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Wick Rotations in Deformation Quantization

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Abstract

We study formal and non-formal deformation quantizations of a family of manifolds that can be obtained by phase space reduction from $\mathbb{C}^{1+n}$ with the Wick star product in arbitrary signature. Two special cases of such manifolds are the complex projective space $\mathbb{CP}^n$ and the complex hyperbolic disc $\mathbb{D}^n$. We generalize several older results to this setting: The construction of formal star products and their explicit description by bidifferential operators, the existence of a convergent subalgebra of “polynomial” functions, and its completion to an algebra of certain analytic functions that allow an easy characterization via their holomorphic extensions. Moreover, we find an isomorphism between the non-formal deformation quantizations for different signatures, linking e.g. the star products on $\mathbb{CP}^n$ and $\mathbb{D}^n$. More precisely, we describe an isomorphism between the (polynomial or analytic) function algebras that is compatible with Poisson brackets and the convergent star products. This isomorphism is essentially given by Wick rotation, i.e. holomorphic extension of analytic functions and restriction to a new domain. It is not compatible with the $^*$-involution of pointwise complex conjugation.

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## 1 Introduction

One way to study the quantization problem arising in physics, which asks how to associate a quantum mechanical system to a classical mechanical one, is formal deformation quantization as introduced in [2]. In this approach, the classical observable algebra is assumed to be the algebra $\mathcal{C}^\infty(M)$ of smooth functions on a Poisson manifold $M$ and one tries to find a so-called formal star product $\star$ that deforms the classical product. More precisely, $\star: \mathcal{C}^\infty(M)[\lambda] \times \mathcal{C}^\infty(M)[\lambda] \to \mathcal{C}^\infty(M)[\lambda]$ is called a formal star product if it is $\mathbb{C}[\lambda]$-bilinear, associative, has the constant 1-function as a unit, and if it can be expanded as $f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g)$ with $\mathbb{C}[\lambda]$-linear extensions of bidifferential operators $C_r: \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ that satisfy that $C_0(f,g) = fg$ is the usual commutative product and that $C_1(f,g) - C_1(g,f) = i\{f,g\}$ is (up to the factor $i$) the Poisson bracket of $f,g \in \mathcal{C}^\infty(M)$. We say that $\star$ deforms in direction of the Poisson bracket $\{\cdot,\cdot\}$. Such a star product is called Hermitian if $\overline{f \star g} = \overline{g} \star \overline{f}$ holds for all $f,g \in \mathcal{C}^\infty(M)$. In a sense, formal deformation quantization transfers the quantization problem to algebra and therefore allows to use powerful algebraic tools in its study. For example, existence and classification results follow from Kontsevich’s formality theorem in the most general case of Poisson manifolds, [16], but were already proven before in the special case of symplectic manifolds by various authors [5, 10, 14, 20] and with the help of different techniques, e.g. the so-called Fedosov construction.
Formal deformation quantizations can also be studied in an equivariant setting. Assume $G$ is a Lie group acting on $M$. Then a star product is called $G$-invariant if all the bidifferential operators $C_r$ are $G$-invariant. For Hamiltonian $G$-actions there is a related notion of $G$-equivariance that considers the quantization of a moment map as well. Existence and classification results are also available in this setting, $[4,11,21]$. Some explicit examples of star products can easily be obtained on $\mathbb{C}^{1+n}$, namely the exponential star products like Moyal / Weyl–Groenewold or Wick star products. There are also some methods to obtain star products on more general spaces, like $\mathbb{C}P^n$ or $\mathbb{D}^n$. $[3,7,8,17]$ use a construction via phase space reduction from one of the aforementioned products on $\mathbb{C}^{1+n}$. Alternatively, one can use Berezin dequantization $[9]$, a Lie algebraic approach $[1]$ or an explicit solution of the recursive equations coming from Fedosov construction, $[18]$. 

The drawback of considering formal power series is that one cannot easily replace the formal parameter $\lambda$ by Planck’s constant $\hbar$, as required in actual physical applications. Therefore strict quantization asks to find a field of well-behaved algebras, usually Fréchet $*$-algebras or $C^*$-algebras, see $[6,19,22]$, that depend nicely on a parameter $\hbar$ ranging over some subset of $\mathbb{C}$, and that reproduce the usual product and Poisson bracket in the zeroth and first order as above for $\hbar \to 0$. Usually, strict quantizations as in $[6,22]$ are constructed by analytical methods, involving oscillatory integrals. If a strict quantization depends smoothly on the parameter $\hbar$, its asymptotic expansion around $\hbar = 0$ yields a formal deformation quantization. Conversely, one can ask to construct strict quantizations that have a given formal deformation quantization as their limit.

Some results in this direction were obtained by Waldmann and collaborators, who try to find some distinguished subalgebra $\mathcal{P}(M)$ of $C^\infty(M)$, on which a star product converges trivially because the formal power series are finite. Such a choice usually comes from some extra structure, for example if $M = T^*Q$ is a cotangent space one can try to use functions that are polynomial in the momenta. One then tries to find some topology with respect to which the star product on $\mathcal{P}(M)$ is continuous, in order to complete $\mathcal{P}(M)$ to a more interesting algebra $\mathcal{A}(M)$, typically consisting of analytic functions. This approach has been worked out e.g. for star products of exponential type on possibly infinite-dimensional vector spaces $[24,26]$, for the Gutt star product on the dual of a Lie algebra $[13]$, for the 2-sphere $[12]$, for the hyperbolic disc $\mathbb{D}^n$ $[3,17]$, and for semisimple coadjoint orbits of semisimple connected Lie groups $[29]$. In the case of the hyperbolic disc the completed algebra $\mathcal{A}$ has a nice geometric interpretation as functions that allow an extension to holomorphic functions on some larger space.

In this article we generalize the approach used in $[17]$ for the hyperbolic disc to obtain formal and non-formal star products on a larger class of certain (pseudo-)Kähler manifolds. These manifolds depend on two parameters, dimension $n$ and signature $s$, and are obtained by using Marsden-Weinstein reduction for the canonical $U(1)$-action on $\mathbb{C}^{1+n}$ endowed with a metric of signature $s$. Focussing on treating all these examples in a uniform way, we construct $U(s,1+n-s)$-invariant, Hermitian formal star products. Using ideas relating to Kähler reduction, we derive an explicit formula in Theorem 5.11:

**Main Theorem I** For any of the reduced (pseudo-)Kähler manifolds $M_{\text{red}}$ described above, the for-
\begin{equation}
f \star_{\text{red}} g = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\lambda^r}{(1-\lambda)(1-2\lambda)\ldots(1-(r-1)\lambda)} \left( (D_{\text{red}}^\text{sym})^r f \otimes (D_{\text{red}}^\text{sym})^r g, H_{\text{red}}^r \right)
\end{equation}

defines a formal star product. Here \( f, g \in C^\infty(M_{\text{red}}) \), \( D_{\text{red}}^\text{sym} \) is the symmetrized covariant derivative associated to the Levi-Civita connection of \( M_{\text{red}} \), and \( H_{\text{red}} \) a certain bivector field on \( M_{\text{red}} \).

This formula was already known in the special case of \( \mathbb{C}P^n \) and \( \mathbb{D}^n \), [18], where it was derived from the Fedosov construction. Our result therefore allows to compare this approach with phase space reduction without appealing to any abstract classification results, and generalizes it to a larger class of manifolds.

It will become clear from the construction that, at least outside of the poles appearing in (1.1), the star product \( \star_{\text{red}} \) converges trivially for a class of functions \( \mathcal{P}(M_{\text{red}}) \) that is obtained by reducing polynomials on \( \mathbb{C}^{1+n} \). All these functions can be (uniquely) extended to holomorphic functions on a larger complex manifold \( \hat{M}_{\text{red}} \) that can be obtained by an analogous reduction procedure from \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \). We define the algebra \( \mathcal{A}(M_{\text{red}}) \) of all functions that can be extended to holomorphic functions on \( \hat{M}_{\text{red}} \), thus obtaining an algebra of certain analytic functions. Using methods from complex analytic geometry, we prove that \( \mathcal{P}(M_{\text{red}}) \) is dense in \( \mathcal{A}(M_{\text{red}}) \) with respect to the topology of locally uniform convergence of the extensions to \( \hat{M}_{\text{red}} \). Then we obtain for all complex \( h \) outside of the poles of (1.1) our Theorem 5.25:

**Main Theorem II** The strict product \( \star_{\text{red},h} \) on \( \mathcal{P}(M_{\text{red}}) \) obtained by replacing the formal parameter \( \lambda \) with \( h \) in (1.1), is continuous with respect to the topology of locally uniform convergence of the holomorphic extensions to \( \hat{M}_{\text{red}} \). It therefore extends uniquely to a continuous product on \( \mathcal{A}(M_{\text{red}}) \).

The geometries of the manifolds \( M_{\text{red}} \) can be quite different (e.g. sometimes compact, sometimes not). However, both the classical and quantum algebras of analytic functions cannot see this difference as we show in Theorem 6.4 and Theorem 6.7 using essentially a generalization of the Wick rotation:

**Main Theorem III** The algebras \( \mathcal{A}(M_{\text{red}}) \) (for the same dimension \( n \) but different signatures \( s \)) with the pointwise product are all isomorphic as unital Fréchet algebras.

**Main Theorem IV** The algebras \( \mathcal{A}(M_{\text{red}}) \) (for the same dimension \( n \) but different signatures \( s \)) with the product \( \star_{\text{red},h} \) and fixed \( h \) are all isomorphic as unital Fréchet algebras.

This can also be proven in a more Lie algebraic context for coadjoint orbits [23]. However, the algebras \( \mathcal{A}(M_{\text{red}}) \) are in general not \( * \)-isomorphic (if \( h \) is real and if one considers the \( * \)-involution of pointwise complex conjugation), which demonstrates the importance of considering \( * \)-algebras in strict deformation quantization. This can be shown by examining positive linear functionals on these \( * \)-algebras, which encode information about their \( * \)-representations on pre-Hilbert spaces.

The article is structured as follows: After discussing some notation in Section 2, we discuss the smooth and complex manifolds occurring at various stages of the construction in Section 3. The classical and quantum phase space reduction allow to construct Poisson brackets and formal star products on a reduced manifold \( M_{\text{red}} \) out of a constant Poisson bracket and the Wick star product
on $\mathbb{C}^{1+n}$. This is achieved essentially by first restricting to the level set $Z$ of a momentum map $J \in C^\infty(\mathbb{C}^{1+n})$ and then dividing out the action of the group $U(1)$ to obtain $M_{\text{red}} \cong Z/U(1)$. Depending on the choice of signature, $M_{\text{red}}$ can e.g. be $\mathbb{CP}^n$ or $\mathbb{D}^n$. In order to be able to construct the spaces of analytic functions on which the non-formal star products can be defined, we introduce complex manifolds $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$, $\hat{Z}$ and $\hat{M}_{\text{red}}$ into which $\mathbb{C}^{1+n}$, $Z$ and $M_{\text{red}}$ can be embedded “anti-diagonally”. The complex structure on $\mathbb{C}^{1+n}$ finally gives rise to a complex structure on $M_{\text{red}}$, which in the special cases of $\mathbb{CP}^n$ and $\mathbb{D}^n$ coincides with the usual one. This also allows to obtain $M_{\text{red}}$ by restricting first to an open subset $\mathbb{C}^{1+n}$ of $\mathbb{C}^{1+n}$ and then dividing out an action of the complexification $\mathbb{C}^* = \{ z \in \mathbb{C} \mid z \neq 0 \}$ of $U(1)$, which simplifies some later considerations.

Section 4 deals with the algebras $C^\infty(\ldots)$, $A(\ldots)$ and $P(\ldots)$ of smooth, certain analytic, and polynomial functions on $\mathbb{C}^{1+n}$, $Z$ and $M_{\text{red}}$. It is also discussed under which conditions and how additional structures given by bidifferential operators on $\mathbb{C}^{1+n}$ can be reduced to $M_{\text{red}}$. This is then applied in Section 5 to the Poisson bracket and Wick star product on $\mathbb{C}^{1+n}$. We obtain the usual Fubini–Study structures as well as explicit formulas for the reduced star products both by means of bidifferential operators and by structure constants.

As the constructions for $\mathbb{CP}^n$, $\mathbb{D}^n$ and the other examples only differ by the choice of certain signs, it is not surprising that they yield closely related results: In Section 4 we construct isomorphisms between various function spaces on the reduced manifolds, which are compatible with both the Poisson brackets and the convergent star products, i.e. with the classical and quantum structures.

Finally, in Appendix A we discuss some details concerning the symmetrized covariant derivatives used for the explicit description of bidifferential operators in Section 5.

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2 Notation and Conventions

There are some conventions that will be used throughout the whole article: We fix two natural numbers $n \in \mathbb{N}$, $s \in \{1, \ldots, 1+n\}$. These will be the complex dimension $n$ of the reduced manifold $M_{\text{red}}$ and the choice of signature $s$. Nearly all objects will depend on this signature, but in order to keep the notation clean this dependence will usually not be made explicit. Only when it is necessary (especially when discussing the Wick rotation in Section 5) the choice of $s$ will be indicated by a superscript in brackets.

For a smooth manifold $M$, we denote by $C^\infty(M)$ the complex-valued smooth functions on $M$ and similarly, $TM$ and $T^*M$ are the complexified tangent and cotangent spaces. The space of smooth sections of a (complex) vector bundle $E \rightarrow M$ over a smooth manifold $M$ is denoted by $\Gamma^\infty(E)$ and is a $C^\infty(M)$-module. Tensor products between such spaces of sections are always tensor products over the ring $C^\infty(M)$. If $M$ is endowed with the action of a group $G$, then $C^\infty(M)^G \subseteq C^\infty(M)$ denotes the $G$-invariant smooth functions on $M$. This notation is also applied to subspaces of $C^\infty(M)$.
The tensor algebra over a vector space $V$ is denoted by $\mathcal{T}^*V$ with $\mathcal{T}^kV$ the linear subspace of homogeneous tensors of degree $k \in \mathbb{N}_0$. The symmetric and antisymmetric tensor algebra are identified with the linear subspaces $S^*V$ and $\Lambda^*V$ of $\mathcal{T}^*V$ consisting of symmetric and antisymmetric tensors, respectively, with symmetric and antisymmetric tensor product $X \vee Y = \text{Sym}^*(X \otimes Y)$ for all $X, Y \in S^*V$ and $X \wedge Y = \text{Asym}^*(X \otimes Y)$ for all $X, Y \in \Lambda^*V$. Here $\text{Sym}^*, \text{Asym}^* : \mathcal{T}^*V \to \mathcal{T}^*V$, the operators of symmetrization and antisymmetrization, are defined as the homogeneous projections onto $S^*V$ and $\Lambda^*V$ fulfilling

$$\text{Sym}^k(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

(2.1)

and

$$\text{Asym}^k(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_\sigma \text{sgn}(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

(2.2)

for $k \in \mathbb{N}_0$ and $v_1, \ldots, v_k \in V$, where the sum is over all permutations $\sigma$ of $\{1, \ldots, k\}$. So especially $v \vee w = \frac{1}{2}(v \otimes w + w \otimes v)$ and $v \wedge w = \frac{1}{2}(v \otimes w - w \otimes v)$ for all $v, w \in V$. Vector bundles and their sections are treated analogously.

By $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{C}$ we denote the dual pairing between a complex vector space $V$ and its algebraic dual $V^*$, $\langle \omega, \alpha \rangle := \omega(\alpha)$ for all $\omega \in V^*$, $\alpha \in V$. This pairing is extended to higher tensor powers by demanding that

$$\langle \omega_1 \otimes \cdots \otimes \omega_k, \alpha_1 \otimes \cdots \otimes \alpha_k \rangle = \langle \omega_1, \alpha_1 \rangle \ldots \langle \omega_k, \alpha_k \rangle$$

(2.3)

for all $k \in \mathbb{N}_0$ and $\omega_1, \ldots, \omega_k \in V^*$, $\alpha_1, \ldots, \alpha_k \in V$. Especially for symmetric tensor products this yields

$$\langle \omega_1 \vee \cdots \vee \omega_k, \alpha_1 \vee \cdots \vee \alpha_k \rangle = \frac{1}{k!} \sum_\sigma \langle \omega_1, \alpha_{\sigma(1)} \rangle \ldots \langle \omega_k, \alpha_{\sigma(k)} \rangle$$

(2.4)

where again the sum is over all permutations $\sigma$ of $\{1, \ldots, k\}$. If $\iota_\beta$ denotes the insertion derivation with a vector $\beta \in V$, i.e. the derivation of degree $-1$ of the symmetric tensor algebra over $V^*$ that fulfills $\iota_\beta \omega = \langle \omega, \beta \rangle$ for all $\omega \in V^*$, then by the above conventions,

$$\frac{1}{k} \langle \iota_\beta(\omega_1 \vee \cdots \vee \omega_k), \alpha_1 \vee \cdots \vee \alpha_{k-1} \rangle = \langle \omega_1 \vee \cdots \vee \omega_k, \beta \vee \alpha_1 \vee \cdots \vee \alpha_{k-1} \rangle$$

(2.5)

holds for all $k \in \mathbb{N}$, $\omega_1, \ldots, \omega_k \in V^*$ and $\alpha_1, \ldots, \alpha_{k-1} \in V$. Like before, vector bundles and their sections are treated analogously.
3 Geometric Background

In this section we will in detail explain the following diagram, that describes the reduction procedures to obtain \( M_{\text{red}} \) and \( \hat{M}_{\text{red}} \).

\[
\begin{array}{ccc}
\mathbb{C}^{1+n} \times \mathbb{C}^{1+n} & \xrightarrow{i} & \hat{Z} \xrightarrow{\hat{pr}} \hat{M}_{\text{red}} \\
\mathbb{C}^{1+n} & \xleftarrow{i} & Z \xrightarrow{pr} M_{\text{red}} \\
\mathbb{C}^{1+n}_+ & \xleftarrow{P_r} \end{array}
\]

(3.1)

Note the similarity to the diagram considered in [17].

Middle Row

The middle row is a typical example of Marsden–Weinstein reduction, even though we will not yet discuss symplectic structures in this section. It consists of (at least) smooth manifolds endowed with an action of the real Lie group \( G_J \), which is defined below, and of \( G_J \)-equivariant smooth maps.

On \( \mathbb{C}^{1+n} \), let \( z^0, \ldots, z^n \) be the standard coordinates, i.e. \( z^k(\rho) = \rho^k \) for all \( k \in \{0, \ldots, n\} \) and \( \rho \in \mathbb{C}^{1+n} \). We define

\[
J := \sum_{k=0}^{n} \nu_k z^k z^k = \sum_{k=0}^{s-1} z^k z^k - \sum_{k=s}^{n} z^k z^k,
\]

(3.2)

where the coefficients \( \nu_k \) are +1 if \( k \in \{0, \ldots, s-1\} \) and -1 if \( k \in \{s, \ldots, n\} \). Note that we drop the dependence of \( J \) and \( \nu_k \) on \( s \) from our notation as explained in the convention at the end of Section 1. The Lie group \( GL(1+n, \mathbb{C}) \) acts from the left on \( \mathbb{C}^{1+n} \) as usual via \( A \triangleright \rho := A \rho \) for all \( A \in GL(1+n, \mathbb{C}) \) and \( \rho \in \mathbb{C}^{1+n} \). This left action \( \cdot \triangleright \cdot \) on \( \mathbb{C}^{1+n} \) induces a right action \( \cdot \triangleleft \cdot \) on smooth functions and tensor fields by pullback. Especially for the coordinate functions, this yields

\[
z^k \triangleleft A = \sum_{\ell=0}^{n} A^{k \ell} z^\ell.
\]

The stabilizer of \( J \), i.e. the set of all \( A \in GL(1+n, \mathbb{C}) \) fulfilling \( J \triangleleft A = J \), is

\[
G_J := U(s, 1+n-s) = \left\{ A \in GL(1+n, \mathbb{C}) \mid \sum_{k=0}^{n} \nu_k A^{k \ell} \overline{A^{k m}} = \delta_{\ell, m} \nu_m \text{ for all } \ell, m \in \{0, \ldots, n\} \right\}
\]

(3.3)

with \( \delta_{\ell, m} \) the usual Kronecker delta. Note that \( G_J \) is a real Lie group and a subgroup of \( GL(1+n, \mathbb{C}) \). Its Lie algebra is

\[
g_J := u(s, 1+n-s) = \left\{ A \in gl_{1+n}(\mathbb{C}) \mid \nu_k A^{k \ell} \overline{A^{k m}} + A^{m \ell} = 0 \text{ for all } \ell, m \in \{0, \ldots, n\} \right\},
\]

(3.4)

which is a real form of \( gl_{1+n}(\mathbb{C}) = \mathbb{C}^{(1+n) \times (1+n)} \).

We define \( Z := J^{-1}(\{1\}) = \left\{ \rho \in \mathbb{C}^{1+n} \mid 1 + \sum_{k=s}^{n} |\rho^k|^2 = \sum_{k=0}^{s-1} |\rho^k|^2 \right\} \), the 1-level set of \( J \),
and \( i : Z \to \mathbb{C}^{1+n} \) as the canonical inclusion. Then the \( G_\mathcal{J} \)-action on \( \mathbb{C}^{1+n} \) restricts to \( Z \) and \( i \) is \( G_\mathcal{J} \)-equivariant.

The second step is to divide out the orbits of the action of the \( U(1) \)-subgroup \( \{ e^{i\phi} \mathbb{1}_{1+n} \mid \phi \in \mathbb{R} \} \) of \( G_\mathcal{J} \), which yields

\[
M_{\text{red}} := Z / U(1). \tag{3.5}
\]

As the \( U(1) \)-subgroup of \( G_\mathcal{J} \) is central, the \( G_\mathcal{J} \)-action remains well-defined on \( M_{\text{red}} \) and the canonical projection \( \text{pr} : Z \to M_{\text{red}} \) is \( G_\mathcal{J} \)-equivariant.

In the special case of the signature \( s = 1 + n \), this construction yields \( M_{\text{red}}^{(1+n)} \cong \mathbb{CP}^n \) with the usual action of \( U(1+n) \) on it. For \( s = 1 \), one obtains the disc \( M_{\text{red}}^{(1)} \cong \mathbb{D}^n = \{ \xi \in \mathbb{C}^n \mid \sum_{i=1}^{n} |\xi_i|^2 < 1 \} \) with the an action of \( U(1,n) \) by M"obius transformations.

We note that, by mapping the \( U(1) \)-equivalence class \([\rho] \in M_{\text{red}} \) of some \( \rho \in Z \) to its \( \mathbb{C}^* \)-equivalence class \([\rho] \in \mathbb{CP}^n \), the real manifold \( M_{\text{red}} \) can be identified with the well-defined open complex submanifold \( \{ [\rho] \in \mathbb{CP}^n \mid h(\rho) > 0 \} \) of \( \mathbb{CP}^n \). Then \( w^1, \ldots, w^n : \{ [\rho] \in M_{\text{red}} \mid \rho^0 \neq 0 \} \to \mathbb{C} \),

\[
w^k ([\rho]) := \frac{\rho^k}{\rho^0} \quad \tag{3.6}
\]

with \( k \in \{1, \ldots, n\} \) define the usual (complex) projective coordinates on \( \{ [\rho] \in M_{\text{red}} \mid \rho^0 \neq 0 \} \subseteq M_{\text{red}} \) and it is easy to obtain an atlas by considering similar coordinates on \( \{ [\rho] \in M_{\text{red}} \mid \rho^\ell \neq 0 \} \) for \( 1 \leq \ell \leq n \). We will later see how the complex structure that \( M_{\text{red}} \) inherits from \( \mathbb{CP}^n \) can also be obtained in a more natural way.

Note that these projective coordinates \( w^1, \ldots, w^n \) describe a chart for \( M_{\text{red}} \) with dense domain of definition. Because of this, it is essentially sufficient to use only these coordinates for the explicit description of some tensors later on, but it is important to keep in mind that they describe \( M_{\text{red}} \) only up to a meagre subset.

**Top Row**

The top row consists of complex manifolds carrying an action of a complex Lie group \( G_\mathcal{J} \), and of \( G_\mathcal{J} \)-equivariant holomorphic maps. These complex manifolds will later be helpful for defining certain algebras of analytic functions on \( \mathbb{C}^{1+n} \) and \( M_{\text{red}} \).

On \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \), the standard complex coordinate functions are denoted by \( x^0, \ldots, x^n, y^0, \ldots, y^n \), and given by \( x^k(\xi, \eta) := \xi^k \) as well as \( y^k(\xi, \eta) := \eta^k \) for all \( k \in \{0, \ldots, n\} \) and \( \xi, \eta \in \mathbb{C}^{1+n} \). Define the holomorphic polynomial

\[
\tilde{\mathcal{J}} := \sum_{k=0}^{n} \nu_k x^k y^k = \sum_{k=0}^{s-1} x^k y^k - \sum_{k=s}^{n} x^k y^k. \tag{3.7}
\]

Note that the polynomial \( \mathcal{J} \) considered before is just the restriction of \( \tilde{\mathcal{J}} \) to the antidiagonal. More precisely, if

\[
\Delta : \mathbb{C}^{1+n} \to \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}, \quad z \mapsto (z, \bar{z}) \tag{3.8}
\]

denotes the embedding along the antidiagonal, then \( \mathcal{J} = \Delta^*(\tilde{\mathcal{J}}) \). Similarly, \( \Delta^*(x^k) = z^k \) and \( \Delta^*(y^k) = \bar{z}^k \) for all \( k \in \{0, \ldots, n\} \).
The complex Lie group $GL(1+n, \mathbb{C}) \times GL(1+n, \mathbb{C})$ acts holomorphically from the left on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ as usual via $(A, B) \triangleright (\xi, \eta) := (A\xi, B\eta)$ for all $A, B \in GL(1+n, \mathbb{C})$ and $\xi, \eta \in \mathbb{C}^{1+n}$, which induces a right action $\cdot \triangleleft *$ by pullback on the spaces of holomorphic functions or holomorphic tensor fields. Especially for the coordinate functions, this yields $x^k \triangleleft (A, B) = \sum_{\ell=0}^n A^k_\ell x^\ell$ and $y^k \triangleleft (A, B) = \sum_{\ell=0}^n B^k_\ell y^\ell$.

