0$. Properties of this test are studied in [7].
In view of Corollary 3.3 we can propose a multitude of alternative tests based on the joint asymptotic distribution of the $k$ largest or smallest off-diagonal entries of $S$ and $R$, respectively. Under the conditions of Theorem 3.2 we have as $n \to \infty$,
\[
\tilde{d}_p \left( \frac{(S(1), \ldots, S(k))}{\sqrt{n}} - \tilde{d}_p \right) \xrightarrow{d} (- \log \Gamma_1, \ldots, - \log \Gamma_k)
\]
and $\Gamma_i = E_1 + \cdots + E_i$ for i.i.d. standard exponential random variables $(E_j)$. For $k \geq 1$ and $\alpha \in (0, 1)$, consider a set $A^\alpha_k \subset \mathbb{R}^k$ such that
\[
\mathbb{P} \left( (- \log \Gamma_1, \ldots, - \log \Gamma_k) \in A^\alpha_k \right) = 1 - \alpha
\]
and define the test $T(A^\alpha_k)$ by
\[
T(A^\alpha_k) = 1 \left( \tilde{d}_p \left( \frac{(S(1), \ldots, S(k))}{\sqrt{n}} - \tilde{d}_p \right) \notin A^\alpha_k \right).
\]
If $T(A^\alpha_k) = 1$, we reject $H_0$. Then $T(A^\alpha_k)$ is an asymptotic independence test with significance level $\alpha$.

Convenient univariate test statistics can be constructed from spacings of $S(1), \ldots, S(k)$. An advantage of using spacings is that one avoids centering by $\tilde{d}_p$. For example, consider for some $k \geq 2$,
\[
T_k^{(1)} = \tilde{d}_p (S(1) - S(k))/\sqrt{n},
\]
\[
T_k^{(2)} = \tilde{d}_p \max_{i=1, \ldots, k-1} (S(i) - S(i+1))/\sqrt{n},
\]
\[
T_k^{(3)} = \tilde{d}_p \frac{1}{n} \sum_{i=1}^{k-1} (S(i) - S(i+1))^2.
\]
Recall the well-known fact that
\[
\left( \frac{\Gamma_1}{\Gamma_{k+1}}, \ldots, \frac{\Gamma_k}{\Gamma_{k+1}} \right) \xrightarrow{d} (U(1), \ldots, U(k)),
\]
where the right-hand vector consists of the order statistics of $k$ i.i.d. uniform random variables on $(0, 1)$. Then we have
\[
T_k^{(1)} \xrightarrow{d} \log(\Gamma_k/\Gamma_1) = \log \frac{\Gamma_k}{\Gamma_{k+1}} \xrightarrow{d} \log(U(1)/U(k)),
\]
\[
T_k^{(2)} \xrightarrow{d} \max_{i=1, \ldots, k-1} \log(\Gamma_{i+1}/\Gamma_i) = \max_{i=1, \ldots, k-1} \log(U_{(k-i)}/U_{(k-i+1)}),
\]
\[
T_k^{(3)} \xrightarrow{d} \sum_{i=1}^{k-1} (\log(\Gamma_{i+1}/\Gamma_i))^2 = \sum_{i=1}^{k-1} (\log(U_{(k-i)}/U_{(k-i+1)}))^2.
\]
Now, choosing $q_\alpha$ as the $(1 - \alpha)$-quantiles of the limiting random variables we have $T(A^\alpha_k) = 1(T_k^{(i)} > q_\alpha)$, $i = 1, 2, 3$.

5. **Proof of Theorem 2.1.** In view of Remark 2.2 it suffices to prove the theorem under (C1). Throughout this proof we assume the standard conditions on $(X_{it})$.

We start with a useful auxiliary result due to Einmahl ([14], Corollary 1(b), page 31, in combination with Remark on page 32).
Lemma 5.1. Consider independent \( \mathbb{R}^d \)-valued random vectors \( \xi_1, \ldots, \xi_n \) with mean zero. Assume that \( \xi_i, i = 1, \ldots, n \), has finite moment generating function in some neighborhood of the origin and that the covariance matrix \( \text{var}(\xi_1 + \cdots + \xi_n) = B_n \mathbf{I}_d \) where \( B_n > 0 \) and \( \mathbf{I}_d \) denotes the identity matrix. Let \( \eta_k \) be independent \( N(0, \sigma^2 \text{var}(\xi_k)) \) random vectors independent of \( (\xi_k) \), and \( \sigma^2 \in (0, 1) \). Let \( \xi_k^* = \xi_k + \eta_k, k = 1, \ldots, n \), and write \( p_n^* \) for the density of \( B_n^{-1/2} (\xi_1^* + \cdots + \xi_n^*) \). Choose \( \alpha \in (0, 0.5) \) such that

\[
\alpha \sum_{k=1}^n \mathbb{E}[|\xi_k|^3 \exp(\alpha|\xi_k|)] \leq B_n,
\]

and write

\[
\beta_n = \beta_n(\alpha) = B_n^{-3/2} \sum_{k=1}^n \mathbb{E}[|\xi_k|^3 \exp(\alpha|\xi_k|)],
\]

where \( |x| \) denotes the Euclidean norm. If

\[
|x| \leq c_1 \alpha B_n^{1/2}, \quad \sigma^2 \geq -c_2 \beta_n^2 \log \beta_n, \quad B_n \geq c_3 \alpha^{-2},
\]

where \( c_1, c_2, c_3 \) are constants only depending on \( d \), then

\[
p_n^*(x) = \varphi_{(1+\sigma^2)}(x) \exp(\bar{\mathcal{T}}_n(x)) \text{ with } |\bar{\mathcal{T}}_n(x)| \leq c_4 \beta_n(|x|^3 + 1),
\]

where \( \varphi_\Sigma \) is the density of a \( N(0, \Sigma) \) random vector and \( c_4 \) is a constant only depending on \( d \).

