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Hansen, Frank

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Frank Hansen

Department of Mathematical Sciences, Copenhagen University, Denmark

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**ABSTRACT**

By using a variational principle we find a necessary and sufficient condition for an operator to majorise the parallel sum of two positive definite operators. This result is then used as a vehicle to create new operator inequalities involving the parallel sum.

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1. Introduction

Anderson and Duffin defined the parallel sum $A : B$ of two positive definite operators $A$ and $B$ by setting

$$A : B = \frac{1}{A^{-1} + B^{-1}},$$

and they proved [1, Lemma 18] that for any vector $\xi$ the inner product

$$((A : B)\xi \mid \xi) = \inf_{\eta} \{(A\eta \mid \eta) + (B(\xi - \eta) \mid \xi - \eta)\}.$$  \(1\)

**E-mail address:** frank.hansen@math.ku.dk.
We begin by giving an intuitive proof of the variational result in (1). The purpose of this note is then to establish that the operator inequality

$$A : B \leq H$$

is valid, if and only if there exists an operator $C$ such that

$$H = C^* AC + (I - C^*) B (I - C).$$

This result then functions as a generator of operator inequalities involving the parallel sum. We refer to [3] for a recent paper on the parallel sum.

2. Preliminaries

We first establish the rule of differentiating an expectation value with respect to a vector,

$$d_x (Ax | x) \xi = 2 \operatorname{Re} (Ax | \xi).$$

Indeed,

$$d_x (Ax | x) \xi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( (A(x + \varepsilon \xi) | x + \varepsilon \xi) - (Ax | x) \right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \varepsilon (Ax | \xi) + \varepsilon (A \xi | x) + \varepsilon^2 (A \xi | \xi) \right) = 2 \operatorname{Re} (Ax | \xi).$$

Let $A, B$ be positive definite matrices and consider to a given vector $x$ the vector function

$$f(\xi) = (A \xi | \xi) + (B(x - \xi) | x - \xi).$$

It is manifestly convex with derivative

$$df(\xi) \eta = 2 \operatorname{Re} (A \xi | \eta) - 2 \operatorname{Re} (B(x - \xi) | \eta)$$

$$= 2 \operatorname{Re} (A \xi - B(x - \xi) | \eta).$$

The derivative vanishes in all $\eta$ if and only if

$$A \xi - B(x - \xi) = 0 \quad \text{or} \quad (A + B) \xi = Bx,$$

and this is equivalent to

$$\xi = (A + B)^{-1} Bx.$$  \hspace{1cm} (2)$$

In addition,
\[ x - \xi = x - (A + B)^{-1}Bx = (A + B)^{-1}((A + B)x - Bx) = (A + B)^{-1}Ax. \]

We thus obtain that
\[ (A\xi | \xi) = (A(A + B)^{-1}Bx | (A + B)^{-1}Bx) \]

and
\[ (B(x - \xi) | x - \xi) = (B(A + B)^{-1}Ax | (A + B)^{-1}Ax). \]

Since \( f \) is convex the global minimum of \( f \) is obtained in \( \xi \) with minimum value
\[ f(\xi) = (A\xi | \xi) + (B(x - \xi) | x - \xi). \]

Since
\[ B(A + B)^{-1}A = (A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B, \]

we calculate the global minimum value to be
\[ f(\xi) = ((A^{-1} + B^{-1})^{-1}x | (A + B)^{-1}Bx + (A + B)^{-1}Ax) \]
\[ = ((A^{-1} + B^{-1})^{-1}x | x) = ((A : B)x | x), \]

where \( A : B \) is the parallel sum of \( A \) and \( B \). It is also half of the harmonic mean. In conclusion, we recover (1) and obtain the inequality
\[ ((A : B)x | x) = f(\xi) \leq f(\eta) \]

for any other vector \( \eta \). For an arbitrary operator \( D \) we set \( \eta = D\xi \) and obtain
\[ ((A : B)x | x) \leq f(D\xi) \]
\[ = (AD\xi | D\xi) + (B(x - D\xi) | x - D\xi) \]
\[ = (AD(A + B)^{-1}Bx | D(A + B)^{-1}Bx) \]
\[ + (B(x - D(A + B)^{-1}Bx) | x - D(A + B)^{-1}Bx), \]

where we used (2). Putting \( C = D(A + B)^{-1}B \) this is equivalent to
\[ ((A : B)x | x) \leq (C^*ACx | x) + ((I - C^*)B(I - C)x | x). \]

We have thus proved the following result.
Theorem 2.1. Let $A$ and $B$ be positive definite operators. Then
\[ A : B \leq C^* AC + (I - C^*) B(I - C) \]
for an arbitrary operator $C$.