The stabilizer $G_\hat{\mathcal{J}}$ of $\hat{\mathcal{J}}$, i.e. the set of $(A, B) \in GL(1+n, \mathbb{C}) \times GL(1+n, \mathbb{C})$ fulfilling $\hat{\mathcal{J}} \triangleleft (A, B) = \hat{\mathcal{J}}$, is explicitly given by

$$G_\hat{\mathcal{J}} = \left\{ (A, B) \in GL(1+n, \mathbb{C}) \times GL(1+n, \mathbb{C}) \mid \sum_{k=0}^n \nu_k A^k_\ell B^k_m = \delta_{\ell, m} \nu_m \text{ for all } \ell, m \in \{0, \ldots, n\} \right\}. \quad (3.9)$$

Note that for all $A \in GL(1+n, \mathbb{C})$ there exists a unique $B \in GL(1+n, \mathbb{C})$ such that $(A, B) \in G_\hat{\mathcal{J}}$, namely $B^k_m = \nu_k \nu_m (A^{-1})^m_k$, so $G_\hat{\mathcal{J}}$ is a complex Lie group and isomorphic to $GL(1+n, \mathbb{C})$.

Similar to the definition of $Z$ we define $\hat{Z}$ as the 1-level set of $\hat{\mathcal{J}}$ in $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$, i.e.

$$\hat{Z} := \hat{\mathcal{J}}^{-1}(\{1\}) = \left\{ (\xi, \eta) \in \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \mid 1 + \sum_{k=0}^n \xi^k \eta^k = \sum_{k=0}^{n-1} \xi^k \eta^k \right\}. \quad (3.10)$$

Then $\hat{Z}$ is a complex submanifold of $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$. The canonical inclusion of $\hat{Z}$ into $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ is denoted by $i$. As $\hat{\mathcal{J}}$ is invariant under the action of $G_\hat{\mathcal{J}}$, this action can be restricted to $\hat{Z}$ and $i$ then is clearly $G_\hat{\mathcal{J}}$-invariant. Moreover the inclusion $\Delta$ restricts to an inclusion $\Delta_Z : Z \to \hat{Z}$, which makes the upper left square in (3.1) commute.

The second step is to divide out the orbits of the Lie group $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, more precisely of the subgroup $\{ (\alpha 1_{1+n}, \alpha^{-1} 1_{1+n}) \mid \alpha \in \mathbb{C}^* \}$ of $G_\hat{\mathcal{J}}$. So define

$$\hat{M}^*_{\text{red}} := \hat{Z} / \mathbb{C}^*, \quad (3.11)$$

then $\hat{M}^*_{\text{red}}$ can be identified with $\{ ([\xi], [\eta]) \in \mathbb{CP}^n \times \mathbb{CP}^n \mid \hat{\mathcal{J}}(\xi, \eta) \neq 0 \}$, a well-defined open and dense complex submanifold of $\mathbb{CP}^n \times \mathbb{CP}^n$, via $\hat{M}^*_{\text{red}} \ni ([\xi], [\eta]) \mapsto ([\xi], [\eta]) \in \mathbb{CP}^n \times \mathbb{CP}^n$. As the $\mathbb{C}^*$-subgroup of $G_\hat{\mathcal{J}}$ is central, the $G_\hat{\mathcal{J}}$-action remains well-defined on $\hat{M}_{\text{red}}$. The canonical projection from $\hat{Z}$ onto the quotient $M_{\text{red}}$ will be denoted by $\hat{\pi}$ and is again $G_\hat{\mathcal{J}}$-equivariant by construction. Finally, one can check that $\Delta_{\text{red}} : M_{\text{red}} \to \hat{M}_{\text{red}},$

$$[\rho] \mapsto \Delta_{\text{red}}([\rho]) := [\Delta_Z(\rho)] = [\rho, \hat{\pi}] \quad (3.12)$$

is well-defined and makes the upper right rectangle of (3.1) commute.

On $\hat{M}_{\text{red}}$, we are going to use the usual projective coordinates coming from $\mathbb{CP}^n \times \mathbb{CP}^n$, denoted by $u^1, \ldots, u^n$: $\{ ([\xi], \eta) \in \hat{M}_{\text{red}} \mid \xi^0 \neq 0 \} \to \mathbb{C}$ and $v^1, \ldots, v^n$: $\{ ([\xi], \eta) \in \hat{M}_{\text{red}} \mid \eta^0 \neq 0 \} \to \mathbb{C}$, and given by

$$u^k([\xi], [\eta]) := \frac{\xi^k}{\xi^0} \quad \text{as well as} \quad v^k([\xi], [\eta]) := \frac{\eta^k}{\eta^0} \quad (3.13)$$

for all $k \in \{1, \ldots, n\}$. Note that it is again easy to obtain an atlas by considering similarly defined coordinates on $\{ ([\xi], \eta) \in \hat{M}_{\text{red}} \mid \xi^i \neq 0 \}$ and $\{ ([\xi], \eta) \in \hat{M}_{\text{red}} \mid \eta^i \neq 0 \}$ and that the relations
\((\Delta_{\text{red}})^* (u^k) = w^k\) and \((\Delta_{\text{red}})^* (v^k) = \overline{w^k}\) hold for all \(k \in \{1, \ldots, n\}\). As before, one should also keep in mind that these coordinates form a chart with dense domain of definition.

**Bottom Node**

It turns out that the complex structure on \(\mathbb{C}^{1+n}\) can be used to simplify the Marsden–Weinstein reduction in the middle row of \(\text{(3.11)}\). First, we define a complex structure on \(M_{\text{red}}\) that is compatible with the complex coordinates defined before. A more general treatment of this procedure can be found in \(\text{[23]}\). Then we find a holomorphic projection map \(\text{Pr} : \mathbb{C}^{1+n}_{+} \rightarrow M_{\text{red}}\) from the open subset

\[
\mathbb{C}^{1+n}_{+} := \{ z \in \mathbb{C}^{1+n} \mid \mathcal{J}(z) > 0 \}
\]

of \(\mathbb{C}^{1+n}\) to \(M_{\text{red}}\) making the bottom right triangle in \(\text{(3.11)}\) commute. Since restricting to an open subset is easy for almost any geometric structure, one can therefore avoid the restriction to a hypersurface that is needed in the Marsden–Weinstein reduction.

For \(A \in \mathfrak{gl}_{1+n}(\mathbb{C})\), let \(X_A\) be the vector field on \(\mathbb{C}^{1+n}\) obtained by differentiating the right action of \(\text{GL}(1+n, \mathbb{C})\) on \(\mathcal{C}^\infty(\mathbb{C}^{1+n})\) in the direction of \(A\), i.e. \(X_A(f) = \frac{d}{dt} |_{t=0} f \circ \exp(tA)\). In particular,

\[
X_i := X_{1i_{1+n}} = \sum_{\ell=0}^{n} \left( i z^\ell \frac{\partial}{\partial z^\ell} - i \overline{z}^\ell \frac{\partial}{\partial \overline{z}^\ell} \right)
\]

is the generator of the (diagonal) \(U(1)\)-symmetry. Let \(\{X_i\}\) be the 1-dimensional vector subbundle of \(T \mathbb{C}^{1+n}_{+}\) spanned by \(X_i|_{\mathbb{C}^{1+n}_{+}}\). Furthermore the differential \(d \mathcal{J}|_{\mathbb{C}^{1+n}_{+}} = \sum_{\ell=0}^{n} \nu_{\ell} (z^\ell \frac{\partial}{\partial z^\ell} + \overline{z}^\ell \frac{\partial}{\partial \overline{z}^\ell})|_{\mathbb{C}^{1+n}_{+}}\) of \(\mathcal{J}\) spans a 1-dimensional real vector subbundle \(\text{d} \mathcal{J}\) of \(T^* \mathbb{C}^{1+n}_{+}\). As \(\mathcal{J}\) is \(U(1)\)-invariant, \(\{X_i\} \subseteq \text{d} \mathcal{J}'\) and \(\text{d} \mathcal{J}' \subseteq \text{d} \mathcal{J}\), where \(\{X_i\}' \subseteq T^* \mathbb{C}^{1+n}_{+}\) and \(\text{d} \mathcal{J}' \subseteq T^* \mathbb{C}^{1+n}_{+}\) denote the annihilators of \(\{X_i\}\) and \(\text{d} \mathcal{J}\). Note that \(\{X_i\}\) and \(\text{d} \mathcal{J}\) are – by this definition – restricted to \(\mathbb{C}^{1+n}_{+}\), even though this is not made explicit by our (simplified) notation.

The standard complex structure \(I\) of \(\mathbb{C}^{1+n}\) allows to obtain a natural complement of the real vector subbundle \(\text{d} \mathcal{J}'\) of \(T \mathbb{C}^{1+n}_{+}\): Consider the vector field \(X_{I\mathbb{C}^{1+n}_{+}} := -IX_i = \sum_{\ell=0}^{n} (z^\ell \frac{\partial}{\partial z^\ell} + \overline{z}^\ell \frac{\partial}{\partial \overline{z}^\ell})\). Note that \(X_{I\mathbb{C}^{1+n}_{+}} = 2 \mathcal{J}\), which implies that \(X_{I\mathbb{C}^{1+n}_{+}} \notin \{\text{d} \mathcal{J}'\}'\) for all \(\rho \in \mathbb{C}^{1+n}_{+}\). Consequently, its linear span \(\{X_{I\mathbb{C}^{1+n}_{+}}\}\) is a complement of \(\{\text{d} \mathcal{J}'\}'\) in \(T^* \mathbb{C}^{1+n}_{+}\) and therefore \(T \mathbb{C}^{1+n}_{+} = \{X_{I\mathbb{C}^{1+n}_{+}}\} + \{\text{d} \mathcal{J}'\}'\), where again \(\{X_{I\mathbb{C}^{1+n}_{+}}\}\) is the 1-dimensional vector subbundle of \(T \mathbb{C}^{1+n}_{+}\) spanned by \(X_{I\mathbb{C}^{1+n}_{+}}\).

Moreover, for \(\rho \in \mathbb{C}^{1+n}_{+}\) define

\[
\Xi_{\rho} := \{ \alpha_{\rho} \in \{\text{d} \mathcal{J}'\}' \mid I \alpha_{\rho} \in \{\text{d} \mathcal{J}'\}' \} \quad \text{and} \quad \Xi := \bigcup_{\rho \in \mathbb{C}^{1+n}_{+}} \Xi_{\rho}.
\]

Then \(\Xi = \{\text{d} \mathcal{J}'\}' \cap I^{-1}(\{\text{d} \mathcal{J}'\}'\) is a linear subbundle of \(\{\text{d} \mathcal{J}'\}'\) that has trivial intersection with \(\{X_{I\mathbb{C}^{1+n}_{+}}\}\), which, by counting dimensions, gives the decomposition \(\{\text{d} \mathcal{J}'\}' = \{X_{I\mathbb{C}^{1+n}_{+}}\} \oplus \Xi\) and therefore:

**Proposition 3.1** The tangent bundle of \(\mathbb{C}^{1+n}_{+}\) can be decomposed as the direct sum

\[
T \mathbb{C}^{1+n}_{+} = \{X_{I\mathbb{C}^{1+n}_{+}}\} \oplus \{X_{I\mathbb{C}^{1+n}_{+}}\} \oplus \Xi.
\]
Moreover, for all \( \rho \in Z \), the map \( T_\rho \text{pr} \circ (T_\rho t)^{-1} : \Xi|_\rho \to T|_\rho M_{\text{red}} \) is a linear isomorphism.

**Proof:** The decomposition of \( T\mathbb{C}^{1+n}_+ \) has already been discussed above. As \( \langle X_\rho |_\rho \rangle \oplus \Xi|_\rho \) coincides with the image of \( T_\rho Z \) under \( T_\rho \), for all \( \rho \in Z \), the map \( T_\rho \text{pr} \circ (T_\rho t)^{-1} \) is well-defined as a map from \( \langle X_\rho |_\rho \rangle \oplus \Xi|_\rho \) to \( T|_\rho M_{\text{red}} \) and is clearly surjective. The kernel of this map is \( \langle X_\rho |_\rho \rangle \), so its restriction to \( \Xi|_\rho \) is an isomorphism. \( \square \)

The U(1)-invariant complex structure \( I \) of \( \mathbb{C}^{1+n}_+ \) thus gives rise to a well-defined (almost) complex structure \( I_{\text{red}} \) on \( M_{\text{red}} \):

**Definition 3.2** Define the vector bundle endomorphism \( I_{\text{red}} : TM_{\text{red}} \to TM_{\text{red}} \), that maps any \( \beta|_\rho \in T|_\rho M_{\text{red}} \) with \( |\rho| \in M_{\text{red}} \) to \( I_{\text{red}}(\beta|_\rho) := (T_\rho \text{pr} \circ (T_\rho t)^{-1} \circ I|_\rho \circ (T_\rho \text{pr} \circ (T_\rho t)^{-1})^{-1})(\beta|_\rho) \).

It is clear that \( I_{\text{red}} \) squares to \( -\text{id}_{TM_{\text{red}}} \) and hence is an almost complex structure. In order to see that it is also integrable, we check that \( I_{\text{red}} \) coincides with the complex structure that \( M_{\text{red}} \) inherits from \( \mathbb{CP}^n \). For a more general discussion, see [25]:

**Definition 3.3** On \( \mathbb{C}^{1+n}_+ \setminus \{ \rho \in \mathbb{C}^{1+n}_+ \mid z^0(\rho) = 0 \} \) we define the vector fields

\[
W_k := z^0 \left( \frac{\partial}{\partial z^k} - \frac{\nu_k z^k}{J} \sum_{j=0}^{n} z^j \frac{\partial}{\partial z^j} \right)_{|\mathbb{C}^{1+n}_+ \setminus \{ \rho \in \mathbb{C}^{1+n}_+ \mid z^0(\rho) = 0 \}}
\]

for all \( k \in \{1, \ldots, n\} \).

Note that, analogously to the projective coordinates \( w^1, \ldots, w^n \) on \( M_{\text{red}} \), the vector fields \( W_1, \ldots, W_n \) are only defined on a dense subset of \( \mathbb{C}^{1+n}_+ \). However, this will be completely sufficient for our purposes.

As \( IW_1 = iW_k \) and \( \langle dJ, W_k \rangle = 0 \) for all \( k \in \{1, \ldots, n\} \) on the domain of definition of \( W_k \), these vector fields \( W_k \), as well as their complex conjugates \( \overline{W_k} \) with \( k \in \{1, \ldots, n\} \), are actually (local, densely defined) sections of \( \Xi \). As one can check that they are pointwise linearly independent and by counting dimensions, they even form a (local, densely defined) frame of \( \Xi \).

**Proposition 3.4** If \( \rho \in Z \), \( z^0(\rho) \neq 0 \), then

\[
(T_\rho \text{pr} \circ (T_\rho t)^{-1})(W_k|_\rho) = \frac{\partial}{\partial w^k}|_{|\rho|} \quad \text{and} \quad (T_\rho \text{pr} \circ (T_\rho t)^{-1})(\overline{W_k}|_\rho) = \frac{\partial}{\partial \overline{w}^k}|_{|\rho|}
\]

for all \( k \in \{1, \ldots, n\} \).

**Proof:** One can check that

\[
\langle dw^k|_\rho, (T_\rho \text{pr} \circ (T_\rho t)^{-1})(W_k|_\rho) \rangle = \langle \epsilon^\ell d(z^\ell/z^0)|_\rho, (T_\rho t)^{-1}(W_k|_\rho) \rangle = \langle d(z^\ell/z^0)|_\rho, W_k|_\rho \rangle,
\]

that \( \langle d(z^\ell/z^0)|_\rho, W_k|_\rho \rangle = \delta_k^\ell \) and \( \langle dw^k|_\rho, (T_\rho \text{pr} \circ (T_\rho t)^{-1})(\overline{W_k}|_\rho) \rangle = 0 \) for all \( k, \ell \in \{1, \ldots, n\} \). \( \square \)

As an immediate consequence we obtain:
Corollary 3.5 The reduced complex structure fulfills $I_{\text{red}}(\frac{\partial}{\partial w^k}) = i \frac{\partial}{\partial u^k}$ and $I_{\text{red}}(\frac{\partial}{\partial w^k}) = -i \frac{\partial}{\partial u^k}$ for all $k \in \{1, \ldots, n\}$, so $I_{\text{red}}$ indeed is just the standard complex structure of $M_{\text{red}}$ interpreted as an open subset of $\mathbb{CP}^n$. In particular $I_{\text{red}}$ is integrable and really a complex structure.

It thus makes sense to talk about holomorphic maps from $\mathbb{C}^{1+n}$ or $\mathbb{C}_+^{1+n}$ to $M_{\text{red}}$:

Lemma 3.6 If a holomorphic complex-valued map $\phi$ from a connected and open subset $S \subseteq \mathbb{C}^{1+n}$ with $S \cap Z \neq \emptyset$ vanishes on $S \cap Z$, then it already vanishes on all of $S$.

Proof: Indeed, as $T_\rho C^{1+n} = \left\langle -IX|_\rho \right\rangle \oplus (T_\rho)(T_\rho Z)$ for all $\rho \in Z$, as $\alpha_\rho(\phi) = 0$ for all $\alpha_\rho \in (T_\rho)(T_\rho Z)$ by assumption and as also $(-IX|_\rho)(\phi) = X|_\rho(-i\phi) = 0$ because $\phi$ is holomorphic, all first order partial derivatives of $\phi$ vanish on $S \cap Z$. This now extends to all arbitrarily high partial derivatives by using the same argument and thus the holomorphic $\phi$ vanishes on whole $S$. \qed

As a consequence, there is at most one holomorphic map $\text{Pr}: \mathbb{C}_+^{1+n} \rightarrow M_{\text{red}}$ whose restriction to $Z$ coincides with $\text{pr}$. In the special case treated here it is not hard to guess this map:

Proposition 3.7 There exists a (necessarily unique) holomorphic map $\text{Pr}: \mathbb{C}_+^{1+n} \rightarrow M_{\text{red}}$ whose restriction to $Z$ coincides with $\text{pr}$. It is explicitly given by

$$\rho \mapsto \text{Pr}(\rho) = [\rho/\sqrt{J(\rho)}].$$

In coordinates, $u^k \circ \text{Pr} = z^k/z^0$.

Proof: It is not hard to check the expression of (3.20) in coordinates, which also shows that $\text{Pr}$ is holomorphic. Its restriction to $Z$ clearly coincides with $\text{pr}$. \qed

We also note that the domain $\mathbb{C}_+^{1+n}$ of $\text{Pr}$, which was chosen rather arbitrarily, is naturally determined from the $U(1)$-action on $\mathbb{C}^{1+n}$ and the complex structure $I$: The action of the corresponding Lie algebra $u(1) \cong \mathbb{R}$ is given by its fundamental vector field $X_1$, and the complex structure $I$ allows to extend this to an action of the complexified Lie algebra $u(1) \otimes \mathbb{C} \cong \mathbb{C}$ via the fundamental vector fields $X_1$ and $X_2$. This action even integrates to a unique holomorphic action of the corresponding complex Lie group $\mathbb{C}^*$ on $\mathbb{C}^{1+n}$, which is just given by multiplication with scalars. The orbit of $Z$ under the action of $\mathbb{C}^*$ is easily seen to be $\mathbb{C}_+^{1+n}$, and $\text{Pr}: \mathbb{C}_+^{1+n} \rightarrow M_{\text{red}}$ is the quotient map that identifies $\mathbb{C}_+^{1+n}/\mathbb{C}^*$ with $M_{\text{red}}$ as complex manifolds. From this point of view, the complex structure on $\mathbb{C}^{1+n}$ allows to replace the two steps of Marsden-Weinstein reduction (restriction to the level set $Z$ and taking $U(1)$-equivalence classes) by restriction to the open complex submanifold $\mathbb{C}_+^{1+n}$ and taking equivalence classes with respect to the action of the complexification $\mathbb{C}^*$ of $U(1)$.

For future use it will be helpful to be able to express the standard coordinate vectors $\frac{\partial}{\partial z^k}$ with $k \in \{0, \ldots, n\}$ in terms of the holomorphic Euler vector field

$$E := -\frac{1}{2}(IX_1+iX_1)|_{\mathbb{C}_+^{1+n}} = \sum_{k=0}^n z^k \frac{\partial}{\partial z^k} |_{\mathbb{C}_+^{1+n}}.$$  (3.21)
and the $W_\ell$, $\ell \in \{1, \ldots, n\}$. On their domain of definition, one gets

$$\frac{\partial}{\partial z^0} = \frac{\pi^0}{\mathcal{J}} E - \sum_{\ell=1}^{n} \frac{z^\ell}{(z^0)^2} W_\ell \quad \text{and} \quad \frac{\partial}{\partial z^k} = \frac{\nu_k z^k}{\mathcal{J}} E + \frac{1}{z^0} W_k = \text{Pr}^*(d\nu^k) \tag{3.22}$$

for all $k \in \{1, \ldots, n\}$ and $(E, W_1, \ldots, W_n)$ is a frame for $T^{(1,0)}(\mathbb{C}^{1+n}_+ \setminus \{ \rho \in \mathbb{C}^{1+n}_+ | z^0(\rho) = 0 \})$. Together with its complex conjugates $(\overline{E}, \overline{W}_1, \ldots, \overline{W}_n)$ we obtain a densely defined frame for the whole tangent space. The dual frames are denoted by $(E^*, W_1^*, \ldots, W_n^*)$ and $(\overline{E}^*, \overline{W}_1^*, \ldots, \overline{W}_n^*)$, and (again only on the domain of definition of the vector fields $W_k$) we have

$$E^* = \frac{\pi^0}{\mathcal{J}} dz^0 + \sum_{k=1}^{n} \nu_k z^k dz^k \quad \text{and} \quad W_k^* = -\frac{z^k}{(z^0)^2} dz^0 + \frac{1}{z^0} dz^k \tag{3.23}$$

$$dz^0 = z^0 E^* - \frac{(z^0)^2}{\mathcal{J}} \sum_{k=1}^{n} \nu_k z^k W_k^* \quad \text{and} \quad dz^k = z^k E^* + z^0 \left( W_k^* - \frac{z^k}{\mathcal{J}} \sum_{\ell=1}^{n} \nu_\ell z^\ell W_\ell^* \right) \tag{3.24}$$

Note that $E$ and $\overline{E}$ are obtained from the symmetry and complex structure of $\mathbb{C}^{1+n}_+$. Similarly, also $E^*$ and $\overline{E}^*$ can be obtained naturally as the $(1,0)$ and $(0,1)$-parts of $d\mathcal{J}/\mathcal{J}$. Only the vector fields $W_1, \ldots, W_n$ as well as their conjugates and duals depend on a choice of coordinates.

## 4 Algebraic Point of View

The general reduction procedure from $\mathbb{C}^{1+n}$ to $M_{\text{red}}$ by first restricting to the level set $Z$ and then dividing out the action of $U(1)$ has a dual version that connects various function algebras on $\mathbb{C}^{1+n}$ and $M_{\text{red}}$: First, one divides out the ideal of functions vanishing on $Z$ and then restricts to $U(1)$-invariant equivalence classes. However, as every $U(1)$-invariant equivalence class of functions also contains at least one $U(1)$-invariant function, which can be obtained by averaging over the compact group $U(1)$, a simplified procedure yields the same results: First, one restricts to $U(1)$-invariant functions and then divides out the ideal of functions vanishing on $Z$. We will use this second approach throughout.

It is well-known that this way one can also construct algebraic structures on $M_{\text{red}}$ out of such structures on $\mathbb{C}^{1+n}$, especially Poisson brackets and star products. In the following we will consider three types of function algebras: All smooth functions, polynomial functions and certain analytic functions. While formal star products are defined on all smooth functions, their non-formal versions can only be defined on polynomial or some analytic functions. All these function algebras on $\mathbb{C}^{1+n}$ will also be endowed with the right-action of the stabilizer group $G_{\mathcal{J}}$.

### 4.1 Smooth Functions

Recall that $\mathcal{C}^\infty(\mathbb{C}^{1+n})$ is the unital subalgebra of $\mathcal{C}^\infty(\mathbb{C}^{1+n})$ whose elements are the $U(1)$-invariant functions. It is easy to see that the following is well-defined:

**Definition 4.1** Let $S$ be an open and $U(1)$-invariant subset of $\mathbb{C}^{1+n}$ such that $S \supseteq Z$. The (classical)
reduction map is \( \cdot \red: \mathcal{C}^\infty(S)^{U(1)} \to \mathcal{C}^\infty(M\red), f \mapsto f\red \), where

\[
\red(f(\rho)) := f(\rho) \tag{4.1}
\]

for all \( \rho \in \mathbb{Z} \).

We will especially be interested in the two cases \( S = \mathbb{C}^{1+n} \) and \( S = \mathbb{C}^{1+n}_+ \). Note that \( f\red \) is the unique smooth function on \( M\red \) that fulfils \( \text{pr}^*(f\red) = \tau^*(f) \). From the algebraic point of view, smooth functions on \( \mathbb{C}^{1+n} \) and \( M\red \) can be related as follows:

**Lemma 4.2** For every \( g \in \mathcal{C}^\infty(M\red) \) there exists an \( f \in \mathcal{C}^\infty(\mathbb{C}^{1+n})^{U(1)} \) such that \( f\red = g \), and \( f \) can even be chosen in such a way that the following locality condition is fulfilled: Whenever \( U \) is an open subset of \( M\red \) such that the restriction of \( g \) to \( U \) vanishes, then there exists an open subset \( V \) of \( \mathbb{C}^{1+n} \) such that \( V \supset \text{pr}^{-1}(U) \) and such that the restriction of \( f \) to \( V \) vanishes.

