Maximal almost disjoint families, determinacy, and forcing

Bakke Haga, Karen; Schrittesser, David; Törnquist, Asger

Published in:
Journal of Mathematical Logic

DOI:
10.1142/S0219061321500264

Publication date:
2022

Document version
Early version, also known as pre-print

Document license:
CC BY

Citation for published version (APA):
Abstract. We study the notion of $J$-MAD families where $J$ is a Borel ideal on $\omega$. We show that if $J$ is an arbitrary $F_\sigma$ ideal, or is any finite or countably iterated Fubini product of $F_\sigma$ ideals, then there are no analytic infinite $J$-MAD families, and assuming Projective Determinacy there are no infinite projective $J$-MAD families; and under the full Axiom of Determinacy $+ V = L(R)$ there are no infinite $J$-mad families. These results apply in particular when $J$ is the ideal of finite sets $\text{Fin}$, which corresponds to the classical notion of MAD families. The proofs combine ideas from invariant descriptive set theory and forcing.

1. Introduction

(A) Let Fin denote the ideal of finite subsets of $\omega$. Classically, a family $A \subseteq \mathcal{P}(\omega)$ is called an almost disjoint family (short: AD family) if the family $A$ consists of infinite subsets of $\omega$ and any two distinct $x, y \in A$ have finite intersection, that is $x \cap y \in \text{Fin}$. A maximal almost disjoint (short: MAD) family is an AD family which is maximal (with respect to $\subseteq$) among AD families. Finite MAD families exist trivially, and using Choice (e.g., Zorn’s Lemma), it is routine to show that there are infinite MAD families. Below we always assume, to avoid trivialities, that MAD families in question are infinite.

The study of the definability of (infinite) MAD families has a long history, but the area has in recent years seen a remarkable blossoming with many new results. The fundamental results in the area go back to Mathias’ famous paper [8], where it was shown that there are no analytic MAD families, and further proved that assuming there is a Mahlo cardinal, ZF+“there are no MAD families” is consistent. Much more recently, Törnquist [16] showed that there are no MAD families in Solovay’s model (from [13]), thus weakening the large cardinal assumption. Horowitz and Shelah [1] then removed the large cardinal assumption completely, showing that one can construct a model of ZF without MAD families just assuming ZF is consistent.
The leading idea of the present paper is to use ideas from invariant descriptive set theory, i.e. the descriptive set theory of definable equivalence relations and invariance properties, in combination with Mathias-like forcings and absoluteness to give uniform proofs of the non-existence of (definable) MAD families in various settings. This approach in turn allows us to prove vastly more general results about the definability of $\mathcal{J}$-MAD families, where the ideal $\text{Fin}$ is replaced by an ideal $\mathcal{J}$ on $\omega$ for $\mathcal{J}$ in a large class of Borel ideals. However, we first give transparent and surprisingly uniform proofs of the following:

**Theorem 1.1.**

1. There are no analytic MAD families.
2. Under $\text{ZF} + \text{Dependent Choice} + \text{Projective Determinacy}$, there are no projective infinite MAD families.
3. Under $\text{ZF} + \text{Determinacy} + V = L(\mathbb{R})$ there are no infinite MAD families.

The first result is originally due to Mathias [8]. The remaining two results were also shown independently by Neeman and Norwood in [12] using somewhat different methods.

We mention that two of the present authors have very recently found a proof (unpublished) that if every (resp., every projective) set is completely Ramsey and Ellentuck-comeager uniformization (resp., for projective relations) holds, then there are no (projective) MAD families.

**B** We now discuss the notion of $\mathcal{J}$-MAD families and the generalization of Theorem 1.1 to this setting.

Given an ideal $\mathcal{J}$ on a countable set $S$, write $\mathcal{J}^{+}$ for $\mathcal{P}(S) \backslash \mathcal{J}$. One may define $\mathcal{J}$-almost disjoint (short: $\mathcal{J}$-AD) sets in $\mathcal{J}^{+}$ and $\mathcal{J}$-maximal almost disjoint (short: $\mathcal{J}$-MAD) families as subsets of $\mathcal{P}(\mathcal{J}^{+})$ in the obvious manner (see Section 2). MAD families are of course the special case $\mathcal{J} = \text{Fin}$, where $\text{Fin}$ denotes the ideal of finite sets (i.e., the Fréchet ideal).

The motivating question for the results we now describe is:

**Question 1.2.** For which Borel ideals $\mathcal{J}$ on $\omega$ can we generalize results about MAD families to the case of $\mathcal{J}$-MAD families?

As a first step we consider $F_{\sigma}$ ideals. It is well known that every $F_{\sigma}$ ideal $\mathcal{J}$ is given as the finite part of a lower semi-continuous (short: lsc) submeasure $\phi: \mathcal{P}(\omega) \to [0, \infty]$ as follows:

$$\mathcal{J} = \text{Fin}(\phi) = \{J \subseteq \omega \mid \phi(J) < \omega\}.$$ 

(See Section 2 for a complete definition of these notions and [9][1.2] for a proof of the claim.) We will see in Section 3 that Theorem 1.1 generalizes to $\mathcal{J}$-MAD families where $\mathcal{J}$ is any $F_{\sigma}$ ideal.
We can reach Borel ideals that are more complex, in the sense of belonging to higher parts of the Borel hierarchy, using Fubini sums and products. If for each $k \in \omega$ we are given an ideal $I_k$ on a countable set $S_k$, and an ideal $I$ on $\omega$, we form an ideal $\bigoplus_k I_k$ on $S = \bigsqcup_{k \in \omega} S_k$, called the Fubini sum of $(I_k)_{k \in \omega}$ over $\mathcal{I}$ as follows:

$$\bigoplus_{I_k} I_k = \{ I \subseteq S \mid \{ k \in \omega \mid I \cap S_k \notin I_k \} \in I \}$$

In Section 4 we show that our methods apply in this generalized setting:

**Theorem 1.3.** Let $\mathcal{J} = \bigoplus_k I_k$ where $\mathcal{I}$ and $I_k$ for each $k \in \omega$ are $F_\sigma$ ideals on $\omega$.

1. There are no analytic infinite $\mathcal{J}$-MAD families.
2. Under ZF + Dependent Choice + Projective Determinacy, there are no projective infinite $\mathcal{J}$-MAD families.
3. Under ZF + Determinacy + $V = L(\mathbb{R})$ there are no infinite $\mathcal{J}$-MAD families.

We note in passing that part (1) of Theorem 1.3 in the special case of $\mathcal{J} = \text{Fin} \otimes \text{Fin}$, the first iteration of the Fréchet ideal, in itself answers a question that seems to have belonged to the folklore of the field for a long time. Here $\text{Fin} \otimes \text{Fin}$ is the Fubini sum of $\text{Fin}$ over $\text{Fin}$, that is, it consists of those $X \subseteq \omega \times \omega$ such that

$$\{ n \in \omega \mid \{ m \in \omega \mid (n, m) \in X \} \text{ is infinite} \} \text{ is finite}.$$ 

Yet more complex ideals are obtained by iterating Fubini products into the transfinite. Namely, given $\alpha < \omega_1$ and a sequence $\vec{\phi}$ of lsc submeasures of length $\alpha$, we will define in a natural manner an ideal $\text{Fin}(\vec{\phi})$ on a countable set $S$, by recursively applying Fubini products. (See Section 5 for the detailed definition.)

One can do this in such a way that $\text{Fin}(\vec{\phi})$ is $\Sigma^0_{\alpha+1}$ but not $\Sigma^0_{\alpha}$. A particular instance of this construction is the iterated Fréchet ideal $\text{Fin}^\alpha$, obtained by iterating the Fubini product construction applied to the ideal $\text{Fin}$ transfinetly, and thus obtaining “higher dimensional” analogues $\text{Fin}^\alpha$ of $\text{Fin} \otimes \text{Fin}$.

We shall see in Section 5 that our methods apply even to the more general class of ideals described in the previous paragraph:

**Theorem 1.4.** Let $\mathcal{J} = \text{Fin}(\vec{\phi})$ be as defined in Section 5.

1. There are no analytic $\mathcal{J}$-MAD families.
2. Under ZF + Dependent Choice + Projective Determinacy, there are no projective infinite $\mathcal{J}$-MAD families.
3. Under ZF + Determinacy + $V = L(\mathbb{R})$ there are no infinite $\mathcal{J}$-MAD families.
By this theorem, the ideals $J$ for which we have the familiar pattern of non-definability of $J$-MAD families lie cofinally in the Borel hierarchy with respect to the complexity of their definition.

(C) The paper is structured as follows: Section 2 collects definitions and facts that will be used throughout—most importantly, some facts from inner model theory which will allow us to assume that our $J$-MAD families are $\kappa$-Suslin witnessed by a tree from a model with small $\mathcal{P}(\mathcal{P}(\omega))$.

In Section 3 we prove Theorem 1.1 for any $F_\sigma$ ideal $J$. We describe Mathias forcing $M^I$ relative to an ideal $I$ and state a crucial fact, Main Proposition 3.5, regarding $M^I$ when $I$ is generated by a $\kappa$-Suslin $J$-MAD family. Theorem 1.1 follows quickly from Main Proposition 3.5 together with the inner model theory lemmas from Section 2.

Section 3.1 collects facts about $M^I$, and Section 3.2 proves Main Proposition 3.5. This proof is based on the definition of an $J$-invariant tree together with the purely combinatorial Branch Lemma 3.14 (stating that the projection of this tree is a singleton).

In Section 4 we introduce the simple Fubini product and a 2-dimensional version $M^I_2$ of $M^I$. Theorem 1.3 follows from Main Proposition 4.6 (the analogue of 3.5 for $M^I_2$) by the same proof as before. Section 4.1 collects facts about $M^I_2$ and Section 4.2 proves Main Proposition 4.6; the proof of the corresponding Branch Lemma 4.11 is much more involved.

Section 5 is structured in the same way: We introduce Fubini products $Fin(\vec{\phi})$ coming from a sequence $\vec{\phi}$ of lsc submeasures, an $\alpha$-dimensional version $M^I_\alpha$ of $M^I$, and state Main Proposition 5.12 for $M^I_\alpha$, from which Theorem 1.4 follows immediately. Section 5.1 collects facts about $M^I_\alpha$. Section 5.2 proves Main Proposition 5.12 via the Branch Lemma 5.19, generalizing Section 4.2.

Finally, in section 6, we briefly discuss the general (and open) problem of characterizing precisely for which Borel ideals on $\omega$ one may hope to obtain an analogue of Theorem 1.4.

Acknowledgments: We owe a great debt to Sandra Müller for helping us find proofs of Lemmas 2.4 and 2.5. The first author thanks for support through Asger Törnquist’s Sapere Aude level 2 grant from the Danish Council for Independent Research. The second and third authors thank the DNRF Niels Bohr Professorship of Lars Hesselholt for support. The second author also gratefully acknowledges the generous support from the Austrian Science Fund (FWF) grant number F29999. The third author thanks the Danish Council for Independent Research for support through grant DFF-7014-00145B.
2. Notation and Preliminaries

We sometimes decorate names in the forcing language with checks and dots with the aim of helping the reader. For notation not defined here we refer to [5, 3, 4, 10]

2.1. Ideals. Fix a countable set $S$. An ideal on $S$ is a family $S \subseteq \mathcal{P}(S)$ satisfying

1. $\emptyset \in \mathcal{J}$;
2. If $A \in \mathcal{J}$, then for any subset $B \subseteq A$ we have $B \in \mathcal{J}$;
3. If $A \in \mathcal{J}$ and $B \in \mathcal{J}$, then $A \cup B \in \mathcal{J}$.

We denote by $\text{Fin}$ the ideal of finite sets.

Given an ideal $\mathcal{J}$, we write $\mathcal{J}^+$ to denote the co-ideal, i.e.,

$$\mathcal{J}^+ = \{ A \subseteq S \mid A \notin \mathcal{J} \}.$$ 

For $A, B \in \mathcal{P}(S)$, we write

$$A \not\subseteq^* \mathcal{J} B \iff (\exists I \in \mathcal{J}) A \subseteq B \cup I.$$ 

We write $A \not\subseteq^* B$ for $A \not\subseteq^* \text{Fin} B$.

We say that a family $A \subseteq \mathcal{P}(S)$ is $\mathcal{J}$-almost disjoint (short: $\mathcal{J}$-AD) if $A \subseteq \mathcal{J}^+$ and for any $A, B \in A$ we have $A \cap B \in \mathcal{J}$. A set $A \subseteq \mathcal{P}(S)$ is said to be a $\mathcal{J}$-MAD family if $A$ is a $\mathcal{J}$-AD family which is maximal with respect to inclusion among $\mathcal{J}$-AD families.

**Definition 2.1.** Let $A \subseteq \mathcal{P}(S)$. By the ideal generated by $A$ we mean the ideal $\mathcal{I}$ on $S$ defined as follows:

$$\mathcal{I} = \{ I \subseteq S \mid (\exists n \in \omega)(\exists A_0, \ldots, A_n \in A) I \subseteq \bigcup_{i \leq n} A_i \},$$

i.e., the smallest (under $\subseteq$) ideal on $S$ containing each set from $A$.

Suppose $A \subseteq \mathcal{P}(S)$ and $\mathcal{J}$ is an ideal on $S$. Then note the ideal generated by $A \cup \mathcal{J}$ is

$$\{ I \subseteq S \mid I \in \mathcal{J} \lor (\exists n \in \omega)(\exists A_0, \ldots, A_n \in A) I \subseteq^*_\mathcal{J} \bigcup_{i \leq n} A_i \}.$$ 

We point out that if $A$ is an infinite $\mathcal{J}$-AD family then $[S]^{<\omega} \subseteq \mathcal{J}$ and $\mathcal{J}$ is proper (i.e., $S \notin \mathcal{J}$; otherwise there are no non-empty, let alone infinite, $\mathcal{J}$-AD families). Moreover we could assume $\bigcup A = S$ (although we shall never need this).

We point out that enlarging an ideal $\mathcal{J}$ by an infinite $\mathcal{J}$-AD family yields a proper ideal.

**Lemma 2.2.** Let $S$ be arbitrary, $\mathcal{J}$ an ideal on $S$ and $A \subseteq \mathcal{P}(S)$ a $\mathcal{J}$-AD family. If $A$ is infinite, the ideal $\mathcal{I}$ generated by $A \cup \mathcal{J}$ is proper. (The other implication holds if $\bigcup A = S$.)


Proof. Suppose towards a contradiction that $S \in \mathcal{I}$. Then there exist $A_0, \ldots, A_n \in \mathcal{A}$, $J \in \mathcal{J}$ such that $S \subseteq \bigcup_{i \leq n} A_i \cup J$. Since $\mathcal{A}$ is $\mathcal{J}$-almost disjoint $\mathcal{A} = \{A_0, \ldots, A_n\}$ is finite.

For the last claim, suppose $\bigcup \mathcal{A} = S$; show the contrapositive. If $\mathcal{A}$ is finite, then $S \subseteq \bigcup_{A \in \mathcal{A}} A$, and thus $S \in \mathcal{I}$. \qed

A submeasure on $\omega$ is a function $\phi: \mathcal{P}(\omega) \to [0, \infty]$ which satisfies

- $\phi(\emptyset) = 0$;
- $\phi(X) \leq \phi(Y)$ for $X \subseteq Y$;
- $\phi(X \cup Y) \leq \phi(X) + \phi(Y)$ for $X, Y \in \mathcal{P}(\omega)$;
- $\phi(n) < \infty$ for every $n \in \omega$.

We say that $\phi$ is lower semi-continuous (lsc) if identifying $\mathcal{P}(\omega)$ with $2^\omega$ carrying product topology, it is lower semi-continuous as a function $\phi: 2^\omega \to [0, \infty]$, i.e., if $X_n \to X$ implies $\liminf_{n \to \infty} \phi(X_n) \geq \phi(X)$. For submeasures, this is equivalent to saying that $\phi(X) = \lim_{n \to \infty} \phi(X \cap n)$.

As stated already in the introduction, given a submeasure $\phi$ on $\omega$

$$\text{Fin}(\phi) = \{X \in \mathcal{P}(\omega) \mid \phi(X) < \infty\}$$

is an $F_\sigma$ ideal on $\omega$ and every $F_\sigma$ ideal $\mathcal{J} \supseteq \text{Fin}$ arises in this way \cite{1} 1.2.

2.2. Trees and Suslin sets of reals. Let $X_0, X_1$ be a sets. We follow established descriptive set theoretic conventions and call a tree $T$ on $X_0 \times X_1$ a subset of $X_0^{<\omega} \times X_1^{<\omega}$ which is closed under initial segments and such that $(t_0, t_1) \in T \Rightarrow \text{lh}(t_0) = \text{lh}(t_1)$ (compare \cite{2} 2.C)). Given $t = (t_0, t_1) \in T$, $\pi(t) = t_0$. For any $s \in T$, $T[s] = \{t \in T \mid t \text{ is compatible with } s\}$. Of course

$$[T] = \{(x_0, x_1) \in X_0^{\omega} \times X_1^{\omega} \mid (\forall n \in \omega) (x_0 \upharpoonright n, x_1 \upharpoonright n) \in T\},$$

and for $w = (x_0, x_1) \in [T]$, $\pi(w) = x_0$. Finally we write

$$\pi[T] = \{x_0 \in X_0^{\omega} \mid (\exists x_1 \in X_1^{\omega}) (x_0, x_1) \in [T]\}.$$

Recall that $\mathcal{A} \subseteq 2^\omega$ is $\kappa$-Suslin if and only if there exists an ordinal $\kappa$ and a tree $T$ on $2 \times \kappa$ such that

$$\mathcal{A} = \pi[T] = \{x \in 2^\omega \mid (\exists f \in \kappa^{<\omega}) (x, f) \in [T]\}.$$  

The analytic subsets of $2^\omega$ are precisely the $\omega$-Suslin sets.

If $S$ is a countable set we will also talk of $\kappa$-Suslin subsets of $\mathcal{P}(S)$. For this purpose we shall identify $S$ with $\omega$ via some fixed bijection $h: \omega \to S$ as well as identify each $x \in S$ with its characteristic function $\chi_x$.