Proof under (C1). We proceed by formulating and proving various auxiliary results. We will use the following notation: \( c \) denotes any positive constant whose value is not of interest, sometimes we write \( c_0, c_1, c_2, \ldots \) for positive constants whose value or size is relevant in the proof,

\[
\bar{X}_i = X_i \mathbb{1}(|X_i| \leq n^{1/s}) - \mathbb{E}[X \mathbb{1}(|X| \leq n^{1/s})], \quad \bar{X}_i = X_i - \bar{X}_i,
\]

\[
\bar{S}_n = n \sum_{i=1}^n \bar{X}_i, \quad \bar{S}_n = S_n - \bar{S}_n.
\]

Next we consider an approximation of the distribution of \( \bar{S}_n \).

Lemma 5.2. Let \( \tilde{p}_n \) be the density of

\[
n^{-1/2} \sum_{i=1}^n (\bar{X}_i + \sigma_n N_i),
\]

where \( (N_i) \) is i.i.d. \( N(0, 1) \), independent of \( (X_i) \) and \( \sigma_n^2 = \text{var}((\bar{X})_{n^2}) \). If \( n^{-2c_6} \log n \leq s_n^2 \leq 1 \) with \( c_6 = 0.5 - (1 - \delta)/s \) for arbitrarily small \( \delta > 0 \), then the relation

\[
\tilde{p}_n(x) = \varphi_{1+\sigma_n^2}(x)(1 + o(1)), \quad n \to \infty,
\]

holds uniformly for \( |x| = o(n^{1/6-1/(3s)}) \).

Proof. We apply Lemma 5.1 to the i.i.d. random variables \( \tilde{\xi}_i = \bar{X}_i, i = 1, \ldots, n \). Notice that \( \mathbb{E}[\bar{X}] = 0 \) and \( B_n = \text{var}(\bar{S}_n) = n \text{ var}(\bar{X}) \). Choose \( \tilde{\alpha} = c_5 n^{-1/s} \). Then

\[
\tilde{\alpha} \sum_{i=1}^n \mathbb{E}[|\bar{X}_i|^3 \exp(\tilde{\alpha} |\bar{X}_i|)] = \tilde{\alpha} n \mathbb{E}[|\bar{X}|^3 \exp(\tilde{\alpha} |\bar{X}|)]
\]

\[
\leq c_5 n^{1-1/s} \mathbb{E}[|\bar{X}|^3] \exp(2c_5)
\]

\[
\leq 8c_5 \exp(2c_5)n^{1-\delta/s} \mathbb{E}[|X|^{2+\delta}],
\]
where $\delta \in (0,1)$ is chosen such that $\mathbb{E}[|X|^{2+\delta}] < \infty$. Hence (5.1) is satisfied for $\alpha = \tilde{\alpha}$ and sufficiently small $c_5$.

Next choose
\begin{equation}
\tilde{\beta}_n = B_n^{-3/2} \sum_{i=1}^{n} \mathbb{E}[|X_i|^3 \exp(\tilde{\alpha}|X_i|)] = B_n^{-3/2} n \mathbb{E}[|X|^3 \exp(\tilde{\alpha}|X|)]
\end{equation}
\begin{equation}
\leq c B_n^{-3/2} n^{1+(1-\delta)/s} \mathbb{E}[|X|^{2+\delta}] \leq c n^{-c_6},
\end{equation}
where $\delta$ is chosen as above and $c_6 = 0.5 - (1-\delta)/s$.

Next we consider (5.2). We can choose $x$ according to the restriction
\begin{equation}
|x| \leq c_1 \tilde{\alpha} B_n^{1/2} \sim c n^{1/2-1/s}.
\end{equation}
By (5.2) and (5.5) we can choose $\sigma^2 = \sigma_n^2$ according to
\begin{equation}
1 \geq \sigma_n^2 \geq c \log nn^{-2c_6}.
\end{equation}
Moreover, $B_n \geq c_3 \tilde{\alpha}^{-2}$. An application of (5.3) yields
\begin{equation}
\tilde{p}_n(x) = \varphi_{1+\sigma_n^2}(x) \exp(\tilde{T}_n(x)) \quad \text{for } |\tilde{T}_n(x)| \leq c_4 \tilde{\beta}_n(|x|^3+1),
\end{equation}
but in view of (5.4) and (5.5), $\tilde{\beta}_n(|x|^3+1) = o(1)$ uniformly for $|x|^3 = o(\min(n^{0.5-1/s}, n^{c_6})) = o(n^{0.5-1/s})$ for arbitrarily small $\delta > 0$. That is, the remainder term $|\tilde{T}_n(x)|$ converges to zero, uniformly for the $x$ considered. This proves the lemma.

We add another auxiliary result.