**Proof:** This is well-known to be true in more generality, but in the present case it is also easy to construct such an \( f \in \mathcal{C}^\infty(\mathbb{C}^{1+n})^{U(1)} \) for every \( g \in \mathcal{C}^\infty(M\red) \): Indeed, one can define \( f(\rho) := 0 \) for all \( \rho \in \mathbb{C}^{1+n} \setminus \mathbb{C}^{1+n}_+ \) and \( f(\rho) := g(\text{Pr}(\rho))\chi(\mathcal{J}(\rho)) \) for all \( \rho \in \mathbb{C}^{1+n}_+ \), where \( \chi : ]0, \infty[ \to [0,1] \) is a smooth function with compact support that fulfils \( \chi(1) = 1 \).

This of course yields analogous results for lifts to open subsets of \( \mathbb{C}^{1+n}_+ \) as well. So we get:

**Proposition 4.3** For every open and \( U(1) \)-invariant subset \( S \subseteq \mathbb{C}^{1+n} \) that contains \( Z \), the reduction map \( \cdot \red: \mathcal{C}^\infty(S)^{U(1)} \to \mathcal{C}^\infty(M\red) \) descends to an isomorphism between the unital *-algebras \( \mathcal{C}^\infty(S)^{U(1)}/\{ v \in \mathcal{C}^\infty(S)^{U(1)} \mid \tau^*(v) = 0 \} \) and \( \mathcal{C}^\infty(M\red) \).

We can now also construct algebraic structures on \( \mathcal{C}^\infty(M\red) \) out of such structures on \( \mathcal{C}^\infty(\mathbb{C}^{1+n}) \) or \( \mathcal{C}^\infty(\mathbb{C}^{1+n}_+) \):

**Proposition 4.4** Let \( S \) be an open and \( U(1) \)-invariant subset of \( \mathbb{C}^{1+n} \) such that \( S \supset Z \), and let \( C: \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(S) \) be a \( U(1) \)-equivariant bilinear map, then the following is equivalent:

- There exists a bilinear map \( C\red: \mathcal{C}^\infty(M\red) \times \mathcal{C}^\infty(M\red) \to \mathcal{C}^\infty(M\red) \) such that
  \[
  (C(f,g))\red = C\red(f\red,g\red)
  \]
  holds for all \( f,g \in \mathcal{C}^\infty(S)^{U(1)} \).

- \( C(f,v)|_Z = 0 = C(v,f)|_Z \) holds for all \( f,v \in \mathcal{C}^\infty(S)^{U(1)} \) with \( \tau^*(v) = 0 \).

If one, hence both of these two conditions are fulfilled, then the bilinear map \( C\red \) from the first point is uniquely determined.

**Proof:** Using the existence of preimages under \( \cdot \red \) from Lemma 4.2, the equivalence of the two points and the uniqueness of \( C\red \) are standard results. \( \square \)
Definition 4.5 Let $S$ be an open and $U(1)$-invariant subset of $\mathbb{C}^{1+n}$ such that $S \supseteq Z$, and let $C : \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(S)$ be a $U(1)$-equivariant bilinear map, then $C$ is called reducible if one, hence both of the equivalent properties from the previous Proposition 4.4 are fulfilled. In this case, we also define the reduced map $C_{\text{red}}$ like in the first point there.

One example is of course the multiplication: Let $C$ be the pointwise multiplication of smooth functions on $\mathbb{C}^{1+n}$, then $C_{\text{red}}$ is the pointwise multiplication of smooth functions on $M_{\text{red}}$. For more interesting examples, however, the second point in Proposition 4.4 can still be hard to check. Luckily, there are some simplifications for bidifferential operators. Note also that in the following it is no loss of generality to consider the special case of a $U(1)$-equivariant bidifferential operator $C : \mathcal{C}^\infty(\mathbb{C}^{1+n}_+) \times \mathcal{C}^\infty(\mathbb{C}^{1+n}_+) \to \mathcal{C}^\infty(\mathbb{C}^{1+n}_+)$: A bidifferential operator on a different domain of definition can always be restricted and extended (in a not necessarily unique way) to a bidifferential operator on $\mathbb{C}^{1+n}_+$ which coincides with the original one in a neighbourhood of $Z$ and thus yields the same reduced map.

Proposition 4.6 Let $C : \mathcal{C}^\infty(\mathbb{C}^{1+n}_+) \times \mathcal{C}^\infty(\mathbb{C}^{1+n}_+) \to \mathcal{C}^\infty(\mathbb{C}^{1+n}_+)$ be a $U(1)$-equivariant bidifferential operator. If $C((J - 1)|_{\mathbb{C}^{1+n}_+} f, f') = 0 = C(f, (J - 1)|_{\mathbb{C}^{1+n}_+} f')$ holds for all $f, f' \in \mathcal{C}^\infty(\mathbb{C}^{1+n}_+)U(1)_1$, then $C$ is reducible and

\[
(C(\Pr^*(g), \Pr^*(g')))_{\text{red}} = C_{\text{red}}(g, g')
\]

holds for all $g, g' \in \mathcal{C}^\infty(M_{\text{red}})$.

Proof: In order to show that $C$ is reducible, let $f, v \in \mathcal{C}^\infty(\mathbb{C}^{1+n}_+)U(1)$ with $\iota^*(v) = 0$ be given. For every $\epsilon \in [0, 1]$ and using a bump function $\chi \in \mathcal{C}^\infty([0, \infty[)$ with support in $[1 - \epsilon, 1 + \epsilon]$ fulfilling $\chi(r) = 1$ for all $r \in [1 - \epsilon/2, 1 + \epsilon/2]$, one can express $v$ as the sum $v = v \cdot (\chi \circ J|_{\mathbb{C}^{1+n}_+}) + (J - 1)|_{\mathbb{C}^{1+n}_+} \tilde{v}$ of a function $v \cdot (\chi \circ J|_{\mathbb{C}^{1+n}_+}) \in \mathcal{C}^\infty(\mathbb{C}^{1+n}_+)U(1)$ with support in $\{ \rho \in \mathbb{C}^{1+n}_+ | - \epsilon \leq J(\rho) - 1 \leq \epsilon \}$ and the product of $(J - 1)|_{\mathbb{C}^{1+n}_+}$ with a function $\tilde{v} \in \mathcal{C}^\infty(\mathbb{C}^{1+n}_+)U(1)$. Then $C(f, v) = C(f, v \cdot (\chi \circ J|_{\mathbb{C}^{1+n}_+}))$ and $C(v, f) = C(v \cdot (\chi \circ J|_{\mathbb{C}^{1+n}_+}), f)$ have support in $\{ \rho \in \mathbb{C}^{1+n}_+ | - \epsilon \leq J(\rho) - 1 \leq \epsilon \}$. As $\epsilon \in [0, 1]$ was arbitrary, even $C(f, f) = 0 = C(f, v)$ holds and $C$ is reducible. For Equation (4.2) we just note that $(\Pr^*(g))_{\text{red}} = g$ for all $g \in \mathcal{C}^\infty(M_{\text{red}})$.

4.2 Polynomial Functions

On polynomial functions it will be possible to construct non-formal star products in Section 5. Here we only discuss the basic definitions and the reduction procedure:

Definition 4.7 We write $\mathcal{P}(\mathbb{C}^{1+n})$ for the unital $^*$-subalgebra of $\mathcal{C}^\infty(\mathbb{C}^{1+n})$ that consists of all (not necessarily holomorphic) polynomial functions. We denote the image of $\mathcal{P}(\mathbb{C}^{1+n})U(1)$ under $\cdot_{\text{red}}$ by $\mathcal{P}(M_{\text{red}})$ and call its elements polynomials on $M_{\text{red}}$.

One can check that $\mathcal{P}(M_{\text{red}})$ is a unital $^*$-subalgebra of $\mathcal{C}^\infty(M_{\text{red}})$ and so the reduction map restricts to a surjective unital $^*$-homomorphism from $\mathcal{P}(\mathbb{C}^{1+n})U(1)$ to $\mathcal{P}(M_{\text{red}})$. Its kernel are all $U(1)$-invariant polynomial functions on $\mathbb{C}^{1+n}$ which vanish on $Z$. So we see that, like in the smooth case, the unital $^*$-algebra $\mathcal{P}(M_{\text{red}})$ is isomorphic to the quotient $\mathcal{P}(\mathbb{C}^{1+n})U(1)/\{ v \in \mathcal{P}(\mathbb{C}^{1+n})U(1) | \iota^*(v) = 0 \}$.  

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A basis of $\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$ yields a generating subset of $\mathcal{P}(M_{\text{red}})$, a subset of which is a basis of $\mathcal{P}(M_{\text{red}})$. We essentially follow [3,17] and just check that the definitions and results there, which were made for the special case $s = 1$, actually work for all signatures:

**Definition 4.8** For every pair of multiindices $P, Q \in \mathbb{N}_0^{1+n}$ we define the monomial on $\mathbb{C}^{1+n}$

$$b_{P,Q} := z^P \mathcal{Z}^Q := (z_0)^{P_0} \cdots (z_n)^{P_n} (\mathcal{Z}_0)^{Q_0} \cdots (\mathcal{Z}_n)^{Q_n} \quad (4.3)$$

where $z^k, \mathcal{Z}^\ell \in \mathcal{P}(\mathbb{C}^{1+n})$ with $k, \ell \in \{0, \ldots, n\}$ are the standard coordinates on $\mathbb{C}^{1+n}$. The linear span in $\mathcal{P}(\mathbb{C}^{1+n})$ of all $b_{P,Q}$ with $|P| + |Q| \leq N$ for fixed $N \in \mathbb{N}_0$ will be denoted by $\mathcal{P}(N)(\mathbb{C}^{1+n})$. Similarly, we write $\mathcal{P}(N)(M_{\text{red}})$ for the image of $\mathcal{P}(2N)(\mathbb{C}^{1+n})^{U(1)}$ under the reduction map $\cdot_{\text{red}}$.

The monomials $b_{P,Q}$ with $P, Q \in \mathbb{N}_0^{1+n}$ are a basis of $\mathcal{P}(\mathbb{C}^{1+n})$, and those monomials with $|P| = |Q|$ are a basis of $\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$. The resulting reduced monomials $b_{P,Q;\text{red}} \in \mathcal{P}(M_{\text{red}})$ are, in the projective coordinates defined in (3.6) (and restricted to the dense domain of definition of these coordinates),

$$b_{P,Q;\text{red}} = \frac{w^{P'} w^Q}{(1 + \sum_{k=1}^n \nu_k w^{k} w^k)^{|P|}} := \frac{(w^1)^{P_1} \cdots (w^n)^{P_n} (\mathcal{Z}^1)^{Q_1} \cdots (\mathcal{Z}^n)^{Q_n}}{(1 + \sum_{k=1}^n \nu_k w^{k} w^k)^{|P|}} \quad (4.4)$$

for all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and with $P' := (P_1, \ldots, P_n) \in \mathbb{N}_0^n$, analogously for $Q$. To check this, note that the pullback with $\text{Pr}$ of the right-hand side coincides with $b_{P,Q;\text{red}} / \mathcal{J}^{|P|}$ on $\mathbb{C}^{1+n}_+$, hence with $b_{P,Q}$ on $Z$. Even though the monomials $b_{P,Q}$ on $\mathbb{C}^{1+n}$ are linearly independent, this does no longer hold for their counterparts $b_{P,Q;\text{red}}$ on $M_{\text{red}}$. Because of this we introduce:

**Definition 4.9** For all multiindices $P, Q \in \mathbb{N}_0^n$ we define the fundamental monomial on $M_{\text{red}}$

$$c_{P,Q} := \begin{cases} b_{(|Q|-|P|, P_1, \ldots, P_n), (0, Q_1, \ldots, Q_n);\text{red}} & \text{if } |P| \leq |Q|, \\ b_{(0, P_1, \ldots, P_n), (|P|-|Q|, Q_1, \ldots, Q_n);\text{red}} & \text{if } |P| \geq |Q|. \end{cases} \quad (4.5)$$

Note that the fundamental monomials on $M_{\text{red}}$—unlike the monomials on $\mathbb{C}^{1+n}$—are determined by $2n$ indices, not $2n + 2$. Using projective coordinates on $M_{\text{red}}$, they can be expressed as

$$c_{P,Q} = \frac{w^{P'} w^Q}{(1 + \sum_{k=1}^n \nu_k w^{k} w^k)\max\{|P|, |Q|\}} \quad (4.6)$$

for all $P, Q \in \mathbb{N}_0^n$. While the usual easy multiplication rules for monomials still hold for the $b_{P,Q;\text{red}}$; i.e. $b_{P,Q;\text{red}} b_{R,S;\text{red}} = b_{P+R,Q+S;\text{red}}$ for all $P, Q, R, S \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and $|R| = |S|$, this is no longer true for the fundamental monomials on $M_{\text{red}}$. Their product can be obtained by rewriting them in terms of the reduced monomials, which can easily be multiplied, and by applying the following:

**Lemma 4.10** For all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$, the identity

$$b_{P,Q;\text{red}} = \sum_{T \in \mathbb{N}_0^n} (-1)^{|T|} \text{sgn}(T) \left(\min\{P_0, Q_0\}\right)^{|T|} T! c_{P+T,Q+T} \quad (4.7)$$

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holds, where \( P' := (P_1, \ldots, P_n) \in \mathbb{N}_0^n \), \( Q' := (Q_1, \ldots, Q_n) \in \mathbb{N}_0^n \) and \( \text{sgn}(T) := \prod_{k=1}^n \nu_k^{T_k} \).

**Proof:** For \( k \in \{0, \ldots, n\} \), let \( E_k := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_0^{1+n} \) be the tuple with 1 at position \( k \). From \( b_{E_0,E_0;\text{red}} = 1 - \sum_{k=1}^n \nu_k b_{E_k,E_k;\text{red}} \) it follows that

\[
(b_{E_0,E_0;\text{red}})^{\min\{P_0,Q_0\}} = \sum_{T \in \mathbb{N}_0^n \atop |T| \leq \min\{P_0,Q_0\}} (-1)^{|T|} \text{sgn}(T) \left( \prod_{i=1}^n \min\{P_i,Q_i\} \right) \frac{|T|!}{T!} c_{T,T}.
\]

Combining this with \( b_{P,Q;\text{red}} = (b_{E_0,E_0;\text{red}})^{\min\{P_0,Q_0\}} c_{P,Q} \) yields the desired result. \( \square \)

Analogous to \( \text{[3, 17]} \), one can show that these fundamental monomials \( c_{P,Q} \) with \( P,Q \in \mathbb{N}_0^n \) are a Hamel basis of \( \mathcal{D}(M_{\text{red}}) \). We will come back to this problem later in Section 6.

### 4.3 Analytic Functions

The polynomial algebras discussed in the previous Subsection 4.2 can be completed to algebras of certain analytic functions. More precisely, we are interested in the pullbacks with \( \Delta : \mathbb{C}^{1+n} \to \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) and \( \Delta_{\text{red}} : M_{\text{red}} \to M_{\text{red}} \) of holomorphic functions:

**Definition 4.11** By \( \mathcal{O}(M) \) we denote the unital complex algebra of holomorphic functions on a complex manifold \( M \). Moreover, we define the following subsets of \( \mathcal{C}^\infty(\mathbb{C}^{1+n}) \) and \( \mathcal{C}^\infty(M_{\text{red}}) \), respectively:

\[
\mathcal{A}(\mathbb{C}^{1+n}) := \left\{ \Delta^*(\hat{f}) \mid \hat{f} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \right\}
\]

and

\[
\mathcal{A}(M_{\text{red}}) := \left\{ \Delta^*_{\text{red}}(\hat{g}) \mid \hat{g} \in \mathcal{O}(\hat{M}_{\text{red}}) \right\}.
\]

It is not hard to check that \( \mathcal{A}(\mathbb{C}^{1+n}) \) and \( \mathcal{A}(M_{\text{red}}) \) are unital *-subalgebras of \( \mathcal{C}^\infty(\mathbb{C}^{1+n}) \) and \( \mathcal{C}^\infty(M_{\text{red}}) \), respectively. Especially for the *-involution one finds: Given \( \hat{f} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \) and \( \hat{g} \in \mathcal{O}(\hat{M}_{\text{red}}) \), then one can define \( \hat{f}' \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \) and \( \hat{g}' \in \mathcal{O}(\hat{M}_{\text{red}}) \) as the functions \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \ni (\xi,\eta) \mapsto \hat{f}'(\xi,\eta) := \hat{f}(\overline{\eta},\overline{\xi}) \in \mathbb{C} \) and \( \mathbb{C}^{1+n} \ni (\xi,\eta) \mapsto \hat{g}'(\xi,\eta) := \hat{g}(\overline{\eta},\overline{\xi}) \in \mathbb{C} \), so that \( \Delta^*(\hat{f}') = \Delta^*\hat{f} \) and \( \Delta^*_{\text{red}}(\hat{g}') = \Delta^*_{\text{red}}\hat{g} \). As algebras, \( \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \) and \( \mathcal{A}(\mathbb{C}^{1+n}) \) as well as \( \mathcal{O}(\hat{M}_{\text{red}}) \) and \( \mathcal{A}(M_{\text{red}}) \) are isomorphic:

**Proposition 4.12** The pullbacks \( \Delta^* : \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \to \mathcal{A}(\mathbb{C}^{1+n}) \) and \( \Delta^*_{\text{red}} : \mathcal{O}(\hat{M}_{\text{red}}) \to \mathcal{A}(M_{\text{red}}) \) are isomorphisms of algebras.

**Proof:** It is easy to check that \( \Delta^* \) and \( \Delta^*_{\text{red}} \) are homomorphisms of algebras, and they are surjective by definition of \( \mathcal{A}(\mathbb{C}^{1+n}) \) and \( \mathcal{A}(M_{\text{red}}) \), so only injectivity remains: Given \( \hat{f} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \) with \( \Delta^*(\hat{f}) = 0 \) or \( \hat{g} \in \mathcal{O}(\hat{M}_{\text{red}}) \) with \( \Delta^*_{\text{red}}(\hat{g}) = 0 \), then, in the coordinates introduced in Section 3

\[
\frac{\partial \hat{f}}{\partial x} \bigg|_{(\rho,\overline{\rho})} = (T_\rho \Delta) \left( \frac{\partial}{\partial z^i} \right)_\rho (\hat{f}) = \frac{\partial}{\partial z^i} \bigg|_{\rho} \Delta^*(\hat{f}) = 0 \quad \text{and} \quad \frac{\partial \hat{g}}{\partial y} \bigg|_{(\rho,\overline{\rho})} = \frac{\partial}{\partial \overline{z}^i} \bigg|_{\rho} \Delta^*_{\text{red}}(\hat{g}) = 0.
\]
or
\[
\frac{\partial \hat{g}}{\partial w_j} \bigg|_{[\rho,\pi]} = (T_{[\rho]} \Delta_{\text{red}} \bigg( \frac{\partial}{\partial w_j} \bigg|_{[\rho]} \bigg) (\hat{g}) = \frac{\partial}{\partial w_j} \bigg|_{[\rho]} \Delta_{\text{red}}^*(\hat{g}) = 0 \quad \text{and} \quad \frac{\partial \hat{g}}{\partial w_j} \bigg|_{[\rho,\pi]} = \frac{\partial}{\partial w_j} \bigg|_{[\rho]} \Delta_{\text{red}}^*(\hat{g}) = 0
\]
hold for all \( \rho \in \mathbb{Z} \) with \( z^0(\rho) \neq 0 \) and all \( i \in \{0,\ldots,n\}, \ j \in \{1,\ldots,n\} \), respectively. By iteration of this argument one finds that also all higher derivatives of \( \hat{f} \) or \( \hat{g} \) vanish, so that \( \hat{f} = 0 \) or \( \hat{g} = 0 \), respectively.

\[\Box\]

It is well-known that the holomorphic functions \( \mathcal{O}(M) \) on a complex manifold \( M \) with the pointwise operations become a Fréchet algebra with the topology of locally uniform convergence (i.e. \( \mathcal{O}(M) \) is complete and the multiplication continuous with respect to this metrizable locally convex topology). This locally convex topology can be described by all the submultiplicative seminorms \( ||\cdot||_K: \mathcal{O}(M) \to [0,\infty]\),

\[\hat{f} \mapsto ||\hat{f}||_K := \max_{z \in K} |\hat{f}(z)| \quad (4.10)\]

with \( K \) a compact subset of \( M \). From this we see immediately that \( \mathcal{A}(\mathbb{C}^{1+n}) \) and \( \mathcal{A}(\hat{M}_{\text{red}}) \) with the topology coming from \( \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \) and \( \hat{\mathcal{M}}_{\text{red}} \), respectively, are Fréchet *-algebras (Fréchet algebras endowed with a continuous *-involution). It is a consequence of the Cauchy integral formula on \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) that every \( f \in \mathcal{A}(\mathbb{C}^{1+n}) \) can be expressed in a unique way as an absolutely convergent series

\[ f = \sum_{P,Q \in \mathbb{N}_0^{1+n}} f_{P,Q} b_{P,Q} \quad (4.11) \]

with complex coefficients \( f_{P,Q} \) fulfilling

\[ ||f||_r := \sum_{P,Q \in \mathbb{N}_0^{1+n}} |f_{P,Q}| r^{|P|+|Q|} < \infty \quad (4.12) \]

for all \( r \in [1,\infty) \), and that the topology of \( \mathcal{A}(\mathbb{C}^{1+n}) \) can equivalently be described by these seminorms \( ||\cdot||_r: \mathcal{A}(\mathbb{C}^{1+n}) \to [0,\infty] \). See e.g. [23, Proposition 3.5] for details. We will later in Proposition 6.11 obtain an analogous result also for \( \mathcal{A}(\hat{M}_{\text{red}}) \). Like for polynomials one also finds that the \( U(1) \)-invariant analytic functions \( f \) are precisely those which fulfil \( f_{P,Q} = 0 \) for all \( P,Q \in \mathbb{N}_0^{1+n} \) with \( |P| \neq |Q| \), e.g. by explicitly calculating the coefficients with the help of the Cauchy integral formula. Note that due to the completeness of \( \mathcal{A}(\mathbb{C}^{1+n}) \), averaging over the \( U(1) \)-action on \( \mathcal{A}(\mathbb{C}^{1+n}) \) is possible and yields for every \( f \in \mathcal{A}(\mathbb{C}^{1+n}) \) an \( f_{av} \in \mathcal{A}(\mathbb{C}^{1+n})^{U(1)} \).

We observe that the reduction map \( \cdot \rightarrow \text{red} \) can be defined analogously as before also for holomorphic functions:

**Lemma 4.13** Let \( \hat{f} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \) be \( \mathbb{C}^* \)-invariant in the sense that \( \hat{f} \circ (\alpha \mathbb{1}_{1+n}, \alpha^{-1} \mathbb{1}_{1+n}) = \hat{f} \) holds for all \( \alpha \in \mathbb{C}^* \), then there exists a unique \( \hat{f}_{\text{red}} \in \mathcal{O}(\hat{M}_{\text{red}}) \) for which

\[ \hat{r}^* (\hat{f}) = \hat{p} \hat{r}^* (\hat{f}_{\text{red}}) \]
holds.

**Proof:** As \( i^\ast(\hat{f}) \) is \( \mathbb{C}^\ast \)-invariant, it descends to a well-defined function \( \hat{f}_{\text{red}} \) on \( \hat{M}_{\text{red}} = \hat{Z}/\mathbb{C}^\ast \), which is automatically holomorphic. \( \square \)

**Proposition 4.14** The reduction map \( \cdot_{\text{red}} \) restricts to a map from \( \mathcal{A}(\mathbb{C}^{1+n})^{U(1)} \) to \( \mathcal{A}(M_{\text{red}}) \). More precisely, given \( f \in \mathcal{A}(\mathbb{C}^{1+n})^{U(1)} \) and \( \hat{f} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \) such that \( \Delta^\ast(\hat{f}) = f \), then \( \hat{f} \) is \( \mathbb{C}^\ast \)-invariant in the sense of the previous Lemma 4.13 and \( f_{\text{red}} = \Delta_{\text{red}}^\ast(\hat{f}_{\text{red}}) \in \mathcal{A}(M_{\text{red}}) \).

**Proof:** Given such \( f \) and \( \hat{f} \), then

\[
\Delta^\ast(\hat{f} \triangleleft (e^{i\phi}1_{1+n}, e^{-i\phi}1_{1+n})) = \Delta^\ast(\hat{f}) \triangleleft e^{i\phi}1_{1+n} = f \triangleleft e^{i\phi}1_{1+n} = f = \Delta^\ast(\hat{f})
\]

holds for all \( \phi \in \mathbb{R} \). But since the action of the complex Lie group \( \mathbb{C}^\ast \) on \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) is holomorphic, this shows not only that \( \hat{f} \) is \( U(1) \)-invariant, but even \( \mathbb{C}^\ast \)-invariance. Using the commutativity of the diagram in Section 3 one can now check that

\[
\text{pr}^\ast(\Delta^\ast_{\text{red}}(\hat{f}_{\text{red}})) = \Delta^\ast_{\hat{Z}}(\text{pr}^\ast(\hat{f}_{\text{red}})) = \Delta^\ast_{\hat{Z}}(i^\ast(\hat{f})) = i^\ast(\Delta^\ast(\hat{f})) = i^\ast(f)
\]

holds, hence \( f_{\text{red}} = \Delta^\ast_{\text{red}}(\hat{f}_{\text{red}}) \in \mathcal{A}(M_{\text{red}}) \). \( \square \)

Using some deep results from complex analysis, the analytic functions on \( M_{\text{red}} \) and on \( \mathbb{C}^{1+n} \) can be related in the same way as smooth or polynomial functions:

**Lemma 4.15** For every \( g \in \mathcal{A}(M_{\text{red}}) \) there exists an \( f \in \mathcal{A}(\mathbb{C}^{1+n})^{U(1)} \) such that \( f_{\text{red}} = g \).