Thus, $\mathcal{A} \subseteq \mathcal{P}(S)$ is $\kappa$-Suslin if and only if there is a tree $T$ on $2 \times \kappa$ such that $\mathcal{A} = \{x \in \mathcal{P}(S) \mid \chi_x \circ h \in \pi[T]\}$. We shall (sloppily and through the identifications of $S$ with $\omega$ and $\chi_x$ with $x$) also write $\mathcal{A} = \pi[T]$ in such a case.

We use both $\subseteq$ and $\sqsubseteq$ for the initial segment relation for sequences.
2.3. Inner model theory. It is well known that under PD the pointclasses \( \Pi^1_{2n+1} \) and \( \Sigma^1_{2n+2} \) are scaled (i.e., they have the prewellordering property—see [10], [5] Chapter 36, or [4] §30 for an introduction to the theory of scales). These scales provide us with tree representations for projective sets while at the same time, each scale can be captured by a ‘small’ model. For the proof of the next lemma, also recall that \( \delta^1_n \) is the supremum of the lengths of \( \Delta^1_n \) prewellorderings of \( \omega^\omega \) (see, e.g., [4] p. 423).

**Lemma 2.4.** Assume PD. Suppose \( A \) is projective. There exists a model \( M \) of ZFC and a tree \( T \in M \) on \( \omega \times \kappa \) (for some ordinal \( \kappa \)) such that \( \pi[T] = A \) and \( \mathcal{P}(\mathcal{P}(\omega))^M \) is countable in \( V \).
Proof. Given a projective set $A$, suppose without loss of generality that $A$ is $\Pi^1_n$, $n \geq 3$, and $n$ is odd. By the scale property let $T_n$ be the tree given by a $\Pi^1_n$ scale on a complete $\Pi^1_n$ set. Fix $a \in \omega^\omega$ such that $A$ is $\Sigma^1_n(a)$ and let $M = L[T_n, a]$.

To see that $M$ satisfies what is claimed in the lemma, let $M_n$ be the class-sized iterable model with $n$ Woodin cardinals obtained by iterating out the top extender of $M_\#$. By PD this model exists and $P_\omega$ is countable in $V$. As is pointed out in [15, p. 12–13] (the proof is said to be implicit in [14]) there is an iterate $Q$ of $M_n$ such that $M_\# = L[T_n, a]$ (for this we need that $n$ is odd and $n \geq 3$). Moreover $P_\omega$ is the same as $P_\omega(Q)$, which is countable in $V$. By the presence of $T_n$ and $a$ in $M$, it is easy to obtain $T$ such that $\pi_r T = A$. □

There is a version of this based on the full Axiom of Determinacy ($\text{AD}$), which we shall also use:

**Lemma 2.5.** Assume $\text{AD}$ holds and $V = L(\mathbb{R})$. Suppose $A$ is $\Sigma^2_1$. There exists a model $M$ of $\text{ZFC}$ and a tree $T \in M$ on $\omega \times \kappa$ (for some ordinal $\kappa$) such that $\pi[T] = A$ and $P(\mathcal{P}(\omega))^M$ is countable in $V$.

**Proof.** As we are working in $L(\mathbb{R})$, $\Sigma^2_1$ and $\Sigma_1(\mathbb{R} \cup \{\mathbb{R}\})$ are the same point-class (see, e.g., [15, p. 13]). Under the hypothesis of the lemma, by [7] this pointclass is scaled; let $T^*$ be the tree coming from this scale. According to [15, p. 13], [14] shows that $L[T^*] = L[Q]$ where $Q$ is an iterate of an initial segment of $M_\omega$ and again it holds that $P(\mathcal{P}(\omega))^L[T^*]$ is countable in $V$. Moreover $A = \pi[T]$ for some tree $T$ such that $T \in L[T^*]$. □

Finally we shall need the following result (due to Woodin) known as Solovay’s Basis Theorem (see [6, Remark 2.29(3)]).

**Fact 2.6.** Assume $\text{AD}$ holds and $V = L(\mathbb{R})$. Then every $\Sigma^2_1$ statement is witnessed by a set $A \subseteq \mathbb{R}$ which is itself $\Delta^2_1$.

### 3. Classical MAD families (and a bit more)

In this section we give proofs of the following:

**Theorem 3.1.** Let $\mathcal{J} = \text{Fin}$, or more generally $\mathcal{J} = \text{Fin}(\phi)$ where $\phi$ is an lsc submeasure on $\omega$.

1. There are no analytic infinite $\mathcal{J}$-MAD families.
2. Under ZF + Dependent Choice + Projective Determinacy, there are no projective infinite $\mathcal{J}$-MAD families.
3. Under ZF + Determinacy + $V = L(\mathbb{R})$ there are no infinite $\mathcal{J}$-MAD families.
The first item was first shown by Mathias [8] (at least in the case of Fin). The next two items are independently, and by a somewhat different method shown by Neeman and Norwood [12] (also in the case of Fin).

We use the following close relative of Mathias forcing:

**Definition 3.2.** Suppose that $\mathcal{I} \supseteq \text{Fin}$ is an ideal on $\omega$, and $\mathcal{I}^+$ its co-ideal. Define

$$M^\mathcal{I} = \{(a, A) \mid a \in [\omega]^{<\omega}, A \in \mathcal{I}^+, \max(a) < \min(A)\}$$

ordered by

$$(a', A') \leq (a, A) \text{ if and only if } a \subseteq a' \cup A' \subseteq a \cup A.$$  

Of course for $X, Y \subseteq \omega$, $X \subseteq Y$ means $X = Y \cap (\max(X) + 1)$. We write $M$ for $M^\text{Fin}$.

We use the following notation, which should be familiar enough:

**Notation 3.3.**

1. Given a filter $G$ on $M^\mathcal{I}$, let

$$x_G = \bigcup\{a \mid (\exists A \in \mathcal{I}^+) (a, A) \in G\}.$$  

Note that if $G$ is $M^\mathcal{I}$-generic then $x_G \notin \text{Fin}$.

2. For $(a, A) \in M^\mathcal{I}$, and $b \subseteq A$ finite, let $A/b = \{n \in A \mid n > \max(b)\}$.

3. For $p \in M^\mathcal{I}$, we write $p = (a(p), A(p))$ when we want to refer to its components.

4. For $p \in M^\mathcal{I}$, we let $M^\mathcal{I}(\leq p) = \{q \in M^\mathcal{I} \mid q \leq p\}$.

**Assumption 3.4.** Until the end of Section 3 let $\mathcal{J} = \text{Fin}$ or more generally $\mathcal{J} = \text{Fin}(\phi)$ and assume $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an infinite $\mathcal{J}$-AD family which is $\kappa$-Suslin. Fix a tree $T$ on $2^{<\kappa}$ such that $\pi[T] = \mathcal{A}$. Let $\mathcal{I}$ be the ideal generated by $\mathcal{A} \cup \mathcal{J}$.

To avoid overly cumbersome notation, we shall phrase our presentation in terms of the ideal Fin. However this section is written so that whenever relevant, the reader may replace Fin (but not the word “finite” or the expression $[\omega]^{<\omega}$) with Fin$(\phi)$, for any lsc submeasure $\phi$ on $\omega$, in which case she must also replace “almost disjoint” by “Fin$(\phi)$-AD”, etc. We will point out how to modify proofs when these trivial substitutions do not suffice.

The main workload in the proof Theorem 3.1 is carried by the following Main Proposition, of which we give a proof in Section 3.2 after we collect some properties of the forcing $M^\mathcal{I}$ in Section 3.1.

**Main Proposition 3.5.** $\vdash_{M^\mathcal{I}} (\forall y \in \pi[T]) y \cap x_G \notin \text{Fin}$. In other words, $\vdash_{M^\mathcal{I}} x_G \notin \pi[T]$ and $\{x_G\} \cup \pi[T]$ is an almost disjoint family.

Before we prove the Main Proposition, we show how easily it leads to Theorem 3.1. Firstly, we give a very short proof of the classical result that there are no analytic MAD families:

**Corollary 3.6** ([8]). There are no analytic MAD families.
Proof. Suppose \( A \) is an analytic almost disjoint family, and fix a tree \( T \) on \( 2 \times \omega \) such that \( A = \pi[T] \) (identifying \( P(\omega) \) with \( 2^\omega \)). By Levy-Shoenfield Absoluteness \( \pi[T]^{V[G]} \) should be maximal in any forcing extension \( V[G] \) of \( V \); but by Main Proposition 3.5 there is a forcing extension \( V[G] \) containing a real which is almost disjoint from any set in \( \pi[T]^{V[G]} \). \( \square \)

We likewise obtain an easy and transparent proof that under projective determinacy, there are no projective MAD families. Here we make use of the inner model theory facts from Section 2.3.

**Corollary 3.7.** Under PD there are no projective MAD families.

**Proof.** Assume PD holds and suppose \( A \) is an infinite almost disjoint family which is projective. Fix a tree \( T \) so that \( A = \pi[T] \) and a model \( M \) as in the previous lemma. Note that \( M(\pi[T]) \) is an infinite almost disjoint family. Working inside \( M \) let \( I \) be the ideal generated by \( \text{Fin}_{\pi[T]} \) and let \( P \) denote \( M[I] \). As \( P(\omega)^M \) is countable in \( V \) we may find \( r \in [\omega]^\omega \) which is \( P \)-generic. By Main Proposition 3.5

\[
M[r] = (\forall y \in \pi[T]) \ y \text{ is almost disjoint from } r.
\]

By Item 3 of Lemma 2.3 the statement on the right is absolute for models of ZFC and therefore holds in \( V \). Thus, \( A \) is not maximal. \( \square \)

A similar proof can be given of the AD analogue:

**Corollary 3.8.** If \( \mathbf{L}(\mathbb{R}) \models \mathbf{AD} \), there are no MAD families in \( \mathbf{L}(\mathbb{R}) \).

**Proof.** Suppose towards a contradiction that \( V = \mathbf{L}(\mathbb{R}) \), AD holds, and there is a MAD family. As the existence of a MAD family is a \( \Sigma^1_2 \) statement, by Lemma 2.5 there is a \( \Sigma^1_2 \) MAD family \( A \). By Lemma 2.5 we may pick an ordinal \( \kappa \) and a tree \( T \) on \( \kappa \times \omega \) such that \( \pi[T] = A \). Moreover, there is a model \( M \) such that \( T \in M \) and \( \mathcal{P}(\mathcal{P}(\omega))^M \) is countable. Now argue precisely as in Corollary 3.7 above to show that \( A \) is not maximal, reaching a contradiction. \( \square \)

3.1. **Properties of Mathias forcing relative to an ideal.** For the proof of the Main Proposition 3.5 in the next section, we need to explore the immediate properties of the forcing notion \( M^T \).

The following lemma holds for any ideal \( I \supseteq \text{Fin} \) and under our Assumption 3.4.

**Lemma 3.9.**

1. \( \models_{M^T} (\forall y \in I) \ x_G \in \text{Fin}_y \cap \text{Fin} \).
2. \( \models_{M^T} x_G \in \text{Fin}_y^+ \).
3. Fix \( A \in I^+ \) and \( a_0, a_1 \in [\omega]^{<\omega} \) with \( \max(a_i) < \min(A) \) for each \( i \in \{0, 1\} \). Let \( p_i = (a_i, A) \). Then \( h \colon M^T(\leq p_0) \to M^T(\leq p_1) \) given by

\[
h(a_0 \cup b, B) = (a_1 \cup b, B),
\]
where \( b \subseteq A \) is finite and \( B \subseteq A/b \) is an isomorphism of partial orders.

(4) For \( p_0, p_1 \) as above, \( \theta \) a formula in the language of set theory, and \( v \in V \) it holds that

\[
p_0 \models \theta(v, [x_G]_{E_0}) \text{ if and only if } p_1 \models \theta(v, [x_G]_{E_0})
\]

Proof. (1) For any \( y \in \mathcal{I} \), the set

\[
D_y = \{ p \in \mathcal{M}^I \mid A(p) \cap y = \emptyset \}
\]

is dense in \( \mathcal{M}^I \), which implies that for any generic \( G \) we have \( x_G \cap y \in \text{Fin} \).

(2) We verify the general case where \( \mathcal{J} = \text{Fin}(\phi) \). Supposing \( p \models x_G \in \text{Fin}(\phi) \) we can find \( p' \preceq p \) and \( n \in \omega \) so that \( p' \models \phi(x_G) < n \). Since \( \phi \) is lower semi-continuous and \( \phi(A(p')) = \infty \) we can find a finite set \( a \) such that \( a(p') \preceq a \preceq A(p') \) and \( \phi(a) > n \). Since \( (a, A(p')/a) \models a \preceq x_G \) we reach a contradiction.

(3) Immediate from the definitions.

(4) Suppose \( p_1 \models \theta(v, [x_G]_{E_0}) \). Let \( G \) be a generic such that \( p_0 \in G \). Use \( h : \mathcal{M}^I(\leq p_0) \to \mathcal{M}^I(\leq p_1) \) from (3) to obtain a generic \( h(G) \) containing \( p_1 \).

Since \( x_G E_0 x_{h(G)} \), we conclude \( \theta(v, [x_G]_{E_0}) \), proving that “if” holds. The proof of “only if” is analogous. \( \square \)

Furthermore, we have the following diagonalization result.

Lemma 3.10. Let \( (A_k)_{k \in \omega} \) be a sequence from \( \mathcal{I}^+ \) satisfying that \( A_{k+1} \subseteq A_k \) for every \( k \in \omega \). Then there is \( A_x \in \mathcal{I}^+ \) such that \( A_x \subseteq^* A_k \) for every \( k \in \omega \).

In both the lemma and its proof for the case \( \mathcal{J} = \text{Fin}(\phi) \) there is no need to replace \( \subseteq^* \) by \( \subseteq^*_{\text{Fin}(\phi)} \).

Also note that it follows that the preorder \( (\mathcal{I}^+, \subseteq^*_\mathcal{J}) \) is \( \sigma \)-closed. In fact, \( \mathcal{I}^+ \) is a selective co-ideal—however, we will only need the statement in the lemma.

Proof. We construct two sequences \( (B_n)_{n \in \alpha} \) and \( (C_n)_{n \in \alpha} \) of length \( \alpha \leq \omega \) such that for each \( n < \alpha \),

- \( B_n \subseteq A_n \);
- for each \( m \), \( B_n \subseteq^*_F A_m \);
- \( C_n \in A \{ A_i \mid i < n \} \);
- \( B_n \cap C_n \notin \text{Fin} \) and \( B_n \cap C_i \in \text{Fin} \) for \( i < n \).

Suppose we have found \( B_i \) and \( C_i \) as above for \( i < n \). Define a sequence \( m_0, m_1, \ldots \) from \( \omega \) by recursion on \( k \) as follows:

\[
m_k = \min \left( A_{n+k} \left( \{ m_i \mid i < k \} \cup \bigcup_{i<n} C_i \right) \right)
\]

and let \( B = \{ m_k \mid k \in \omega \} \).
In the case of \( \text{Fin}(\phi) \), instead chose finite sets \( M_0, M_1, \ldots \) such that \( M_k \subseteq A_{\alpha+k}\backslash(\bigcup_{i<n} C_i \cup M_i) \) and \( \phi(M_k) > 0 \) for each \( k \in \omega \). This is possible since for each \( k \), \( A_{\alpha+k}\backslash(\bigcup_{i<n} C_i \cup M_i) \in \text{Fin}(\phi)^+ \). Then let \( B = \bigcup_{k \in \omega} M_k \).

If \( B \in \mathcal{I}^+ \), we let \( \alpha = n - 1 \) and we are done, since \( B \subseteq A_i \) for every \( i \in \omega \). If on the other hand \( B \notin \mathcal{I}^+ \), we let \( B_n = B \); since \( B \in \text{Fin}^+ \) we can pick \( C_n \in \mathcal{A}\{C_i \mid i < n\} \) such that \( B_n \cap C_n \notin \text{Fin} \).

Supposing that the construction does not end at a finite stage, let \( A_\omega = \bigcup_{n \in \omega} B_n \cap C_n \). It is clear by construction that \( A_\omega \subseteq^* A_m \) for every \( m \in \omega \). Furthermore, since \( A_\omega \) is an infinite union of sets not in \( \text{Fin} \) which are also subsets of distinct elements in \( \mathcal{A} \), and the latter is an almost disjoint family, we conclude that \( A_\omega \in \mathcal{I}^+ \). \( \square \)

**Lemma 3.11.** Let \( \text{HVD}(X) \) denote the sets which are hereditarily definable using parameters from \( V \cup \{X\} \). Then the following holds:

\[
\models_{\mathcal{M}^2} (\mathcal{ON}^\omega)^{\text{HVD}(x_G)_{E_0}} \subseteq V.
\]

**Proof.** Suppose \( \theta(x_1, x_2, x_3, x_4) \) is a formula with all free variables shown, \( p_0 \in \mathcal{M}^2 \), \( a \) is arbitrary, and \( \dot{x} \) is a \( \mathcal{M}^2 \)-name such that

\[
p_0 \Vdash \dot{x} : \omega \to \mathcal{ON} \land (\forall n \in \omega)(\forall \alpha \in \mathcal{ON}) \dot{x}(n) = \alpha \iff \theta(n, \alpha, \dot{a}, [x_G]_{E_0}).
\]

Let \( A_0 = A(p_0) \), and build \( A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \) and \( \alpha_0, \alpha_1, \alpha_2, \ldots \) a sequence of ordinals as follows: given \( A_n \), find \( (b, A_{n+1}) \leq (a(p_0), A_n) \) and \( \alpha_n \) such that

\[
(b, A_{n+1}) \Vdash \theta(n, \alpha_n, \dot{a}, [x_G]_{E_0}).
\]

Finally, find \( A_\omega \) such that \( A_\omega \subseteq^* A_n \) for every \( n \in \omega \).