**Lemma 5.3.** Assume that $p = p_n \rightarrow \infty$ and $p = O(n^{(s-2)/2})$. Then for $x \in \mathbb{R}$, $c_6$ as in Lemma 5.2, an i.i.d. $N(0,1)$ sequence $(N_i)$ and $\sigma_n^2 = c \log nn^{-2c_6}$, we have
\begin{equation}
p \mathbb{P}(n^{-1/2} \sum_{i=1}^{n} (X_i + \sigma_n N_i) > d_p + x/d_p) \rightarrow e^{-x}, \quad n \rightarrow \infty.
\end{equation}

**Proof.** Write $y_n = \sqrt{(s-2)\log n}$. By virtue of Lemma 5.2 we observe that for any $C > 1$,
\begin{align*}
P_1 &= \mathbb{P} \left( d_p + x/d_p < n^{-1/2} \sum_{i=1}^{n} (X_i + \sigma_n N_i) \leq y_n \right) \sim \int_{d_p+x/d_p}^{y_n} \Phi_{1+\sigma_n^2}(y) \, dy, \\
P_2 &= \mathbb{P} \left( y_n < n^{-1/2} \sum_{i=1}^{n} (X_i + \sigma_n N_i) \leq Cy_n \right) \sim \int_{y_n}^{Cy_n} \varphi_{1+\sigma_n^2}(y) \, dy.
\end{align*}
However, using Mill’s ratio and the definition of $d_p$, we have that
\begin{align*}
p P_1 &\sim e^{-x} \frac{\Phi_{d_p+x/d_p}}{\Phi(d_p + x/d_p)} - p \frac{y_n}{\sqrt{1 + \sigma_n^2}} \\
&\sim e^{-x} \exp \left( 0.5(d_p + x/d_p)^2 \frac{\sigma_n^2}{1 + \sigma_n^2} \right) - \frac{1}{\sqrt{2\pi\sqrt{(s-2)\log n}}} pn^{-(s-2)/2},
\end{align*}
but the right-hand side converges to $e^{-x}$ since $(d_p + x/d_p)^2 \sigma_n^2 / (1 + \sigma_n^2) = o(1)$, $pn^{-(s-2)/2} = O(1)$ and $(\log n)^2 n^{-2c_6} = o(1)$. A similar argument shows $p P_2 \rightarrow 0$. 

If we have another auxiliary result.

**Lemma 5.4.** Assume that $p = p_n \rightarrow \infty$ and $p = O(n^{(s-2)/2})$. Then for $x \in \mathbb{R}$, $c_6$ as in Lemma 5.2, an i.i.d. $N(0,1)$ sequence $(N_i)$ and $\sigma_n^2 = c \log nn^{-2c_6}$, we have
\begin{equation}
p \mathbb{P}(n^{-1/2} \sum_{i=1}^{n} (X_i + \sigma_n N_i) > d_p + x/d_p) \rightarrow e^{-x}, \quad n \rightarrow \infty.
\end{equation}

**Proof.** Write $y_n = \sqrt{(s-2)\log n}$. By virtue of Lemma 5.2 we observe that for any $C > 1$,
\begin{align*}
P_1 &= \mathbb{P} \left( d_p + x/d_p < n^{-1/2} \sum_{i=1}^{n} (X_i + \sigma_n N_i) \leq y_n \right) \sim \int_{d_p+x/d_p}^{y_n} \Phi_{1+\sigma_n^2}(y) \, dy, \\
P_2 &= \mathbb{P} \left( y_n < n^{-1/2} \sum_{i=1}^{n} (X_i + \sigma_n N_i) \leq Cy_n \right) \sim \int_{y_n}^{Cy_n} \varphi_{1+\sigma_n^2}(y) \, dy.
\end{align*}
However, using Mill’s ratio and the definition of $d_p$, we have that
\begin{align*}
p P_1 &\sim e^{-x} \frac{\Phi_{d_p+x/d_p}}{\Phi(d_p + x/d_p)} - p \frac{y_n}{\sqrt{1 + \sigma_n^2}} \\
&\sim e^{-x} \exp \left( 0.5(d_p + x/d_p)^2 \frac{\sigma_n^2}{1 + \sigma_n^2} \right) - \frac{1}{\sqrt{2\pi\sqrt{(s-2)\log n}}} pn^{-(s-2)/2},
\end{align*}
but the right-hand side converges to $e^{-x}$ since $(d_p + x/d_p)^2 \sigma_n^2 / (1 + \sigma_n^2) = o(1)$, $pn^{-(s-2)/2} = O(1)$ and $(\log n)^2 n^{-2c_6} = o(1)$. A similar argument shows $p P_2 \rightarrow 0$. 