**Proof:** Given \( g \in \mathcal{A}(M_{\text{red}}) \) and corresponding \( \hat{g} \in \mathcal{O}(\hat{M}_{\text{red}}) \) such that \( \Delta^\ast_{\text{red}}(\hat{g}) = g \), then \( \text{pr}^\ast(\hat{g}) \) is a holomorphic function on \( \hat{Z} \). Now note that \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) is a Stein manifold by [15] Section 5.1 and that \( \hat{Z} \) is – in the language of [15] Definition 6.5.1 – an analytic submanifold thereof because it is the set of zeros of a holomorphic function on \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \). So [15] Theorem 7.4.8 applies and shows that there exists an extension \( \hat{f} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \) of \( \text{pr}^\ast(\hat{g}) \), i.e. \( i^\ast(\hat{f}) = \text{pr}^\ast(\hat{g}) \). Therefore \( f := \Delta^\ast(\hat{f}) \) fulfills \( i^\ast(f) = \text{pr}^\ast(g) \). By averaging over the \( U(1) \)-action on \( \mathcal{A}(\mathbb{C}^{1+n}) \) we can even arrange that \( f \) is \( U(1) \)-invariant. \( \square \)

For an alternative proof one can also generalize the more constructive results obtained in [17] Sec. 3.2 for the case of signature \( s = 1 \), or use these results and the Wick rotation as discussed later in Section 6.

Clearly, \( \{ f \in \mathcal{A}(\mathbb{C}^{1+n})^{U(1)} \mid i^\ast(f) = 0 \} \) is the kernel of \( \cdot_{\text{red}} \) restricted to \( \mathcal{A}(\mathbb{C}^{1+n}) \) and therefore a closed \( * \)-ideal of \( \mathcal{A}(\mathbb{C}^{1+n})^{U(1)} \). Similarly to the case of smooth or polynomial functions we get:

**Proposition 4.16** The reduction map \( \cdot_{\text{red}} \) induces a homeomorphic isomorphism between the Fréchet \( * \)-algebras \( \mathcal{A}(\mathbb{C}^{1+n})^{U(1)}/\{ f \in \mathcal{A}(\mathbb{C}^{1+n}) \mid i^\ast(f) = 0 \} \) and \( \mathcal{A}(M_{\text{red}}) \).

**Proof:** Using Lemma 4.13 it is clear that that \( \cdot_{\text{red}} \) induces an isomorphism. As \( ||\hat{f}_{\text{red}}||_K = ||\hat{f}||_{B^\ast \text{pr}^\ast(K)} \) holds for every \( \hat{f} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})^C \) with \( B \subseteq \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) a sufficiently large closed ball, the map \( \cdot_{\text{red}} : \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})^C \to \mathcal{O}(\hat{M}_{\text{red}}) \) from Lemma 4.13 is continuous with respect to the topologies of locally uniform convergence, thus \( \cdot_{\text{red}} : \mathcal{A}(\mathbb{C}^{1+n})^{U(1)} \to \mathcal{A}(M_{\text{red}}) \) is continuous as well. It follows from the open mapping theorem that it is a homeomorphism. \( \square \)
As the U(1)-invariant polynomials \( \mathcal{P}(\mathbb{C}^{1+n})^{U(1)} \) are dense in \( \mathcal{A}(\mathbb{C}^{1+n})^{U(1)} \), this immediately yields:

**Corollary 4.17** The polynomials \( \mathcal{P}(M_{\text{red}}) \) are dense in \( \mathcal{A}(M_{\text{red}}) \).

5 Poisson Brackets and Star Products

In this section we introduce a Poisson bracket and star product on \( \mathbb{C}^{1+n} \) and discuss their reduction to \( M_{\text{red}} \). First we consider formal star products, which make sense for formal power series of smooth functions. We present a method for reducing the (pseudo-)Wick product on \( \mathbb{C}^{1+n} \) to \( M_{\text{red}} \) in Subsection 5.1 and derive more explicit formulas in Subsection 5.2. The other two sections deal with strict star products. In order to make the formal power series convergent, we restrict ourselves to polynomials in Subsection 5.3 and extend these results to analytic functions in Subsection 5.4.

5.1 The Smooth Case

We will now introduce the Wick star product on \( \mathbb{C}^{1+n} \). The antisymmetrization of its first order gives rise to a Poisson structure on \( \mathbb{C}^{1+n} \). Let \( \nabla \) be the Euclidean covariant derivative of \( \mathbb{C}^{1+n} \), \( D \) its exterior covariant derivative and \( D_{\text{sym}} \) the corresponding symmetrized covariant derivative, see Appendix A. We define

\[
H := \sum_{k=0}^{n} \nu_k \frac{\partial}{\partial z^k} \otimes \frac{\partial}{\partial z^k} \in \Gamma^\infty (T^{(0,1)} \mathbb{C}^{1+n} \otimes T^{(1,0)} \mathbb{C}^{1+n}).
\]  

(5.1)

It is easy to see that \( H \) is U(1)-invariant, so that \( H_{\text{red}} \in \Gamma^\infty (T^{(0,1)} M_{\text{red}} \otimes T^{(1,0)} M_{\text{red}}) \) can be defined as

\[
H_{\text{red}}|_{[\rho]} := (T_{\rho} \Pr)^{\otimes 2} (H|_{[\rho]})
\]

(5.2)

for all \([\rho] \in M_{\text{red}} \) with representative \( \rho \in Z \). An explicit formula for \( H_{\text{red}} \) in projective coordinates will be given later in Lemma 5.7. Using \( H \) and symmetrized covariant derivatives, we can now define the well-known Wick star product:

**Definition 5.1** The product \(* : \mathcal{C}^\infty(\mathbb{C}^{1+n})[[\lambda]] \times \mathcal{C}^\infty(\mathbb{C}^{1+n})[[\lambda]] \to \mathcal{C}^\infty(\mathbb{C}^{1+n})[[\lambda]], \)

\[
(f, g) \mapsto f \ast g := \sum_{r=0}^{\infty} \frac{\mathcal{N}^r}{r!} \left\langle (D_{\text{sym}})^r(f) \otimes (D_{\text{sym}})^r(g), H^r \right\rangle
\]

(5.3)

is the (pseudo-)Wick star product on \( \mathbb{C}^{1+n} \) for the (pseudo-)metric \( g := \sum_{k=0}^{n} \nu_k dz^k \wedge dz^k \). Here \( H^r \) denotes the \( r \)-th power of \( H \) as an element of degree \( (1,1) \) in the algebra \( \mathcal{S}^\bullet(\mathbb{C}^{1+n}) \otimes \mathcal{S}^\bullet(\mathbb{C}^{1+n}) \) with \( \mathcal{S}^\bullet(\mathbb{C}^{1+n}) := \bigoplus_{k=0}^{\infty} \Gamma^\infty (S^k T \mathbb{C}^{1+n}) \) the usual algebra of symmetric multivector fields.

Note that one can check that \( \ast \) is actually an \( G_J \)-invariant Hermitian formal star product constructed out of the bidifferential operators

\[
C_r(f, g) = \frac{1}{r!} \left\langle (D_{\text{sym}})^r(f) \otimes (D_{\text{sym}})^r(g), H^r \right\rangle.
\]

(5.4)
It deforms in direction of the standard Poisson bracket with signature $s$

$$\frac{1}{i} (C_1(f, g) - C_1(g, f)) = \frac{1}{i} \sum_{k=0}^{n} \nu_k \left( \frac{\partial f}{\partial z^k} \frac{\partial g}{\partial z^k} - \frac{\partial g}{\partial z^k} \frac{\partial f}{\partial z^k} \right) = \{ f, g \} \quad (5.5)$$
on $\mathbb{C}^{1+n}$ with Poisson tensor

$$\pi = -2i \sum_{k=0}^{n} \nu_k \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^k} = 2 \text{Im}(H), \quad (5.6)$$

where, as usual, $\{ f, g \} = \langle df \otimes dg, \pi \rangle = \langle D^{\text{sym}}f \otimes D^{\text{sym}}g, \pi \rangle$. Note that (5.6) implies that $\pi$ is a real tensor.

**Lemma 5.2** The Poisson bracket (5.5) fulfills the equivalent conditions of Proposition 4.6.

**Proof:** First, $\{ \cdot, \cdot \}$ is bidifferential, hence can be restricted to $\mathbb{C}^{1+n}_+$. One can check that for all $\phi \in \mathbb{R}$ the identity $\{ f \circ e^{i\phi}, g \circ e^{i\phi} \} = \{ f, g \} \circ e^{i\phi}$ holds, so that $\{ \cdot, \cdot \}$ is U(1)-invariant. Second, one finds that $\{ f, \mathcal{J} \} = X_i(f)$ for all $f \in \mathcal{C}^{\infty}(\mathbb{C}^{1+n})$, with $X_i$ the generator of the U(1)-symmetry as before. So if $f, g, \mathcal{J} \in \mathcal{C}^{\infty}(\mathbb{C}^{1+n})^{U(1)}$ are U(1)-invariant, then $\{ f, \mathcal{J} \} = \{ f, g \}(\mathcal{J} - 1) - f X_i(g) = \{ f, g \}(\mathcal{J} - 1)$ vanishes on $Z$, and similarly $\{ f, g(\mathcal{J} - 1) \}_Z \equiv 0$ as well. □

Thus we can construct a reduced Poisson bracket on $M_{\text{red}}$ by application of Definition 4.5 and get:

**Proposition 5.3** The reduced Poisson bracket $\{ \cdot, \cdot \}_{\text{red}} : \mathcal{C}^{\infty}(M_{\text{red}}) \times \mathcal{C}^{\infty}(M_{\text{red}}) \to \mathcal{C}^{\infty}(M_{\text{red}})$ is given for all $f, g \in \mathcal{C}^{\infty}(M_{\text{red}})$ and $\rho \in Z$ by

$$\{ f, g \}_{\text{red}}([\rho]) = \{ \text{Pr}^*(f), \text{Pr}^*(g) \}(\rho) = \langle df \otimes dg, (\mathcal{T}_\rho \text{Pr})^{\otimes 2} \pi |_{\rho} \rangle \quad (5.7)$$

and the corresponding Poisson tensor $\pi_{\text{red}}$ on $M_{\text{red}}$ is simply

$$\pi_{\text{red}} = 2 \text{Im}(H_{\text{red}}) \quad (5.8)$$

**Proof:** Equation (5.7) is clear and (5.8) then follows from (5.2) and (5.6). □

However, the situation is a bit more difficult if one tries to reduce the bidifferential operators $C_r$ defining the Wick star product. One immediately sees that Proposition 4.4 cannot be applied directly: For example, $C_1(\mathcal{J}, \mathcal{J}) = \mathcal{J} \neq 0$. Following [1], this problem can be overcome by restricting to $\mathbb{C}^{1+n}_+$ and performing an equivalence transformation $S = \text{id} + \sum_{k=1}^{\infty} \lambda^k S_k$, with differential operators $S_k : \mathcal{C}^{\infty}(\mathbb{C}^{1+n}_+) \to \mathcal{C}^{\infty}(\mathbb{C}^{1+n}_+)$ that vanish on constant functions, from $\star$ to a suitable new star product $\hat{\star}$, i.e. $f \hat{\star} f' := S(S^{-1}(f) \star S^{-1}(f'))$, in such a way that $\hat{\star}$ is reducible to a star product $\star_{\text{red}}$ on $M_{\text{red}}$ by application of Proposition 4.6. If this can be achieved, then $\text{pr}^*(g \star_{\text{red}} g') = (\text{Pr}^*(g) \hat{\star} \text{Pr}^*(g'))|_Z$ for all $g, g' \in \mathcal{C}^{\infty}(M_{\text{red}})$. For this we require the following:

i.) $S$ should commute with $\pi$, since then $\hat{\star}$ is again a Hermitian star product.

ii.) $S$ should be $G_\mathcal{J}$-equivariant, since then $\hat{\star}$ is again $G_\mathcal{J}$-equivariant.
iii.) Moreover, \( \hat{*} \) should fulfil \( J \hat{*} f = J f \) for all \( f \in C^\infty(\mathbb{C}^{1+n})^{U(1)} \), hence also \( f \hat{*} J = \overline{J f} = \overline{\overline{J f}} = J f \) for all \( f \in C^\infty(\mathbb{C}^{1+n})^{U(1)} \). As a consequence, Proposition 4.6 can be applied to the bidifferential operator defining the \( r \)-th order of \( \hat{*} \) for any \( r \), so that \( \hat{\star}_\text{red} \) as described above is indeed well-defined.

iv.) Finally, it would be helpful if \( S \) (hence also \( S^{-1} \)) acts as the identity on \( \mathbb{C}^* \)-invariant functions, because this has the consequence that the formula for \( \hat{\star}_\text{red} \) simplifies to

\[
\text{pr}^s(g \hat{\star}_\text{red} g') = \left( \text{pr}^s(g) \hat{\star} \text{pr}^s(g') \right) |_Z = \left( S(\text{pr}^s(g) \ast \text{pr}^s(g')) \right) |_Z \quad (5.9)
\]

for all \( g, g' \in C^\infty(M_\text{red}) \).

Let us define the rescaled vector field

\[
\frac{\partial}{\partial J} := \frac{1}{2J} X_1 \in \Gamma^\infty(T\mathbb{C}_+^{1+n}) \quad (5.10)
\]

on \( \mathbb{C}_+^{1+n} \), which satisfies \( \frac{\partial}{\partial J} J = 1 \). Then Properties (i), (ii) and (iii) are fulfilled if all the differential operators \( S_k \) with \( k \in \mathbb{N} \) are of the form

\[
S_k = \sum_{\ell=1}^\infty (S_{k,\ell} \circ J) \left( \frac{\partial}{\partial J} \right)^\ell \quad \text{with smooth functions } S_{k,\ell} : \mathbb{R} \to \mathbb{R} \text{ such that for every fixed } k \in \mathbb{N} \text{ there are only finitely many } \ell \in \mathbb{N} \text{ for which } S_{k,\ell} \neq 0.
\]

We are interested in the inverse equivalence transformation \( T = S^{-1} \), which should only contain derivatives \( \frac{\partial}{\partial J} \) and coefficient functions dependent on \( J \), i.e. \( T = \text{id} + \sum_{k=1,\ell=1}^\infty \lambda^\ell (T_{k,\ell} \circ J) \left( \frac{\partial}{\partial J} \right)^\ell \).

**Proposition 5.4** Let \( T \) be a \( U(1) \)-equivariant equivalence transformation on \( \mathbb{C}_+^{1+n} \) from a new star product \( \hat{\star} \) to \( \ast \), then the following is equivalent:

- \( T(J) = J \) and \( J \hat{\star} f = J f \) for all \( f \in C^\infty(\mathbb{C}_+^{1+n})^{U(1)} \).
- \( [T, J](f) = \lambda J \frac{\partial}{\partial J} T(f) \) for all \( f \in C^\infty(\mathbb{C}_+^{1+n})^{U(1)} \).

If \( T \) fulfils one, hence both of these conditions, then

\[
T\left( \lambda^r (J / \lambda)_{\lambda, r} \right) = J^r \quad \text{and} \quad T\left( J^{\lambda+1} (J / \lambda)_{\lambda+1, \lambda+1} \right) = J^{-r} \quad (5.11)
\]

for all \( r \in \mathbb{N}_0 \), where \( (\xi)_\lambda := \xi(\xi - 1) \ldots (\xi - (r - 1)) \) is the falling factorial, and similarly \( (\xi)_{\lambda+1} := \xi(\xi + 1) \ldots (\xi + (r - 1)) \) is the rising factorial.

**Proof:** Assume \( J \hat{\star} f = J f \) and \( T(J) = J \), then

\[
T(J f) = T(J \hat{\star} f) = T(J) \ast T(f) = J \ast T(f) = \left( J + \lambda J \frac{\partial}{\partial J} \right) T(f)
\]

and so \( [T, J](f) = \lambda J \frac{\partial}{\partial J} T(f) \). Conversely, if \( [T, J](f) = \lambda J \frac{\partial}{\partial J} T(f) \) for all \( f \in C^\infty(\mathbb{C}_+^{1+n})^{U(1)} \), then especially for \( f = 1 \) one gets \( T(J) - JT(1) = \lambda J \frac{\partial}{\partial J} T(1) \), i.e. \( T(J) - J = 0 \) because \( T(1) = 1 \) for the equivalence transformation \( T \). Then one also checks that

\[
J \hat{\star} f = T^{-1} \left( \left( J + \lambda J \frac{\partial}{\partial J} \right) T(f) \right) = T^{-1}(JT(f) + [T, J](f)) = J f.
\]
Moreover, by induction one finds that indeed \( T(\lambda'({\mathcal J}/\lambda)_{\downarrow,r}) = \mathcal{J}^r \) for all \( r \in \mathbb{N}_0 \): For \( r = 0 \) this is just \( T(1) = 1 \), and if it holds for one \( r \in \mathbb{N}_0 \), then

\[
T(\lambda'^{r+1}({\mathcal J}/\lambda)_{\downarrow,r+1}) = \lambda T(\lambda'({\mathcal J}/\lambda)_{\downarrow,r+1}({\mathcal J}/\lambda - r)) \\
= [T, \mathcal{J}](\lambda'({\mathcal J}/\lambda)_{\downarrow,r}) + J T(\lambda'({\mathcal J}/\lambda)_{\downarrow,r}) - \lambda r T(\lambda'({\mathcal J}/\lambda)_{\downarrow,r}) \\
= (\lambda \mathcal{J} \frac{\partial}{\partial \mathcal{J}} + \mathcal{J} - \lambda r)T(\lambda'({\mathcal J}/\lambda)_{\downarrow,r}) \\
= (\lambda \mathcal{J} \frac{\partial}{\partial \mathcal{J}} + \mathcal{J} - \lambda r)\mathcal{J}^r \\
= \mathcal{J}^{r+1}.
\]

To check the formula for the rising factorial, we note first that

\[
(\mathcal{J} + r \lambda)T(\mathcal{J}(\lambda'^{r+1}(\mathcal{J}/\lambda)_{\downarrow,r+1})^{-1}) = [\mathcal{J} + r \lambda, T](\lambda'^{r+1}(\mathcal{J}/\lambda)_{\downarrow,r+1})^{-1}) + T(\mathcal{J}(\lambda'^{r+1}(\mathcal{J}/\lambda)_{\downarrow,r})^{-1}) \\
= -\lambda \mathcal{J} \frac{\partial}{\partial \mathcal{J}}T(\mathcal{J}(\lambda'^{r+1}(\mathcal{J}/\lambda)_{\downarrow,r+1})^{-1}) + T(\mathcal{J}(\lambda'^{r+1}(\mathcal{J}/\lambda)_{\downarrow,r})^{-1}),
\]

so

\[
(\mathcal{J} + r \lambda + \lambda \mathcal{J} \frac{\partial}{\partial \mathcal{J}})T(\mathcal{J}(\lambda'^{r+1}(\mathcal{J}/\lambda)_{\downarrow,r+1})^{-1}) = T(\mathcal{J}(\lambda'^{r+1}(\mathcal{J}/\lambda)_{\downarrow,r})^{-1}).
\]

Since \( \mathcal{J} \) is an invertible function on \( C^{1+n}_+ \) it follows that \( \mathcal{J} + r \lambda + \lambda \mathcal{J} \frac{\partial}{\partial \mathcal{J}} \) is invertible on \( C^{\infty}(C^{1+n}_+) \). Since

\[
(\mathcal{J} + r \lambda + \lambda \mathcal{J} \frac{\partial}{\partial \mathcal{J}})^{-1} \mathcal{J}^{-r} = \mathcal{J}^{-r+1} + r \lambda \mathcal{J}^{-r} - r \lambda \mathcal{J}^{-r} = \mathcal{J}^{-r+1}
\]

we obtain \( (\mathcal{J} + r \lambda + \lambda \mathcal{J} \frac{\partial}{\partial \mathcal{J}})^{-1}(\mathcal{J}^{-r+1}) = \mathcal{J}^{-r} \). The statement now follows by induction because the base case \( r = 0 \) reduces to \( T(1) = 1 \) and is therefore fulfilled. \( \Box \)

**Proposition 5.5** There exists a unique equivalence transformation \( T \) on \( C^{1+n}_+ \) of the form

\[
T = \text{id} + \sum_{k=1}^{2k} \sum_{\ell=1}^{\ell} \lambda^n (T_{k,\ell} \circ \mathcal{J}) \left( \frac{\partial}{\partial \mathcal{J}} \right)^{\ell}
\]

with \( T_{k,\ell} \in C^{\infty}([0, \infty]) \) that has the properties from the previous Proposition \( \Box \). Its inverse \( S = T^{-1} \) thus has all the properties \( \Box \) to \( \Box \) above the fifth requirement that \( S(\mathcal{J}) = \mathcal{J} \). \( \Box \)

**Proof:** The identity \( [T, \mathcal{J}] = \lambda \mathcal{J} \frac{\partial}{\partial \mathcal{J}} T \) together with \( T = \text{id} + O(\lambda) \) and \( T(1) = 1 \) is equivalent (by collecting terms in \( \lambda^k \) and \( \left( \frac{\partial}{\partial \mathcal{J}} \right)^{\ell} \)) to

\[
T_{k+1,\ell+1} = \frac{\mathcal{J}}{\ell+1} (T_{k,\ell} + T_{k,\ell-1})
\]

for all \( k \in \mathbb{N}_0, \ell \in \mathbb{N}_0 \) with initial conditions \( T_{0,0} = 1 \) and \( T_{0,\ell} = 0 = T_{k,0} \) for all \( k \in \mathbb{N}, \ell \in \mathbb{N} \), where \( T_{k,\ell}' \in C^{\infty}([0, \infty]) \) is the derivative of \( T_{k,\ell} \) and where \( T_{k,-1} := 0 \) for all \( k \in \mathbb{N}_0 \). \( \Box \)

So the equivalence transformation \( S \) exists and is uniquely determined if we add to the four requirements \( \Box \) to \( \Box \) above the fifth requirement that \( S(\mathcal{J}) = \mathcal{J} \), which is just a convenience. We can
now construct the reduced star product on $M_{\text{red}}$:

**Definition 5.6** The transformed star product $\tilde{*}$ on $\mathbb{C}^1$ is the one obtained from $*$ by application of the equivalence transformation $S = T^{-1}$ with $T$ like in Proposition 5.5. Explicitly,

$$f \tilde{*} g = S(T(f) \star T(g))$$

for all $f, g \in \mathcal{C}^\infty(\mathbb{C}^1)$. Moreover, the reduced star product $*_{\text{red}}$ on $M_{\text{red}}$ is defined as

$$f *_{\text{red}} g := \sum_{r=0}^{\infty} \lambda^r \tilde{C}_{r,\text{red}}(f, g)$$

for all $f, g \in \mathcal{C}^\infty(M_{\text{red}})$ and extended to formal power series in $\lambda$, where $\tilde{C}_{r,\text{red}}$ on $M_{\text{red}}$ are the reductions like in Definition 4.5 of the bidifferential operators $\tilde{C}_r$ on $\mathbb{C}^1$ that describe the transformed star product $\tilde{*}$ on $\mathbb{C}^1$.

Using the defining properties of the reduced bilinear maps $\tilde{C}_{r,\text{red}}$ it is easy to check that $*_{\text{red}}$ is again associative and it is clear that the constant 1-function is the neutral element. In the next subsection, we will give an explicit formula as bidifferential operators on $M_{\text{red}}$. As $T$ and thus also $S$ commute with the pointwise complex conjugation and the action of $G_J$, both $\tilde{*}$ and $*_{\text{red}}$ are Hermitian and $G_J$-invariant. Note also that $*_{\text{red}}$ still deforms in direction of the original Poisson bracket $\{\cdot, \cdot\}$ (or rather, its restriction to $\mathbb{C}^1$), so that it is easy to check that $*_{\text{red}}$ deforms in direction of the reduced Poisson bracket $\{\cdot, \cdot\}_{\text{red}}$ on $M_{\text{red}}$.

### 5.2 Explicit Formulae

We want to find an explicit expression for the reduced Poisson bracket $\{\cdot, \cdot\}_{\text{red}}$ and star product $*_{\text{red}}$ in terms of bidifferential operators on $M_{\text{red}}$.

**Lemma 5.7** The restriction to $\mathbb{C}^1$ of the tensor $H$ can alternatively be expressed as

$$H|_{\mathbb{C}^1} = \frac{1}{J} E \otimes E + H_\Xi$$

with some $H_\Xi \in \Gamma^\infty(\Xi \otimes \Xi)$. Explicitly,

$$H_\Xi = \frac{1}{z_0^{2n} \prod_{k=1}^{n} w_k} \left( \sum_{k,\ell=1}^{n} \bar{w}_k \otimes W_\ell + \sum_{k=1}^{n} \nu_k \bar{w}_k \otimes W_k \right)$$

on the domain of definition of the vector fields $W_1, \ldots, W_n$ and consequently

$$H_{\text{red}} = \left( 1 + \sum_{k=1}^{n} \nu_k \bar{w}_k w_k \right) \left( \sum_{k,\ell=1}^{n} \bar{w}_k w_\ell \frac{\partial}{\partial w_\ell} \otimes \frac{\partial}{\partial w_k} + \sum_{k=1}^{n} \nu_k \frac{\partial}{\partial w_k} \otimes \frac{\partial}{\partial w_k} \right)$$

in projective coordinates on $M_{\text{red}}$. 

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We will give a more explicit characterization of the corresponding covariant derivative on \( M^{(1+n)} \).

As an immediate consequence we have:

**Proposition 5.8** The reduced Poisson tensor \( \pi_{\text{red}} \) that determines \( \{ \cdot, \cdot \}_{\text{red}} \) is

\[
\pi_{\text{red}} = -2i \left( 1 + \sum_{k=1}^{n} \nu_k w^k w^\dagger k \right) \left( \sum_{k,l=1}^{n} \frac{\partial}{\partial w^k} \wedge \frac{\partial}{\partial w^l} + \sum_{k=1}^{n} \nu_k \frac{\partial}{\partial w^k} \wedge \frac{\partial}{\partial w^k} \right)
\]

(5.17)
in projective coordinates.