We claim that \( (a(p_0), A_\omega) \Vdash (\forall n \in \omega) \dot{x}(n) = \alpha_n \), and thus \( \dot{x} \in V \). To prove this, suppose towards a contradiction that there is \( n \in \omega \) such that \( (a(p_0), A_\omega) \not\Vdash \dot{x}(n) = \alpha_n \), and find \( (b, B) \leq (a(p_0), A_\omega) \) such that \( (b, B) \Vdash \dot{x}(n) \neq \alpha_n \). That is, \( (b, B) \Vdash \neg \theta(n, \alpha_n, \dot{a}, [x_G]_{E_0}) \). By Fact 3.9(4), also \( (a(p_0), B) \Vdash \neg \theta(n, \alpha_n, \dot{a}, [x_G]_{E_0}) \). However, since \( B \subseteq A_\omega \subseteq^* A_{n+1} \) we know that \( (a(p_0), B \cap A_{n+1}) \leq (a(p_0), B), (a(p_0), A_{n+1}) \). This contradicts the fact that \( (a(p_0), A_{n+1}) \Vdash \theta(n, \alpha_n, \dot{a}, [x_G]_{E_0}) \). \( \square \)

### 3.2. The Branch Lemma

In this section we shall finally prove the Main Proposition [5.5]. We make a crucial definition (imported from [16]), followed by some fairly straightforward observations:

**Definition 3.12.** For \( x \subseteq \omega \), let

\[
T^x = \{ t \in T \mid (\exists w \in [T|t]) \pi(w) \cap x \notin \text{Fin} \}.
\]

**Facts 3.13.**

1. If \( x E_0 z \), then \( T^x = T^z \). This means that for a generic \( G \), the tree \( T^{x_G} \) is definable from \([x_G]_{E_0}\).
2. \( T^x \) is a pruned tree on \( 2 \times \kappa \).
(3) \( t \in T^x \) if and only if there is some \( y \in \pi[T^x_{[t]}]\) such that \( y \cap x \notin \text{Fin} \).

(4) \( \emptyset \notin T^x \) is equivalent to \( T^x = \emptyset \), as well as to \( [T^x] = \emptyset \), as well as to that \( \{x\} \cup A \) is not an AD family.

(5) Since \( T^x \) is a subtree of \( T \), \( \pi[T^x] \subseteq A \).

The proof of the Main Proposition is actually based on the following Branch Lemma.

**The Branch Lemma 3.14.** \( \models_{M^x} |\pi[T^{x_G}]| \leq 1 \).

Momentarily assuming the Branch lemma, we can very quickly show the Main Proposition 3.15, i.e., that

\[ \models_{M^x} (\forall y \in \pi[T]) y \cap x_G \in \text{Fin} \]

as follows.

**Proof of the Main Proposition 3.15.** Towards a contradiction, suppose \( G \) is \( M^x \)-generic and we have \( y \in \pi[T]^{V[G]} \) such that \( y \cap x_G \notin \text{Fin} \). By the Branch Lemma \( \pi[T^{x_G}] = \{y\} \). Thus, since \( y \) is definable from \( x_G \mid E_0 \), we have \( y \in \pi[T] \cap V \subseteq \mathcal{I} \) by Lemma 3.11. But then by 3.9(1), \( x_G \cap y \notin \text{Fin} \), contradiction.

Main Proposition 3.15 □

For the proof of Theorem 3.1, it remains but to prove the Branch Lemma.

**Proof of the Branch Lemma 3.14.** Towards a contradiction, suppose \( G \) is \( M^x \)-generic and we have distinct \( x_0, x_1 \in \pi[T^{x_G}] \). Fix \( n \) such that \( x_0 \mid n 
eq x_1 \mid n \), and let \( s_i = w_i \mid n \) where \( x_i = \pi(w_i) \) and \( w_i \in [T] \).

**Claim 3.15.** There exists \( t_0, t_1 \in T^{x_G} \) such that

(1) \( s_i \subseteq t_i \) for \( i \in \{0, 1\} \);

(2) \( \forall x_0^*, x_1^* \) such that \( x_i^* \in \pi[T_{[t_i]}^{x_G}] \) it holds that \( x_0^* \cap x_1^* \subseteq \pi(t_0) \cap \pi(t_1) \).

**Proof of claim.** Suppose otherwise. Then for all \( t_0, t_1 \in T^{x_G} \) extending \( s_0, s_1 \) respectively, there exists \( x_0^*, x_1^* \) such that \( x_i^* \in \pi[T_{[t_i]}^{x_G}] \) and \( \pi(t_0) \cap \pi(t_1) \subseteq x_0^* \cap x_1^* \). We may build branches \( w_0^*, w_1^* \in [T^{x_G}] \) such that \( s_i \subseteq w_i^* \) and \( \pi(w_0^*) \cap \pi(w_1^*) \notin \text{Fin} \). This however contradicts the fact that \( \pi[T^{x_G}] \subseteq \pi[T] \), which is an almost disjoint family.

Claim 3.15 □

Thus, pick \( t_0, t_1 \in T^{x_G} \) as in the claim, and let

\[ y_i = \bigcup \pi[T_{[t_i]}^{x_G}] \quad i \in \{0, 1\} \]

It must be the case that \( y_0 \in V \) since \( y_0 \) is definable from \( x_G \mid E_0 \) (the same is true of \( y_1 \)). Noting \( y_0 \in \text{Fin}^+ \), one of the two following cases occurs:

**Case 1:** \( x_G \cap y_0 \in \text{Fin} \). This, however, is a contradiction; indeed, since \( y_0 = \bigcup \pi[T_{[t_0]}^{x_G}] \) where \( t_0 \in T^{x_G} \), Facts 3.14 yields the existence of a set \( y \in \pi[T_{[t_0]}^{x_G}] \) such that \( y \cap x_G \notin \text{Fin} \).

**Case 2:** If the first case fails, since \( \{p \in M^x \mid A(p) \subseteq y_0 \} \) is dense in \( M^x \) we have \( x_G \subseteq y_0 \). But then \( x_G \cap y_1 \in \text{Fin} \). This is also a contradiction, for the same reasons as above.

Lemma 3.13 □
4. Simple Fubini products

The ideas from the previous section can be used to prove similar results about ideals that are further up the Borel hierarchy. In this section, we will take one step up the ladder, whilst in the following section we see that we can go all the way.

Recall from Section 1 that given ideals \( J_\ast, J_k \) on \( \omega \) (for each \( k \in \omega \)) we can form the ideal \( \bigoplus J_\ast J_k \) on \( \omega \times \omega \). If \( J_k = J' \) for each \( k \in \omega \), one writes \( \bigoplus J_\ast J_k \) (called the Fubini product of \( J_\ast \) with \( J' \)).

We will study ideals of the form \( J = \bigoplus_{\text{Fin}} \text{Fin}(\phi_k) \), where \( \phi \) and \( \phi_k \) for each \( k \in \omega \) are lsc submeasures on \( \omega \). Clearly this includes \( \text{Fin} \circ \text{Fin} \), which is \( \text{Fin}(\phi) \circ \text{Fin}(\phi) \) where \( \phi \) is the counting measure. For \( X \subseteq \omega \times \omega \) we write

\[ X(n) = \{ k \in \omega \mid (n, k) \in X \}, \]
\[ \text{dom}_X^\varphi(X) = \{ n \in \omega \mid X(n) \notin \text{Fin}(\phi_n) \}. \]

We write \( \text{dom}_X \) for \( \text{dom}_{\text{Fin} \circ \text{Fin}} \), and note that

\[ \text{Fin} \circ \text{Fin} = \{ X \subseteq \omega \times \omega \mid \text{dom}_X(X) \in \text{Fin} \}. \]

We will use the two following orderings on \( \mathcal{P}(\omega \times \omega) \). For \( X \subseteq \omega \times \omega \) finite and \( Y \subseteq \omega \times \omega \) we say

\[ X \sqsubseteq Y \iff \text{dom}(X) \subseteq \text{dom}(Y) \land (\forall n \in \text{dom}(X)) X(n) \subseteq Y(n), \]

and

\[ X \sqsubset Y \iff \text{dom}(X) \subseteq \text{dom}(Y) \land (\forall n \in \text{dom}(X)) X(n) \subseteq Y(n) \]

(of course \( X \subseteq Y \iff X = Y \cap (\max(X) + 1) \)). In the general case where \( \mathcal{J} = \bigoplus_{\text{Fin}(\phi)} \text{Fin}(\phi_k) \), let

\[ X \sqsubseteq Y \iff X \sqsubseteq Y \land \phi(\text{dom}(X)) < \phi(\text{dom}(Y)) \land \\
(\forall n \in \text{dom}(X)) \phi_n(X(n)) < \phi_n(Y(n)). \]

This section was written so that most proofs generalize almost mechanically from \( \text{Fin} \circ \text{Fin} \) to the above more general case; often this is made possible by the definition of \( \sqsubseteq \) given above.

We let as usual \( (\text{Fin} \circ \text{Fin})^+ \) (resp., \( \mathcal{J}^+ \)) denote the co-ideal.

**Definition 4.1.** Let \( (\text{Fin} \circ \text{Fin})^{++} \) denote the set of \( A \in (\text{Fin} \circ \text{Fin})^+ \) such that for all \( k \in \text{dom}(A) \), \( A(k) \notin \text{Fin} \).

Conditions of the forcing notion \( \mathcal{M}_2 \) are pairs \((a, A)\) where

(a) \( a \subseteq \omega \times \omega \) and is finite;
(b) \( A \in (\text{Fin} \circ \text{Fin})^{++} \);
(c) \( \max(a(k)) < \min(A(k)) \) for every \( k \in \text{dom}(a) \);
(d) \( \text{dom}(a) \subseteq \text{dom}(A) \).
We let \((a', A') \leq (a, A)\) just in case \(A' \subseteq A\), and \(a \subseteq_2 a' \subseteq a \cup A\).

In the general case when \(\mathcal{J} = \bigoplus_{\text{Fin}(\phi)} \text{Fin}(\phi_k)\), \(\mathcal{J}^{++}\) denotes the set of \(A \in \mathcal{J}^+\) such that for all \(k \in \text{dom}(A)\), \(A(k) \notin \text{Fin}(\phi_k)\). Moreover, replace \((b)\) in the definition\(^1\) of \(\mathcal{M}_2\) by \(A \in \mathcal{J}^{++}\).

Note that if \((a, A)\) is a condition in \(\mathcal{M}_2\) then for every \(k \in \text{dom}(a)\), the pair \((a(k), A(k))\) is a Mathias forcing condition (resp., a condition in \(\mathcal{M}^{\text{Fin}(\phi)}\)). Moreover, the pair \((\text{dom}(a), \text{dom}(A))\) is a Mathias forcing condition (resp., a condition in \(\mathcal{M}^{\text{Fin}(\phi)}\)) as well.

As in the 1-dimensional case, a relativized forcing notion is needed.

**Definition 4.2.** If \(\mathcal{I}^+\) is a co-ideal of an ideal \(\mathcal{I} \supseteq \text{Fin} \otimes \text{Fin}\), then we write \(\mathcal{I}^{++}\) for \(\mathcal{I}^+ \cap (\text{Fin} \otimes \text{Fin})^{++}\). We let

\[
\mathcal{M}_2^\mathcal{I} = \{(a, A) \in \mathcal{M}_2 : A \in \mathcal{I}^{++}\}
\]

equipped with the ordering inherited from \(\mathcal{M}_2\).

Note that if \(\mathcal{I} = \mathcal{J}\) then \(\mathcal{M}_2^\mathcal{I} = \mathcal{M}_2\). Note furthermore that if \(A \in \mathcal{I}^+\), then we can always find a subset \(B \subseteq A\) such that \(B \in \mathcal{I}^{++}\). We need to establish some notation:

**Notation 4.3.**

1. Given a filter \(G\) on \(\mathcal{M}_2^\mathcal{I}\), let

\[
x_G = \bigcup \{a : (\exists A)(a, A) \in G\}.
\]

It is easy to see \(\models x_G \in (\mathcal{J})^{++}\) when \(\mathcal{J} = \text{Fin} \otimes \text{Fin}\); we will check this more carefully for \(\bigoplus_{\text{Fin}(\phi)} \text{Fin}(\phi_k)\) below, see Lemma 4.8(3).

2. For \(p \in \mathcal{M}_2^\mathcal{I}\), we write \(p = (a(p), A(p))\) when we want to refer to the components of \(p\).

3. For \((a, A) \in \mathcal{M}_2^\mathcal{I}\) and \(b \subseteq a \cup A\) finite, let

\[
A/b = \bigcup_{n \in N} A(n) \setminus \{m \in \omega : m \leq \max(b(n))\}
\]

where \(N = \text{dom}(b) \cup \{\max(\text{dom}(b)) + 1, \infty\}\).

4. For \(p \in \mathcal{M}_2^\mathcal{I}\), let \(\mathcal{M}_2^\mathcal{I}(< p) = \{q \in \mathcal{M}_2 : q \leq p\}\).

**Remark 4.4.** Note that in order to meaningfully talk about \(\kappa\)-Suslin sets in \(\mathcal{P}(\omega \times \omega)\), we identify \(\omega \times \omega\) with \(\omega\) (via some fixed bijection), sets with their characteristic functions, and in effect, \(\mathcal{P}(\omega \times \omega)\) with \(2^\omega\) (see also Section 2.2).

**Assumption 4.5.** Until the end of Section 2 let \(\mathcal{J} = \text{Fin} \otimes \text{Fin}\), or more generally let \(\mathcal{J} = \bigoplus_{\text{Fin}(\phi)} \text{Fin}(\phi_k)\) as above. Moreover suppose \(\mathcal{A} \subseteq \mathcal{P}(\omega \times \omega)\) to be a \(\mathcal{J}\)-almost disjoint family which is \(\kappa\)-Suslin and fix a tree \(T\) on \(2 \times \kappa\) such that \(\pi[T] = \mathcal{A}\). Finally, let \(\mathcal{I}\) be the ideal generated by \(\mathcal{A} \cup \mathcal{J}\).

\(^1\)This makes the designation somewhat ambiguous; i.e., \(\mathcal{M}_2\) depends on the ideal \(\mathcal{J}\) being considered. Note that the \(\mathcal{M}_2^\mathcal{I}\) notation does not provide a way to refer to these variants, unlike in the previous section.
To ease the notation, we will focus our attention on $\mathcal{J} = \text{Fin} \otimes \text{Fin}$. However, our proofs work for $\mathcal{J} = \bigoplus_{\text{Fin}(\phi)} \text{Fin}(\phi_k)$ as above. For the general case, substitute $\text{Fin} \otimes \text{Fin}$ (but not the word finite or the expression $[\omega^2]^{<\omega}$) by $\bigoplus_{\text{Fin}(\phi)} \text{Fin}(\phi_k), \text{dom}_x$ by $\text{dom}_{\text{Fin}}^J$, etc. wherever relevant, unless we provide commentary.

Now we are ready to state the Main Proposition regarding $\mathcal{M}_I^2$ from which Theorem 1.3 follows as a corollary, precisely analogous to the previous section. The proof of the Main Proposition will again rely on a Branch Lemma and will be postponed until Section 4.2.

**Main Proposition 4.6.** $\Vdash_{\mathcal{M}_I^2} (\forall y \in \pi[T]) y \cap x_\mathcal{G} \in \text{Fin} \otimes \text{Fin}.$

As in the one-dimensional case, our main result about $\text{Fin} \otimes \text{Fin}$ also follows directly from the Main Proposition.

**Corollary 4.7.** Theorem 1.3 holds.

**Proof.** The proofs are essentially identical to those of Corollary 3.6, Corollary 3.7, and Corollary 3.8, simply substituting $\mathcal{M}_I^2$ for $\mathcal{M}_2^1$. □

### 4.1. Properties of the two-dimensional forcing.

Before we can prove the Main Proposition, we shall collect some of the necessary facts about the forcing $\mathcal{M}_I^2$.

**Lemma 4.8.**

1. For any $A \in \mathcal{I}$, $\Vdash_{\mathcal{M}_I^2} x_\mathcal{G} \cap \check{A} \in \text{Fin} \otimes \text{Fin}$.
2. For any $k \in \omega$ the partial order $\mathcal{M}_I^2$ is isomorphic to the product $\mathcal{M}_k^k \times \mathcal{M}_I^2 (\leq (\emptyset, (\omega \setminus k) \times \omega))$, where $\mathcal{M}_k$ is the set of $k$-tuples of classical (1-dimensional) Mathias forcing conditions. In the general case where $\mathcal{J} = \bigoplus_{\text{Fin}(\phi)} \text{Fin}(\phi_k)$ we have $\mathcal{M}_I^2 \cong \left( \prod_{i < k} \mathcal{M}_{\text{Fin}}(\phi_i) \right) \times \mathcal{M}_I^2 \left( \leq (\emptyset, (\omega \setminus k) \times \omega) \right)$.
3. $\Vdash_{\mathcal{M}_I^2} x_\mathcal{G} \in (\text{Fin} \otimes \text{Fin})^{++}$.

**Proof.**

1. By an easy density argument.

2. Define a map $\phi: \mathcal{M}_k^k \times \mathcal{M}_I^2 \to \mathcal{M}_I^2$ by $((c_i, C_i)_{i < k}, (a, A)) \mapsto (\bigcup_{i < k} \{i\} \times c_i \cup a, \bigcup_{i < k} \{i\} \times C_i \cup A)$ This map is easily seen to be bijective and order preserving. The same definition works in the general case.

3. Work in the general case where $\mathcal{J} = \bigoplus_{\text{Fin}(\phi)} \text{Fin}(\phi_k)$. First note that $\Vdash_{\mathcal{M}_I^2} \text{dom}(x_\mathcal{G}) = \text{dom}_{\mathcal{J}}(x_\mathcal{G})$. For let $n$ and $p$ be such that $p \Vdash \check{n} \in \text{dom}(x_\mathcal{G})$. It must hold that $n \in \text{dom}(a(p))$. By Lemma 3.9(2),

\[ (a(p)(n), A(p)(n)) \Vdash_{\mathcal{M}} x_\mathcal{G} \notin \text{Fin}(\phi_n) \]
so by item (2) of the present lemma, \( p \models \text{dom}(x_{\check{G}}) \).