THE OFF-DIAGONAL POINT PROCESS 551
We also have
\[
\mathbb{P}\left(n^{-1/2} \sum_{i=1}^{n} (X_i + \sigma_n N_i) > C y_n \right) \\
\leq \mathbb{P}(n^{-1/2} \mathcal{S}_n > 0.5 C y_n) + \Phi(0.5 C y_n / \sigma_n) \\
= P_3 + P_4.
\]
It is easy to see that \( p P_4 \to 0 \). We observe that
\[
|X_i| \leq n^{1/s} (1 + o(n^{1/s})) = c_n, \quad \text{a.s.}
\]
\[
\text{var}(X) \leq \mathrm{E}[X^2 1(|X| \leq n^{1/s})] \leq \text{var}(X) = 1.
\]
We apply Prokhorov’s inequality (Petrov [33], Chapter III.5) for any \( C > 1 \),
\[
p \mathbb{P}(S_n > C \sqrt{n} y_n) \\
\leq p \exp\left(- \frac{C \sqrt{(s - 2) n \log n}}{2 c_n} \log\left(1 + \frac{C \sqrt{(s - 2) n \log n c_n}}{2 n \text{var}(X)}\right)\right) \\
\leq p \exp\left(- \frac{C^2 (s - 2) \log n}{4}\right) \\
= p n^{-C^2(s-2)/8}.
\]
The right-hand side converges to zero for sufficiently large \( C \). This proves the lemma. \( \square \)

Write \((X_{it})_{t \geq 1}\) for the i.i.d. sequence of the summands constituting \( S_n^{(i)} \) and
\[
\mathcal{S}_n^{(i,N)} = \sum_{t=1}^{n} (X_{it} + \sigma_n N_{it}) =: S_n^{(i)} + \sigma_n \sqrt{n} \tilde{N}_i,
\]
where \((\tilde{N}_i)\) are i.i.d. standard normal random variables independent of everything else. Then by Lemma 5.3,
\[
\mathbb{P}\left(\max_{i=1,\ldots,p} d_p (S_n^{(i,N)}/\sqrt{n} - d_p) \leq x \right) \to \Lambda(x), \quad x \in \mathbb{R}, n \to \infty.
\]
We have
\[
d_p n^{-1/2} \max_{i=1,\ldots,p} |S_n^{(i)} - \mathcal{S}_n^{(i,N)}| \\
\leq d_p \max_{i=1,\ldots,p} |\sigma_n (\tilde{N}_i - d_p)| + \sigma_n d_p^2 \\
\leq d_p \sigma_n \max_{i=1,\ldots,p} |\tilde{N}_i - d_p| + \sigma_n d_p^2 \\
= O_p(d_p^2 \sigma_n) = o_p(1), \quad n \to \infty.
\]
Therefore
\[
\mathbb{P}(d_p \left(\max_{i=1,\ldots,p} (\mathcal{S}_n^{(i)}/\sqrt{n} - d_p) \leq x \right) \to \Lambda(x), \quad x \in \mathbb{R}, n \to \infty,
\]
and the latter relation is equivalent to
\[
p \mathbb{P}(S_n/\sqrt{n} > d_p + x/d_p) \to e^{-x}, \quad x \in \mathbb{R}, n \to \infty.
\]
Our next goal is to prove that we can replace \( \mathcal{S}_n \) by \( S_n \) in the latter relation. In view of the equivalence between (5.7) and (5.8) it suffices to show (5.7) with \( \mathcal{S}_n^{(i)} \) replaced by \( S_n^{(i)} \). Therefore we will show that
\[
\frac{d_p}{\sqrt{n}} \max_{i=1,\ldots,p} |S_n^{(i)}| \to 0.
\]
We have by the Fuk–Nagaev inequality ([34], page 78), for \( y > 0 \) and suitable constants \( c_0, c_1 > 0 \),
\[
\mathbb{P}
\left(
\max_{i=1,\ldots,p} |S^{(i)}_n/\sqrt{n}| > y \right)
\leq p \mathbb{P}(|S_n| > \sqrt{ny/d_p})
\]
(5.9)
\[
\leq c_0 n \mathbb{E}[|X|^p] \left(\frac{\sqrt{ny}}{d_p}\right)^{-s} + \exp\left(-c_1 \frac{y^2}{d_p \operatorname{var}(X)}\right).
\]
Using partial integration and Markov’s inequality of order \( s \), we find that \( \operatorname{var}(X) \leq cn^{-0.5+1/(2s)} \) holds if \( \mathbb{E}[|X|^p] < \infty \). Combining this bound with the rate \( p = O(n^{-(s-2)/2}) \), we see that \( d_p^2 \operatorname{var}(X) \to 0 \) and therefore the exponential term in (5.9) vanishes. The polynomial term in (5.9) converges to zero for the same reason. This proves (5.7) with \( S^{(i)}_n \) replaced by \( S^{(i)}_n \) and finishes the proof of the theorem. \( \Box \)

6. Proofs of sample covariance and correlation results.

6.1. Proof of Theorem 3.2. By Kallenberg’s criterion for the convergence of simple point processes (see for instance [16], p. 233, Theorem 5.2.2) it suffices to verify the following conditions:

(i) For any \(-\infty < a < b < \infty\), one has \( \mathbb{E}[N_n^S(a, b)] \to \mathbb{E}[N(a, b)] = \mu(a, b) \) as \( n \to \infty \).

(ii) For \( B = \bigcup_{i=1}^d (b_i, c_i] \subset (-\infty, \infty) \) with \(-\infty < b_1 < c_1 < \cdots < b_d < c_d < \infty\), one has \( \mathbb{P}(N_n^S(B) = 0) \to \mathbb{P}(N(B) = 0) = e^{-\mu(B)} \) as \( n \to \infty \).