For the signature \( s = 1 + n \), this is the usual Poisson tensor associated to the symplectic Fubini-Study form on \( M^{(1+n)}_\text{red} \cong \mathbb{C}P^n \). If \( s = 1 \), then one obtains (up to a sign) the Poisson tensor to the symplectic Fubini-Study form on the hyperbolic disc \( M^{(1+n)}_\text{red} \cong \mathbb{D}^n \).

Similarly to the Wick star product from Definition 5.1, the bidifferential operators defining the reduced star product should be expressed using symmetrized covariant derivatives. In order to define reduced symmetrized covariant derivatives we need the following:

**Definition 5.9** We write \( \Theta_\Xi : \Gamma^\infty(T\mathbb{C}^{1+n}_\Xi) \to \Gamma^\infty(T\mathbb{C}^{1+n}_\Xi) \) for the projection on the subbundle \( \Xi \) of \( T\mathbb{C}^{1+n}_\Xi \) associated to the decomposition \( T\mathbb{C}^{1+n}_\Xi = \{(\xi_i) \oplus \{\overline{\xi_i}\} \oplus \Xi \) from Proposition 3.1. Moreover, its dual will be denoted by \( \Theta_\Xi^* : \Gamma^\infty(T^* \mathbb{C}^{1+n}_\Xi) \to \Gamma^\infty(T^* \mathbb{C}^{1+n}_\Xi) \).

Like in Proposition A.6 and Lemma A.8 we can construct a reduced exterior covariant derivative and a reduced symmetrized covariant derivative on \( M_\text{red} \) out of \( D \) and \( D^\text{sym} \) on \( \mathbb{C}^{1+n}_\Xi \):

**Definition 5.10** By \( D_\text{red} : (\mathcal{A} \otimes \mathcal{E})^{k,\ell}(M_\text{red}) \to (\mathcal{A} \otimes \mathcal{E})^{k+1,\ell+1}(M_\text{red}) \) we denote the reduced exterior covariant derivative on \( M_\text{red} \), which is the one that fulfills

\[
\text{Pr}^* \left( D_\text{red} \Omega \right) = (\Theta_\Xi^*)^{\otimes (k+1+\ell)} D \text{Pr}^* \left( \Omega \right)
\]

(5.18)

for all \( \Omega \in (\mathcal{A} \otimes \mathcal{E})^{k,\ell}(M_\text{red}) \), \( k, \ell \in \mathbb{N}_0 \), and analogously, the reduced symmetrized covariant derivative on \( M_\text{red} \) is determined by

\[
\text{Pr}^* \left( D^\text{sym}_\text{red} \omega \right) = (\Theta_\Xi^*)^{\otimes (k+1)} D^\text{sym} \text{Pr}^* \left( \omega \right)
\]

(5.19)

for all \( \omega \in \mathcal{E}^{k}(M_\text{red}) \), \( k \in \mathbb{N}_0 \).

We will give a more explicit characterization of the corresponding covariant derivative on \( M_\text{red} \) later in Proposition 5.17.

**Theorem 5.11** The reduced Wick star product is

\[
f \ast \text{red} g = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{1}{(1/\lambda)_{1,r}} \left\langle (D^\text{sym}_\text{red})^r f \otimes (D^\text{sym}_\text{red})^r g, H_\text{red}^r \right\rangle
\]

(5.20)

for all \( f, g \in \mathcal{C}^\infty(M_\text{red}) \), where \( H_\text{red} |_{\rho} = (\text{Pr} \otimes \mathbb{I}) H_{\text{red}} |_{\rho} \) was computed in Equation (5.16).
Note that for complex projective spaces and hyperbolic discs this formula coincides (up to rescaling the formal parameter) with the formula derived in [13, Thm. 3.2.4] for a Fedosov star product with form $\Omega = 0$.

For the proof of Theorem 5.11 we have to collect some intermediate results. Since the Euclidean covariant derivative $\nabla$ is compatible with the complex structure in the sense of Definition A.9, the associated symmetric covariant derivative $D$ splits into its holomorphic and antiholomorphic part, $D^{\text{sym}} = D^{\text{hol}} + D^{\text{hol}}$ as explained in Definition A.10.

**Lemma 5.12** On $\mathbb{C}^*$-invariant functions $f, g \in \mathcal{C}^\infty(\mathbb{C}^+_1) \mathcal{C}^r$ the transformed Wick star product can be expressed as

$$f \star g = S(f \star g) = \sum_{r=0}^{\infty} \frac{\lambda}{r!} \langle (D^{\text{sym}})^r f \otimes (D^{\text{sym}})^r g, H^r \nabla^r \rangle$$  \hspace{1cm} (5.21)

**Proof:** The first equality in (5.21) follows from requirement (22) for the equivalence transformation. For the second one we use that we can express $f \star g$ as

$$f \star g = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \langle (D^{\text{sym}})^r f \otimes (D^{\text{sym}})^r g, H^r \nabla^r \rangle$$

with $\mathbb{C}^*$-invariant $\langle (D^{\text{sym}})^r f \otimes (D^{\text{sym}})^r g, H^r \nabla^r \rangle$ (on which all derivatives $\frac{\partial}{\partial \sigma}$ vanish) and that $\langle (D^{\text{sym}})^r f \otimes (D^{\text{sym}})^r g, H^r \nabla^r \rangle = \langle (D^{\text{sym}})^r f \otimes (D^{\text{sym}})^r g, H^r \nabla^r \rangle$ because of Lemma A.11 and since the first tensor factor of $H$ lies in $\mathbf{T}^{*,(0,1)} \mathbb{C}^1$ and the second one in $\mathbf{T}^{*,(1,0)} \mathbb{C}^1$. Then it only remains to apply the formula for $S(\nabla^r)$ from Proposition 5.4. \hspace{1cm} (22)

If we restrict (5.21) to $Z$, we can substitute $\nabla$ by $1$. In order to express $\langle (D^{\text{sym}})^r f \otimes (D^{\text{sym}})^r g, H^r \rangle|_Z$ with $\mathbb{C}^*$-invariant functions $f$ and $g$ by differential operators on $M_{\text{red}}$, we use formula (5.14) for $H$ and explicitly calculate the contribution of the vertical directions $E$ and $\overline{E}$. For this we define $\mathcal{S}^{(k,0)}(\mathbb{C}^+_1) := \Gamma^\infty(S^k \mathbf{T}^{*,(1,0)} \mathbb{C}^1_1)$ and set $\mathcal{S}^{(\bullet,0)}(\mathbb{C}^+_1) := \bigoplus_{k=0}^{\infty} \mathcal{S}^{(k,0)}(\mathbb{C}^+_1)$. Then we define the degree

$$\text{deg}: \mathcal{S}^{(\bullet,0)}(\mathbb{C}^+_1) \to \mathcal{S}^{(\bullet,0)}(\mathbb{C}^+_1)$$

by extending the assignment $\omega = k \omega$ for $\omega \in \Gamma^\infty(S^k \mathbf{T}^{*,(1,0)} \mathbb{C}^1_1)$ linearly. Clearly $\text{deg}$ is a derivation.

**Lemma 5.13** For $\mathbb{C}^*$-invariant $\sigma \in \mathcal{S}^{(\bullet,0)}(\mathbb{C}^+_1) \mathcal{C}^r$ we have $[\iota_E, D^{\text{sym}}] (\sigma) = -2 \text{deg}(\sigma)$.

**Proof:** Note that both $\iota_E$ and $D^{\text{hol}}$ are derivations on $\mathcal{S}^{(\bullet,0)}(\mathbb{C}^+_1) \mathcal{C}^r$, so $[\iota_E, D^{\text{sym}}]$ is a derivation and it suffices to check $[\iota_E, D^{\text{hol}}] = -2 \text{deg}$ on generators, i.e. on elements $f \in \mathcal{C}^\infty(\mathbb{C}^+_1) \mathcal{C}^*$ and $g dz^\ell \in \Gamma^\infty(T^{*,(1,0)} \mathbb{C}^1_1)$ for all $\ell \in \{0, \ldots, n\}$ and with suitable smooth functions $g$. Note that $\mathbb{C}^*$-invariance of these elements implies $E(f) = 0$ and $E(g) = -g$. As $[\iota_E, D^{\text{sym}}] (h) = \iota_E D^{\text{sym}} h = E(h)$ for all $h \in \mathcal{C}^\infty(\mathbb{C}^+_1)$ we have $[\iota_E, D^{\text{hol}}] (f) = 0 = -2 \text{deg}(f)$ and

$$[\iota_E, D^{\text{sym}}] (g dz^\ell) = -g dz^\ell + g [\iota_E, D^{\text{sym}}] dz^\ell = -g dz^\ell - g D^{\text{sym}} \iota_E dz^\ell = -2g dz^\ell = -2 \text{deg}(g dz^\ell)$$

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for all $\ell \in \{0, \ldots, n\}$. □

**Lemma 5.14** For $f,g \in C^\infty(C^{1+n})^{\mathbb{C}^*}$ and $r \in \mathbb{N}$ we have

$$\tau(E)(D_{\text{hol}}^{\text{sym}})^rf = -r(r-1)(D_{\text{hol}}^{\text{sym}})^{r-1}f \quad \text{and} \quad \iota_E(D_{\text{hol}}^{\text{sym}})^rg = -r(r-1)(D_{\text{hol}}^{\text{sym}})^{r-1}g$$

(5.22)
as well as

$$\left\langle (D_{\text{hol}}^{\text{sym}})^rf \otimes (D_{\text{hol}}^{\text{sym}})^rg, H^r \right\rangle|_Z = \sum_{k=1}^{r} \frac{k^r}{k!} \left[ (D_{\text{hol}}^{\text{sym}})^r f \otimes (D_{\text{hol}}^{\text{sym}})^r g \right] \left( (D_{\text{hol}}^{\text{sym}})^k f \otimes (D_{\text{hol}}^{\text{sym}})^k g, (H_Z)^k \right) |_Z,$$

(5.23)

where $H_Z$ is the component of $H$ in $\Xi \otimes \Xi$, defined in (5.13).

**Proof:** For (5.22) it suffices to prove the second statement since the first one then follows by taking complex conjugates. Note that $(D_{\text{hol}}^{\text{sym}})^k g$ is $\mathbb{C}^*$-invariant for all $k \in \mathbb{N}_0$, so the previous Lemma 5.13 yields

$$\iota_E((D_{\text{hol}}^{\text{sym}})^rg) = \sum_{k=0}^{r-1} (D_{\text{hol}}^{\text{sym}})^{r-k-1} [\iota_E, D_{\text{hol}}^{\text{sym}}](D_{\text{hol}}^{\text{sym}})^k g = \sum_{k=0}^{r-1} (-2k)(D_{\text{hol}}^{\text{sym}})^{r-1}g = -r(r-1)g.$$  

With this and $H|_Z = \overline{E} \otimes E + H_Z$ from Lemma 5.7, we can now calculate

$$\left\langle (D_{\text{hol}}^{\text{sym}})^rf \otimes (D_{\text{hol}}^{\text{sym}})^rg, H^r \right\rangle|_Z =$$

$$\sum_{k=0}^{r} \binom{r}{k} \left\langle (D_{\text{hol}}^{\text{sym}})^r f \otimes (D_{\text{hol}}^{\text{sym}})^r g, (\overline{E} \otimes E)^{r-k}(H_Z)^k \right\rangle|_Z$$

$$= \sum_{k=0}^{r} \binom{r}{k} \frac{(k!)}{(r!)}^2 \left\langle (\overline{E})^{r-k}(D_{\text{hol}}^{\text{sym}})^r f \otimes (\iota_E)^{r-k}(D_{\text{hol}}^{\text{sym}})^r g, (H_Z)^k \right\rangle|_Z$$

$$= \sum_{k=1}^{r} \binom{r}{k} \frac{(k!)}{(r!)}^2 \left\langle (\overline{E})^{r-k}(D_{\text{hol}}^{\text{sym}})^r f \otimes (\iota_E)^{r-k}(D_{\text{hol}}^{\text{sym}})^r g, (H_Z)^k \right\rangle|_Z$$

$$= \sum_{k=1}^{r} \frac{r! (r-k)k!}{(k-1)!} \left\langle ((D_{\text{hol}}^{\text{sym}})^k f \otimes (D_{\text{hol}}^{\text{sym}})^k g, (H_Z)^k \right\rangle|_Z.$$  

The factors appearing in step (1) are due to our conventions for the symmetric product, the dual pairing and the insertion derivation, see Equation (2.6). In (2) we used

$$(\iota_E)^{r-k}(D_{\text{sym}})^rg = (-1)^{r-k} r! \frac{(r-1)!}{k! (k-1)!} = (-1)^{r-k} \frac{r!}{k! (k-1)!} (r-1)! \binom{r}{k} \left( \frac{r-1}{k-1} \right) ((r-k)!)^2$$

and its complex conjugate, which can be obtained by applying Lemma 5.14 several times. In the special case $k = 0$, Lemma 5.14 yields $(\iota_E)^r(D_{\text{sym}})^rg = 0$. □

The resulting combinatorial factors can be simplified using:
Lemma 5.15 For any \( k \geq 1 \) we have

\[
\sum_{s=0}^{\infty} \frac{1/\lambda}{(1/\lambda)_{\uparrow, k+s+1}} \binom{k+s-1}{k-1}^2 \frac{1}{s!} = \frac{1}{(1/\lambda)_{\downarrow, k}}
\]

(5.24)

where the equality is understood as equality of the series expansion in the formal parameter \( \lambda \).

Proof: Recall the definition of the hypergeometric series \( {}_2F_1(a, b; c \mid z) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s} \frac{z^s}{s!} \) for \( a, b \in \mathbb{C} \) and \( c \in \mathbb{C} \setminus (-\mathbb{N}_0) \). If \( \Re(c-a-b) > 0 \) the series converges for \( z = 1 \) and the well-known Gauss identity yields

\[
{}_2F_1(a, b; c \mid 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.
\]

Let \( x > k - 1 \). Then we compute

\[
\sum_{s=0}^{\infty} \frac{x}{(x)_{\uparrow, k+s+1}} \binom{k+s-1}{k-1}^2 \frac{1}{s!} = \sum_{s=0}^{\infty} \frac{1}{(x+1)_{\uparrow, k}(x+k+1)_{\uparrow, s}} \frac{((k+s-1)!)^2}{(k-1)!^2 s!}
\]

\[
= \frac{1}{(x+1)_{\uparrow, k}} \sum_{s=0}^{\infty} \frac{(k)_{\uparrow, s}^2}{(x+k+1)_{\uparrow, s}} \frac{1^s}{s!}
\]

\[
= \frac{1}{(x+1)_{\uparrow, k}} {}_2F_1(k, k; x+k+1 \mid 1)
\]

\[
= \frac{1}{(x+1)_{\uparrow, k}} \frac{\Gamma(x+k+1)\Gamma(x-k+1)}{\Gamma(x+1)^2}
\]

\[
= \frac{1}{(x+1)_{\uparrow, k}} \frac{(x+1)_{\uparrow, k}}{(x)_{\downarrow, k}}
\]

Replacing \( x \) by \( 1/h \) and taking the Taylor series expansion around \( h = 0 \) (corresponding to the expansion for large \( x \) where the above identity holds), proves the statement.

The last, crucial step is the following observation:

Lemma 5.16 We have

\[
D_{\text{hol}}^{\text{sym}} E^* = - (E^*)^2 \quad \text{as well as} \quad D_{\text{hol}}^{\text{sym}} E^* = - (E^*)^2
\]

(5.25)

and consequently

\[
(\Theta^k_\mathcal{E})^{\otimes (k+1)} D_{\text{hol}}^{\text{sym}} (\Theta^k_\mathcal{E})^{\otimes k} \omega = (\Theta^k_\mathcal{E})^{\otimes (k+1)} D_{\text{hol}}^{\text{sym}} \omega
\]

(5.26)

as well as

\[
(\Theta^k_\mathcal{E})^{\otimes (k+1)} D_{\text{hol}}^{\text{sym}} (\Theta^k_\mathcal{E})^{\otimes k} \omega = (\Theta^k_\mathcal{E})^{\otimes (k+1)} D_{\text{hol}}^{\text{sym}} \omega
\]

(5.27)

for all \( \omega \in \mathcal{S}^{(k,0)}(\mathbb{C}_+^{1+n}) = \Gamma^\infty(S^k T^{\ast_1 (1,0)} \mathbb{C}_+^{1+n}) \) with \( k \in \mathbb{N}_0 \).

Proof: Again, it suffices to prove the second equalities since the first ones then follow by taking
complex conjugates. Using (3.23) and (3.24), an easy computation shows

\[ D_{\text{hol}}^\text{sym} E^* = D_{\text{hol}}^\text{sym} \left( \frac{1}{f} \sum_{k=0}^{n} \nu_k z^k d^k \right) = - \sum_{k=0}^{n} \nu_k z^k d^k J E^* = -(E^*)^2. \]

For (5.27) it is sufficient to consider the case \( k = 1 \), the general case then follows from the algebraic properties of \( \Theta_\omega \) and \( D_{\text{hol}}^\text{sym} \) (i.e. being a projection and a derivation). If \( k = 1 \), then there is an \( f \in C^\infty(\mathbb{C}_+^{1+n}) \) such that \( \Theta_\omega^* \omega - \omega = f E^* \), and thus

\[ D_{\text{hol}}^\text{sym} \Theta_\omega^* \omega - D_{\text{hol}}^\text{sym} \omega = D_{\text{hol}}^\text{sym} (\Theta_\omega^* \omega - \omega) = D_{\text{hol}}^\text{sym} f E^* = df \vee E^* - f(E^*)^2 \]

is in the kernel of \( (\Theta_\omega^*)^{\otimes 2} \). □

**Proof of Theorem 5.11.** By Proposition 4.6 the reduced star product on \( M_{\text{red}} \) fulfills

\[ \text{pr}^\ast(f \ast_{\text{red}} g) = \left( S(\text{Pr}^\ast(f) \ast \text{Pr}^\ast(g)) \right) \big|_{Z} \]

for all \( f, g \in C^\infty(M_{\text{red}}) \). Application of first Lemma 5.12 and then Lemma 5.14 now yields

\[
\text{pr}^\ast(f \ast_{\text{red}} g) = \\
= \sum_{r=0}^{\infty} \frac{1}{r!} \frac{1}{(1/\lambda)_{r+1}} \left( (D_{\text{hol}}^\text{sym})^r \text{Pr}^\ast(f) \otimes (D_{\text{hol}}^\text{sym})^r \text{Pr}^\ast(g), H^r \right) |_{Z} \\
= fg |_{Z} + \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{(1/\lambda)_{r+1}} \frac{1}{k!} \frac{1}{k} \frac{1}{(r-k)!} \frac{1}{(r-1)!} \frac{1}{k!} \frac{1}{k} \left( (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(f) \otimes (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(g), (H\Xi)^k \right) |_{Z}
\]

for all \( f, g \in C^\infty(M_{\text{red}}) \). By collecting the k-th derivatives and using Lemma 5.15 this leads to

\[
\text{pr}^\ast(f \ast_{\text{red}} g) = \\
= fg |_{Z} + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{(1/\lambda)_{k+1}} \frac{1}{k!} \frac{1}{k} \frac{1}{(k-s+1)!} \frac{1}{(k-1)!} \frac{1}{k!} \frac{1}{k} \left( (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(f) \otimes (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(g), (H\Xi)^k \right) |_{Z} \\
= fg |_{Z} + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{(1/\lambda)_{k+1}} \frac{1}{k!} \frac{1}{k} \frac{1}{(k-s+1)!} \frac{1}{(k-1)!} \frac{1}{k!} \frac{1}{k} \left( (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(f) \otimes (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(g), (H\Xi)^k \right) |_{Z} \\
= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(1/\lambda)_{k+1}} \frac{1}{k!} \frac{1}{k} \frac{1}{(k-s+1)!} \frac{1}{(k-1)!} \frac{1}{k!} \frac{1}{k} \left( (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(f) \otimes (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(g), (H\Xi)^k \right) |_{Z}.
\]

Finally, as \( (H\Xi)|_{\rho} \in \Xi_{\rho} \otimes \Xi_{\rho} \) for all \( \rho \in \mathbb{C}_+^{1+n} \), we may insert projections \( \Theta_\omega^* \) and get

\[
\left( (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(f) \otimes (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(g), (H\Xi)^k \right) = \left( (\Theta_\omega^*)^k (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(f) \otimes (\Theta_\omega^*)^k (D_{\text{hol}}^\text{sym})^k \text{Pr}^\ast(g), (H\Xi)^k \right).
\]

As a consequence of Proposition 3.1 the projection \( \Theta_\omega^* \) commutes with the complex structure \( I \) and
therefore $(\Theta^*_{\Xi})^{(p+q)}$ commutes with the projection onto symmetric tensors of degree $(p, q)$. The projection onto such tensors also commutes with $Pr^*$ since $Pr$ is holomorphic. Therefore $D_{\text{red}}$ is compatible with the complex structure and $(\Theta^*_{\Xi})^{(k+1)}D_{\text{hol}}^\text{sym} Pr^*(\omega) = Pr^*(D_{\text{hol,red}}^\text{sym})$ for all $\omega \in \mathscr{J}^*(M_{\text{red}})$. So using Lemma 5.16 we obtain

$$\left(\Theta^*_{\Xi}\right)^{(k)}(D_{\text{hol}}^\text{sym}) Pr^*(g) = (\Theta^*_{\Xi})^{(k)}D_{\text{hol}}^\text{sym} (\Theta^*_{\Xi})^{(k-1)}D_{\text{hol}}^\text{sym} \ldots \Theta^*_{\Xi}D_{\text{hol}}^\text{sym} Pr^*(g) = Pr^*(D_{\text{hol,red}}^\text{sym}^k g)$$

and analogously for $f$, so that

$$\text{pr}^*(f \ast_{\text{red}} g) = \sum_{k=0}^{\infty} \frac{1}{k! \frac{1}{(1/\lambda)}} \left( (D_{\text{hol}}^\text{sym})^k Pr^*(f) \otimes (D_{\text{hol}}^\text{sym})^k Pr^*(g), (H_{\Xi})^k \right) |_Z$$

$$= \sum_{k=0}^{\infty} \frac{1}{k! \frac{1}{(1/\lambda)}} \left( Pr^* ((D_{\text{hol,red}}^\text{sym})^k f) \otimes Pr^* ((D_{\text{hol,red}}^\text{sym})^k g), (H_{\Xi})^k \right) |_Z$$

$$= \sum_{k=0}^{\infty} \frac{1}{k! \frac{1}{(1/\lambda)}} \text{pr}^* \left( ((D_{\text{hol,red}}^\text{sym})^k f \otimes (D_{\text{hol,red}}^\text{sym})^k g, H_{\text{red}}^k \right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k! \frac{1}{(1/\lambda)}} \text{pr}^* \left( (D_{\text{red}}^\text{sym})^k f \otimes (D_{\text{red}}^\text{sym})^k g, H_{\text{red}}^k \right).$$

In the last step we used Lemma A.11 and that the first tensor factor of $H_{\text{red}}$ lies in $T^{*,(0,1)}M_{\text{red}}$ whereas the second lies in $T^{*,(1,0)}M_{\text{red}}$.

Finally, we can also characterize the reduced covariant derivative as follows:

**Proposition 5.17** The reduced exterior covariant derivative $D_{\text{red}}$ on $M_{\text{red}}$ is the one for the Levi-Civita connection associated to the (not necessarily definite) reduced metric $g_{\text{red}} \in \mathscr{J}^2(M_{\text{red}})$, which is defined by

$$\text{Pr}^*(g_{\text{red}}) = (\Theta^*_{\Xi})^{\otimes 2} \left( \sum_{k=0}^{n} \nu_k d\tilde{z}^k \vee d\bar{z}^k \bigg|_J \right)_{G_{1+n}^1}. \tag{5.28}$$

**Proof:** As $\sum_{k=0}^{n} \nu_k d\tilde{z}^k \vee d\bar{z}^k / J$ is $C^*$-invariant, $g_{\text{red}}$ is indeed well-defined. As $D$ is torsion-free, $D_{\text{red}}$ is torsion free as well (see Proposition A.6). Now we calculate

$$\text{Pr}^* \left( D_{\text{red}}(g_{\text{red}}) \right) = (\Theta^*_{\Xi})^{\otimes 3} D \text{Pr}^*(g_{\text{red}}) = (\Theta^*_{\Xi})^{\otimes 3} D(\Theta^*_{\Xi})^{\otimes 2} \left( \sum_{k=0}^{n} \nu_k d\tilde{z}^k \vee d\bar{z}^k \bigg|_J \right)_{G_{1+n}^1}. \tag{5.28}$$

Like in Lemma 5.7 one can check that $\sum_{k=0}^{n} \nu_k d\tilde{z}^k \vee d\bar{z}^k - \sum_{k=0}^{n} \nu_k (\Theta^*_{\Xi} d\tilde{z}^k) \vee (\Theta^*_{\Xi} d\bar{z}^k) = J E^* \vee E^*$ so that

$$\text{Pr}^* \left( D_{\text{red}}(g_{\text{red}}) \right) = (\Theta^*_{\Xi})^{\otimes 3} D \left( \sum_{k=0}^{n} \nu_k d\tilde{z}^k \vee d\bar{z}^k \bigg|_J \right)_{G_{1+n}^1} - (\Theta^*_{\Xi})^{\otimes 3} D(E^* \vee E^*) = 0$$

because $\Theta^*_{\Xi} d(J^{-1}) = -J^{-2} \Theta^*_{\Xi} dJ = 0$ and because $(\Theta^*_{\Xi})^{\otimes 3} D(E^* \vee E^*) = 0$ by Lemma 5.16. This shows $D_{\text{red}}(g_{\text{red}}) = 0$. \(\square\)
Note that $g_{\text{red}}$ can equivalently be obtained from the standard (pseudo-)metric $g := \sum_{k=0}^{n} \nu_k d^k \cdot d^k$ on $\mathbb{C}^{1+n}$ in signature $s$ by first restricting $(\Theta_\Xi)^{\otimes 2}g$ to $Z$ and then projecting down on $M_{\text{red}}$. In coordinates one finds that

$$
\sum_{k=0}^{n} \frac{\nu_k d^k \cdot d^k}{J} \bigg|_{\mathbb{C}_+^{1+n}} = E^* \cdot E^* + \frac{|z|^2}{J} \sum_{k=1}^{n} \nu_k W_k^* \cdot W_k^* - \frac{|\bar{z}|^2}{J} \sum_{k,\ell=1}^{n} \nu_k \nu_\ell \bar{z}^k \cdot \bar{z}^\ell \cdot W_k \cdot W_\ell
$$

(5.29)

and hence that

$$
g_{\text{red}} = \frac{\sum_{k=0}^{n} \nu_k d^k \cdot d^k}{1 + \sum_{k=1}^{n} \nu_k d^k \cdot d^k} - \frac{\sum_{k,\ell=1}^{n} \nu_k \nu_\ell \bar{z}^k \cdot \bar{z}^\ell \cdot W_k \cdot W_\ell}{(1 + \sum_{k=1}^{n} \nu_k d^k \cdot d^k)^2}.
$$

(5.30)

In particular, for the signature $s = 1 + n$ one obtains for $g_{\text{red}}$ the usual Fubini-Study metric on $M_{\text{red}}^{(1+n)} \cong \mathbb{C}P^n$, and for $s = 1$ the negative of the usual Fubini-Study metric on $M_{\text{red}}^{(1)} \cong \mathbb{D}^n$.