It remains to show \( \models \text{dom}(x_{\check{G}}) \notin \text{Fin}(\phi) \). Towards a contradiction, suppose there is \( n \) and \( p \) so that \( p \models \phi(\text{dom}(x_{\check{G}})) < n \). Find a finite set \( d \) such that \( \text{dom}(a(p)) \subseteq d \subseteq \text{dom}(a(p)) \cup \text{dom}(A(p)) \) and \( \phi(d) > n \), and \( a \) such that \( a(p) \subseteq a(p) \cup A(p) \) and \( \text{dom}(a) = d \). We reach a contradiction since \( (d, A(p)/d) \models \phi(d) > n \).

We prove a general diagonalization result (which shall be put to use in Lemma 4.14 below):

**Lemma 4.9.** Let \((A_k)_{k \in \omega}\) be a sequence from \(\mathcal{I}^+\) satisfying that \(A_{k+1} \subseteq A_k\) for every \(k \in \omega\). Then there is \(A_x \in \mathcal{I}^+\) such that \(A_x \subseteq \text{Fin} \otimes \text{Fin} A_k\) for every \(k \in \omega\).

In other words, \((\mathcal{I}^+, \subseteq^*)\) is \(\sigma\)-closed. In a certain sense \(\mathcal{I}^+\) is even a selective co-ideal, a fact which will be more or less implicit in the proof of Lemma 4.13 below.

**Proof.** As in the previous section, we construct two sequences \((B_n)_{n \in \alpha}\) and \((C_n)_{n \in \alpha}\) of length \(\alpha \leq \omega\) such that for each \(n < \alpha\),

- \(B_n \in (\text{Fin} \otimes \text{Fin})^+\);
- \(B_n \subseteq A_n\) and \((\forall k \in \omega) B_n \subseteq \text{Fin} \otimes \text{Fin} A_k\);
- \(C_n \in A \setminus \{C_i \mid i < n\}\);
- \(B_n \cap C_n \in (\text{Fin} \otimes \text{Fin})^+\) and \(B_n \cap C_i \in \text{Fin} \otimes \text{Fin}\) for \(i < n\).

Suppose we have found \(B_i\) and \(C_i\) as above for \(i < n\). Define a sequence \(m_0^n, m_1^n, \ldots\) from \(\omega\) by recursion on \(k\) as follows:

\[
m_k^n = \min \left( \text{dom} (A_{n+k} \setminus \{(m_i \mid i < k) \cup \bigcup_{i<n} C_i\}) \right)
\]

and let \(B = \bigcup_{k \in \omega} A_{n+k}(m_k^n)\).

In the case of \(\text{Fin}(\phi)\), instead chose finite sets \(M_0^n, M_1^n, \ldots\) such that \(M_k^n \subseteq \text{dom} (A_{n+k} \setminus \bigcup_{i<n} (C_i \cup M_i^n))\) and \(\phi(M_k^n) > 0\) for each \(k \in \omega\). Then let

\[
B = \bigcup_{k \in \omega} \bigcup_{m \in M_k^n} A_{n+k}(m).
\]

The remainder of the proof is essentially identical to the 1-dimensional case, i.e., Lemma 3.10 simply replacing \(\text{Fin}\) by \(\text{Fin} \otimes \text{Fin}\) everywhere. We leave this to the reader.

\[\square\]

**4.2. The two-dimensional Branch Lemma.** The crucial definition is again that of an invariant tree, analogous to Definition 3.12.

**Definition 4.10.** For \(x \subseteq \omega \times \omega\), let

\[
T^x = \{ t \in T \mid (\exists y \in \pi[T_{|t}])(y \cap x) \notin \text{Fin} \otimes \text{Fin} \}
\]
As in Section 3.2, it is easy to see that whenever \( x \Delta x' \in \text{Fin} \otimes \text{Fin}, \ T^x = T^{x'} \). Moreover Facts 5.13.2 and 5.13.3 hold here as well.

We are now ready to state the main lemma of this section.

**The Branch Lemma 4.11.** \( \models_{M^F} |\pi[T^{x_G}]| \leq 1. \)

We postpone the proof of the Branch Lemma and first give the proof of the Main Proposition 4.6 assuming the lemma. The proof is not quite as straightforward as in the previous section, but the idea remains the same.

**Proof of the Main Proposition 4.6.** Suppose towards a contradiction there is \( p_0 \in M^F \) forcing that there is \( A \in \pi[T]^{V[G]} \) with \( A \cap x_G \notin \text{Fin} \otimes \text{Fin} \). By the Branch Lemma 4.11, \( p_0 \) forces that \( \pi[T^{x_G}] \) has precisely one element; let \( \hat{A} \) be a name for it.

**Claim 4.12.** There is \( q \in M^F_2 \) and \( A' \in V \) such that \( q \models \hat{A} = \hat{A}' \).

**Proof of Claim.** It suffices to show that if \( p \leq p_0 \) and \( p \) decides \((n, m) \in \hat{A}\) then in fact \((a(p_0), A(p)) \) decides \((n, m) \in \hat{A}\): For then we may pick \( A_0 \supseteq A_1 \supseteq \ldots \) such that for each pair \((n, m) \in \omega \times \omega\), some \((a(p_0), A_k)\) decides \((n, m) \in \hat{A}\); by Lemma 4.9, we can find \( A_\omega \) diagonalizing \((A_k)_{k \in \omega}\). Any condition below \( q = (a(p_0), A_\omega) \) is compatible with each \((a(p_0), A_k)\), and so \( q \) decides all of \( \hat{A} \).

So suppose \( p \leq q \) decides \((n, m) \in \hat{A}\); we must show \((a(p_0), A(p)) \) decides \((n, m) \in \hat{A}\). Let us suppose that \( p \models \neg (n, m) \in \hat{A} \); the proof is similar in case \( p \models (n, m) \notin \hat{A} \) and we leave this case to the reader.

Fix any \( M^{F_2}_{\omega} \)-generic \( G \) such that \((a(p_0), A(p)) \in G\). By Lemma 4.8.2, we can decompose \( G \) as \( G_0 \times G_1 \) where \( G_1 \) is \( M^{F_2}_{\omega} \)-generic and \( G_0 \) is \( M^F_{\omega} \)-generic for \( k \) large enough so that \( \text{dom}(a(p)) \subseteq k \). Note that as \( x_G \Delta x_{G_1} \in \text{Fin} \otimes \text{Fin}, T^{x_G} = T^{x_{G_1}} \in V[G_1] \). Since \( V[G] \models \pi[T^{x_G}] = \{ \hat{A}^G \} \), by a simple absoluteness argument the same must hold in \( V[G_1] \), i.e., \( \hat{A}^G \in V[G_1] \) and \( V[G_1] \models \pi[T^{x_G}] = \{ \hat{A}^G \} \).

Since \( \text{dom}(a(p)) \subseteq k \) we can find \( G' \) which is \( M^{F_2}_{\omega} \)-generic over \( V \) such that \( G' = G'_0 \times G_1 \) and \( p \in G' \). Clearly \((n, m) \in \hat{A}^G \) (since \( p \models \neg (n, m) \in \hat{A} \)). Arguing as before using absoluteness, this time between \( V[G'] \) and \( V[G_1] \), \( \hat{A}^G \) must equal the unique element of \( \pi[T^{x_{G_1}}] \), i.e., \( \hat{A}^G = \hat{A}^G \) and so \((n, m) \in \hat{A}^G \). Since \( G \) was arbitrary, \((a(p_0), A(p)) \models \neg (n, m) \in \hat{A}^G \).

**Main Proposition 4.6.**

We now gradually work towards the proof of the Branch Lemma, for which it is necessary to introduce some notation. Firstly, write

\[
U = [\omega \times \omega]^{<\omega} \times T.
\]
Given a pair $\vec{u} \in U$, we write it as $(a(\vec{u}), t(\vec{u}))$ if we want to refer to the components of $\vec{u}$.

We define a partial order $\leq_U$ on $U$ as follows:

$$\vec{u}_1 \leq_U \vec{u}_0 \iff a(\vec{u}_1) \supseteq a(\vec{u}_0) \land t(\vec{u}_1) \supseteq t(\vec{u}_0).$$

Now secondly assume $G$ is $M^2_2$-generic over $V$; working in $V[G]$ for the moment and for a fixed $x \in P(\omega \times \omega)$, define the set $U^x \subseteq U$ consisting of those pairs $(a, t) \in U$ such that there is $w \in [T_t]$ with

1. $\pi(w) \cap x \notin \text{Fin} \otimes \text{Fin}$,
2. $\text{dom}(a) \subseteq \text{dom}_x(\pi(w) \cap x)$ and
3. for each $k \in \text{dom}(a)$, $a(k) \subseteq \pi(w)(k) \cap x(k)$.

Intuitively, $U^x$ searches for a branch through $T$ whose projection has large intersection with $x$ and a subset of this intersection in $\text{Fin} \otimes \text{Fin}^+$ to witness its largeness.

In analogy to the tree $T^x$, when $\vec{u}_0 \in U$ write $U^x_{[\vec{u}_0]}$ for $\{\vec{u} \in U \mid \vec{u} \leq_U \vec{u}_0\}$.

The following three lemmas gather some observations concerning $U^x_G$ which will be important in the proof of the Branch Lemma.

**Lemma 4.13.** Suppose $(a, A) \models G \in U^{x_G}$.

1. It holds that $a \equiv a(\vec{u})$.
2. The set $A' \subseteq \omega \times \omega$ defined by
   $$A' = \{(k, l) \mid (\exists \vec{u}' \leq (a, A)) (\exists \vec{u} \leq_U \vec{u}) (k, l) \in a(\vec{u}') \land p' \models \vec{u}' \in U^{x_G}\}$$
   is not in $\mathcal{I}$.
3. For any $k \in \text{dom}(a(\vec{u}))$, the set $A_k \subseteq \omega$ defined by
   $$\{l \mid (\exists \vec{u}' \leq (a, A)) (\exists \vec{u} \leq_U \vec{u}) l \in a(\vec{u}')(k) \land p' \models \vec{u}' \in U^{x_G}\}$$
   is not in $\text{Fin}$ (resp., in $\text{Fin}(\phi_k)$).

**Proof.**

1. Immediate from the definition of $U^{x_G}$.

2. Assume to the contrary that $A' \in \mathcal{I}$. Then $A \setminus A' \in \mathcal{I}^+$, so take $B \subseteq A \setminus A'$ such that $B \in \mathcal{I}^{++}$ and set $p = (a, B) \in M^{x}_2$. Since $p \models G \in U^{x_G}$ we can find a name $\hat{w}$ such that
   $$p \models \hat{w} \in \pi[T_{t(\vec{u})}] \land \hat{w} \cap \chi^G \notin \text{Fin} \otimes \text{Fin}. $$
   (In fact, all we need here is that $p \models T^{x_G} \neq \emptyset$). Thus we can extend $p$ to $p'$ to force a pair $(k, l)$ into $\hat{w} \cap \chi^G \setminus a(p)$. But it has to be the case that $(k, l) \in a(p')$, whence $(k, l) \in A'$ by definition of $A'$, contradicting that also $(k, l) \in B$ which is disjoint from $A'$.

3. Assume to the contrary that $k \in \text{dom}(a(\vec{u}))$ and $A_k \in \text{Fin}$. Take $B \subseteq A \setminus (\{k\} \times A_k)$ such that $B \in \mathcal{I}^{+++}$, and set $p = (a, B) \in M^{x}_2$. Since $p \models G \in U^{x_G}$ we can find a name $\hat{w}$ such that
   $$p \models \hat{w} \in \pi[T_{t(\vec{u})}] \land \hat{w} \cap \chi^G \in (\text{Fin} \otimes \text{Fin})^+. $$
and
\[ p \Vdash \text{dom}(a(\vec{u})) \subseteq \text{dom}_G(\vec{w} \cap x_G). \]

As \( k \in \text{dom}_G(\vec{w} \cap x_G) \), we can extend \( p \) to \( p' \) to force a pair \((k, l)\) into \( \vec{w} \cap x_G \setminus a(p) \). But as in the proof of the previous item, it has to be the case that \((k, l) \in a(p')\), whence \( l \in A_k \) by definition of \( A_k \), contradicting that also \( l \in B(k) \) which is disjoint from \( A_k \). \( \square \)

In order to prove the two-dimensional Branch Lemma, we also need to introduce the partially ordered set \( \Gamma \) defined as follows:
\[ \Gamma = \{(p, \vec{u}_0, \vec{u}_1) \in M_2^f \times U \times U \mid (\forall i \in \{0, 1\}) \ p \Vdash \vec{u}_i \in U^{\pi_G} \}. \]

This set carries a weak and a strict order, defined as follows:
\[ (p_1, \vec{u}_1, \vec{u}_1') \preceq_{\Gamma} (p_0, \vec{u}_0, \vec{u}_0') \]
if and only if \( p_1 \leq p_0 \), and for each \( i \in \{0, 1\} \), \( a(u_i') \supseteq a(\vec{u}_i) \) and \( t(\vec{u}_i) \supseteq t(u_i) \) (that is, \( u_i^1 \leq U u_i^0 \)); and
\[ (p_1, \vec{u}_1, \vec{u}_1') <_{\Gamma} (p_0, \vec{u}_0, \vec{u}_0') \]
if and only in addition, \( a(\vec{u}_0) \cap a(\vec{u}_1') \supseteq a(\vec{u}_0') \cap a(\vec{u}_1') \).

Note that \( \Gamma \) is well-founded with respect to the second, strict ordering \(<_{\Gamma}\); indeed, suppose towards a contradiction that there is an infinite \(<_{\Gamma}\)-descending sequence
\[ \ldots <_{\Gamma} (p_3, \vec{u}_3, \vec{u}_3') <_{\Gamma} (p_2, \vec{u}_2, \vec{u}_2') <_{\Gamma} (p_1, \vec{u}_1, \vec{u}_1') \]
from \( \Gamma \). Define
\[ y^i = \bigcup_{n>1} t(\vec{u}_n^i) \]
for \( i \in \{0, 1\} \) and
\[ A = \bigcup_{n>1} a(\vec{u}_n^0) \cap a(\vec{u}_n^1). \]
Since the sequence is \(<_{\Gamma}\)-decreasing and from \( \Gamma \), \( A \in (\text{Fin} \otimes \text{Fin})^{++} \) and \( A \subseteq \pi(y^0) \cap \pi(y^1) \), contradicting that \( \pi[T] \) is \( \text{Fin} \otimes \text{Fin} \)-almost disjoint.

The following lemma says that we can approximate \( U^{\pi_G} \) reasonably well in the ground model. A very similar proof shows that \( M_2^f \) is proper.

**Lemma 4.14.** For each \( \vec{u}_0 \in U \) the set \( D(\vec{u}_0) \) is dense and open in \( M_2^f \), where we define \( D(\vec{u}_0) \) to be the set of \( p \in M_2^f \) such that for all \( p' \leq p \) and any \( \vec{u} \in U \),
\[ p' \Vdash \vec{u} \in U^{\pi_G}_{\vec{u}_0} \Rightarrow (a(p'), A(p)/a(p')) \Vdash (\exists t \in T) (a(\vec{u})), t) \in U^{\pi_G}_{\vec{u}_0}. \]
The proof follows the same strategy as Lemma 4.9 (the diagonalization lemma) to build a set in $\mathcal{I}^{++}$. While we build this set, we carefully anticipate each of its finite subsets $a$ to see if there is some $t \in T$ and some forcing condition $q \in \mathcal{M}_2$ which forces $(a, t)$ to be in $U_x \hat{\cdot}$. If so, we make sure that our final set is contained in $a \cup A(q)$. We succeed as there are only countably many finite $a \subseteq \omega \times \omega$ to consider. Note though that due to the nature of the proof of Lemma 4.9 we have to consider each finite $a$ again and again, and the construction potentially takes $\omega \times \omega$ stages.

Proof. Fix $q_0 \in \mathcal{M}_2$ and $\bar{u}_0 \in U$ such that $q \Vdash \bar{u}_0 \in U_x \hat{\cdot}$. We construct $q \leq q_0$ such that $q \in D(\bar{u}_0)$.

As in the proof of Lemma 4.9 we construct sequences $B_0, B_1, \ldots$, and $C_0, C_1, \ldots$ both of which are possibly finite, such that whenever defined

- $B_n \in (\text{Fin} \otimes \text{Fin})^{++}$;
- $C_n \in \mathcal{A}\{C_i \mid i < n\}$;
- $B_n \cap C_n \in (\text{Fin} \otimes \text{Fin})^{+}$ while for $i < n$, $B_n \cap C_i \in \text{Fin} \otimes \text{Fin}$.

Suppose $B_1$ and $C_1$ have been defined for $i < n$ (this includes the case $n = 0$). In $\omega$-many steps we define a descending sequence of conditions $(b_{n}^{k}, B_{n}^{k})_{k}$ from $\mathcal{M}_2$ and at the end let

$$B_n = \bigcup_{k \in \omega} b_{n}^{k}. \tag{4.1}$$

If $n = 0$, let $b_{0}^{0} = a(q_0)$ and $B_{0}^{0} = A(q_0)$. Otherwise, let

$$b_{n}^{0} = \bigcup_{i,j < n} b_{j}^{i}$$

and

$$B_{n}^{0} = (B_{n-1}^{n-1} \restriction \text{dom}(b_{n}^{0})) \cup B_{n-1}^{n-1} \setminus \{C_i \mid i < n\} \tag{4.2}$$

noting that $B_{n}^{0} \in (\text{Fin} \otimes \text{Fin})^{+}$ since $B_{n-1}^{n-1} \in \mathcal{I}^{++}$ by induction hypothesis. So $(b_{n}^{0}, B_{n}^{0}) \in \mathcal{M}_2$ and $B_{n}^{0} \cap C_i \in \text{Fin} \otimes \text{Fin}$ for $i < n$.