We start with (i). Note that \( \mu(a, b) = e^{-a} - e^{-b} \). Since the assumptions of Theorem 2.1 hold it follows from (2.5) (with \( p \) replaced by \( p(p-1)/2 \)), that as \( n \to \infty \)
\[
\mathbb{E}[N_n^S(a, b)] = \frac{p(p-1)}{2} \mathbb{P}(\tilde{d}_p + a/\tilde{d}_p < S_{12}/\sqrt{n} < \tilde{d}_p + b/\tilde{d}_p) \to \mu(a, b).
\]

To show (ii), we consider
\[
1 - \mathbb{P}(N_n^S(B) = 0) = \mathbb{P}\left(\bigcup_{1 \leq i < j \leq p} A_{ij}\right), \quad A_{ij} = \{\tilde{d}_p(S_{ij}/\sqrt{n} - \tilde{d}_p) \in B\}.
\]

By an inclusion-exclusion argument we get for \( k \geq 1 \),
\[
\sum_{d=1}^{2k} (-1)^{d-1} W_d \leq \mathbb{P}\left(\bigcup_{1 \leq i < j \leq p} A_{ij}\right) \leq \sum_{d=1}^{2k-1} (-1)^{d-1} W_d,
\]
(6.1)

where
\[
W_d = \sum_{(I, J) \in I_d} \mathbb{P}(A_{i_1j_1} \cap \cdots \cap A_{i_dj_d}) =: \sum_{(I, J) \in I_d} q(I, J)
\]
and the summation runs over the set
\[
I_d = \{(I, J) = ((i_1, j_1), \ldots, (i_d, j_d)) \text{ such that } 1 \leq i_t < j_t \leq p, t = 1, \ldots, d, \text{ and } (i_1, j_1) < (i_2, j_2) < \cdots < (i_d, j_d)\}.
\]

In the definition of \( I_d \), we use the lexicographic ordering of pairs \((i_s, j_s), (i_t, j_t)\):
\[
(i_s, j_s) < (i_t, j_t) \quad \text{if and only if} \quad i_s < i_t \text{ or } (i_s = i_t \text{ and } j_s < j_t).
\]

A combinatorial argument yields
\[
|I_d| = \left(\frac{p(p-1)}{2}\right)^d \sim \frac{1}{d!} \left(\frac{p^2}{2}\right)^d, \quad n \to \infty.
\]
(6.2)
Proof of (ii) under (C1). Consider the set $\hat{I}_d$ consisting of all elements $(I, J) \in I_d$ such that all $i_t, j_t, t = 1, \ldots, d$ are mutually distinct. For $(I, J) \in I_d$, the random variables $S_{n,i_t,j_t}, t = 1, \ldots, d$, are i.i.d. and therefore
\begin{equation}
q(I, J) = \left(\mathbb{P}(A_{12})\right)^d.
\end{equation}
For $(I, J) \in I_d \setminus \hat{I}_d$ we write

\begin{equation}
S_{n}(I, J) = (S_{11,1}, \ldots, S_{id,jd})^\top = \sum_{t=1}^n (X_{i1}X_{j1,t}, \ldots, X_{i_d,j_d})^\top =: \sum_{t=1}^n \xi_t,
\end{equation}

and also $1 = (1, \ldots, 1)^\top \in \mathbb{R}^d$. The i.i.d. $\mathbb{R}^d$-valued summands $\xi_t$ with generic element $\xi$ have mean zero and covariance matrix $I_d$. We have
\begin{equation}
q(I, J) = \mathbb{P}(n^{-1/2}S_{n}(I, J) \in \tilde{B}_p 1 + B/d/\tilde{d}_p).
\end{equation}
We will apply Lemma 5.1 to $(\xi_t)$. We will prove it under (C1); the proof under (C2') and (C3') is analogous; we will indicate some necessary changes. In this case, $\mathbb{E}[|\xi_t|^s] < \infty$ for some $s > 2$. Write
\begin{equation}
\widetilde{\xi}_t = (\widetilde{\xi}_t^{(l)}) \mathbb{I}(\{\xi_t^{(l)} | \leq n^{1/s}\}) - \mathbb{E}[\xi_t^{(l)} \mathbb{I}(\{\xi_t^{(l)} | \leq n^{1/s}\})],
\end{equation}
\begin{equation}
\xi_t = \widetilde{\xi}_t - \xi_t,
\end{equation}
\begin{equation}
\tilde{S}_n(I, J) = \sum_{t=1}^n \widetilde{\xi}_t, \quad S_n(I, J) = S_n(I, J) - \tilde{S}_n(I, J) = \sum_{t=1}^n \xi_t.
\end{equation}
Proceeding as in the proof of Lemma 5.2, we obtain the following result.

**Lemma 6.1.** Let $\tilde{p}_n$ be the density of
\begin{equation}
n^{-1/2} \sum_{i=1}^n (\tilde{\xi}_i + \sigma_n N_i),
\end{equation}
where $(N_i)$ is i.i.d. $N(0, I_d)$, independent of $(\xi_t)$ and $\sigma_n^2 = \text{var}(\xi_t^{(l)}) n^{2}$. If $n^{-2c_6} \log n \leq s_n^2 \leq 1$ with $c_6 = 0.5 - (1 - \delta)/s$ for arbitrarily small $\delta > 0$, then the relation
\begin{equation}
\tilde{p}_n(x) = \varphi_{(1 + \sigma_n^2)}(x)(1 + o(1)), \quad n \to \infty,
\end{equation}
holds uniformly for $|x| = o(n^{1/6 - 1/(3s)})$.