### 5.3 The Polynomial Case

In this section we will replace the formal parameter $\lambda$ by a complex number $\hbar$. In order to make sense of the convergence of the formal power series describing the star product, we restrict ourselves to polynomial functions.

From the definition of the Poisson bracket in (5.5) it is clear that it restricts to a well-defined map

$$\{ \cdot, \cdot \} : \mathcal{P}(\mathbb{C}^{1+n}) \times \mathcal{P}(\mathbb{C}^{1+n}) \to \mathcal{P}(\mathbb{C}^{1+n})$$

that is given by the same formula, and similarly for the Wick star product:

**Lemma 5.18** In the basis $b_{P,Q}$ defined in Definition 4.8, the Poisson bracket is

$$\{ b_{P,Q}, b_{R,S} \} = \frac{1}{n} \sum_{k=0}^{n} \nu_k (Q_k R_k - P_k S_k) b_{P+R-E_k, Q+S-E_k}$$

(5.31)

with $E_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_0^{1+n}$ having the 1 at position $k$, and the product $\star$ from Definition 5.7 is

$$b_{P,Q} \star b_{R,S} = \sum_{T=0}^{\min\{Q,R\}} \text{sgn}(T) \lambda^{|T|} T! \binom{Q}{T} \binom{R}{T} b_{P+R-T, Q+S-T}$$

(5.32)

with $\text{sgn}(T) = \prod_{k=0}^{n} v_k^{T_k} = \prod_{k=1}^{n} v_k^{T_k}$ like in Lemma 4.10.

So by setting $\lambda$ to $\hbar \in \mathbb{C}$, this yields a well-defined map $\star_{\hbar} : \mathcal{P}(\mathbb{C}^{1+n}) \times \mathcal{P}(\mathbb{C}^{1+n}) \to \mathcal{P}(\mathbb{C}^{1+n})$.

Next, we consider the equivalence transformation $S$ from Proposition 5.25.

**Lemma 5.19** For $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$, the equivalence transformation $S$ is given by

$$S(b_{P,Q}) = \left( \frac{\lambda}{\bar{\lambda}} \right)^{|P|} \binom{\mathcal{J}}{\lambda} \sum_{P} b_{P,Q}$$

(5.33)

**Proof:** As $\mathcal{J}^{-|P|} b_{P,Q}$ is $\mathbb{C}^*$-invariant, we get

$$S(b_{P,Q}) = S(\mathcal{J}^{-|P|} \mathcal{J}^{-|P|} b_{P,Q}) = S(\mathcal{J}^{-|P|}) \mathcal{J}^{-|P|} b_{P,Q} = (\mathcal{J}/\lambda) \sum_{P} (\mathcal{J}/\lambda)^{|P|} b_{P,Q}.$$
Note that this is indeed a well-defined formal power series in \( \lambda \) as the term \( \mathcal{J}/\lambda \) that occurs in \((\mathcal{J}/\lambda)_{\downarrow,|P|}\) if \(|P| \geq 1\) is cancelled.

Replacing \( \lambda \) by \( \hbar \in \mathbb{C} \) yields a rational expression in \( \hbar \) and we have to be aware of some poles:

**Definition 5.20** We define the open subset \( \Omega \) of \( \mathbb{C} \) as

\[
\Omega := \mathbb{C} \setminus \left\{ \frac{1}{k} \mid k \in \mathbb{N} \right\} \cup \{0\}.
\]

(5.34)

**Proposition 5.21** For \( P, Q, R, S \in \mathbb{N}^{1+n}_{0} \) with \(|P| = |Q|\) and \(|R| = |S|\), the reduced star product from Definition 5.6 is given by

\[
b_{P,Q;\text{red}} *_{\text{red}} b_{R,S;\text{red}} = \sum_{T=0}^{\min\{Q,R\}} \text{sgn}(T) \frac{(1/\lambda)_{\downarrow,|Q+R-T|}}{(1/\lambda)_{\downarrow,|Q|}(1/\lambda)_{\downarrow,|R|}} T! \left( Q \right) \left( R \right) b_{P+R-T,Q+S-T;\text{red}}.
\]

(5.35)

Replacing \( \lambda \) by \( \hbar \in \Omega \) yields a strict (associative) product \( *_{\text{red},\hbar} : \mathcal{P}(M_{\text{red}}) \times \mathcal{P}(M_{\text{red}}) \to \mathcal{P}(M_{\text{red}}) \) and \( \mathcal{P}(M_{\text{red}}) \) with this product and pointwise complex conjugation becomes a unital \( * \)-algebra if \( \hbar \in \Omega \cap \mathbb{R} \).

At the poles we obtain a strict (associative) product \( *_{\text{red},1/k} : \mathcal{P}(k)(M_{\text{red}}) \times \mathcal{P}(k)(M_{\text{red}}) \to \mathcal{P}(k)(M_{\text{red}}) \) by replacing \( \lambda \) with \( \hbar = \frac{1}{k} \), \( k \in \mathbb{N} \), explicitly:

\[
b_{P,Q;\text{red},1/k} *_{\text{red},1/k} b_{R,S;\text{red}} = \sum_{T \in \tau} \text{sgn}(T) \frac{(k-|Q|)(k-|R|)T!}{k!(k+|T|-|Q+R|)!} \left( Q \right) \left( R \right) b_{P+R-T,Q+S-T;\text{red}}
\]

(5.36)

with the sum ranging over \( \tau := \{T \in \mathbb{N}^{1+n}_{0} \mid T \leq Q \text{ and } T \leq R \text{ and } k + |T| \geq |Q + R| \} \). Again, \( \mathcal{P}(k)(M_{\text{red}}) \) with \( *_{\text{red},1/k} \) becomes a unital \( * \)-algebra.

**Proof:** First we note that Lemma 5.18 and Lemma 5.19 show that

\[
\left( \frac{\lambda}{\mathcal{J}} \right)^{\uparrow Q} \left( \frac{\mathcal{J}}{\lambda} \right)^{\downarrow R} b_{P,Q;\text{red}} *_{\text{red}} \left( \frac{\lambda}{\mathcal{J}} \right)^{\uparrow R} \left( \frac{\mathcal{J}}{\lambda} \right)^{\downarrow Q} b_{R,S;\text{red}} = \sum_{T=0}^{\min\{Q,R\}} \text{sgn}(T) \lambda^{T}|T|! \left( Q \right) \left( R \right) \left( \frac{\lambda}{\mathcal{J}} \right)^{\downarrow,|Q+R-T|} \left( \frac{\mathcal{J}}{\lambda} \right)^{\downarrow,|Q+R-T|} b_{P+R-T,Q+S-T;\text{red}}
\]

holds for the transformed star product \( \check{*} \) and all \( P, Q, R, S \in \mathbb{N}^{1+n}_{0} \) with \(|P| = |Q|\) and \(|R| = |S|\). As

\[
\text{pr}^{*} \left( \lambda^{Q}(1/\lambda)_{\downarrow,|Q|} b_{P,Q;\text{red}} \right) = \iota^{*} \left( (\lambda/\mathcal{J})^{\uparrow Q} (\mathcal{J}/\lambda)^{\downarrow,|Q|} b_{P,Q} \right)
\]

we find that

\[
\left( \lambda^{Q}(1/\lambda)_{\downarrow,|Q|} b_{P,Q;\text{red}} \right) *_{\text{red}} \left( \lambda^{R}(1/\lambda)_{\downarrow,|R|} b_{R,S;\text{red}} \right) = \sum_{T=0}^{\min\{Q,R\}} \text{sgn}(T) \lambda^{Q+R-T}|T|! \left( Q \right) \left( R \right) \left( 1/\lambda \right)^{\downarrow,|Q+R-T|} \left( \mathcal{J} \right)^{\downarrow,|Q+R-T|} b_{P+R-T,Q+S-T;\text{red}}
\]

which yields (5.35) by \( \mathbb{C}[\lambda] \)-linearity of \( *_{\text{red}} \). Note that the right-hand side of (5.35) is indeed a well-defined formal power series in \( \lambda \) because the factor \( 1/\lambda \) that occurs in \((1/\lambda)_{\downarrow,|Q+R-T|}\) for \(|Q+R-T| \geq 1\)
is cancelled.

We can now substitute $\lambda$ by $\hbar \in \mathbb{C}$: If $\hbar \in \Omega$, the falling factorials in the nominator are non-zero, thus (5.35) defines a well-defined product on the whole algebra $\mathcal{P}(M_{\text{red}})$. If $\hbar = 1/k$, $k \in \mathbb{N}$, then the falling factorials in the denominator are still non-zero as long as $f, g \in \mathcal{P}^{(k)}(M_{\text{red}})$ and the numerator vanishes if $T \notin \tau$. As $(k)_{\pm \ell} = k!/(k - \ell)!$ for all $k, \ell \in \mathbb{N}_0$ with $k \geq \ell$, and $(k)_{\pm \ell} = 0$ for all $k, \ell \in \mathbb{N}_0$ with $k < \ell$, this yields Equation (5.36). Associativity and compatibility with pointwise complex conjugation follow from the properties of the Hermitian formal star product $\star_{\text{red}}$, and the unit is the constant 1-function.

Equation (5.35) immediately yields:

**Corollary 5.22** For two fixed polynomials $f, g \in \mathcal{P}(M_{\text{red}})$, the map $\mathbb{C} \to \mathbb{C}$, $\hbar \mapsto f \star_{\text{red}, \hbar} g$ is rational and $\lim_{\hbar \to 0} f \star_{\text{red}, \hbar} g = fg$.

**Proposition 5.23** For two polynomials $f, g \in \mathcal{P}(M_{\text{red}})$, we have

$$\lim_{\hbar \to 0} \frac{1}{i\hbar} \left( f \star_{\text{red}, \hbar} g - g \star_{\text{red}, \hbar} f \right) = \{ f, g \}_{\text{red}}$$

(5.37)

Pointwise and with the reduced Poisson bracket $\{ \cdot, \cdot \}_{\text{red}}$ on $M_{\text{red}}$.

**Proof:** All terms with $|T| \geq 2$ in Equation (5.35) are of order at least $\hbar^2$ and the $T = 0$ term cancels out when taking the commutator. The first order in $\hbar$ of the terms with $|T| = 1$ produces

$$\lim_{\hbar \to 0} \frac{1}{i\hbar} \left( b_{P,Q,\text{red}} \star_{\text{red}, \hbar} b_{R,S,\text{red}} - b_{R,S,\text{red}} \star_{\text{red}, \hbar} b_{P,Q,\text{red}} \right) = \frac{1}{i} \sum_{k=0}^{n} \nu_k (Q_k R_k - S_k P_k) b_{P+R-E_k, Q+S-E_k, \text{red}}$$

with $E_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_{0}^{1+k}$ having the 1 at position $k$, which coincides by Definition 4.5 and Lemma 5.18 with $\{ b_{P,Q,\text{red}}, b_{R,S,\text{red}} \}_{\text{red}}$.

### 5.4 The Analytic Case

The aim of this section is to obtain a strict star product on the algebra $\mathcal{A}(M_{\text{red}})$. We achieve this by proving the continuity of the star product $\star_{\text{red}, \hbar}$ on $\mathcal{P}(M_{\text{red}})$ with respect to the locally convex topology that $\mathcal{P}(M_{\text{red}})$ inherits from $\mathcal{A}(M_{\text{red}})$, i.e. the topology of locally uniform convergence of the holomorphic extensions to $\hat{M}_{\text{red}}$. This then implies that $\star_{\text{red}, \hbar}$ extends uniquely to a continuous star product on $\mathcal{A}(M_{\text{red}})$.

Recall from Proposition 4.16 that the topology on $\mathcal{A}(M_{\text{red}})$ is just the quotient topology of the topology on $\mathcal{A}(\mathbb{C}^{1+n})^{U(1)}$ defined by locally uniform convergence of the holomorphic extensions to $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$. For $\hbar \in \Omega$ define a product $\star_{\hbar}$ on $\mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$ by bilinearly extending

$$b_{P,Q} \star_{\hbar} b_{R,S} := \sum_{T=0}^{\min(Q,R)} \text{sgn}(T) \left( \frac{1}{i\hbar} \right)_{Q+R-T} \left( \frac{1}{i\hbar} \right)_{Q+R-T} T! \left( \frac{Q}{T} \right) \left( \frac{R}{T} \right) b_{P+R-T,Q+S-T}$$

(5.38)

for all $P, Q, R, S \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$ and $|R| = |S|$. Note that this product might be badly behaved, for example it does not need to be associative. However, from Proposition 5.21 it follows
immediately that dividing out the vanishing ideal of $J - 1$ is possible and reproduces the product $*_\text{red,}h$. Consequently, continuity of $*_h$ with respect to the seminorms $|| \cdot ||_r$ with $r \in [1, \infty[$ defined in Equation (4.12) implies continuity of $*_\text{red,}h$.

**Proposition 5.24** The product $*_h$ is continuous with respect to the locally convex topology defined by the seminorms $|| \cdot ||_r$ with $r \in [1, \infty[$ as in Equation (4.12). More precisely, for every $r \in [1, \infty[$ and every compact subset $K$ of $\Omega$ there exists $r' \in [1, \infty[$ such that

$$||f \ast_h g||_{r'} \leq ||f||_{r'} ||g||_{r'} \quad (5.39)$$

holds for all $h \in K$ and all $f, g \in \mathcal{P}(\mathbb{C}^{1+n})^{U(1)}$.

**Proof:** It is well-known that for any compact set $K' \subseteq \mathbb{C} \setminus \mathbb{N}_0$ there are constants $c > 0$ and $C > 0$ such that

$$c^n n! \leq |(z)_{\perp, n}| \leq C^n n!$$

holds for all $z \in K$ and all $n \in \mathbb{N}_0$. For a compact set $K \subseteq \Omega$ also $K' := \{ z \in \mathbb{C} \setminus \{0\} \mid z^{-1} \in K \}$ is compact and a subset of $\mathbb{C} \setminus \mathbb{N}_0$. Therefore we obtain for any $r \in [1, \infty[$ and $P, Q, R, S \in \mathbb{N}_0^{n+1}$ with $|P| = |Q|$ and $|R| = |S|$ that

$$||b_{P,Q} \ast_h b_{R,S}||_{r} = \left| \sum_{T=0}^{\min\{Q,R\}} \frac{\text{sgn}(T)}{(1/h)^{Q+R-T}} \left( \frac{|Q|}{|R|} \right) |T|^{r} b_{P+R-T,Q+R-S} \right|_{r} \leq \left( \sum_{T=0}^{\min\{Q,R\}} \frac{\text{sgn}(T)}{(1/h)^{Q+R-T}} \left( \frac{|Q|}{|R|} \right) |T|^{r} b_{P+R-T,Q+R-S} \right) \leq \left( \sum_{T=0}^{\min\{Q,R\}} (c^{-1} C)^{|Q+R-T|} (1/h)^{|Q+R-T|} |T|^{r} \right) \leq \frac{(8e^{-1} C)^{|Q+R-T|} (1/h)^{|Q+R-T|}}{4^{|Q+R-T|} |T|^{r}} \leq \frac{(8e^{-1} C)^{|Q+R-T|} (1/h)^{|Q+R-T|}}{4^{|Q+R-T|} |T|^{r}} \leq (8e^{-1} C)^{|Q+R+T|}.$$

So given $U(1)$-invariant polynomials $f = \sum_{P,Q} f_{P,Q} b_{P,Q}$ and $g = \sum_{R,S} g_{R,S} b_{R,S}$ on $\mathbb{C}^{1+n}$ with complex coefficients $f_{P,Q}$ and $g_{R,S}$, then

$$||f \ast_h g||_{r} \leq \sum_{P,Q \in \mathbb{N}_0^{1+n}} |f_{P,Q}| (8e^{-1} C)^{|P+Q|} \sum_{R,S \in \mathbb{N}_0^{1+n}} |g_{R,S}| (8e^{-1} C)^{|R+S|} = ||f||_{8e^{-1} C^{r}} ||g||_{8e^{-1} C^{r}} \Box$$

We would like to remark that, similar as in [12], one can also use the description of the star product using bidifferential operators to prove its continuity.

**Theorem 5.25** For every $h \in \Omega$, the product $*_\text{red,}h$ on $\mathcal{A}(M_{\text{red}})$ extends to a continuous associative product on $\mathcal{A}(M_{\text{red}})$. Moreover $\mathcal{A}(M_{\text{red}})$ becomes a unital Fréchet-°-algebra with this product and pointwise complex conjugation as °-involution in the case that $h \in \Omega \cap \mathbb{R}$. Finally, for any two fixed elements $f, g \in \mathcal{A}(M_{\text{red}})$ and $[\rho] \in M_{\text{red}}$, the map $\Omega \to \mathbb{C}, h \mapsto (f \ast_{\text{red,}h} g)([\rho])$ is holomorphic.
Proof: By the previous Proposition 5.24 and the discussion above, the associative product $\star_{\text{red}, \hbar}$ is continuous on $\mathcal{D}(M_{\text{red}})$ with respect to the topology inherited from $\mathcal{A}(M_{\text{red}})$, and thus extends to an associative and continuous product on $\mathcal{A}(M_{\text{red}})$ because $\mathcal{D}(M_{\text{red}})$ is dense in $\mathcal{A}(M_{\text{red}})$ by Corollary 4.17. The constant 1-function remains the unit like on polynomials. Compatibility with the $^*$-involution is clear as well if $\hbar$ is additionally real.

Now recall that for polynomials $p, q \in \mathcal{D}(M_{\text{red}})$, the map $\hbar \mapsto (p \star_{\text{red}, \hbar} q)([\rho])$ is rational by Corollary 5.22. Since the estimates in Proposition 5.24 are locally uniform in $\hbar$, it follows that $\hbar \mapsto (f \star_{\text{red}, \hbar} g)([\rho])$ is a locally uniform limit of rational functions and therefore holomorphic. □

Note that $0 \notin \Omega$, so one would like to understand whether in the limit $\hbar \to 0$, the product $\star_{\text{red}, \hbar}$ yields the pointwise one, and whether its commutator yields the Poisson bracket also on $\mathcal{A}(M_{\text{red}})$. Despite the results from Corollary 5.22 and Proposition 5.24 in the polynomial case, this is not so obvious because $0$ is an accumulation point of the poles of $\star_{\text{red}, \hbar}$. We will come back to this question later in Proposition 6.24.

6 Wick Rotation

The dependence on the choice of signature $s$ will now always be made explicit by a superscript $^{(s)}$.

We have already seen that the construction of the formal and non-formal star products on $M^{(s)}_{\text{red}}$ works completely independent of $s \in \{1, \ldots, 1+n\}$. We will see now that the non-formal star product algebras are even all isomorphic as unital complex algebras. This will be proven by construction of a Wick transformation: A holomorphic isomorphism between the complex manifolds $\hat{M}^{(s)}_{\text{red}}$ for different values of $s$ which gives rise to isomorphisms of the algebras $\mathcal{D}(M^{(s)}_{\text{red}})$ as well as $\mathcal{A}(M^{(s)}_{\text{red}})$ (with the pointwise product) and which are also compatible with the Poisson brackets and the non-formal star products, i.e. describe isomorphisms of Poisson algebras and associative algebras, respectively. However, we will also see that these isomorphisms are not compatible with the $^*$-involution which is given by pointwise complex conjugation, hence are not $^*$-isomorphisms. This demonstrates how important it is to consider $^*$-algebras and not just algebras in non-formal deformation quantization: After all, one would surely want to be able to distinguish the quantization of the complex projective space $\mathbb{CP}^n$ from the one of the hyperbolic disc $\mathbb{D}^n$.

6.1 Geometric Wick Rotation

We start first with discussing the complex manifolds $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ and then proceed to $\hat{M}^{(s)}_{\text{red}}$.

The Lie group $\text{GL}(1+n, \mathbb{C}) \times \text{GL}(1+n, \mathbb{C})$ acts on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ from the left via $\cdot \triangleright \cdot$, which induces a right action $\cdot \triangleleft \cdot$ on $\mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})$. It is easy to check that $\Delta(A \triangleright \rho) = (A, \overline{A}) \triangleright \Delta(\rho)$ for all $\rho \in \mathbb{C}^{1+n}$ and all $A \in \text{GL}(1+n, \mathbb{C})$, where $\overline{A}$ denotes the elementwise complex conjugate of $A$. For all $s \in \{1, \ldots, 1+n\}$, let $W^{(s)} \in \text{GL}(1+n, \mathbb{C})$ be $(W^{(s)})^k_{\ell} := 1$ if $k = \ell \in \{0, \ldots, s-1\}$ and $(W^{(s)})^k_{\ell} := i$ if $k = \ell \in \{s, \ldots, n\}$, and otherwise $(W^{(s)})^k_{\ell} := 0$. Then the action of $(W^{(s)}, W^{(s)})$ on $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}$ does not come from an action on $\mathbb{C}^{1+n}$, except in the trivial case that $s = 1+n$. However, the identity

$$\mathcal{J}^{(s)} \triangleleft (W^{(s)}, W^{(s)}) = \mathcal{J}^{(1+n)}$$ (6.1)
holds and thus the holomorphic automorphism of \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) that is given by the action of 
\((W^{(s)}, W^{(s)})\) restricts to a holomorphic isomorphism from \( \hat{Z}^{(1+n)} \) to \( \hat{Z}^{(s)} \). It is then immediate
that this restriction even descends to a holomorphic isomorphism from \( \hat{M}_{\text{red}}^{(1+n)} \) to \( \hat{M}_{\text{red}}^{(s)} \), because
\((W^{(s)}, W^{(s)})\) commutes with all elements of the \( \mathbb{C}^*\)-subgroup of \( \text{GL}(1 + n, \mathbb{C}) \times \text{GL}(1 + n, \mathbb{C}) \).

**Definition 6.1** For every \( s \in \{1, \ldots, 1+n\} \) we define the map \( \alpha^{(s)} : \hat{M}_{\text{red}}^{(1+n)} \to \hat{M}_{\text{red}}^{(s)} \),

\[
[(\xi, \eta)] \mapsto \alpha^{(s)}([(\xi, \eta)]) := [(W^{(s)}, W^{(s)})(\xi, \eta)].
\] (6.2)

The above discussion shows that \( \alpha^{(s)} \) is well-defined and even more:

**Proposition 6.2** The maps \( \alpha^{(s)} : \hat{M}_{\text{red}}^{(1+n)} \to \hat{M}_{\text{red}}^{(s)} \) are holomorphic isomorphisms of complex mani-

folds for all \( s \in \{1, \ldots, 1+n\} \).

Moreover, Equation (6.1) also shows that the inner automorphism of the Lie group \( \text{GL}(1 + n, \mathbb{C}) \times \text{GL}(1 + n, \mathbb{C}) \) that is given by conjugation with \( (W^{(s)}, W^{(s)}) \), i.e.

\[
(A, B) \mapsto (W^{(s)} A (W^{(s)})^{-1}, W^{(s)} B (W^{(s)})^{-1}),
\] (6.3)

restricts to an isomorphism from \( G_{\gamma^{(1+n)}} \) to \( G_{\gamma^{(s)}} \). Note that we have already seen in Section 3 that
\( G_{\gamma^{(s)}} \) is isomorphic to \( \text{GL}(1 + n, \mathbb{C}) \) for all \( s \in \{1, \ldots, 1+n\} \). However, this way we can see the holomorphic isomorphism \( (x, y) \mapsto (W^{(s)} x, W^{(s)} y) \) of \( \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \) as an \( \text{GL}(1 + n, \mathbb{C}) \)-equivariant holomorphic isomorphism along the automorphism (6.2) of \( \text{GL}(1 + n, \mathbb{C}) \).

As a final remark, we note that the isomorphisms of \( \hat{M}_{\text{red}}^{(s)} \) with different signature \( s \) clearly do not descend to isomorphisms of \( M_{\text{red}}^{(s)} \). For example, \( M_{\text{red}}^{(1+n)} \cong \mathbb{C}P^n \) is compact while \( M_{\text{red}}^{(1)} \cong \mathbb{D}^n \) is not.

### 6.2 Algebraic Wick Rotation

The isomorphism of the complex manifolds \( \hat{M}_{\text{red}}^{(s)} \) for different signatures from Proposition 6.2 immediate-

ly shows that the corresponding unital associative algebras \( \mathcal{O}(\hat{M}_{\text{red}}^{(s)}) \) are also isomorphic. By
Proposition 4.12 the algebras \( \mathcal{A}(\hat{M}_{\text{red}}^{(s)}) \) for different signatures are isomorphic as unital associative
algebras as well (but not necessarily as \(*\)-algebras).