Supposing we have already defined $(b_{n}^{k}, B_{n}^{k}) \in \mathcal{M}_2$ we thin out $B_{n}^{k}$ to $B^{*} \in \mathcal{I}^{++}$ in finitely many steps such that whenever $a \subseteq b_{n}^{k}$ and

$$(\exists p' \leq (a(q_0), B_{n}^{k}))((\exists \bar{u} \in U) a(\bar{u})) = a \land a(p') \subseteq b_{n}^{k} \land p' \Vdash \bar{u} \in U_{\bar{u} \in \hat{\cdot}} \tag{4.3}$$

then for some $t' \in T$

$$(a(p'), B^{*}) \Vdash (a(\bar{u}), t') \in U_{\bar{u} \in \hat{\cdot}}. \tag{4.4}$$

Extend $b_{n}^{k}$ to the some finite (we mean finite also in the general case!) set $b_{n}^{k+1} \subseteq \omega \times \omega$ satisfying

$$b_{n}^{k} \subseteq b_{n}^{k+1} \subseteq b_{n}^{k} \cup B^{*} \tag{4.4}$$

and let

$$B_{n}^{k+1} = B^{*}/b_{n}^{k+1}. \tag{4.4}$$
Assuming we have defined $b_n^k$ for each $k \in \omega$ and letting $B_n$ be defined by (4.1), note that (4.3) ensures that $B_n \in (\text{Fin} \otimes \text{Fin})^+$. Should it be the case that $B_n \in {\mathcal I}^+$ the construction terminates and we let

$$q = (a(q_0), B_n).$$

Otherwise, we may chose $C_n \in {\mathcal A}\setminus\{C_i \mid i < n\}$ such that $C_n \cap B_n \in (\text{Fin} \otimes \text{Fin})^+$ as in Lemma 4.9 and continue the construction.

If the construction does not terminate at any stage $n < \omega$, let

$$B_\omega = \bigcup_{n \in \omega} b_n^n.$$

Note that $B_\omega = \bigcup_{k \in \omega} B_k$ and thus since $B_\omega \cap C_k \in (\text{Fin} \times \text{Fin})^+$ for each $k \in \omega$, it must be the case that $B_\omega \in {\mathcal I}^{++}$ (as in the proof of Lemma 4.9).

So we obtain a condition in $M_2^\omega$ by letting

$$q = (a(q_0), B_\omega).$$

To see that $q \in D(\bar{u}_0)$, let $p' \leq q$, $\bar{u} \in U$ such that $p' \models \bar{u} \in U_{\bar{u}_0}^{x, \hat{c}}$ be given. Let us first assume that the construction did not stop at any stage $n < \omega$ and that $B_\omega$ is defined. We can find $n > 0$ so that $a(p') \subseteq b_{n-1}^n$. Thus, at stage $k = n$ in the construction of $B_n$, (4.2) was satisfied for $a = \bar{u}$, and so (4.3) is also satisfied. By construction $B_\omega \setminus b_{n-1}^n \subseteq B_n^n$. Thus any condition below $(a(p'), B_\omega) = (a(p'), A(q))$ is compatible with $(a(p'), B_n^n)$, and so we may replace $B^*$ by $A(q)$ in (4.3), obtaining

$$(\exists t \in T) (a(p'), A(q)) \models \bar{u} \in U_{\bar{u}_0}^{x, \hat{c}}$$

and showing that $q \in D(\bar{u})$.

If the construction of $B_0, B_1, \ldots$ terminated with $B_n \in {\mathcal I}^{++}$, we may find $k$ such that $a(p') \subseteq b_{n-1}^n$ and argue similarly with $B_n$ in place of $B_\omega$.

The proof of the Branch Lemma will crucially depend on the following simple lemma. It plays the same role as Lemma 3.9[4] in that it allows us to change the finite part of a condition while maintaining that something is forced about $U^{x, \hat{c}}$.

**Lemma 4.15.** For any $p \in M_2^\omega$, $\bar{u} \in U$ such that $p \models \bar{u} \in U^{x, \hat{c}}$, any $a \subseteq a(p)$ and $a' \subseteq a \cap a(\bar{u})$, it holds that

$$(a, A(p)/a) \models (a', t(\bar{u})) \in U^{x, \hat{c}}.$$

**Proof.** Let $G$ be a generic over $V$ with $(a, A(p)/a) \in G$, and let $I = \text{dom}(a(p)) \setminus \text{dom}(a)$.

Suppose $H$ is $\prod_{j \in I} M$-generic over $V[G]$ such that $(a(p)(j), A(p)(j))_{j \in I} \in H$.

Then $G \times H$ is generic over $V$ for

$$M_2^\omega \times \prod_{j \in I} M.$$
We define a bijection
\[ \phi : M_2^x \left( \preceq (a', A(p)/a') \right) \times \prod_{j \in I} M \left( \preceq (a(p)(j), A(p)(j)) \right) \rightarrow M_2^x \left( \preceq p \right) \]
by \((b, B, (c_j, C_j)_{j \in I}) \rightarrow (a^*, A^*)\) where
\[ a^*(k) = \begin{cases} a(p)(k) \cup (b(k) \setminus a(k)) & \text{if } k \in \text{dom}(a); \\ c_k & \text{if } k \in I; \\ b(k) & \text{otherwise}. \end{cases} \]
and
\[ A^*(k) = \begin{cases} C_k & \text{if } k \in I; \\ B(k) & \text{otherwise}. \end{cases} \]

Note that \(p \in \phi(G \times H)\), so \(\bar{u} \in U^{x_{\phi(G \times H)}}\) in \(V[G][H]\). By definition of \(U^{x}\) this means that in \(V[G]\) we can find \(w \in [T_{\Gamma[G]}]\) so that
\[ (\exists u \in (\text{Fin} \otimes \text{Fin})^{++}) \ a \subseteq u \subseteq \pi(w) \cap x_{\phi(G \times H)}. \]
Since \(a'' \subseteq x_G\) and since \(x_G \Delta x_{\phi(G \times H)} \in \text{Fin} \otimes \text{Fin}\) we may replace \(a\) by \(a'\) and then \(x_{\phi(G \times H)} \) by \(x_G \) in \([1.10]\), and thus
\[ (\exists x \in \pi[T_{\Gamma[G]}])(\exists u \in (\text{Fin} \otimes \text{Fin})^{++}) \ a'' \subseteq u \subseteq \pi(x) \cap x_G. \]
It is easy to find a tree \(S \in V[G]\) such that \([S]\) consists of the \((\text{codes for})\) pairs \((x, u)\) witnessing the two existential quantifiers in \([1.10]\). Since being well-founded is absolute between models of ZFC, we conclude \([1.10]\) holds in \(V[G]\). But \([1.10]\) implies (in fact, is equivalent to) \((a', t(\bar{u})) \in U^{x_G}\), so since \(G\) was arbitrary, we have shown that \((a', A/a') \models (a'', t(\bar{u})) \in U^{x_G}. \)

With this notation and the lemmas at our disposal, we are ready to prove
\[ \models M_2^x \left| \pi[T^{x_G}] \right| \leq 1, \]
i.e., the Branch Lemma \([4.11]\).

**Proof of the Branch Lemma \([4.11]\).** Assume towards a contradiction that the lemma is false, whence we may find \(p \in M_2^x\) and a pair of \(M_2^x\)-names \(\dot{w}^0\) and \(\dot{w}^1\) so that
\[ p \models (\forall i \in \{0, 1\}) \ \dot{w}^i \in [T^{x_G}] \land x_{\dot{G}} \cap \pi(\dot{w}^i) \notin \text{Fin} \otimes \text{Fin} \]
and \(p \models \pi(\dot{w}^0) \neq \pi(\dot{w}^1)\). Then clearly we may also find \((p_0, \bar{v}_0^0, \bar{v}_0^1) \in \Gamma\) such that
\[ \pi(t(\bar{v}_0^0)) \neq \pi(t(\bar{v}_0^1)) \ \text{(}a(\bar{v}_0^0)\text{) plays no role here).} \]

**Claim 4.16.** One of the following holds:

1. There is \(n^* \in \omega\) and \((p_1, \bar{v}_1^0, \bar{v}_1^1) \leq_G (p_0, \bar{v}_0^0, \bar{v}_0^1)\) in \(\Gamma\) such that for any \((p_2, \bar{v}_2^1, \bar{v}_2^1) \leq_G (p_1, \bar{v}_1^0, \bar{v}_1^1)\) from \(\Gamma\), \(\text{dom}(a(\bar{v}_2^0)) \cap \text{dom}(a(\bar{v}_2^1)) \leq n^*; \)
or
We may also assume that $p^k \vDash \alpha \Gamma$, reach a contradiction: Define

$$a(\bar{u}^0_k)(k^*) \cap a(\bar{u}^1_2)(k^*) \subseteq l^*.$$ 

**Proof of Claim.** Suppose that both Items 1 and 2 above fail; we show that there is a $\prec_{\Gamma}$-descending sequence in $\Gamma$, which contradicts the well-foundedness of $(\Gamma, \prec_{\Gamma})$.

It suffices to show that any $(p, \bar{u}^0, \bar{u}^1) \leq_{\Gamma} (p_0, \bar{u}_0^0, \bar{u}_0^1)$ has a $\prec_{\Gamma}$-extension. That Items 1 and 2 above fail means precisely that

1. For each $n \in \omega$ and $(p_1, \bar{u}^0_1, \bar{u}^1_1) \leq_{\Gamma} (p_0, \bar{u}^0_0, \bar{u}^1_0)$ in $\Gamma$ there is $n^* > n$ and $(p_2, \bar{u}^0_2, \bar{u}^1_2) \leq_{\Gamma} (p_1, \bar{u}^0_1, \bar{u}^1_1)$ from $\Gamma$ such that $n \in dom(a(\bar{u}^0_2)) \cap dom(a(\bar{u}^1_2))$; and

2. For each $(p_1, \bar{u}^0_1, \bar{u}^1_1) \leq_{\Gamma} (p_0, \bar{u}^0_0, \bar{u}^1_0)$, $k^* \in dom(a(\bar{u}^0_1)) \cap dom(a(\bar{u}^1_1))$ and $l \in \omega$ there is $l^* > l$ and $(p_2, \bar{u}^0_2, \bar{u}^1_2) \leq_{\Gamma} (p_1, \bar{u}^0_1, \bar{u}^1_1)$ from $\Gamma$ such that

$$l \in a(\bar{u}^0_2)(k^*) \cap a(\bar{u}^1_2)(k^*).$$

This means that in finitely many steps, we can extend any $(p, \bar{u}^0, \bar{u}^1) \leq_{\Gamma} (p_0, \bar{u}_0^0, \bar{u}_0^1)$ to some $(q, \bar{v}^0, \bar{v}^1) \leq_{\Gamma} (p, \bar{u}^0, \bar{u}^1)$ so that

$$a(\bar{u}^0_0) \cap a(\bar{u}^1_0) \sqsubset_{\omega} a(\bar{v}^0_0) \cap a(\bar{v}^1_0)$$

by applying (2) once for each vertical in $a(\bar{u}^0_0) \cap a(\bar{u}^1_0)$ and (1) once for the domain. Thus $(q, \bar{v}^0, \bar{v}^1) \prec_{\Gamma} (p, \bar{u}^0, \bar{u}^1)$.  

Finally, having established that one of Items 1 and 2 above must hold, we use Lemmas 4.13 and 4.14 to finish the proof of Lemma 4.11 by case distinction.

**Case 1:** If Item 2 holds, we may fix $(p_1, \bar{u}^0_1, \bar{u}^1_1) \in \Gamma$, $l^* \in \omega$ and $k^* \in dom(a(\bar{u}^0_1)) \cap dom(a(\bar{u}^1_1))$ such that for any $(p_2, \bar{u}^0_2, \bar{u}^1_2) \leq_{\Gamma} (p_1, \bar{u}^0_1, \bar{u}^1_1)$ from $\Gamma$,

$$a(\bar{u}^0_2)(k^*) \cap a(\bar{u}^1_2)(k^*) \subseteq l^*.$$ 

We may also assume that $p_1 \in D(\bar{u}^0_1) \cap D(\bar{u}^1_1)$ (see Lemma 4.14). We now reach a contradiction: Define $A \subseteq \omega \times \omega$ by letting $A(k) = A(p_1)(k)$ for each $k \neq k^*$, and letting

$$A(k^*) = \{ l \in \omega \mid (\exists p \subseteq p_1)(\exists \bar{u} \subseteq \bar{u}^0_1) l \in a(\bar{u})(k^*) \land p \models \bar{u} \in U^{\bar{u}} \bar{u} \}. $$

Lemma 4.13 ensures that $A \in \mathcal{I}^+$. Let $p^* = (a(p_1), A)$.

Since $p^* \models \bar{u}^1_1 \in U^{\bar{u} \bar{u}}$, we can find $p \preceq p^*$, $l \in \omega \setminus \{ l^* \}$ and $\bar{u}$ such that

$$l \in a(\bar{u})(k^*) \land p \models \bar{u} \in U^{\bar{u} \bar{u}}_{\bar{u}^1_1}.$$ 

It follows that $l \in A(k^*)$ and so by definition of $A(k^*)$ we can find $p' \preceq p_1$ and $\bar{u}'$ such that

$$l \in a(\bar{u}')(k^*) \land p' \models \bar{u}' \in U^{\bar{u} \bar{u}}_{\bar{u}^1_1}.$$
Then, as \( p, p' \leq p_1 \) and \( p_1 \in D(\vec{w}_1^0) \cap D(\vec{w}_1^1) \), we can find \( \vec{u}_0 \) and \( \vec{u}_1 \) such that
\[
l \in a(\vec{u}_0^1)(k^\ast) \land (a(p), A(p_1)/a(p)) \models \vec{u}_0 \in U^{x, \vec{u}_1}_{[\vec{u}_1]}.
\]
and
\[
l \in a(\vec{u}_1^1)(k^\ast) \land (a(p'), A(p_1)/a(p')) \models \vec{u}_1 \in U^{x, \vec{u}_1}_{[\vec{u}_1]}.
\]
Note that \( \{ (k^\ast, l) \mid a(\vec{u}_1^1) \subseteq a(p_1) \subseteq a(p) \cap a(p') \} \) for each \( i \in \{0, 1\} \). Let \( \alpha = a(p) \cap a(p') \), let \( p_2 = (a, A(p_1)/a) \) and let \( a^\ast = a(\vec{u}_1^1) \cup \{ (k^\ast, l) \} \) for each \( i \in \{0, 1\} \). By Lemma 4.13, we conclude
\[
p_2 \models (a^\ast, t(a^\ast)) \in U^{x, \vec{u}_1}_{[\vec{u}_1]}.
\]
which contradicts the choice of \( (p_1, \vec{w}_1^0, \vec{w}_1^1) \) and \( l^\ast \).

**Case 2:** Otherwise, Item 1 holds and we may fix \( n^\ast \in \omega \) and \( (p_1, \vec{w}_1^0, \vec{w}_1^1) \in \Gamma \) such that for any \( (p_2, \vec{w}_2^0, \vec{w}_2^1) \leq \Gamma (p_1, \vec{w}_1^0, \vec{w}_1^1) \) from \( \Gamma \),
\[
\text{dom}(a(\vec{w}_2^0)) \cap \text{dom}(a(\vec{w}_2^1)) \subseteq n^\ast.
\]

We now argue entirely analogously to the previous case, but in the domain instead of in one of the verticals. To this end, set
\[
A' = \{ (k, l) \mid (3p \leq p_1)(3\vec{u} \leq U \vec{u}_1) (k, l) \in a(\vec{u}) \land p \models \vec{u} \in U^{x, \vec{u}_1}_{[\vec{u}_1]} \}.
\]
Note that \( A' \subseteq A(p_1) \) and \( A \in \mathcal{I}^+ \) by Lemma 4.13. Let \( A \subseteq A' \) be the largest subset satisfying \( A \in \mathcal{I}^+ \). Letting \( p^\ast = (a(p_1), A) \) we reach a contradiction almost exactly as in the previous case; details are left to the reader.

5. Iterated Fubini Products

In this section we will look at iterated Fubini products of Fin(\( \phi \))-ideals. In order to study these, we will first recursively define sets \( M^\alpha \):

**Definition 5.1.** Set \( M^1 = \omega \). For a successor ordinal, set \( M^{\alpha + 1} = \omega \times M^\alpha \). For \( \alpha \) limit ordinal, fix once and for all a sequence \( (\alpha_n)_{n \in \omega} \leq \alpha \) which is cofinal in \( \alpha \), and set \( M^\alpha = \bigcup_{n \in \omega} \{ n \} \times M^{\alpha n} \).

We will fix some notation concerning the sets \( M^\alpha \):

**Notation 5.2.** Let \( X \subseteq M^\alpha \). We set
\[
X(n) = \{ x \in \bigcup_{\beta < \alpha} M_\beta \mid (n, x) \in X \}.
\]

For \( \alpha > 1 \), we let as usual \( \text{dom}(X) = \{ n \in \omega \mid X(n) \neq \emptyset \} \). If \( (n_0, \ldots, n_k) \in \omega^{k+1} \) satisfies that \( n_0 \in \text{dom}(X) \) and for every \( 1 \leq i < k \) we have \( n_i \in \text{dom}(X(n_0) \cdots (n_{i-1})) \) and \( n_k \in X(n_0) \cdots (n_k) \subseteq \omega \), we say that \( (n_0, \ldots, n_k) \) is a terminal sequence. Any proper initial segment of a terminal sequence in \( X \) is called a domain sequence in \( X \). Note that we allow a domain sequence to be empty, and set \( X(\emptyset) = X \). We will often refer to domain sequences and terminal sequences as vectors, \( \vec{n} = (n_0, \ldots, n_k) \), and we will write \( X(\vec{n}) \).
for $X(n_0) \cdots (n_k), \vec{n}(i)$ for $n_i, \vec{n} \upharpoonright l$ for $(n_0, \ldots, n_{l-1})$ when $1 \leq l \leq k + 1$ and $\text{lh}(\vec{n})$ for $k + 1$, of course setting $\vec{n} \uparrow 0 = \emptyset$ and $\text{lh}(\emptyset) = 0$. We also set

$$\text{dom}_\alpha(X) = \{\vec{n} \uparrow l \mid l < \text{lh}(\vec{n}) \land \vec{n} \in X\}$$

i.e., the set of domain sequences in $X$. We denote by $\delta_\alpha(\vec{n})$ the ordinal $\delta \leq \alpha$ such that $X(\vec{n}) \subseteq M^\delta$. If the origin of the domain sequence $\vec{n}$ is unambiguous, we will often just write $\delta(\vec{n})$.