Following the lines of the proof of Lemma 5.3, we obtain the following result.

**Lemma 6.2.** Assume that $p = p_n \to \infty$ and $p^2 = O(n^{(s-2)/2})$. Then for $\sigma_n^2 = c \log nn^{-2c_6}$ and an i.i.d. $N(0, 1)$ sequence $(\tilde{N}_i)$, uniformly for $(I, J)$ in $I_d$,
\begin{equation}
\left(\frac{p^2}{2}\right)^d q(I, J) \sim \left(\frac{p^2}{2}\mathbb{P}\left(n^{-1/2} \sum_{i=1}^n (\tilde{\xi}_i + \sigma_n \tilde{N}_i) \in B\right)\right)^d \sim (\mu(B))^d.
\end{equation}

Finally, we need to prove that $\tilde{S}_n(I, J)$ in (6.4) can be replaced by $S_n(I, J)$. However, this follows in the same way as the corresponding steps in the proof of Theorem 2.3. Indeed, since we need to show that $n^{-1/2} \tilde{S}_n(I, J)$ does not contribute asymptotically to $n^{-1/2} S_n(I, J)$ it suffices to prove this fact for each of the components of $n^{-1/2} \tilde{S}_n(I, J)$.
We conclude that as \( n \to \infty \)

\[
W_d = \left( \sum_{(I,J) \in I_d \backslash \hat{I}_d} + \sum_{(I,J) \in \hat{I}_d} \right) q(I,J) \sim \frac{1}{d!} \left( \frac{p^2}{2} \right)^d \left( \mathbb{P}(A_{12}) \right)^d \sim \frac{(\mu(B))^d}{d!}.
\]

We recall that (6.1) provides an upper and lower bound for \( \mathbb{P}(N_n(B) = 0) \). Letting first \( n \to \infty \) and then \( k \to \infty \), thanks to (6.5) we see that both bounds converge to the same limit. More precisely, we have

\[
\lim_{n \to \infty} P(N_n(B) = 0) = 1 - \sum_{d=1}^{\infty} (-1)^{d-1} \frac{(\mu(B))^d}{d!} = \sum_{d=0}^{\infty} (-\mu(B))^d d! = e^{-\mu(B)}.
\]

The proof of (ii) is complete.

**Proof of (ii) under (C2'), (C3').** Write \( b_0 = \min_{1 \leq q \leq \ell} b_q, c_0 = \max_{1 \leq q \leq \ell} c_q \) and for \( (I,J) \in I_d \backslash \hat{I}_d \),

\[
\tilde{S}_n = S_{i_1j_1} + \cdots + S_{i_dj_d} = \sum_{t=1}^{n} (X_{i_1t} X_{j_1t} + \cdots + X_{i_dt} X_{j_dt}).
\]

We have

\[
q(I,J) \leq \mathbb{P}\left( \left( \frac{\tilde{S}_n}{d \sqrt{n}} - \tilde{d}_p \right) \tilde{d}_p \in (b_0, c_0) \right) = \mathbb{P}\left( \sqrt{d}(b_0 / \tilde{d}_p + \tilde{d}_p) < \frac{\tilde{S}_n}{d \sqrt{n}} < \sqrt{d}(c_0 / \tilde{d}_p + \tilde{d}_p) \right).
\]

Note that \( \tilde{S}_n / \sqrt{d} \) has i.i.d. summands with mean zero and unit variance. Since \( \sqrt{d}(c_0 / \tilde{d}_p + \tilde{d}_p) = o(n^{1/6}) \) under (C3') and \( \sqrt{d}(c_0 / \tilde{d}_p + \tilde{d}_p) = o(n^{1/6} \wedge g_n) \) under (C2') applications of [33], Theorem 1 in Section VIII.2, and [31], Theorem 4, respectively, yield

\[
q(I,J) \leq c \left( \Phi(\sqrt{d}(b_0 / \tilde{d}_p + \tilde{d}_p)) - \Phi(\sqrt{d}(c_0 / \tilde{d}_p + \tilde{d}_p)) \right) = O(p^{-2d+\varepsilon}),
\]

for an arbitrarily small \( \varepsilon > 0 \). This shows that (6.5) holds. Now one can proceed as under condition (C1).

6.2. **Proof of Theorem 4.1.** We proceed as in the proof of Theorem 3.2 and show (i), (ii) therein. For \( -\infty < a < b < \infty \), it follows from (2.5) that as \( n \to \infty \)

\[
\mathbb{E}[N_n^{(m)}(a, b)] = \binom{p}{m} \mathbb{P}(d_{p,m} + a / d_{p,m} < S_{12} / \sqrt{n} < d_{p,m} + b / d_{p,m})
\] \[
\to e^{-a} - e^{-b}.
\]

This proves condition (i). The proof of (ii) is completely analogous to the proof of Theorem 3.2. The main difference is that \( i < j \) needs to be replaced with \( i_1 < i_2 < \cdots < i_m \). For example, instead of the index set \( I_d \) whose elements are \( d \) distinct \( m \)-tuples, with \( |I_d| = \left( \binom{p}{d} \right) \); see (6.2); one would get an index set \( I_d^{(m)} \) of \( d \) distinct \( m \)-tuples satisfying \( |I_d^{(m)}| = \left( \binom{p}{d} \right) \). We omit details.