**Definition 6.3** For every \( s \in \{1, \ldots, 1+n\} \) we define the maps \( \Phi^{(s)} : \mathcal{A}(\mathbb{C}^{1+n}) \to \mathcal{A}(\mathbb{C}^{1+n}) \),

\[
f \mapsto \Phi^{(s)}(f) := \Delta^* (\hat{f} \circ (W^{(s)}, W^{(s)}))
\] (6.4)

as well as \( \Phi_{\text{red}}^{(s)} : \mathcal{A}(M_{\text{red}}^{(s)}) \to \mathcal{A}(M_{\text{red}}^{(1+n)}) \),

\[
g \mapsto \Phi_{\text{red}}^{(s)}(g) := (\Delta_{\text{red}}^{(1+n)})^* (\hat{g} \circ \alpha^{(s)})
\] (6.5)

where \( \hat{f} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n}) \) and \( \hat{g} \in \mathcal{O}(\hat{M}_{\text{red}}^{(1+n)}) \) are such that \( \Delta^* (\hat{f}) = f \) and \( (\Delta_{\text{red}}^{(s)})^* (\hat{g}) = g \). We will refer to \( \Phi^{(s)} \) and \( \Phi_{\text{red}}^{(s)} \) as the Wick rotation and the reduced Wick rotation, respectively.

Proposition 4.12 and Proposition 6.2 together with the observation that \( (W^{(s)}, W^{(s)}) \) commutes with
the whole \( \mathbb{C}^*\)-subgroup of \( \text{GL}(1 + n, \mathbb{C}) \times \text{GL}(1 + n, \mathbb{C}) \) immediately shows:
Theorem 6.4 The Wick rotation $\Phi^{(s)}$ is a well-defined homeomorphic automorphism of the unital associative Fréchet algebra $\mathcal{A}(\mathbb{C}^{1+n})$ that restricts to an automorphism of $\mathcal{A}(\mathbb{C}^{1+n})^{U(1)}$, and the reduced Wick rotation $\Phi^{(s)}_{\text{red}}: \mathcal{A}(M^{(s)}_{\text{red}}) \rightarrow \mathcal{A}(M^{(1+n)}_{\text{red}})$ is a well-defined homeomorphic isomorphism of unital associative Fréchet algebras.

The Wick rotations are also compatible with the reduction procedure:

Proposition 6.5 Given $f \in \mathcal{A}(\mathbb{C}^{1+n})^{U(1)}$, then

$$
(\Phi^{(s)}(f))_{\text{red}} = \Phi^{(s)}_{\text{red}}(f_{\text{red}}).
$$

Proof: By Proposition 4.14 there exists an $\hat{f} \in \mathcal{O}(\mathbb{C}^{1+n} \times \mathbb{C}^{1+n})^{\mathbb{Z}^*}$ like in Lemma 4.13 such that $\Delta^*(\hat{f}) = f$ and $(\Delta^*_{\text{red}})^*(\hat{f}_{\text{red}}) = f_{\text{red}}$. Using the commutativity of the diagram from Section 3 and the properties of the action of $(W^{(s)}, W^{(s)})$ one finds:

$$
(\iota^{(1+n)}(\Phi^{(s)}(f))) = (\iota^{(1+n)}(\Delta^*(\hat{f} \triangleleft (W^{(s)}, W^{(s)})))
= (\Delta_{\mathbb{Z}}^{(1+n)})^*(\iota^{(1+n)}(\hat{f} \triangleleft (W^{(s)}, W^{(s)})))
= (\Delta_{\mathbb{Z}}^{(1+n)})^*((\iota^{(s)}(\hat{f} \triangleleft W^{(s)})))
= (\Delta_{\mathbb{Z}}^{(1+n)})^*((\pi^{(s)}(\hat{f}_{\text{red}} \triangleleft (W^{(s)}, W^{(s)})))
= (\pi^{(1+n)}(\Delta_{\text{red}}^{(1+n)})(\hat{f}_{\text{red}} \triangleleft \alpha^{(s)})))
= (\pi^{(1+n)}(\Delta_{\text{red}}^{(1+n)})(\hat{f}_{\text{red}} \triangleleft \alpha^{(s)})))
= (\pi^{(1+n)}(\Phi_{\text{red}}^{(s)}(f_{\text{red}}))).
$$

In the following we will see that the Wick rotations are not only isomorphisms of unital associative algebras, but also compatible with Poisson brackets and star products:

Lemma 6.6 Given $s \in \{1, \ldots, 1+n\}$, then the identity

$$
\Phi^{(s)}(b_{P,Q}) = i^{\sum_{k=1}^{n}(P_k+Q_k)}b_{P,Q}
$$

holds for all $P, Q \in \mathbb{N}_0^{1+n}$,

$$
\Phi_{\text{red}}^{(s)}(b_{P,Q;\text{red}}) = i^{\sum_{k=1}^{n}(P_k+Q_k)}b_{P,Q;\text{red}}^{(1+n)}
$$

holds for all $P, Q \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|$, and

$$
\Phi_{\text{red}}^{(s)}(c_{P,Q}) = i^{\sum_{k=1}^{n}(P_k+Q_k)}c_{P,Q}^{(1+n)}
$$

holds for all $P, Q \in \mathbb{N}_0^n$. Moreover, $\Phi^{(s)}$ restricts to an automorphism of the unital subalgebra $\mathcal{P}(\mathbb{C}^{1+n})$ of $\mathcal{A}(\mathbb{C}^{1+n})$, and $\Phi_{\text{red}}^{(s)}$ restricts to a an isomorphism from the unital subalgebra $\mathcal{P}(M^{(s)}_{\text{red}})$ of $\mathcal{A}(M^{(s)}_{\text{red}})$ to the unital subalgebra $\mathcal{P}(M^{(1+n)}_{\text{red}})$ of $\mathcal{A}(M^{(1+n)}_{\text{red}})$.
PROOF: Using $b_{P,Q} = \Delta^\ast(x^P y^Q)$ with $x^0, \ldots, x^n, y^0, \ldots, y^n: \mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \to \mathbb{C}$ the standard coordinates, it is easy to check that Equation (6.6) holds. Equation (6.7) then follows by applying the previous Proposition 6.5 which gives Equation (6.8) as a special case. The rest is clear. \hfill \Box

Theorem 6.7 The Wick rotations remain isomorphisms of unital associative algebras also for the deformed products. More precisely, given $s \in \{1, \ldots, 1+n\}$, then the identities

$$\Phi^{(s)}(f \ast^h_s g) = \Phi^{(s)}(f) \ast^h_s \Phi^{(s)}(g)$$

(6.9)

and

$$\Phi^{(s)}(\{ f, g \}) = \{ \Phi^{(s)}(f), \Phi^{(s)}(g) \}^{(1+n)}$$

(6.10)

hold for all $f, g \in \mathcal{A}(\mathbb{C}^{1+n})$ and all $h \in \Omega$. Similarly, the identities

$$\Phi^{(s)}_{\text{red}}(f \ast^h_{\text{red},h} g) = \Phi^{(s)}_{\text{red}}(f) \ast^h_{\text{red},h} \Phi^{(s)}_{\text{red}}(g)$$

(6.11)

and

$$\Phi^{(s)}(\{ f, g \})_{\text{red}} = \{ \Phi^{(s)}(f), \Phi^{(s)}(g) \}_{\text{red}}^{(1+n)}$$

(6.12)

hold for all $f, g \in \mathcal{A}(M^{(s)}_{\text{red}})$ and all $h \in \Omega$ as well as for all $f, g \in \mathcal{A}^{(k)}(M^{(s)}_{\text{red}})$ and $h = \frac{1}{k}$ with $k \in \mathbb{N}$.

PROOF: First note that as a consequence of the previous Lemma 6.6 the identity

$$\Phi^{(s)}(\text{sgn}^{(s)}(T)b_{P+R-T,Q+S-T}) = i^{\sum_{k=1}^n(P_k+Q_k+R_k+S_k)}b_{P+R-T,Q+S-T}$$

holds for all $P, Q, R, S, T \in \mathbb{N}_0^{1+n}$ with $T \leq \min\{Q, R\}$, and similarly,

$$\Phi^{(s)}_{\text{red}}(\text{sgn}^{(s)}(T)b_{P+R-T,Q+S-T;\text{red}}) = i^{\sum_{k=1}^n(P_k+Q_k+R_k+S_k)}b^{(1+n)}_{P+R-T,Q+S-T;\text{red}}$$

holds for all $P, Q, R, S, T \in \mathbb{N}_0^{1+n}$ with $|P| = |Q|, |R| = |S|$ and $T \leq \min\{Q, R\}$. Using this and the explicit formulas from Lemma 5.18 and Proposition 5.21 and noting that $\text{sgn}^{(1+n)}(T) = 1$ for all $T \in \mathbb{N}_0^{1+n}$, it is easy to check the identities for the star products in the special case that $f$ and $g$ are monomials. The identities for the Poisson brackets are an immediate consequence thereof due to the representation of the Poisson brackets as a limit of the star product commutator like in Proposition 5.22. The general case then follows by bilinearity and continuity of the star product and the Poisson bracket. \hfill \Box

While it is completely clear that the Wick rotations do not commute with the $*$-involution given by pointwise complex conjugation, it is somewhat harder to show that the algebras $\mathcal{A}(M^{(s)}_{\text{red}})$ with product $\ast^{(s)}_{\text{red},h}$ are in general not $*$-isomorphic, not even via some other isomorphism. One possibility to prove this is to examine their positive linear functionals: A linear functional $\phi: \mathcal{A}(M^{(s)}_{\text{red}}) \to \mathbb{C}$ is called positive for the product $\ast^{(s)}_{\text{red},h}$ with $h \in \Omega \cap \mathbb{R}$ if

$$\phi(T \ast^{(s)}_{\text{red},h} f) \geq 0$$

(6.13)
holds for all \( f \in \mathcal{A}(M_{\text{red}}(s)). \) It is easy to see that the pullback of a positive linear functional with a \(*\)-homomorphism between two \(*\)-algebras yields again a positive linear functional. In the special case of \( s = 1, \) i.e. \( M_{\text{red}}(1) \cong \mathbb{D}^n, \) the existence of non-trivial positive linear functionals for negative \( \hbar \) is known:

**Proposition 6.8** The evaluation functionals \( \delta_{[\rho]}^{(1)} : \mathcal{A}(\mathbb{D}^n) \to \mathbb{C}, \)

\[
f \mapsto \delta_{[\rho]}^{(1)}(f) := f([\rho])
\]

with \( [\rho] \in M_{\text{red}}(1) \cong \mathbb{D}^n \) are positive linear functionals on \( \mathcal{A}(\mathbb{D}^n) \) with product \( \star_{\text{red},\hbar}^{(1)} \) for all \( \hbar \in ]-\infty, 0[. \)

**Proof:** Positivity of evaluation functionals has been proven in [3, Sec. 5.4] on an algebra containing (at least) \( \mathcal{P}(\mathbb{D}^n) \) with a product \( \star_{\hbar} \) fulfilling \( f \star_{\text{red},\hbar}^{(1)} g = g \star_{-\hbar} f \) for all \( f, g \in \mathcal{P}(\mathbb{D}^n). \) By continuity of the evaluation functionals, the pointwise complex conjugation and the product \( \star_{\text{red},\hbar}^{(1)}, \) this extends to whole \( \mathcal{A}(\mathbb{D}^n). \)

However, there are some limitations to the existence of positive linear functional in the special case of \( s = 1 + n, \) i.e. \( M_{\text{red}}(1+n) \cong \mathbb{C}F^n, \) at least if \( n = 1: \)

**Lemma 6.9** Consider only the case \( n = 1 \) and \( s = 1 + n = 2. \) Then the identity

\[
\sum_{i,j=0}^{1} b_{E_i,E_j,\text{red}}^{(2)} \star_{\text{red},\hbar}^{(2)} b_{E_i,E_j,\text{red}}^{(2)} = 1 + \hbar \quad (6.14)
\]

holds for all \( \hbar \in \mathbb{R} \setminus \{0\}, \) where \( E_0 = (1,0) \in \mathbb{N}_0^{1+1} \) and \( E_1 = (0,1) \in \mathbb{N}_0^{1+1}. \)

**Proof:** Because of the low degree of the involved polynomials, their \( \star_{\text{red},\hbar}^{(2)} \)-product is indeed well-defined for all \( \hbar \in \mathbb{R} \setminus \{0\} \) (including the poles that usually occur for polynomials of higher degree) and Proposition [5.21] yields

\[
b_{E_i,E_j,\text{red}}^{(2)} \star_{\text{red},\hbar}^{(2)} b_{E_i,E_j,\text{red}}^{(2)} = \frac{(1/\hbar)_{\perp 2}}{(1/\hbar)_{\perp 1}} b_{E_i+E_j,E_i+E_j,\text{red}}^{(2)} + \frac{(1/\hbar)_{\perp 1}}{(1/\hbar)_{\perp 1}} b_{E_i,E_j,\text{red}}^{(2)} = (1 - \hbar) b_{E_i+E_j,E_i+E_j,\text{red}}^{(2)} + \hbar b_{E_i,E_j,\text{red}}^{(2)}
\]

for all \( i, j \in \{0,1\}. \) By summation over \( i \) and \( j \) we get

\[
\sum_{i,j=0}^{1} b_{E_i,E_j,\text{red}}^{(2)} \star_{\text{red},\hbar}^{(2)} b_{E_i,E_j,\text{red}}^{(2)} = (1 - \hbar) \left( b_{E_0+E_1,E_0+E_1,\text{red}}^{(2)} + 2b_{E_0+E_1,E_0+E_1,\text{red}}^{(2)} + b_{2E_1,2E_1,\text{red}}^{(2)} \right) + 2\hbar \left( b_{E_0,E_0,\text{red}}^{(2)} + b_{E_1,E_1,\text{red}}^{(2)} \right).
\]

Keeping in mind that the reduced monomials are not linearly independent, this can be simplified: We find that \( b_{E_0,E_0,\text{red}}^{(2)} + b_{E_1,E_1,\text{red}}^{(2)} = \varphi^{(2)}(\mathcal{R}) \) is the constant \( 1\)-function, and the same is true for their pointwise square \( b_{E_0,E_0,\text{red}}^{(2)} + b_{E_1,E_1,\text{red}}^{(2)} \)

\[
(6.14) = b_{E_0,E_0,\text{red}}^{(2)} + 2b_{E_0+E_1,E_0+E_1,\text{red}}^{(2)} + b_{2E_1,2E_1,\text{red}}^{(2)} = b_{2E_0,E_0,\text{red}}^{(2)} + 2b_{E_0+E_1,E_0+E_1,\text{red}}^{(2)} + b_{2E_1,2E_1,\text{red}}^{(2)}.
\]

\[\square\]

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Proposition 6.10 Consider only the case \( n = 1 \) and \( h \in [-\infty, -1[\). For signature \( s = 2 \), the only linear functional \( \phi: \mathcal{A}(\mathbb{C}^1) \to \mathbb{C} \), which is positive for the product \( \star_{\text{red}, h}^{(2)} \), is \( \phi = 0 \). But for signature \( s = 1 \), the evaluation functionals from Proposition 6.8 are non-trivial positive linear functionals for the product \( \star_{\text{red}, h}^{(1)} \) on \( \mathcal{A}(\mathbb{D}) \).

As a consequence, the \( \ast \)-algebra \( \mathcal{A}(\mathbb{D}) \) with product \( \star_{\text{red}, h}^{(1)} \) and pointwise complex conjugation as \( \ast \)-involution is not \( \ast \)-isomorphic to the \( \ast \)-algebra \( \mathcal{A}(\mathbb{C}^1) \) with product \( \star_{\text{red}, h}^{(2)} \) and pointwise complex conjugation as \( \ast \)-involution.

**Proof:** Let \( \phi: \mathcal{A}(\mathbb{C}^1) \to \mathbb{C} \) be a positive linear functional for the product \( \star_{\text{red}, h}^{(2)} \). Then the previous Lemma 6.9 shows that there exist functions \( f_1, \ldots, f_4 \in \mathcal{A}(\mathbb{C}^1) \) such that

\[
0 \leq \sum_{k=1}^{4} \phi(f_k) = (1 + h) \phi(1) = (1 + h) \phi(T_{\star_{\text{red}, h}}^{(2)} 1) \leq 0
\]

holds because \( h < -1 \), so \( \phi(1) = 0 \). But then the Cauchy Schwarz inequality applied to the (possibly degenerate) inner product \( \mathcal{A}(\mathbb{C}^1) \ni (f, g) \mapsto \phi(T_{\star_{\text{red}, h}}^{(2)} g) \in \mathbb{C} \) shows that

\[
|\phi(f)|^2 = |\phi(T_{\star_{\text{red}, h}}^{(2)} f)|^2 \leq \phi(T_{\star_{\text{red}, h}}^{(2)} 1) \phi(T_{\star_{\text{red}, h}}^{(2)} f) = \phi(1) \phi(T_{\star_{\text{red}, h}}^{(2)} f) = 0
\]

holds for all \( f \in \mathcal{A}(\mathbb{C}^1) \) and therefore \( \phi = 0 \). The rest is clear. \( \square \)

6.3 Applications

In this section we use the reduced Wick rotation to transfer some of the results obtained in [17] for the special case of the hyperbolic disc, i.e. \( s = 1 \), to general signatures. Note that one could also check that all the proofs in [17] work for an arbitrary signature, but the Wick rotation provides a more elegant way to generalize these results. We will again drop the superscripts \(^{(s)}\) most of the time, the following is valid for every choice of signature \( s \in \{1, \ldots, 1+n\} \).

Proposition 6.11 The fundamental monomials \( c_{P,Q} \) with \( P, Q \in \mathbb{N}_0^n \) form an absolute Schauder basis of \( \mathcal{A}(M_{\text{red}}) \). More precisely, every \( f \in \mathcal{A}(M_{\text{red}}) \) can be expanded in a unique way as a series

\[
f = \sum_{P,Q \in \mathbb{N}_0^n} f_{P,Q} c_{P,Q} \tag{6.15}
\]

with complex coefficients \( f_{P,Q} \) that fulfill the estimate

\[
||f||_{\text{red}, r} := \sum_{P,Q \in \mathbb{N}_0^n} |f_{P,Q}|^r |P| + |Q| < \infty \tag{6.16}
\]

for all \( r \in [1, \infty[ \). Moreover, the topology of \( \mathcal{A}(M_{\text{red}}) \) (i.e. the topology of locally uniform convergence of the holomorphic extensions to \( M_{\text{red}} \)) can equivalently be described by these seminorms \( ||f||_{\text{red}, r} \) and
the coefficients $f_{P,Q}$ can be calculated explicitly by means of the integral formula

$$f_{P,Q} = \frac{1}{(-4\pi^2)^n} \int_C \cdots \int_C \frac{1}{u^{P+1}\cdots u^{n+1}} \left(1 + \sum_{k=1}^n \nu_k u^k v^k\right)^{\max(|P|,|Q|)-1} d^n u \wedge d^n v.$$  

(6.17)

Here $f \in \mathcal{A}(M_{\text{red}})$ and $\hat{f} \in \mathcal{O}(M_{\text{red}})$ satisfies $\Delta^*_{\text{red}}(\hat{f}) = f$. The coordinates $u$ and $v$ were defined in Equation (3.13) and $C \subseteq \mathbb{C}$ is, in projective coordinates, a circle around zero with radius in $[0, 1/\sqrt{n}]$.

**PROOF:** For $s = 1$ this is exactly the statement of [17.3.16]. Because the Wick rotation is a homeomorphic isomorphism and using Lemma 6.6, the generalization to arbitrary signatures is immediately clear for everything except the integral formula. In order to prove that (6.17) holds, we have to check that it is compatible with the holomorphic isomorphisms $\alpha^{(s)}$:

We have to use superscripts $(s)$ again to indicate the signature $s$. As $u^{(s)}, k \circ \alpha^{(s)} = u^{(1+n), k}$ and $v^{(s)}, k \circ \alpha^{(s)} = v^{(1+n), k}$ for all $k \in \{1, \ldots, s-1\}$ as well as $u^{(s)}, k \circ \alpha^{(s)} = iv^{(1+n), k}$ and $v^{(s)}, k \circ \alpha^{(s)} = iv^{(1+n), k}$ for all $k \in \{s, \ldots, n\}$ hold, we get

$$ (\alpha^{(s)})^* (d^n u^{(s)} \wedge d^n v^{(s)}) = (-1)^{n+1-s} d^n u^{(1+n)} \wedge d^n v^{(1+n)} $$

as well as

$$ (\alpha^{(s)})^* \left(1 + \sum_{k=1}^n v^{(s), k} u^{(s), k} v^{(s), k}\right) = 1 + \sum_{k=1}^n v^{(1+n), k} u^{(1+n), k} v^{(1+n), k} $$

and

$$ (\alpha^{(s)})^* \left((u^{(s)})^{P+1} v^{(1+n)} Q^{(1+n)}\right) = \sum_{k=0}^n (P_k + Q_k) (u^{(1+n)})^{P+(1+n)} (v^{(1+n)})^{Q+(1+n)} $$

for all $P, W \in \mathbb{N}_0^n$. So given $\hat{f} \in \mathcal{O}(M_{\text{red}}^{(s)})$, then the right-hand side of (6.17) for $\hat{f}$ in signature $s$ gives the same result as for $\hat{f} \circ \alpha^{(s)}$ in signature $1+n$ times the factor $(-i)\sum_{k=0}^n (P_k + Q_k)$. This matches precisely with Lemma 6.6 which shows that

$$ \Phi^{(s)}_{\text{red}}(f) = \sum_{P,Q} f_{P,Q} \Phi^{(s)}_{\text{red}}(b^{(s)}_{P,Q}) = \sum_{P,Q} f_{P,Q} \sum_{k=0}^n (P_k + Q_k) b^{(1+n)}_{P,Q} $$

for all $f \in \mathcal{A}(M_{\text{red}})$ with expansion coefficients $f_{P,Q}$, i.e. that the $(P,Q)$-coefficient of $f$ is the same as the $(P,Q)$-coefficient of $\Phi^{(s)}_{\text{red}}(f)$ times the factor $(-i)\sum_{k=0}^n (P_k + Q_k)$. This way, one first sees that (6.17) holds not only for signature $s = 1$ but also for $s = 1+n$, and then that it even holds for all $s \in \{1, \ldots, n+1\}$. \square

We would now like to generalize Corollary 5.22 and Proposition 5.23 for analytic functions. For some function $f : \mathbb{R} \to \mathbb{C}$ the limit when $h$ approaches 0 from the left is denoted by $\lim_{h \to 0^-} f(h)$ (if it exists).

**Proposition 6.12** For any two analytic functions $f, g \in \mathcal{A}(M_{\text{red}})$ the limits $\lim_{h \to 0^-} f \ast_{\text{red}, h} g$ and $\lim_{h \to 0^-} \frac{1}{h} (f \ast_{\text{red}, h} g - g \ast_{\text{red}, h} f)$ exist. They are given by

$$ \lim_{h \to 0^-} f \ast_{\text{red}, h} g = fg $$

(6.18a)
and

\[ \lim_{h \to 0^-} \frac{1}{ih}(f \ast_{\text{red},h} g - g \ast_{\text{red},h} f) = \{ f, g \}_\text{red} \]  

(6.18b)

with the reduced Poisson bracket \( \{ \cdot, \cdot \}_\text{red} \) on \( M_{\text{red}} \).

**Proof:** This was proven in [17, Thm. 4.5] in the special case of signature \( s = 1 \) for a product \( \ast_{h} \) with \( -h \in \Omega \) fulfilling \( f \ast_{\text{red},h} g = g \ast_{-h} f \) for all \( f, g \in \mathcal{A}(M_{\text{red}}) \) and the corresponding Poisson bracket \( \{ \cdot, \cdot \}_s = -\{ \cdot, \cdot \}_\text{red} \). The statements for arbitrary signatures \( s \) follow immediately from Theorem 6.4 and Theorem 6.7. \( \square \)

A Symmetrized Covariant Derivatives

On a smooth manifold \( M \) we define the spaces of tensor fields

\[ \mathcal{S}^t(M) := \Gamma^\infty(S^t T^* M) \quad \text{and} \quad (\mathcal{A} \otimes \mathcal{S})^{k,\ell}(M) := \Gamma^\infty(\Lambda^k T^* M \otimes S^\ell T^* M) \]

for all \( k, \ell \in \mathbb{Z} \), as well as the \( \mathbb{Z} \)-graded algebra \( \mathcal{S}^\bullet(M) := \bigoplus_{\ell \in \mathbb{Z}} \mathcal{S}^\ell(M) \) with the usual pointwise symmetric tensor product \( \vee \) as well as the \( \mathbb{Z}^2 \)-graded algebra \( (\mathcal{A} \otimes \mathcal{S})^{\bullet,\bullet}(M) := \bigoplus_{k, \ell \in \mathbb{Z}} (\mathcal{A} \otimes \mathcal{S})^{k,\ell}(M) \) with product \( \circ \) given by the combination of the pointwise antisymmetric and symmetric tensor products. In order to define graded commutators, a \( \mathbb{Z}_2 \)-grading on these two algebras is needed: In the case of \( \mathcal{S}^\bullet(M) \), this is the trivial one, in which all elements of \( \mathcal{S}^\bullet(M) \) have even degree, and on \( (\mathcal{A} \otimes \mathcal{S})^{\bullet,\bullet}(M) \) we consider the antisymmetric degree only, i.e. all elements of \( (\mathcal{A} \otimes \mathcal{S})^{k,\ell}(M) \) with \( k \in 2\mathbb{Z}, \ell \in \mathbb{Z} \) have even degree and all elements of \( (\mathcal{A} \otimes \mathcal{S})^{k,\ell}(M) \) with \( k \in 1 + 2\mathbb{Z}, \ell \in \mathbb{Z} \) have odd degree. This way, both \( \mathcal{S}^\bullet(M) \) and \( (\mathcal{A} \otimes \mathcal{S})^{\bullet,\bullet}(M) \) are generated as graded commutative. For later use we also define the total degree \( \text{Deg} \) on \( (\mathcal{A} \otimes \mathcal{S})^{\bullet,\bullet}(M) \) by setting

\[ \text{Deg} \Omega = (k + \ell)\Omega \]  

(A.1)

for all \( \Omega \in (\mathcal{A} \otimes \mathcal{S})^{k,\ell}(M) \) with \( k, \ell \in \mathbb{Z} \). Clearly \( \text{Deg} \) is a graded derivation of degree \((0, 0)\). Note that \( \mathcal{S}^0(M) \cong \mathcal{C}^\infty(M) \cong (\mathcal{A} \otimes \mathcal{S})^{0,0}(M) \) and that \( \mathcal{S}^\bullet(M) \) is generated as a complex algebra by \( \mathcal{S}^0(M) \oplus \mathcal{S}^1(M) \), whereas \( (\mathcal{A} \otimes \mathcal{S})^{\bullet,\bullet}(M) \) is generated as a complex algebra by \( (\mathcal{A} \otimes \mathcal{S})^{0,0}(M) \oplus (\mathcal{A} \otimes \mathcal{S})^{1,0}(M) \oplus (\mathcal{A} \otimes \mathcal{S})^{0,1}(M) \).