We now define a hierarchy of ideals which complexity-wise lies cofinal in the Borel hierarchy:

**Definition 5.3.** We define an ideal $\text{Fin}^\alpha$ on $M^\alpha$ for $\alpha \in \omega_1 \setminus \{0\}$ by recursion as follows:

- $\text{Fin}^1 = \text{Fin}$.
- For a successor ordinal $\alpha + 1 > 1$, set
  $$A \in \text{Fin}^{\alpha+1} \iff \{n \in \omega \mid A(n) \notin \text{Fin}^\alpha\} \in \text{Fin}.$$  
- For a limit ordinal $\alpha$ with cofinal sequence $(\alpha_n)_{n \in \omega}$, set
  $$A \in \text{Fin}^\alpha \iff \{n \in \omega \mid A(n) \notin \text{Fin}^{\alpha_n}\} \in \text{Fin}.$$

Generalizing the previous definition, we also define iterated Fubini products of a sequence of $F_\sigma$ ideals on $\omega$ (given as the finite part of a submeasure):

**Definition 5.4.** We define an ideal $\text{Fin}^\alpha(\vec{\phi})$ on $M^\alpha$, where $\vec{\phi} = (\phi_\beta)_{0 < \beta \leq \alpha}$ is a sequence of lsc submeasures on $\omega$ and $\alpha \in \omega_1 \setminus \{0\}$. The definition is again by recursion on $\alpha$:

- For $\alpha = 1$ and $A \subseteq M^1$ set
  $$A \in \text{Fin}(\vec{\phi}) \iff A \in \text{Fin}(\phi_1).$$
- For a successor ordinal $\alpha > 1$ and $A \subseteq M^\alpha$ set
  $$A \in \text{Fin}(\vec{\phi}) \iff \{n \in \omega \mid A(n) \notin \text{Fin}(\vec{\phi} \uparrow \alpha)\} \in \text{Fin}(\phi_\alpha).$$
- For a limit ordinal $\alpha$ with cofinal sequence $(\alpha_n)_{n \in \omega}$ and $A \subseteq M^\alpha$ set
  $$A \in \text{Fin}(\vec{\phi}) \iff \{n \in \omega \mid A(n) \notin \text{Fin}(\phi \uparrow \alpha_n + 1)\} \in \text{Fin}(\phi_\alpha).$$

Clearly $\text{Fin}^\alpha = \text{Fin}(\vec{\phi})$ where for each $\beta$, $\phi_\beta$ is just the counting measure.

(One could think of defining yet more general ideals of the form $\text{Fin}(\vec{\phi})$ on $M^\alpha$ where $\vec{\phi} = (\phi_\beta)_{\beta \in D(\alpha)}$ is an assignment of submeasures to the set $D(\alpha)$ of domain sequences in $M^\alpha$, letting $D(1) = \{\emptyset\}$. Write $\vec{\phi}(n) = (\phi_{n - l})_{l \in D(\alpha_n)}$, where if $\alpha$ is a limit ordinal, $(\alpha_n)_{n \in \omega}$ is its cofinal sequence and if $\alpha$ is a successor, we let $\alpha_0 = \alpha - 1$. We can define $\text{Fin}(\vec{\phi})$ by recursion on $\alpha$ as follows: For $\alpha = 1$, let $X \in \text{Fin}(\vec{\phi}) \iff \phi_\emptyset(X) < \infty$; for $\alpha > 1$, let
\(X \in \text{Fin}(\bar{\phi}) \iff \phi_\omega(\{n \in \omega \mid X(n) \in \text{Fin}(\bar{\phi}(n))\}) < \infty\). We conjecture all our proofs go through.)

Since we can view any element in \(M^\alpha\) as a finite sequence in \(\omega\), the set \(M^\alpha\) can be identified with a subset of \(\omega^{<\omega}\)—to be precise, with the set of terminal sequences in \(M^\alpha\). Note that a set \(a \subseteq M^\alpha\) is finite if and only if there are finite sets \(K_0, \ldots, K_{n-1}\) with \(K_i \subseteq \omega\) such that \(a \subseteq K_0 \times K_1 \times \cdots \times K_{n-1}\) under this identification. Furthermore, there is a natural ordering on \(M^\alpha\), namely the lexicographical ordering, \(\leq_{\text{lex}}\), inherited from \(\omega^{<\omega}\). We will also consider several other orderings on \(M^\alpha\):

**Definition 5.5.** We recursively define \(\sqsubseteq_\alpha\) on \(M^\alpha\) as follows:

- Set \(X \sqsubseteq_1 Y\) if and only if \(X \subseteq Y\), i.e. if \(X\) is an initial segment of \(Y\).
- Set \(X \sqsubseteq_\alpha+1 Y\) if and only if \(\text{dom}(X) \subseteq \text{dom}(Y)\) and for every \(i \in \text{dom}(Y)\) we have \(X(i) \sqsubseteq_\alpha Y(i)\):
- For \(\alpha\) a limit ordinal with cofinal sequence \((\alpha_n)_{n \in \omega}\), we set \(X \sqsubseteq_\alpha Y\) if and only if \(\text{dom}(X) \subseteq \text{dom}(Y)\) and for every \(i \in \text{dom}(Y)\) we have \(X(i) \sqsubseteq_\alpha, Y(i)\).

In order to determine if a set properly extends another set, we need a strict ordering \(\sqsubset_\alpha\) on \(M^\alpha\) to be a version of \(\sqsubseteq_\alpha\) which is strict at every level. For the case \(J = \text{Fin}^\alpha\) we make the following definition:

- Set \(X \subset_1 Y\) if and only if \(X \not\subseteq Y\), i.e. if \(X\) is a proper initial segment of \(Y\).
- Set \(X \subset_\alpha+1 Y\) if and only if \(\text{dom}(X) \not\subseteq \text{dom}(Y)\) and for every \(i \in \text{dom}(Y)\) we have \(X(i) \subset_\alpha Y(i)\):
- For \(\alpha\) a limit ordinal with cofinal sequence \((\alpha_n)_{n \in \omega}\), we set \(X \subset_\alpha Y\) if and only if \(\text{dom}(X) \not\subseteq \text{dom}(Y)\) and for every \(i \in \text{dom}(Y)\) we have \(X(i) \subset_\alpha, Y(i)\).

In the general case of an ideal \(J = \text{Fin}(\bar{\phi})\) on \(M^\alpha\), we define \(\sqsubset_\alpha\) on \(M^\alpha\) by recursion on \(\alpha\) as follows:

- Set \(X \subset_1 Y\) if and only if \(X \not\subseteq Y\) and \(\phi_1(X) < \phi_1(Y)\).
- Set \(X \subset_\alpha+1 Y\) if and only if \(\text{dom}(X) \not\subseteq \text{dom}(Y)\),

\[\phi_{\alpha+1}(\text{dom}(X)) < \phi_{\alpha+1}(\text{dom}(Y)),\]

and for every \(i \in \text{dom}(X)\) we have \(X(i) \subset_\alpha Y(i)\):
- For \(\alpha\) a limit ordinal with cofinal sequence \((\alpha_n)_{n \in \omega}\), we set \(X \subset_\alpha Y\) if and only if \(\text{dom}(X) \not\subseteq \text{dom}(Y)\), \(\phi_\alpha(\text{dom}(X)) < \phi_\alpha(\text{dom}(Y))\), and for every \(i \in \text{dom}(Y)\) we have \(X(i) \subset_\alpha Y(i)\).

As was the case for the previous section, the material of the present section generalizes almost mechanically from \(\text{Fin}\) to \(\text{Fin}^\alpha\). Often this is made possible by of the above definition of \(\subset_\alpha\).

When defining the \(\alpha\)-dimensional Mathias forcing notion, we will need an ordering \(<_\alpha\) on \(M^\alpha\) defined as follows:
• Set \( X <_1 Y \) if and only if \( \max(X) < \min(Y) \).
• Set \( X <_{\alpha+1} Y \) if and only if \( \text{dom}(X) \subseteq \text{dom}(Y) \), and for every \( i \in \text{dom}(X) \) we have \( X(i) <_{\alpha} Y(i) \).
• For \( \alpha \) a limit ordinal with cofinal sequence \( (\alpha_n)_{n<\omega} \), we set \( X <_{\alpha} Y \) if and only if \( \text{dom}(X) \subseteq \text{dom}(Y) \) and for every \( i \in \text{dom}(Y) \) we have \( X(i) <_{\alpha_i} Y(i) \).

We let as usual \((\text{Fin}^\alpha)^{++}\) denote the co-ideal.

The \( \alpha \)-dimensional forcing notion is now defined as follows:

**Definition 5.6.** Let \((\text{Fin}^\alpha)^{++}\) denote the set of \( A \subseteq M^\alpha \) such that for every \( \vec{n} \in \text{dom}_\alpha(A) \) we have \( A(\vec{n}) \notin \text{Fin}_{\delta_\alpha(\vec{n})} \). Conditions of \( M_\alpha \) are pairs \((a, A)\) where

(a) \( a \subseteq M^\alpha \) is finite;
(b) \( A \in (\text{Fin}^\alpha)^{++} \);
(c) \( a <_{\alpha} A \).

We let \((a', A') \leq (a, A)\) if and only if \( A' \subseteq A \) and \( a \subseteq_{\alpha} a' \subseteq a \cup A \).

For the general case, define \( \text{Fin}(\vec{\delta})^{++} \) to be the set of \( A \subseteq M^\alpha \) such that such that for every \( \vec{n} \in \text{dom}_\alpha(A) \) we have \( A(\vec{n}) \notin \text{Fin}(\vec{\delta} \upharpoonright \delta(\vec{n}) + 1) \) and replace \([b]\) by \( A \in \text{Fin}(\vec{\delta})^{++} \) in the definition of \( M_\alpha \).

Note that for any \( \vec{n} \in \text{dom}_\alpha(a) \), the pair \((a(\vec{n}), A(\vec{n}))\) is a forcing condition in \( M_{\delta_\alpha(\vec{n})} \). The pair \((\text{dom}(a), \text{dom}(A))\) is a classical (1-dimensional) Mathias forcing condition. As before, we need a relativized forcing notion:

**Definition 5.7.** If \( \mathcal{I}^{+} \) is the co-ideal of an ideal \( \mathcal{I} \supseteq \text{Fin}^\alpha \), then we write \( \mathcal{I}^{++} \) for \( \mathcal{I}^{+} \cap (\text{Fin}^\alpha)^{++} \) and we let

\[
M_\alpha^{\mathcal{I}} = \{(a, A) \in M_\alpha \mid A \in \mathcal{I}^{++}\}.
\]

Note that if \( \mathcal{I} = \text{Fin}^\alpha \) then \( M_\alpha^{\mathcal{I}} = M_\alpha \). Note furthermore that if \( A \in \mathcal{I}^{+} \), then we can always find \( B \subseteq A \) such that \( B \in \mathcal{I}^{++} \).

**Notation 5.8.**

1. For any \( X \in M^\alpha \), we define the generalized infinity domain by \( \text{dom}_{\vec{\alpha}}(X) = \{ \vec{n} \in \text{dom}_\alpha(X) \mid X(\vec{n}) \notin \text{Fin}_{\delta_\alpha(\vec{n})} \} \), and note that \( A \in (\text{Fin}^\alpha)^{++} \) if and only if \( \text{dom}_\alpha(A) = \text{dom}_{\vec{\alpha}}(A) \).
2. Given a filter \( G \) on \( M_\alpha^{\mathcal{I}} \), let
\[
x_G = \bigcup\{a \mid (\exists A)(a, A) \in G\}
\]

We will see that for \( M_\alpha^{\mathcal{I}}\)-generic \( G \), \( x_G \in (\text{Fin}^\alpha)^{++} \) holds in \( V[G] \).
3. For a condition \( p \in M_\alpha^{\mathcal{I}} \), we write \((a(p), A(p))\) when we want to refer to its components.
4. For \((a, A) \in M_\alpha^{\mathcal{I}} \) and \( b \subseteq A \) finite, let
\[
A/b = \bigcup_{\vec{n} \in \text{dom}_\alpha(A)} A(\vec{n}) \setminus \{ x \in M^{\delta_\alpha(\vec{n})} \mid (\exists \vec{m} \in b(\vec{n})) x \leq_{\text{lex}} \vec{m} \}.
\]
5. For \( p \in M_\alpha^{\mathcal{I}} \), we let \( M_\alpha^{\mathcal{I}}(\leq p) = \{ q \in M_\alpha^{\mathcal{I}} \mid q \leq p \} \).
Remark 5.9. The definition of $A/b$ was made to guarantee $b \prec_\alpha A/b$. Note that $\vec{n} \in A/b$ if and only if $\vec{n} \notin b$ and letting $\vec{n} \upharpoonright l$ be the longest common initial segment of $\vec{n}$ with some element of $b$, then there is no $\vec{m} \in b$ with $\vec{n} \upharpoonright l \subseteq \vec{m}$ and $\vec{n}(l) < \vec{m}(l)$.

Following the same strategy as in previous sections, our main pursuit will be a generalization of the Main Proposition 4.6.

Remark 5.10. Recall that in order to meaningfully talk about $\kappa$-Suslin sets in $\mathcal{P}(M^\alpha)$, we identify $M^\alpha$ with $\omega$ (via some fixed arbitrary bijection), sets with their characteristic functions, and in effect, $\mathcal{P}(M^\alpha)$ with $2^\omega$ (as described in Section 2.2).

Assumption 5.11. For the remainder of this article, let $\mathcal{J} = \text{Fin}^\alpha$ where $\alpha \geq 2$ (or more generally, $\mathcal{J} = \text{Fin}(\phi)$). Suppose $A \subseteq \mathcal{P}(M^\alpha)$ to be a $\mathcal{J}$-almost disjoint family which is $\kappa$-Suslin. Moreover, fix a tree $T$ on $2 \times \kappa$ such that $\pi[T] = A$. Finally, let $\mathcal{I}$ be the ideal generated by $A \cup \text{Fin}^\alpha$. We leave it to the reader to make trivial substitutions to adapt the proofs to the case of $\text{Fin}(\phi)$-AD families, but do give details when the proofs differ substantially.

Although the proofs in this section work for $\text{Fin}(\phi)$ as above we will of notational concern only consider the case where $\phi_\beta$ is the counting measure for $0 < \beta \leq \alpha$, i.e., where $\mathcal{J} = \text{Fin}^\alpha$. Whenever relevant, we either make an explicit comment or the reader can substitute $\text{Fin}^\alpha$ by $\text{Fin}(\phi)$ (but again, do not substitute for the word finite).

Main Proposition 5.12. $\models_{M^\alpha} (\forall y \in \pi[T]) \ y \cap x_G \in \text{Fin}^\alpha$.

The Main Proposition will be proved in Section 5.2 below. All our results about $\text{Fin}^\alpha$ from Theorem 1.4 follow from the Main Proposition 5.12 as a corollary:

Corollary 5.13. Assuming the Main Proposition 5.12, Theorem 1.4 holds.

Proof. It suffices to replace $M^T$ by $M^T_\alpha$ in the proofs of Corollaries 3.6, 3.7, and 3.8 (just as we did in Corollary 4.7 in the two-dimensional case). □

5.1. Properties of the general higher-dimensional forcing. Before we prove the Main Proposition 5.12, we collect the necessary facts about $M^T_\alpha$.

Lemma 5.14.

1. For any $A \in \mathcal{I}$, $\models_{M^\alpha_G} x_G \cap \hat{A} \in \text{Fin}^\alpha$.
2. Let $k \in \omega$. The partial order $M^\alpha_{k+1}$ is isomorphic to the product $M^\alpha_k \times_k (\mathcal{A}, A)$, where $A = \{\vec{n} \in M^\alpha \mid \vec{n}(0) \geq k\}$, and by $(M^\alpha_k)^k$ we mean $k$-fold (side-by-side) product of $\alpha$-dimensional Mathias forcing $M^\alpha$. If $\alpha$ is a limit ordinal, $M^T_\alpha$ is isomorphic to $M^\alpha_\alpha \times_k (\mathcal{A}, A)$.
Proof. (1) Follows by an obvious density argument.

(2) First we consider the successor case. Define a map
\[ \phi: M_{\alpha+1}^L(\leq (\varnothing, A)) \times (M_\alpha)^k \rightarrow M_{\alpha+1}^L \]
by
\[ ((b, B), (c_i, C_i)_{i<k}) \mapsto \left( \bigcup_{i<k} \{i\} \times c_i \cup b, \bigcup_{i<k} \{i\} \times C_i \cup B \right). \]

For a limit ordinal \( \alpha \) the map can be defined in exactly the same way. Both of these maps are easily seen to be bijective and order preserving.

(3) This is shown easily by induction, slightly adapting the general case of the proof of 4.8(3). We leave this to the reader. \( \square \)

We shall need a more sophisticated way of decomposing the forcing as a product. Towards this, let us regard \( M_\alpha \) and \( a \) as trees, ordered by the initial segment relation \( \subseteq \). Given \( \bar{n} \in A \), let us see how we can characterize the “type” of \( \bar{n} \) in relation to \( a \) with respect to \( \leq_{lex} \).

First note that since \( a \subseteq A \) it is enough to characterize the type of \( \bar{n} \upharpoonright (\text{lh}(\bar{n}) - 1) \) relative to the following set of domain sequences
\[ a^* = \{ \bar{n}' \upharpoonright (\text{lh}(\bar{n}') - 1) \mid \bar{n}' \in a \} \]
(for if \( \bar{n} \) extends \( \bar{n}' \in a^* \), \( \bar{n}' \leq_{lex} \bar{n} \) for every \( \bar{n}' \in a \) which extends \( \bar{n}' \)).