6.3. **Proof of Theorem 3.7.** First, assume \( \text{var}(X^2) = 0 \). Then \( S_{ii} = n \) a.s. for all \( i \) and hence \( \sqrt{n} R_{ij} \to S_{ij} / \sqrt{n} \) so that the claim follows immediately from Theorem 3.2.

In the remainder of this proof, we therefore assume \( \text{var}(X^2) > 0 \). By Theorem 3.2, we already know that the point processes \( N_n^S \) converge to a Poisson random measure with mean
measure \( \mu(x, \infty) = e^{-x}, \ x > 0 \). Our idea is to transfer the convergence of \( N^S_n \) onto \( N^R_n \). To this end, it suffices to show that (see [24], Theorem 4.2) \( N^R_n - N^S_n \overset{p}{\to} 0 \) as \( n \to \infty \), or equivalently that for any continuous function \( f \) on \( \mathbb{R} \) with compact support,

\[
\int f dN^R_n - \int f dN^S_n \overset{p}{\to} 0, \ n \to \infty.
\]

Suppose the compact support of \( f \) is contained in \( [K + \gamma_0, \infty) \) for some \( \gamma_0 > 0 \) and \( K \in \mathbb{R} \). Since \( f \) is uniformly continuous, \( \omega(\gamma) := \sup\{|f(x) - f(y)| : x, y \in \mathbb{R}, |x - y| \leq \gamma\} \) tends to zero as \( \gamma \to 0 \). We have to show that for any \( \varepsilon > 0 \),

\[
(6.7) \lim_{n \to \infty} P\left( \left| \sum_{1 \leq i < j \leq p} \left( f(\sqrt{n}R_{ij} - \tilde{d}_p) - f((S_{ij}/\sqrt{n} - \tilde{d}_p)\tilde{d}_p) \right) \right| > \varepsilon \right) = 0.
\]

On the sets

\[
(6.8) A_{n,\gamma} = \left\{ \max_{1 \leq i < j \leq p} \tilde{d}_p(\sqrt{n}R_{ij} - S_{ij}/\sqrt{n}) \leq \gamma \right\}, \quad \gamma \in (0, \gamma_0),
\]

we have

\[
|f(\sqrt{n}R_{ij} - \tilde{d}_p) - f((S_{ij}/\sqrt{n} - \tilde{d}_p)\tilde{d}_p)| \leq \omega(\gamma) \mathbb{I}(\{(S_{ij}/\sqrt{n} - \tilde{d}_p)\tilde{d}_p > K\})
\]

Therefore, we see that, for \( \gamma \in (0, \gamma_0) \),

\[
\begin{align*}
&\mathbb{P}\left( \left| \sum_{1 \leq i < j \leq p} \left( f(\sqrt{n}R_{ij} - \tilde{d}_p) - f((S_{ij}/\sqrt{n} - \tilde{d}_p)\tilde{d}_p) \right) \right| > \varepsilon, A_{n,\gamma} \right) \\
&\quad \leq \mathbb{P}(\omega(\gamma)\#\{1 \leq i < j \leq p : (S_{ij}/\sqrt{n} - \tilde{d}_p)\tilde{d}_p > K\} > \varepsilon) \\
&\quad \leq \frac{\omega(\gamma)}{\varepsilon} \mathbb{E}[\#\{1 \leq i < j \leq p : (S_{ij}/\sqrt{n} - \tilde{d}_p)\tilde{d}_p > K\}] \\
&\quad = \frac{\omega(\gamma)}{\varepsilon} \frac{p(p - 1)}{2} \mathbb{P}(\{(S_{12}/\sqrt{n} - \tilde{d}_p)\tilde{d}_p > K\}) \to e^{-K} \text{ by Theorem 2.1}
\end{align*}
\]

Moreover, we have

\[
\mathbb{P}(A_{n,\gamma}^c) = \mathbb{P}\left( \max_{1 \leq i < j \leq p} \tilde{d}_p(\sqrt{n}R_{ij} - S_{ij}/\sqrt{n}) > \gamma \right) \\
= \mathbb{P}\left( \max_{1 \leq i < j \leq p} \tilde{d}_p \left| \frac{S_{ij}}{\sqrt{n}} - \frac{n}{\sqrt{S_{ii}S_{jj}}} - 1 \right| > \gamma \right).
\]

Since \( \max_{1 \leq i < j \leq p}(S_{ij}/\sqrt{n} - \tilde{d}_p)\tilde{d}_p \to \Lambda \), we get that \( \max_{1 \leq i < j \leq p} \tilde{d}_p \left| \frac{S_{ij}}{\sqrt{n}} - \frac{n}{\sqrt{S_{ii}S_{jj}}} - 1 \right| = O_p(\tilde{d}_p^2) \). Thus,

\[
(6.10) \lim_{n \to \infty} \mathbb{P}(A_{n,\gamma}^c) = 0
\]

is implied by

\[
(6.11) \lim_{n \to \infty} \mathbb{P}\left( \tilde{d}_p^2 \max_{1 \leq i < j \leq p} \left| \frac{n}{\sqrt{S_{ii}S_{jj}}} - 1 \right| > \beta \right) = 0, \quad \beta > 0.
\]