We will need two other operators, the Koszul differentials: There are unique \( \mathbb{C} \)-linear graded derivations \( \delta, \delta^* \) of \( (\mathcal{A} \otimes \mathcal{S})^{\bullet,\bullet}(M) \) of degree \((+1, -1)\) and \((-1, +1)\) that fulfill

\[ \delta(1 \otimes \omega) = \omega \otimes 1 \quad \text{as well as} \quad \delta^*(\rho \otimes 1) = 1 \otimes \rho, \]

(A.2)

respectively, for all \( \rho, \omega \in \Gamma^\infty(T^* M) \). In local coordinates, \( \delta(\rho \otimes \omega) = (dx^i \wedge \rho) \otimes (\partial/\partial x^i \omega) \) and \( \delta^*(\rho \otimes \omega) = (\partial/\partial x^i \rho) \otimes (dx^i \vee \omega) \) hold for all \( \rho \in \Gamma^\infty(\Lambda^k T^* M) \) and \( \omega \in \Gamma^\infty(S^\ell T^* M) \). Of course, \( \delta \) and \( \delta^* \) are not only \( \mathbb{C} \)-linear but even \( \mathcal{C}^\infty(M) \)-linear.

**Lemma A.1** For the graded commutators we have

\[ [\delta, \delta] = 2\delta^2 = 0, \quad [\delta^*, \delta^*] = 2(\delta^*)^2 = 0, \quad [\delta, \delta^*] = [\delta^*, \delta] = \text{Deg}, \]

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\[ [\deg, \delta] = -[\delta, \deg] = 0 \quad \text{and} \quad [\delta^*, \deg] = -[\deg, \delta^*] = 0. \]

**Proof:** One checks easily that this holds on \((\mathcal{A} \otimes \mathcal{J})^{0,0}(M), (\mathcal{A} \otimes \mathcal{J})^{1,0}(M)\) and \((\mathcal{A} \otimes \mathcal{J})^{0,1}(M)\). But graded derivations are already uniquely determined by how they act on these spaces. \(\square\)

One can also check that \(\delta\) and \(\delta^*\) commute with pull-backs. That is, whenever \(\Psi : M \to N\) is smooth, then \(\delta \circ \Psi^* = \Psi^* \circ \delta\) and \(\delta^* \circ \Psi^* = \Psi^* \circ \delta^*\) where \(\Psi^* : (\mathcal{A} \otimes \mathcal{J})^{**}(N) \to (\mathcal{A} \otimes \mathcal{J})^{**}(M)\) denotes the usual pull-back.

Next we consider the insertion of vector fields into the antisymmetric and symmetric part: Given \(X \in \Gamma^\infty(TM)\), then there exist unique \(\mathbb{C}\)-linear graded derivations \(\iota_X^{(a)}, \iota_X^{(s)}\) of \((\mathcal{A} \otimes \mathcal{J})^{**}(M)\) of degree \((-1,0)\) and \((0,-1)\) that fulfil

\[
\iota_X^{(a)}(\rho \otimes 1) = \langle \rho, X \rangle \quad \text{as well as} \quad \iota_X^{(s)}(1 \otimes \omega) = \langle \omega, X \rangle,
\]

respectively, for all \(\rho, \omega \in \Gamma^\infty(T^*M)\). Clearly, \(\iota_X^{(a)}\) and \(\iota_X^{(s)}\) are even \(\mathcal{C}^\infty(M)\)-linear and:

**Lemma A.2** For the graded commutators we have

\[
[\iota_X^{(a)}, \iota_Y^{(a)}] = [\iota_X^{(s)}, \iota_Y^{(s)}] = [\iota_X^{(s)}, \iota_Y^{(a)}] = [\iota_X^{(a)}, \delta^*] = [\iota_X^{(a)}, \delta] = 0,
\]

\[
[\iota_X^{(a)}, \delta] = \iota_X^{(s)}, \quad [\iota_X^{(s)}, \delta^*] = \iota_X^{(a)}, \quad [\deg, \iota_X^{(a)}] = -\iota_X^{(a)}, \quad [\deg, \iota_X^{(s)}] = -\iota_X^{(s)}
\]

for all \(X, Y \in \mathcal{C}^\infty(M)\).

**Proof:** These identities are easy to check on \((\mathcal{A} \otimes \mathcal{J})^{0,0}(M), (\mathcal{A} \otimes \mathcal{J})^{1,0}(M)\) and \((\mathcal{A} \otimes \mathcal{J})^{0,1}(M)\). \(\square\)

We see that the \(\mathbb{C}\)-linear span of \(\delta, \delta^*, \deg\) and all \(\iota_X^{(a)}\) and \(\iota_X^{(s)}\) with \(X \in \mathcal{C}^\infty(M)\) in the graded Lie algebra of \(\mathbb{C}\)-linear graded derivations of \((\mathcal{A} \otimes \mathcal{J})^{**}(M)\) is a graded Lie subalgebra. Now we can define exterior covariant derivatives:

**Definition A.3** A \(\mathbb{C}\)-linear graded derivation \(D\) of \((\mathcal{A} \otimes \mathcal{J})^{**}(M)\) of degree \((+1,0)\) that fulfils

\[
D(\rho \otimes 1) = d\rho \otimes 1 \quad \text{for all} \quad \rho \in \Gamma^\infty(\Lambda^1 \cdot T^*M)
\]

is called an exterior covariant derivative on \(M\).

For every covariant derivative \(\nabla\) on \(M\) there exists a unique exterior covariant derivative \(D^\nabla\) on \(M\) that fulfils

\[
\iota_X^{(a)} D^\nabla (1 \otimes \omega) = 1 \otimes \nabla_X \omega
\]

for all \(\rho \in \Gamma^\infty(\Lambda^1 \cdot T^*M), \omega \in \mathcal{J}^\ast(M)\) and \(X \in \Gamma^\infty(TM)\). In local coordinates,

\[
D^\nabla(\rho \otimes \omega) = d\rho \otimes \omega + (dx^i \wedge \rho) \otimes \nabla_{\partial_i} \omega
\]

for all \(\rho \in \Gamma^\infty(\Lambda^1 \cdot T^*M)\) and \(\omega \in \mathcal{J}^\ast(M)\). Conversely, every exterior covariant derivative \(D\) on \(M\) determines a unique covariant derivative \(\nabla^D\) on \(M\) that fulfils

\[
\langle \omega, \nabla^D_X Y \rangle = X(\langle \omega, Y \rangle) - \langle \nabla_X^D \omega, Y \rangle = X(\langle \omega, Y \rangle) - \iota_Y^{(a)} \iota_X^{(a)} D(1 \otimes \omega)
\]

(A.6)
for all \( X, Y \in \Gamma^\infty(TM) \) and all \( \omega \in \Gamma^\infty(T^*M) \). One can check that \( \nabla^{\mathcal{D}} \nabla = \nabla \) for every covariant derivative \( \nabla \) on \( M \) and that \( D^{\mathcal{D}} = D \) for every exterior covariant derivative on \( M \). So there is a 1-to-1 correspondence between covariant derivatives and exterior covariant derivatives.

We say that an exterior covariant derivative \( D \) is torsion-free if the associated covariant derivative \( \nabla^D \) is torsion-free.

**Lemma A.4** An exterior covariant derivative \( D \) on \( M \) is torsion-free if and only if \([D, \delta] = 0\).

**Proof:** Denote the torsion of \( \nabla^D \) by \( T \). We compute

\[
\iota_Y \iota_X [D, \delta](1 \otimes \omega) = \iota_Y \iota_X (d\omega \otimes 1) + \iota_Y \iota_X \delta (1 \otimes \omega) = 2(\iota_Y (d\omega), X \wedge Y) - \iota_Y \delta(1 \otimes \nabla_X \omega) + \iota_X \delta(1 \otimes \nabla_Y \omega) = 2(\iota_Y (d\omega), X \wedge Y) - \sigma_\nabla_X Y - \sigma_\nabla_Y X = \langle \omega, -[X, Y] + \sigma_\nabla X Y - \sigma_\nabla Y X \rangle.
\]

In particular, if \([D, \delta] = 0\), then \( \nabla^D \) is torsion-free. Conversely, if \( \nabla^D \) is torsion-free, then \([D, \delta] = 0\) vanishes on \((\mathcal{A} \otimes \Omega)^{0,1}(M)\) by the above calculation. But \([D, \delta] \) is a \( \mathbb{C} \)-linear graded derivation of \((\mathcal{A} \otimes \Omega)^{0,\bullet}(M)\) of degree \((-2, -1)\), so \([D, \delta] = 0\) in this case. \( \square \)

If \( g \in \Omega^2(M) \) is non-degenerate, then there exists a unique exterior covariant derivative \( D \) on \( M \) that fulfills \( D(1 \otimes g) = 0 = [D, \delta] \), namely the one corresponding to the Levi-Civita connection. This exterior Levi-Civita connection will be interesting for us:

**Lemma A.5** Let \( M \) be a smooth manifold, \( g \in \Gamma^\infty(S^2 T^*M) \) a non-degenerate symmetric tensor with Levi-Civita connection \( \nabla \), and \( \Phi : M \rightarrow M \) a diffeomorphism. If \( \nabla \Phi^*(g) = 0 \), then the exterior Levi-Civita connection \( D \) associated to \( g \) commutes with the pullback \( \Phi^* \), i.e. \( D \Phi^*(\Omega) = \Phi^*(D\Omega) \) for all \( \Omega \in (\mathcal{A} \otimes \Omega)^{0,\bullet}(M) \).

**Proof:** It suffices to show that \( D' : (\mathcal{A} \otimes \Omega)^{0,\bullet}(M) \rightarrow (\mathcal{A} \otimes \Omega)^{0,\bullet}(M), \Omega \rightarrow D'\Omega := (\Phi^{-1})^*(D\Phi^*(\Omega)) \) is an exterior covariant derivative and fulfills \( D'(1 \otimes g) = 0 = [D', \delta] \). It is easy to see that \( D' \) is a \( \mathbb{C} \)-linear graded derivation of \((\mathcal{A} \otimes \Omega)^{0,\bullet}(M)\) of degree \((-1, 0)\) that fulfills \( D'(\rho \otimes 1) = d\rho \otimes 1 \) for all \( \rho \in \Gamma^\infty(\Lambda^* T^*M) \), hence an exterior covariant derivative. It commutes with \( \delta \) (in the graded sense) because \( \delta \) commutes with \( D \) and all pullbacks. Finally, \( D'(1 \otimes g) \) holds because of \( \nabla \Phi^*(g) = 0 \). \( \square \)

Note that the condition \( \nabla \Phi^*(g) = 0 \) is fulfilled e.g. if \( \Phi^*(g) = g \), but also more generally if \( \Phi^*(g) = \lambda g \) with \( \lambda \in \mathbb{C} \).

**Proposition A.6** Let \( M \) be a smooth manifold endowed with a free and proper action \( \cdot : G \rightarrow M \) of a Lie group \( G \) and a \( G \)-equivariant exterior covariant derivative \( D \) on \( M \) (i.e. \( D \) commutes with the action of \( G \) on \((\mathcal{A} \otimes \Omega)^{0,\bullet}(M) \) by pull-backs). Moreover, write \( \Pr : M \rightarrow M/G \) for the canonical projection.
onto the quotient manifold \( M/G \) and assume we have chosen a smooth \( G \)-invariant complement \( \Xi = \bigcup_{p \in M} \Xi_p \) of \( \ker(T \Pr) \), i.e. a linear subbundle of \( TM \) such that \( TM = \Xi \oplus \ker(T \Pr) \) and such that \( \Xi_{p \in M} = T_p \Phi_g(\Xi_p) \) for all \( p \in M \). Let \( \Theta_{\Xi} : \Gamma^\infty(TM) \to \Gamma^\infty(TM) \) be the corresponding projection on this subbundle \( \Xi \) and \( \Theta_{\Xi}^* : \Gamma^\infty(T^*M) \to \Gamma^\infty(T^*M) \) its dual endomorphism. Then

\[
\Pr^* \left( D_{\text{red}} \Omega \right) := (\Theta_{\Xi}^*)^{\otimes (k+1+\ell)} D \Pr^*(\Omega)
\]

for all \( \Omega \in (\mathcal{A} \otimes \mathcal{J})^{k,\ell}(M/G) \), \( k, \ell \in \mathbb{N}_0 \) defines an exterior covariant derivative on \( M/G \). If \( D \) is torsion-free, then \( D_{\text{red}} \) also remains torsion-free.

**Proof:** Since \( D \) is \( G \)-equivariant and \( \Xi \) is \( G \)-invariant, it follows that \( (\Theta_{\Xi}^*)^{\otimes (k+1+\ell)} D \Pr^*(\Omega) \) is \( G \)-invariant, so (A.7) does describe a well-defined \( \mathbb{C} \)-linear endomorphism \( D_{\text{red}} \) of \( (\mathcal{A} \otimes \mathcal{J})^{*,*}(M/G) \) of degree \((1, 0)\). From the linearity of \( \Pr^* \) and \( \Theta_{\Xi}^* \) it follows that \( D_{\text{red}} \) is again a graded derivation and one can easily check that \( D_{\text{red}}(\rho \otimes 1) = d\rho \otimes 1 \) holds for all \( \rho \in \Gamma^\infty(\Lambda^*TM) \). Using that \( \delta \) commutes with pullbacks one also finds that \( D_{\text{red}} \) (graded) commutes with \( \delta \) if \( D \) does. \( \square \)

Next we consider the graded commutator of an exterior covariant derivative \( D \) on a smooth manifold \( M \) with \( \delta^* \), which is a \( \mathbb{C} \)-linear graded derivation of \( (\mathcal{A} \otimes \mathcal{J})^{*,*}(M) \) of degree \((0, +1)\) and satisfies \([D, \delta^*](f) = \delta^* D f = 1 \otimes df \) for all \( f \in \mathcal{C}^\infty(M) \). So \([D, \delta^*]\) restricts to a \( \mathbb{C} \)-linear derivation \( D^{\text{sym}} \) of \( \mathcal{J}^*(M) \) of degree 1.

**Definition A.7** A \( \mathbb{C} \)-linear derivation \( \Delta \) of \( \mathcal{J}^*(M) \) of degree 1 that fulfils \( \Delta f = df \) for all \( f \in \mathcal{C}^\infty(M) \) is called a symmetrized covariant derivative, and for every exterior covariant derivative \( D \) of \( M \) we define its induced symmetrized covariant derivative \( D^{\text{sym}} : \mathcal{J}^*(M) \to \mathcal{J}^*(M) \) by

\[
1 \otimes D^{\text{sym}} \omega := [D, \delta^*](1 \otimes \omega)
\]

for all \( \omega \in \mathcal{J}^*(M) \).

Given an exterior covariant derivative \( D \), we compute that its induced symmetrized covariant derivative \( D^{\text{sym}} \) fulfils

\[
\iota_Y \iota_X D^{\text{sym}} \omega = \iota_Y^{(s)} \iota_X^{(s)} [D, \delta^*](1 \otimes \omega)
\]

\[
= \iota_Y^{(s)} \iota_X^{(s)} \delta^* D(1 \otimes \omega)
\]

\[
= \iota_Y^{(a)} \iota_X^{(a)} D(1 \otimes \omega) + \iota_Y^{(a)} \iota_X^{(s)} D(1 \otimes \omega)
\]

\[
= \iota_Y \nabla_X^D \omega + \iota_Y^{(s)} \nabla_X^D \omega
\]

\[
= \langle \nabla_X^D \omega, Y \rangle + \langle \nabla_X^D \omega, X \rangle
\]

for all \( \omega \in \Gamma^\infty(T^*M) \). So in local coordinates, \( D^{\text{sym}} \omega = dx^i \vee \nabla_{\partial_i x^a} \omega \).

Conversely, every \( \mathbb{C} \)-linear derivation \( \Delta \) of \( \mathcal{J}^*(M) \) that fulfils \( \Delta f = df \) for all \( f \in \mathcal{C}^\infty(M) \) defines
a covariant derivative $\nabla^A$ on $M$ by
\[
\langle \nabla^A_\omega, Y \rangle := \langle \Delta \omega, X \vee Y \rangle + \frac{1}{2} \langle X(\langle \omega, Y \rangle) - Y(\langle \omega, X \rangle) - \langle \omega, [X, Y] \rangle \rangle
\] (A.9)
for all $\omega \in \Gamma^\infty(T^*M)$ and all $X, Y \in \Gamma^\infty(TM)$. This covariant derivative $\nabla^A$ then is torsion-free because
\[
\langle \nabla^A_\omega, Y \rangle - \langle \nabla^A_\omega, X \rangle = X(\langle \omega, Y \rangle) - Y(\langle \omega, X \rangle) - \langle \omega, [X, Y] \rangle
\] (A.10)
and fulfills
\[
\langle \nabla^A_\omega, Y \rangle + \langle \nabla^A_\omega, X \rangle = 2\langle \Delta \omega, X \vee Y \rangle = \iota_Y \iota_X \Delta \omega.
\] (A.11)
Consequently there is a 1-to-1-correspondence between torsion-free covariant derivatives (or their exterior covariant derivatives) and symmetrized covariant derivatives. For the reduction of symmetrized covariant derivatives we get:

**Lemma A.8** Let $M$, $G$, $\Pr$ and $\Xi$ be as in Proposition A.6 and $D$ a $G$-equivariant exterior covariant derivative on $M$. Then $D^\text{sym}_{\text{red}}$, the symmetrized covariant derivative on $M/G$ constructed out of the reduced exterior covariant derivative $D_{\text{red}}$, fulfills
\[
\Pr^* (D^\text{sym}_{\text{red}} \omega) = (\Theta^2_\Xi)^{\otimes (k+1)} D^\text{sym} \Pr^* (\omega)
\] (A.12)
for all $\omega \in \mathfrak{S}^k(M/G)$, $k \in \mathbb{N}_0$.

**Proof:** As $\delta^*$ commutes with the pullback $\Pr^*$ and the projection $(\Theta^2_\Xi)^{\otimes (k+1)}$, this follows immediately from (A.7). \qed

Being an endomorphism of $\mathfrak{S}^\ast(M)$, a symmetrized covariant derivative $D^\text{sym}$ can be iterated. Given $k \in \mathbb{N}_0$, $X_0 \in \Gamma^\infty(S^0 TM), \ldots, X_k \in \Gamma^\infty(S^k TM)$, then
\[
\mathcal{C}^\infty(M) \ni f \mapsto \sum_{r=0}^k \langle (D^\text{sym})^r f, X_r \rangle \in \mathcal{C}^\infty(M)
\]
is a differential operator of degree $k$. Conversely, by induction over their symbols, one can show that all differential operators of degree $k$ on $\mathcal{C}^\infty(M)$ are of this form. So symmetrized covariant derivatives yield a way to describe differential operators independent of a choice of coordinates.

Finally, if $M$ is a complex manifold, then its tangent and cotangent space split into $(1,0)$ and $(0,1)$ parts. Consequently $\mathfrak{S}^k = \bigoplus_{p+q=k} \mathfrak{S}^{(p,q)}$ and $\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{(p,q)}$ also split into subspaces $\mathfrak{S}^{(k,\ell)}(\mathbb{C}^{1+n}_+) := \Gamma^\infty(S^k T^\ast(1,0) \mathbb{C}^{1+n}_+ \vee S^\ell T^\ast(0,1) \mathbb{C}^{1+n}_+)$ and $\mathcal{A}^{(k,\ell)}(\mathbb{C}^{1+n}_+) := \Gamma^\infty(\Lambda^k T^\ast(1,0) \mathbb{C}^{1+n}_+ \wedge \Lambda^\ell T^\ast(0,1) \mathbb{C}^{1+n}_+)$. If a symmetrized covariant derivative is compatible with the complex structure as defined below, then we can split it in its holomorphic and antiholomorphic parts:

**Definition A.9** Let $M$ be a complex manifold and $\nabla$ a covariant derivative on $M$. Then $\nabla$ is said to be compatible with the complex structure if for any $X \in \Gamma^\infty(T \mathbb{C}^{1+n})$ the covariant derivative $\nabla_X$ pre-
serves the holomorphic and antiholomorphic parts of the tangent bundle, i.e. \( \nabla_X(\Gamma^\infty(T^{(1,0)}\mathbb{C}^{1+n})) \subseteq \Gamma^\infty(T^{(1,0)}\mathbb{C}^{1+n}) \) and \( \nabla_X(\Gamma^\infty(T^{(0,1)}\mathbb{C}^{1+n})) \subseteq \Gamma^\infty(T^{(0,1)}\mathbb{C}^{1+n}) \).

As an example, it is well-known that the Levi-Civita covariant derivative on a Kähler manifold is compatible with the complex structure in this sense.

Using the local description of the exterior covariant derivative \( D^\nabla \) associated to \( \nabla \) and Equation (A.6) one can prove that \( \nabla \) is compatible with the complex structure if and only if

\[
D^\nabla \left( (\mathcal{A} \otimes \mathcal{S})(p,q),(r,s)(M) \right) \subseteq (\mathcal{A} \otimes \mathcal{S})(p+1,q),(r,s)(M) \oplus (\mathcal{A} \otimes \mathcal{S})(p,q+1),(r,s)(M)
\]

for all \( p,q,r,s \in \mathbb{N}_0 \). Furthermore, in this case the associated symmetrized covariant derivative fulfills

\[
D^\text{sym}(\mathcal{S}(p,q)(M)) \subseteq \mathcal{S}(p+1,q)(M) \oplus \mathcal{S}(p,q+1)(M).
\]

**Definition A.10** Let \( M \) be a complex manifold and \( D \) an exterior covariant derivative satisfying (A.13). Then we define

\[
D^\text{hol}, D^\text{hol}_*: (\mathcal{A} \otimes \mathcal{S})^\bullet\bullet(M) \to (\mathcal{A} \otimes \mathcal{S})^\bullet\bullet(M)
\]

as well as

\[
D^\text{sym}_\text{hol}, D^\text{sym}_\text{hol}_*: \mathcal{S}^\bullet(M) \to \mathcal{S}^\bullet(M)
\]

as the \((1,0)\) and \((0,1)\)-components of \( D \) and \( D^\text{sym} \), respectively, i.e.

\[
D^\text{hol} := \sum_{p,q,r,s \in \mathbb{N}_0} \Theta^*,(p+1,q),(r,s)D\Theta^*,(p,q),(r,s), \quad D^\text{hol}_* := \sum_{p,q,r,s \in \mathbb{N}_0} \Theta^*,(p,q+1),(r,s)D\Theta^*,(p,q),(r,s)
\]

as well as

\[
D^\text{sym}_\text{hol} := \sum_{p,q \in \mathbb{N}_0} \Theta^*,(p+1,q)D^\text{sym}\Theta^*,(p,q), \quad D^\text{sym}_\text{hol}_* := \sum_{p,q \in \mathbb{N}_0} \Theta^*,(p,q+1)D^\text{sym}\Theta^*,(p,q)
\]

with \( \Theta^*,(p,q),(r,s) : (\mathcal{A} \otimes \mathcal{S})^\bullet\bullet(\bullet\bullet)(M) \to (\mathcal{A} \otimes \mathcal{S})^\bullet\bullet(\bullet\bullet)(M) \) and \( \Theta^*,(p,q) : \mathcal{S}^\bullet(\bullet\bullet)(M) \to \mathcal{S}(\bullet\bullet)(M) \) the projections on graded subspaces.

It is obvious that \( D = D^\text{hol} + D^\text{hol}_* \) and \( D^\text{sym} = D^\text{sym}_\text{hol} + D^\text{sym}_\text{hol}_* \). Furthermore,

\[
\left[ D^\text{hol}, \, \delta^* \right](1 \otimes \omega) = 1 \otimes D^\text{sym}_\text{hol}_* \omega
\]

holds for all \( \omega \in \mathcal{S}^\bullet(M) \), and analogously for the antiholomorphic part. This clearly has an important consequence:

**Lemma A.11** Let \( M \) be a complex manifold and \( \nabla \) a covariant derivative on \( M \) and compatible with the complex structure, as well as \( D^\text{sym} := (D^\nabla)^\text{sym} \) its associated symmetrized covariant derivative. Then

\[
\Theta^*,(k,0)(D^\text{sym})^k f = (D^\text{sym}_\text{hol})^k f \quad \text{and} \quad \Theta^*,(0,k)(D^\text{sym})^k f = (D^\text{sym}_\text{hol}_*)^k f
\]

hold for all \( f \in \mathcal{C}^\infty(M) \) and all \( k \in \mathbb{N}_0 \), with \( \Theta^*,(k,0) \) like in the previous Definition A.10.

**Proof:** This follows immediately from the decomposition \( D^\text{sym} = D^\text{sym}_\text{hol} + D^\text{sym}_\text{hol}_* \).
References