Let \( \bar{n}_0, \ldots, \bar{n}_k \) enumerate \( a^* \) in lexicographically increasing order, and let \( \bar{n}_i \) be lexicographically maximal in \( a^* \) such that \( \bar{n}_i \leq_{lex} \bar{n} \). We then know by \( a \subseteq A \) that \( \bar{n} \) must have a longer initial segment in common with \( \bar{n}_i \) than it does with \( \bar{n}_{i+1} \), provided \( i < k \).

Let therefore \( \bar{m}_i \) be the shortest initial segment of \( \bar{n}_i \) such that \( \bar{m}_i \leq_{lex} \bar{n}_{i+1} \) for \( i < k \), and let \( \bar{m}_k = \varnothing \). We have just seen that \( \bar{m}_i \subseteq \bar{n} \). Moreover if \( j < i, \bar{m}_j \nsubseteq \bar{n} \) (for \( \bar{m}_j \leq_{lex} \bar{n}_{j+1} \leq_{lex} \bar{n}_i \) and so \( \bar{m}_j \leq_{lex} \bar{n} \)).

We have thus shown the following lemma:

Lemma 5.15. Suppose \( (a, A) \in M^L_\alpha \). Let \( \bar{n}_0, \ldots, \bar{n}_k \) enumerate
\[ a^* = \{ \bar{n}' \upharpoonright (\text{lh}(\bar{n}') - 1) \mid \bar{n}' \in a \} \]
in lexicographically ascending order, let \( \bar{m}_k = \varnothing \) and for \( i < k \) let \( \bar{m}_i \) be the shortest initial segment of \( n_i \) such that \( \bar{m}_i \leq_{lex} \bar{n}_{i+1} \) (just as above).

Then for each \( \bar{n} \in A \) there is precisely one \( i \) such that \( \bar{m}_i \subseteq \bar{n} \) and \( \bar{n}_i \leq_{lex} \bar{n} \) (namely the maximal \( i \) such that \( \bar{n}_i \leq_{lex} \bar{n} \)).

Technical as the previous lemma may be, it allows us to decompose the forcing as a product in a very useful manner.
Lemma 5.16. Suppose \((a, A) \in M^\alpha_\omega\), and \(\vec{m}_0, \ldots, \vec{m}_k\) and \(\vec{n}_0, \ldots, \vec{n}_k\) are defined as in the previous lemma. Then \(M^\alpha_\omega(\leq (a, A))\) is isomorphic to
\[
\left( \prod_{i < k} M_{\delta(\vec{m}_i)}(\leq (a(\vec{n}_i), A_i(\vec{m}_i))) \right) \times M^\alpha_\omega(\leq (a(\vec{n}_k), A_k))
\]
where
\[
A_i = A \cap \{ \vec{n} \mid \vec{m}_i \subseteq \vec{n} \land \vec{n}_i \leq_{\text{lex}} \vec{n} \}
\]
for each \(i \leq k\).

Proof. The crucial observation is that by Lemma 5.15, \(A\) may be written as a disjoint union
\[
A = \bigcup_{i < k} A_i
\]
Define a map \(\phi\) from \(M^\alpha_\omega(\leq (a, A))\) to the forcing in (5.1) as follows: For \((b, B) \leq (a, A)\) define
\[
\phi(b, B) = ((b \cap A_i)(\vec{m}_i), (B \cap A_i)(\vec{m}_i))_{i < k}.
\]
Using the partition from (5.2), it is straightforward to verify that this map is an isomorphism of partial orders. □

Of course we also have a diagonalization lemma (compare Lemmas 3.10 resp. 4.9) for \(M^\alpha_\omega\).

Lemma 5.17. Let \((A_k)_{k \in \omega}\) be a sequence from \(\mathcal{I}^\omega\) satisfying \(A_{k+1} \subseteq A_k\) for every \(k \in \omega\). Then there is \(A_\infty \in \mathcal{I}^\omega\) such that \(A_\infty \equiv_{\text{Fin}^\alpha} A_k\) for every \(k \in \omega\).

Proof. The proof of Lemma 4.9 can be transcribed completely mechanically by replacing \(\text{Fin} \otimes \text{Fin}\) by \(\text{Fin}^\alpha\) everywhere; we leave this to the reader. □

5.2. The Branch Lemma for general higher dimensions. The reader will find that our line of argumentation in this section is remarkably close to that of the previous section; of course this is only true since the proofs there were written with the general case in mind.

Yet again, the crucial definition is that of an invariant tree, analogous to Definitions 3.12 and 4.10.

Definition 5.18. For \(x \subseteq M^\alpha\), let
\[
T^x = \{ t \in T \mid (\exists w \in \pi[T_1]) w \cap x \notin \text{Fin}^\alpha \}
\]
As in Sections 3.2 and 4.2, it is easy to see that whenever \(x \Delta x' \in \text{Fin}^\alpha\), \(T^x = T^{x'}\). Moreover Facts 3.13(2)–(5) hold here as well.

We are now ready to state the main lemma of this section.

The Branch Lemma 5.19. \(\models_{M^\alpha_\omega} |\pi[T^{x \Diamond}]| \leq 1.\)
In keeping with the pattern established in previous sections, we postpone the proof of the Branch Lemma and first give the proof of the Main Proposition 5.19 assuming the lemma. The proof is verbatim the proof of Main Proposition 1.6 except that we use Lemma 5.14(2) to decompose the forcing; we repeat it for the incredulous reader.

Proof of Main Proposition 5.12. Suppose towards a contradiction that some $p_0 \in M_\alpha^2$ forces that there is $A \in \pi[T]^V[G]$ with $A \cap x_G \notin \text{Fin}_\alpha$. The Branch Lemma 5.19 lets us choose a name $\hat{A}$ so that $p_0 \Vdash \pi[T^{x_G}] = \{\hat{A}\}$.

As in the proof of Main Proposition 1.6 on p. 16, we show the following claim:

Claim 5.20. There is $q \in M_\alpha^2$ and $A^r \in \pi[T]$ such that $q \Vdash \hat{A} = \hat{A}^r$.

Proof of Claim. By the generalized diagonalization lemma (Lemma 5.17), it suffices to show that if $p \leq p_0$ and $p$ decides $\bar{n} \in \hat{A}$ then in fact $(a(p_0), A(p))$ decides $\bar{n} \in A$. So let us assume $p \Vdash \bar{n} \in A$ (if $p \Vdash \bar{n} \notin A$ the proof is similar). We must show that for an arbitrary $M_\alpha^2$-generic $G$ with $(a(p_0), A(p)) \in G$, it holds that $\bar{n} \in \hat{A}^G$.

Fix $k$ large enough so that $\text{dom}(a(p)) \subseteq k$. By Lemma 5.14(2) we can decompose $G$ as $G_0 \times G_1$ where $G_0$ is generic for $\prod_{i<k} M_{\alpha_i}^{\text{Fin}_{\alpha_i}}$ and $G_1$ is $M_\alpha^2$-generic. As $x_G \Delta x_{G_1} \in \text{Fin}_{\alpha_i}$, $T^{x_G} = T^{x_{G_1}} \in V[G_1]$.

By absoluteness, $\hat{A}^G \in V[G_1]$ and $\pi[T^{x_G}] = \{\hat{A}^G\}$ holds in both $V[G]$ and $V[G_1]$.

Since $a(p) \subseteq \prod_{i<k} \{i\} \times M_{\alpha_i}$, we can find $G_0$ which is $(\prod_{i<k} M_{\alpha_i}, V[G_1])$-generic over $V[G_1]$ so that letting $G' = G_0 \times G_1$, $p \in G'$. Again by $\text{Fin}_{\alpha_i}$-invariance of $T^x$ and by absoluteness, $\hat{A}^{G'} \in V[G_1]$ and $\pi[T^{x_{G_1}}] = \{\hat{A}^{G'}\}$ and so $\hat{A}^{G'} = \hat{A}^G$ and $\bar{n} \in \hat{A}^G$.

Claim 5.20. □

Just as in the proof of Main Proposition 1.6 we conclude that $A^r \in T$ by absoluteness while $q \Vdash x_G \cap \hat{A}^r \notin \text{Fin}_\alpha$, contradicting Lemma 5.14(1).

Main Proposition 5.12. □

Gradually working towards a proof of the Branch Lemma 5.19 we start by introducing some notation. Set

$$U = \{(a, t) \in P(M^\alpha) \times T \mid a \text{ is finite}\}.$$ 

For $\bar{u} \in U$, we will often write $\bar{u} = (a(\bar{u}), t(\bar{u}))$. Define an ordering $\leq_U$ on $U$ by

$$\bar{u}_1 \leq_U \bar{u}_0 \iff a(\bar{u}_1) \sqsupseteq a(\bar{u}_0) \quad \text{and} \quad t(\bar{u}_1) \sqsupseteq t(\bar{u}_0).$$

Assume for a moment that $G$ is $M_\alpha^2$-generic over $V$ and work in $V[G]$. For a fixed $x \in P(M^\alpha)$, define a set $U^x \subseteq U$ consisting of those pairs $(a, t) \in U$ such that there is $w \in [T_0]$ with

1. $\pi(w) \cap x_G \notin \text{Fin}_{\alpha}$,
2. $\text{dom}_\alpha(a) \subseteq \text{dom}_\alpha^\infty(x \cap \pi(w))$ and
(3) \( (\forall \vec{n} \in \text{dom}_a(a)) \ a(\vec{n}) \subseteq x(\vec{n}) \cap \pi(w)(\vec{n}). \)

Note that \( U^x \) is closed under initial segments with respect to \( \leq_U \), and that an infinite chain through \( U^x \) will give a set \( A \in \pi[T] \) with a large intersection with \( x \), and a \((\text{Fin}_a)^{++}\)-subset of this intersection to witness its largeness in a useful manner.

In analogy to trees, when \( \vec{u}_0 \in U \) we again write \( U^x_{[\vec{u}_0]} \) for \( \{ \vec{u} \in U \mid \vec{u}_0 \leq \vec{u} \} \).

Finally working in \( V \) again, we note the following about \( U^{x^c} \):

**Lemma 5.21.** Suppose \( (a, A) \models \vec{u} \in U^{x^c} \).

1. It holds that \( a(\vec{u}) \subseteq a \) and moreover,
   \[
   (a(\vec{u}), A) \models \vec{u} \in U^{x^c}.
   \]
   Similarly, if \( (a, A) \models \vec{u}' \notin U^{x^c} \) and \( a(\vec{u}') \subseteq a \) then
   \[
   (a(\vec{u}), A) \models \vec{u}' \notin U^{x^c}.
   \]

2. If \( A' \subseteq A \) such that \( (a, A') \in M^x_{a} \), then also \( (a, A') \models \vec{u} \in U^{x^c} \).

3. The set \( A' \subseteq M^a \) defined by
   \[
   A' = \{ \vec{n} \mid (\exists p' \leq (a, A))(\exists \vec{u}' \leq_U \vec{u}) \vec{n} \in a(\vec{u}') \land p' \models \vec{u}' \in U^{x^c} \}
   \]
   is not in \( I \).

4. For a non-empty domain sequence \( \vec{n} \in \text{dom}_a(a(\vec{u})) \), the set \( A_{\vec{n}} \subseteq M^a_{\vec{n}} \) defined by
   \[
   A_{\vec{n}} = \{ \vec{m} \mid (\exists p' \leq (a, A))(\exists \vec{u}' \leq_U \vec{u}) \vec{m} \in a(\vec{u}')(\vec{n}) \land p' \models \vec{u}' \in U^{x^c} \}
   \]
   is in \( (\text{Fin}^a_{\vec{n}})^+ \).

**Proof.** (1) Immediate from the definition of \( U^{x^c} \).

(2) Suppose that \( (a, A') \not\models \vec{u} \in U^{x^c} \). Then there is some \( (b, B) \leq (a, A') \) such that \( (b, B) \models \vec{u} \notin U^{x^c} \). Since \( A' \setminus A \in I \), there is some \( B' \subseteq B \cap A \) such that \( B' \in I^{++} \). However, \( (b, B') \leq (b, B) \) and \( (b, B') \leq (a, A) \), which is a contradiction.

(3) Although the proof is practically identical to that of Lemma 4.13(2), we give the details for the reader’s convenience. Assume to the contrary that \( A' \in I \). Then \( A' \setminus A \in I^{++} \), so take \( B \subseteq A \setminus A' \) such that \( B \in I^{++} \) and set \( p = (a, B) \in M^x_{a} \). Since \( p \models \vec{u} \in U^{x^c} \) we can find a name \( \dot{w} \) such that
   \[
   p \models \dot{w} \in \pi[T_{e(\vec{u})}] \land \dot{w} \cap x_G \notin \text{Fin}^a .
   \]
   (As in Lemma 4.13(2) it would suffice if \( p \models T^{x^c} \neq \emptyset \).) Thus we can extend \( p \) to \( p' \) to force some terminal sequence \( \vec{n} \) into \( \dot{w} \cap x_G \setminus a(p) \). But it has to be the case that \( \vec{n} \in a(p') \). Whence \( \vec{n} \in A' \) by definition of \( A' \), contradicting that also \( \vec{n} \in B \) which is disjoint from \( A' \).

(4) The proof is identical that of Lemma 4.13(3) in essence, but differs substantially in notation. Assume to the contrary that \( A_{\vec{n}} \in \text{Fin}^a_{\vec{n}} \). Then
we can find $p \leq (a, A)$ such that $A(p)(\vec{n})$ is disjoint from $A_R$. Since $p \models \vec{n} \in U^{x_G}$ we can find a name $\vec{w}$ such that
\[
p \models \vec{w} \in \pi[T_{\iota(a)}] \land \vec{w} \cap x_G \in (\text{Fin}^\alpha)^+.
\]
and
\[
p \models \text{dom}_\alpha(a(\vec{n})) \subseteq \text{dom}_\alpha(\vec{w} \cap x_G).
\]
Therefore $\vec{n} \in \text{dom}_\alpha(\vec{w} \cap x_G)$ and we can extend $p$ to $p'$ to force a terminal sequence $\vec{n} \vdash \vec{n}'$ into $\vec{w} \cap x_G \vdash a(p)$. But as in the proof of the previous item, it has to be the case that $\vec{n} \vdash \vec{n}' \in a(\vec{n}')$, whence $\vec{n}' \in A_R$ by definition of $A_R$, contradicting that also $\vec{n}' \in A(p')(\vec{n})$ which is disjoint from $A_R$.  

Define a set $\Gamma$ as follows:
\[
\Gamma = \{(p, \vec{u}^0, \vec{u}^1) \in M^\alpha \mid (\forall i \in \{0, 1\})p \models \vec{u}^i \in U^{x_G}\}.
\]
Define two orderings on $\Gamma$:
\[
(p_1, \vec{u}^0_1, \vec{u}^1_1) \leq_\Gamma (p_0, \vec{u}^0_0, \vec{u}^1_0) \iff p_1 \leq p_0 \land \vec{u}^0_1 \leq_U \vec{u}^0_0
\]
for $i \in \{0, 1\}$, and
\[
(p_1, \vec{u}^0_1, \vec{u}^1_1) <_\Gamma (p_0, \vec{u}^0_0, \vec{u}^1_0) \iff p_1 \leq p_0 \land [a(\vec{u}^0_0) \cap a(\vec{u}^1_0) \subset a(\vec{u}^0_1) \cap a(\vec{u}^1_1)].
\]
Note that $\Gamma$ is well-founded with respect to the second ordering, $<_\Gamma$. Indeed, suppose towards a contradiction that there is an infinite sequence $(p_0, \vec{u}^0_0, \vec{u}^1_0) <_\Gamma (p_1, \vec{u}^0_1, \vec{u}^1_1) <_\Gamma \cdots$. Set
\[
y^i = \bigcup_{n \in \omega} t(\vec{u}^i_n)
\]
for $i \in \{0, 1\}$, and
\[
A = \bigcup_{n \in \omega} a(\vec{u}^0_n) \cap a(\vec{u}^1_n).
\]
The sequence is $<_\Gamma$-decreasing and from $\Gamma$, hence $A \in (\text{Fin}^\alpha)^+$ and $A \subseteq \pi(\vec{u}^0) \cap \pi(\vec{y}^1)$, contradicting $\text{Fin}^\alpha$-almost disjointness of $\pi[T]$.

The following is the analogue of Lemma 4.14, saying that $U^{x_G}$ can be approximated reasonably well in the ground model. Also as for Lemma 4.14, a very similar proof shows that $M^\alpha_\lambda$ is proper.

**Lemma 5.22.** For each $\vec{u}_0 \in U$ the set $D(\vec{u}_0)$ is dense and open in $M^\alpha_\lambda$, where we define $D(\vec{u}_0)$ to be the set of $p \in M^\alpha_\lambda$ such that for all $p' \leq p$ and any $\vec{u} \leq_U \vec{u}_0 \in U$,
\[
\left[p' \models \vec{u} \in U^{x_G}_{\{\vec{u}_0\}}\right] \Rightarrow \left(a(p'), A(p)/a(p') \models (\exists t \in T)(a(\vec{u}), t) \in U^{x_G}_{\{\vec{u}_0\}}\right).
\]

**Proof.** The proof from the two-dimensional case, i.e., of Lemma 4.14 applies exactly as written once we make the following adaptations: Firstly replace $\text{Fin} \otimes \text{Fin}$ by $\text{Fin}_\alpha$. Secondly, replace $\subset_2$ by $\subset_\alpha$. Thirdly, adapt the definition of $B^0_n$ as follows:
\[
B^0_n = \{\vec{n} \in B^{n-1}_{n-1} \mid (\exists \vec{n}' \in \text{dom}_\alpha(\vec{u}^0_n)) \vec{n}' \leq \vec{n}\} \cup B^{n-1}_{n-1} \setminus \bigcup \{C_i \mid i < n\}.
\]
Then \((b_n^0, B_n^0) \in M_{\alpha, I}^Z\), and \(B_n^0 \cap C_i \in \text{Fin}^\alpha\) for each \(i < n\). With these changes, the remainder of the argument for Lemma 4.14 applies verbatim. □

The following technical lemma is crucial to the proof of the branch Lemma. It plays the same role as Lemma 3.9(4) and Lemma 4.15, giving us some freedom in tampering with the finite parts of conditions while maintaining that something is forced about \(U^{x(i)}\).