Then taking the limits \( n \to \infty \) followed by \( \gamma \to 0^+ \) in (6.9) and (6.10) establishes (6.7).
It remains to prove (6.11). By the law of large numbers, \(|S_{ii}/n| \xrightarrow{a.s.} 1\) as \(n \to \infty\). We have
\[
\max_{1 \leq i < j \leq p} \left| \frac{n}{\sqrt{S_{ii}S_{jj}}} - 1 \right|
= \left( \frac{n}{\min_{1 \leq i < j \leq p} \sqrt{S_{ii}S_{jj}}} - 1 \right) \vee \left( 1 - \frac{n}{\max_{1 \leq i < j \leq p} \sqrt{S_{ii}S_{jj}}} \right)
\leq \max_{1 \leq i \leq p} \left| \frac{n}{S_{ii}} - 1 \right|
\]
so that (6.11) follows from
\[
\lim_{n \to \infty} \mathbb{P} \left( \bar{d}_p^2 \max_{1 \leq i \leq p} \left| \frac{S_{ii}}{n} - 1 \right| > \beta \right) = 0, \quad \beta > 0.
\] We have
\[
\mathbb{P} \left( \bar{d}_p^2 \max_{1 \leq i \leq p} \left| \frac{S_{ii}}{n} - 1 \right| > \beta \right) \leq p\mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (X_{1t}^2 - 1) > \beta \sqrt{n} \right) := \Psi_n.
\]
It remains to prove that \(\Psi_n \to 0\) under each of the conditions (C1), (C2'), (C3').

First, assume (C1). Thus we have \(\mathbb{E}[|X|^s] < \infty\) and \(p = O(n^{(s-2)/4})\) for some \(s > 2\). An application of Markov's inequality yields
\[
\Psi_n \leq c \bar{d}_p^s n^{-(s+2)/4} \mathbb{E}[|S_{11} - n|^{s/2}] .
\] By [11], Lemma A.4, one has
\[
\mathbb{E}[|S_{11} - n|^{s/2}] \leq cn^{\max(1, s/4)}
\]
and therefore it is easy to conclude that \(\Psi_n = O((\log n)^{s/2} n^{-(1/4)\min(s-2,2)}) \to 0\) as \(n \to \infty\).

Next, assume condition (C3'). By [33], Section VIII.4, No. 8, we have for \(0 \leq x \leq n^\alpha/\rho(n)\) with \(0 < \alpha \leq 1/6\) and \(\rho(n) \to \infty\) arbitrarily slowly that
\[
\mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (X_{1t}^2 - 1) > x \right) \sim 2\Phi(x/\sqrt{\text{var}(X^2)}), \quad n \to \infty,
\]
if \(\mathbb{E}[\exp(|X_{11}^2 - 1|^{4\alpha/(2\alpha+1)})] < \infty\). We apply this result with \(\alpha = 1/6\). Then the latter moment requirement reads \(\mathbb{E}[\exp(|X_{11}^2 - 1|^{1/2})] < \infty\) which in view of Lemma 6.3 is implied by (C3'). By definition of \(\bar{d}_p\) and \(p = \exp(o(n^{1/3}))\), we have
\[
\frac{\sqrt{n}}{\bar{d}_p^2} \sim \frac{\sqrt{n}}{4\log p} > \frac{n^{1/6}}{\rho(n)}
\]
for any \(\rho(n) \to \infty\). Using (6.14), applying Mill's ratio and (6.13) yield for a sequence \(\rho(n) \to \infty\) sufficiently slowly that as \(n \to \infty\)
\[
\Psi_n \leq p\mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (X_{1t}^2 - 1) > n^{1/6} \right) \sim 2p\Phi \left( \frac{n^{1/6}}{\rho(n)\sqrt{\text{var}(X^2)}} \right) \to 0.
\]
Finally, assume (C2') and \(p = \exp(o(n^{1/3} \wedge g_n^2))\). We can proceed in the same way as under (C1). By Lemma 6.3, we have \(\mathbb{E}[\exp(g(|X|))] < \infty\). For any \(\rho(n) \to \infty\) we have
\[
\frac{\sqrt{n}}{\bar{d}_p^2} \geq \frac{n^{1/6}}{\rho(n)} \geq \frac{n^{1/6} \wedge g_n'}{\rho(n)}.
\]
An application of [31], Theorem 3, shows that \(\Psi_n \to 0\). The proof is complete.
LEMMA 6.3. Let $Z, Z' \geq 0$ be i.i.d. random variables, $h$ a positive constant and $g$ an increasing function on $(0, \infty)$ such that $\mathbb{E}[\exp(g(hZZ'))] < \infty$. Then we have $\mathbb{E}[\exp(g(Z))] < \infty$.

PROOF. If $Z$ is bounded, the claim is trivial. Otherwise there exists $\alpha > 1/h$ such that $\mathbb{P}(Z \leq \alpha) < 1$. Writing $F$ for the distribution function of $Z$, we have

$$\mathbb{E}[e^{g(Z)}](1 - F(\alpha)) = \int_{\alpha}^{\infty} \mathbb{E}[e^{g(Zt)}] dF(t) \leq \int_{\alpha}^{\infty} \mathbb{E}[e^{g(hZZ')}] dF(t) \leq \mathbb{E}[e^{g(hZZ')}]$$

This implies $\mathbb{E}[e^{g(Z)}] \leq \mathbb{E}[e^{g(hZZ')}] / (1 - F(\alpha)) < \infty$. □

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