**Lemma 5.23.** Suppose we are given \(\vec{u} \in U\), \((a, A) \in M_{\alpha, I}^Z\), and \(a' \subseteq a\) so that \((a', A/a') \in M_{\alpha, I}^Z\), and so that the lexicographically maximal element of \(a'\) and is also the lexicographically maximal element of \(a\). Suppose further \((a, A) \vdash \vec{u} \in U^{x(i)}\). Let \(a'' \subseteq a' \cap a(\vec{u})\) be arbitrary. Then

\[
(a', A/a') \vdash (a'', t(\vec{u})) \in U^{x(i)}.
\]

**Proof.** We will decompose the forcing as a product. Let \(\vec{n}_0, \ldots, \vec{n}_k\) and \(\vec{m}_0, \ldots, \vec{m}_k\) be defined as in Lemma 5.15. Then by Lemma 5.16,

\[
(5.3) \quad M_{\alpha, I}^Z(\leq (a, A)) \cong \prod_{i<k} M_{\delta(\vec{m}_i)}^Z(\leq (a(\vec{n}_i), A_i(\vec{m}_i))) \times M_{\alpha, I}^Z(\leq (a(\vec{n}_k), A_k))
\]

with \(A_i\) defined as in the lemma.

Let \(D\) consist of those \(i < k\) such that some element of \(a'\) extends \(\vec{n}_i\). Then writing \(A' = A/a'\), Lemma 5.16 also gives us an isomorphism

\[
(5.4) \quad M_{\alpha, I}^Z(\leq (a', A')) \cong \prod_{i \in D} M_{\delta(\vec{m}_i)}^Z(\leq (a'(\vec{n}_i), A'_i(\vec{m}_i))) \times M_{\alpha, I}^Z(\leq (a'(\vec{n}_k), A'_k))
\]

with \(A'_i\) defined analogously as in the lemma. We have \(A'_i = A_i\) for each \(i \in D \cup \{k\}\) and so it is easy to see—e.g., using Lemma 5.14(2), a finite induction, and Lemma 3.9(3)—that

\[
M_{\delta(\vec{m}_i)}^Z(\leq (a(\vec{n}_i), A_i(\vec{m}_i))) \cong M_{\delta(\vec{m}_i)}^Z(\leq (a'(\vec{n}_i), A'_i(\vec{m}_i)))
\]

for \(i \in D\) and

\[
M_{\alpha}^Z(\leq (a(\vec{n}_k), A_k)) \cong M_{\alpha}^Z(\leq (a'(\vec{n}_k), A'_k)).
\]
Write

\[ P_+ = \prod_{i \in D} M_i(\bar{m}_i) \left( \preceq (a(\bar{m}_i), A_i(\bar{m}_i)) \right), \]
\[ P_- = \prod_{i \in D} M_i(\bar{m}_i) \left( \preceq (a(\bar{m}_i), A_i(\bar{m}_i)) \right), \]
\[ P'_- = \prod_{i \in D} M_i(\bar{m}_i) \left( \preceq (a'(\bar{m}_i), A'_i(\bar{m}_i)) \right), \]
\[ P_\infty = M_\alpha^\hat{t} \left( \preceq (a(\bar{m}), A_k) \right), \]
\[ P'_\infty = M_\alpha^\hat{t} \left( \preceq (a'(\bar{m}), A'_k) \right). \]

noting that we have established

(5.5) \[ M_\alpha^\hat{t} \left( \preceq (a', A') \right) \cong P'_- \times P'_\infty \cong P_- \times P_\infty \]

and

(5.6) \[ M_\alpha^\hat{t} \left( \preceq (a, A) \right) \cong P_+ \times P_- \times P_\infty. \]

Now finally, let \( G' \) be \( M_\alpha^\hat{t} \)-generic over \( V \) with \( (a', A') \in G' \). We must show \( (a'', t(\bar{u})) \in U^{\hat{xG}} \). Using (5.5) transform \( G' \) into a \( P_- \times P_\infty \) generic \( H_- \times H_\infty \). Find a \( P_+ \)-generic \( H_+ \) over \( V[H_-][H_\infty] \) and let \( G \) be the \( M_\alpha^\hat{t} \)-generic given by \( H_+ \times H_- \times H_\infty \) using (5.6). By construction \( (a, A) \in G \), whence \( (a(\bar{u}), t(\bar{u})) \in U^{\hat{xG}} \).

By definition of \( U^{\hat{xG}} \) this means that in \( V[G] \) we can find \( w \in \pi[T[t(\bar{u})]] \) so that

(5.7) \[ (\exists u \in (\text{Fin}^0)^{++}) \ a'' \subseteq u \subseteq \pi(w) \cap x_G. \]

Since \( a'' \subseteq x_{G'} \) and since \( x_{G} \Delta x_{G'} \in \text{Fin}^0 \) we may replace \( x_G \) by \( x_{G'} \) in (5.7), and thus

(5.8) \[ (\exists x \in \pi[T[t(\bar{w})]])(\exists u \in (\text{Fin}^0)^{++}) \ a'' \subseteq u \subseteq \pi(x) \cap x_{G'}. \]

Just as in the two-dimensional case (i.e., the proof of Lemma 4.13) an absoluteness argument easily shows that (5.8) and hence \( (a'', t(\bar{u})) \in U^{\hat{xG'}} \) must hold in \( V[G'] \), proving \( (a', A/a') \parallel (a'', t(\bar{u})) \in U^{\hat{xG'}} \). \( \square \)

After all these preparations, we are finally ready to prove our last and most general instance of the Branch Lemma.

Proof of the Branch Lemma 5.19. Suppose towards a contradiction we have \( p \in M_\alpha^\hat{t} \) and a pair of \( M_\alpha^\hat{t} \)-names \( \bar{w}^0, \bar{w}^1 \) such that

\[ p \models (\forall i \in \{0, 1\}) \bar{w}^i \in \pi[T] \land x_G \cap \bar{w}^i \notin \text{Fin}^0 \]

and \( p \models \bar{w}^0 \neq \bar{w}^1 \). By definition of \( \Gamma \) we may find \( (p_0, \bar{u}_0^0, \bar{u}_0^1) \in \Gamma \) such that

\[ \pi(t(\bar{u}_0^0)) \neq \pi(t(\bar{u}_0^1)). \]
Claim 5.24. There is \((p_1, \vec{u}_1^0, \vec{u}_1^1) \leq_{\Gamma} (p_0, \vec{u}_0^0, \vec{u}_0^1)\), a terminal sequence \(\vec{n}^* \in a(\vec{u}_1^0) \cap a(\vec{u}_1^1)\), and numbers \(l < \lh(\vec{n}^*)\) and \(k^* \in \omega\) such that for any \((p_2, \vec{u}_2^0, \vec{u}_2^1) \leq_{\Gamma} (p_1, \vec{u}_1^0, \vec{u}_1^1)\) and any
\[
\vec{n} \in a(\vec{u}_2^0) \cap a(\vec{u}_2^1)
\]
such that \(\vec{n}^* \upharpoonright l \subseteq \vec{n}\), we have \(\vec{n}(l) \leq k^*\).

Proof of Claim. We show that if the claim fails, there is a \(<_{\Gamma}\)-descending sequence in \(\Gamma\). It suffices to show that any \((p_1, \vec{u}_1^0, \vec{u}_1^1) \leq_{\Gamma} (p_0, \vec{u}_0^0, \vec{u}_0^1)\) has a \(<_{\Gamma}\)-extension. So let \((p_1, \vec{u}_1^0, \vec{u}_1^1) \leq_{\Gamma} (p_0, \vec{u}_0^0, \vec{u}_0^1)\) be given.

Since the claim fails, given any terminal sequence \(\vec{n} \in a(\vec{u}_1^0) \cap a(\vec{u}_1^1)\), any \(k < \lh(\vec{n})\), and
\[
(p, \vec{u}^0, \vec{u}^1) \leq_{\Gamma} (p_1, \vec{u}_1^0, \vec{u}_1^1)
\]
we can form an extension
\[
(q, \vec{v}^0, \vec{v}^1) \leq_{\Gamma} (p, \vec{u}^0, \vec{u}^1)
\]
such that there is \(\vec{n}' \in a(v_1) \cap a(v_1)\) with \(n' \upharpoonright k = \vec{n} \upharpoonright k\) and \(\vec{n}'(k) > \vec{n}(k)\).

In finitely many steps, construct a (finite) descending sequence
\[
(p_1, \vec{u}_1^0, \vec{u}_1^1) \geq_{\Gamma} (p_2, \vec{u}_2^0, \vec{u}_2^1) \geq_{\Gamma} \ldots \geq_{\Gamma} (p_m, \vec{u}_m^0, \vec{u}_m^1),
\]
at each step taking an extension of the previous element as just described. We can deal with each \(\vec{n} \in a(\vec{u}_1^0) \cap a(\vec{u}_1^1)\) and each \(k < \lh(\vec{n})\), so that at the end
\[
a(\vec{u}_1^0) \cap a(\vec{u}_1^1) \subset_{\alpha} a(\vec{u}_m^0) \cap a(\vec{u}_m^1)
\]
Thus we have found \((p_m, \vec{u}_m^0, \vec{u}_m^1) \leq_{\Gamma} (p_1, \vec{u}_1^0, \vec{u}_1^1)\). \(\blacksquare\)

Let \((p_1, \vec{u}_1^0, \vec{u}_1^1) \leq_{\Gamma} (p_0, \vec{u}_0^0, \vec{u}_0^1), \vec{n}^* \in a(\vec{u}_1^0) \cap a(\vec{u}_1^1), l < \lh(\vec{n}^*)\) and \(k^* \in \omega\) be as in the claim. By Lemma 5.22 and by replacing \(p_1\) by a stronger condition if necessary, we may assume that \(p_1 \in D(\vec{u}_1^0) \cap D(\vec{u}_1^1)\).

Case 1: Assume first that \(l = 0\). Let \(A' \subseteq M^\alpha\) be defined as in Lemma 5.24[3], namely
\[
A' = \{\vec{n} \mid (\exists p' \leq p_1)(\exists \vec{u}' \subseteq_U \vec{u}_1^0) \ n \in a(\vec{u}') \wedge p' \forces \vec{u}' \in U^{z_{\vec{n}}}\}
\]
By Lemma 5.24[3], \(A' \in \mathcal{I}^+\). Find \(A \subseteq A(p_1)\) such that \(A \in \mathcal{I}^{++}\), and letting \(k^{**} = \max(\dom(a(p_1)))\),
\[
A(p_1) \cap \left( \bigcup_{i \in \dom(p_1)} \{i \leq k^*\} \times M^{\delta(i)} \right) \subseteq A
\]
and
\[
A \cap \left( \bigcup_{i > k^{**}} \{i\} \times M^{\delta(i)} \right) \subseteq A'.
\]
Letting \(p^* = (a(p_1), A)\) we obtain a condition in \(M^\alpha_{\vec{u}_1^1}\) such that \(p^* \leq p_1\). Since \(p^* \forces \vec{u}_1^1 \in U^{z_{\vec{n}}_{\vec{u}_1^1}}\), and we can find \(p \leq p^*, \vec{u}\), and \(\vec{n}\) with \(\vec{n}(0) > k^*, k^{**}\) such that
\[
\vec{n} \in a(\vec{u}) \wedge p \forces \vec{u} \in U^{z_{\vec{n}}_{[\vec{u}_1^1]}}.
\]
It follows that $\vec{n} \in A'$ and so by definition of $A'$ we can find $p' \leq p_1$ and $\vec{u}'$ such that

$$\vec{n} \in a(\vec{u}') \wedge p' \models U_{[\vec{u}']_{A_1}}^{x,\vec{n}}.$$ 

By extending $p, p'$ if necessary, we can assume that $a(p)$ and $a(p')$ have the same lexicographically maximal element. As $p_1 \in D(\vec{u}_1^0) \cap D(\vec{u}_1)$ and $p, p' \leq p_1$, we can find $\vec{u}^0$ and $\vec{u}^1$ such that

$$\vec{n} \in a(\vec{u}^0) \wedge (a(p), A(p_1)/a(p)) \models a(\vec{u}^0) \in U_{[\vec{u}^0]_{A_1}}^{x,\vec{n}}$$

and

$$\vec{n} \in a(\vec{u}^1) \wedge (a(p'), A(p_1)/a(p')) \models a(\vec{u}^1) \in U_{[\vec{u}^1]_{A_1}}^{x,\vec{n}}.$$ 

Let $a = a(p) \cap a(p')$ (whose lexicographically maximal element is also that of $a(p')$). For each $i \in \{0, 1\}$ we have

$$a(\vec{u}^i) \cup \{\vec{n}\} \subseteq a \subseteq a(p), a(p')$$

and so by Lemma 5.23

$$(a, A(p_1)/a)) \models (a(\vec{u}^i) \cup \{\vec{n}\}, t(\vec{u}^i)) \in U_{[\vec{u}^i]_{A_1}}^{x,\vec{n}}$$

for each $i \in \{0, 1\}$. Letting

$$p_2 = (a, A(p_1)/a)$$

and

$$u_2^i = (a(\vec{u}^i) \cup \{\vec{n}\}, t(\vec{u}^i))$$

for $i \in \{0, 1\}$ we obtain $(p_2, u_2^0, u_2^1) \leq^F (p_1, u_1^0, u_1^1)$ with $\vec{n} \in a(\vec{u}^0) \cap a(\vec{u}^1)$ and $\vec{n}(0) > k^*$, which contradicts the choice of $(p_1, u_1^0, u_1^1)$, $\vec{n}$, and $k^*$.

**Case 2:** In case $l > 0$, let $\vec{n} = \vec{n}^* \upharpoonright l$ and consider the set $A_{\vec{n}}$ defined as in Lemma 5.21(4), namely

$$A_{\vec{n}} = \{\vec{m} \mid (\exists p \leq p_1)(\exists \vec{u} \leq U \vec{u}_1) \vec{m} \in a(\vec{u})(\vec{n}) \wedge p \models U_{[\vec{u}]_{A_1}}^{x,\vec{n}}\}.$$

Lemma 5.21(4) ensures $A_{\vec{n}} \in (\text{Fin}^{\delta_{\vec{n}}(\vec{n})})^+$. Let $k^{**} = \max\{\vec{n}(l) \mid \vec{n} \in a(p_1)\}$. Find $A \subseteq A(p_1)$ such that $A \in (\text{Fin}^{\delta(\vec{n})})^{++}$,

$$A(p_1)(\vec{n}) \cap \{\vec{m} \in \omega^{<\omega} \mid \vec{m}(0) \leq k^{**}\} \subseteq A$$

and

$$A(\vec{n}) \cap \{\vec{m} \in \omega^{<\omega} \mid \vec{m}(0) > k^{**}\} \subseteq A_{\vec{n}}.$$

By choice of $k^{**}$, letting $p^* = (a(p_1), A)$ we obtain a condition in $M_3^n$, $p^* \leq p_1$.

Since $p^* \models \vec{u}_1 \in U_{\vec{u}}^{x,\vec{n}}$ we can find $p \leq p^*$ and $\vec{u} \leq U \vec{u}_1$ such that there exists a terminal sequence $\vec{m} \in a(\vec{u})$ with $\vec{m}(l) > k^*, k^{**}$ and such that

$$p \models \vec{u} \in T_{[\vec{u}^1]_{A_1}}^{x,\vec{n}}.$$ 

By definition of $A$ we infer $\vec{m} = \vec{m} \cap \vec{m}'$ for some $\vec{m}' \in A_{\vec{n}}$, and so we can find $p' \leq p_1$ and $\vec{u}' \leq U \vec{u}_1^0$ such that

$$\vec{m} \in a(\vec{u}') \wedge p' \models \vec{u}' \in T_{[\vec{u}^1]_{A_1}}^{x,\vec{n}}.$$
Using that $p_1 \in D(\vec{u}_1^0) \cap D(\vec{u}_1^1)$ and Lemma 5.23, argue verbatim as in the previous case to construct $(p_2, \vec{u}_2^0, \vec{u}_2^1) \leq_P \vec{m}$ with $\vec{m} \in a(\vec{u}_2^0) \cap a(\vec{u}_2^1)$. Since $\vec{m}(l) > k^*$, this contradicts the choice of $(p_1, \vec{u}_1^0, \vec{u}_1^1)$, $\vec{n}^*$, $l$, and $k^*$.

6. Postscript: A dichotomy for Borel ideals?

The results of this paper show that for $\mathcal{J}$ in a rather vast class of Borel ideals in $\omega$, one can prove that there are no definable $\mathcal{J}$-MAD families, under suitable assumptions on either what definable means, or what background theory is adopted.

It is worth noting that it is not the case that such a theorem is true for every Borel ideal on $\omega$. Indeed, the ideal on $\omega \times \omega$, defined by

$$\mathcal{J} = \{x \subseteq \omega \times \omega : (\forall n \in \omega)\{m : (n, m) \in x\} \text{ is finite}\}$$

clearly admits the $\mathcal{J}$-MAD family, namely $\{|i| \times i : i \in \omega\}$.

It remains an interesting open problem if it is possible to characterize the Borel ideals for which an analogue of Theorem 1.4 is true, and for which that type of theorem fails. In other words:

**Question 6.1.** Is there a dichotomy for Borel ideals on $\omega$ which characterizes when there are/are no definable MAD families with respect to a given Borel ideal?

We note that in the more general setting of finding maximal discrete sets for Borel graphs, a result of Horowitz and Shelah [2] shows there is no reasonable dichotomy. It is, however, not clear that the obstruction found there also apply to the special case of $\mathcal{J}$-mad families, when $\mathcal{J}$ is a Borel ideal.

**References**


14. John R. Steel, HOD\(^{(R)}\) is a core model below \( \Theta \), Bull. Symbolic Logic 1 (1995), no. 1, 75–84. MR 1324625


Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark

Kurt Gödel Research Center, Universität Wien, Währingerstrasse 26, 1090 Vienna, Austria

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark

E-mail address: karenbhaga@math.ku.dk
E-mail address: david@logic.univie.ac.at
E-mail address: asgert@math.ku.dk