We next investigate the range of the operator function
\[ F(C) = C^* AC + (I - C^*) B(I - C) \]
to given positive definite operators $A$ and $B$. We consider the operator equation $F(C) = H$ and rewrite the equation as
\[ C^*(A + B)C + B - C^* B - BC = H. \]

By multiplying with $(A + B)^{-1/2}$ from the left and from the right the equation is equivalent to
\[
(A + B)^{-1/2}C^*(A + B)C(A + B)^{-1/2} + (A + B)^{-1/2}B(A + B)^{-1/2} \\
- (A + B)^{-1/2}C^* B(A + B)^{-1/2} - (A + B)^{-1/2}BC(A + B)^{-1/2} \\
= (A + B)^{-1/2}H(A + B)^{-1/2}.
\]

We now set
\[ X = (A + B)^{1/2}C(A + B)^{-1/2} \text{ and } Y = (A + B)^{-1/2}B(A + B)^{-1/2} \]
and rewrite the equation as
\[ X^* X + Y - X^* Y - YX = (A + B)^{-1/2}H(A + B)^{-1/2}, \]
which again may be written as
\[ (X - Y)^* (X - Y) - Y^2 + Y = (A + B)^{-1/2}H(A + B)^{-1/2} \]
or
\[ (X - Y)^* (X - Y) = (A + B)^{-1/2}H(A + B)^{-1/2} + Y^2 - Y \\
= (A + B)^{-1/2}(H - B + B(A + B)^{-1} B)(A + B)^{-1/2} \\
= (A + B)^{-1/2}(H - B(A + B)^{-1}(A + B - B))(A + B)^{-1/2} \\
= (A + B)^{-1/2}(H - (A : B))(A + B)^{-1/2}. \]
The equation can thus be solved if and only if
\[ H \geq A : B. \]

Under this condition we may find positive definite solutions in \( X \) given by

\[ X = Y + \left( (A + B)^{-1/2}(H - (A : B))(A + B)^{-1/2} \right)^{1/2} \]

and then obtain

\[
C = (A + B)^{-1/2}X(A + B)^{1/2} = (A + B)^{-1/2}Y(A + B)^{1/2} \\
+ (A + B)^{-1/2}\left( (A + B)^{-1/2}(H - (A : B))(A + B)^{-1/2} \right)^{1/2}(A + B)^{1/2} \\
= (A + B)^{-1}B + \left( (A + B)^{-1}(H - (A : B)) \right)^{1/2}.
\]

Note that the operator appearing inside the square root in the last formula line may not be self-adjoint. It is however similar to a positive semi-definite operator and therefore has a unique square root with positive spectrum. We have obtained.

**Theorem 2.2.** Let \( A, B \) and \( H \) be positive definite operators. The operator equation

\[ F(C) = C^*AC + (I - C^*)B(I - C) = H \]

has solutions in \( C \) if and only if \( H \geq A : B \). One of the solutions is then given by

\[ C = (A + B)^{-1}B + \left( (A + B)^{-1}(H - (A : B)) \right)^{1/2}. \]

### 3. Generating operator inequalities

Theorem 2.1 may serve as a generator for operator inequalities by suitably choosing the operator \( C \). For \( C = \lambda I \), where \( 0 \leq \lambda \leq 1 \), we obtain

\[ A : B \leq \lambda^2 A + (1 - \lambda)^2 B. \]

By setting \( \lambda = 0 \), \( \lambda = 1/2 \) or \( \lambda = 1 \) we obtain the well-known inequalities

\[ A : B \leq B, \quad A : B \leq \frac{A + B}{4}, \quad A : B \leq A. \]

Setting \( C = (A + B)^{-1}B \) we obtain equality

\[ A : B = F(C). \]

Indeed, we note that
\[ I - C = I - (A + B)^{-1}B = (A + B)^{-1}(A + B - B) = (A + B)^{-1}A. \]

Therefore,

\[ F(C) = B(A + B)^{-1}A(A + B)^{-1}B + A(A + B)^{-1}B(A + B)^{-1}A \]
\[ = B(A + B)^{-1}A(A + B)^{-1}B + A(A + B)^{-1}A(A + B)^{-1}B \]
\[ = (B^{-1} + A^{-1})^{-1} = A : B. \]

We next use Theorem 2.1 to obtain new operator inequalities.

**Theorem 3.1.** Let \( A, B \) be positive definite operators.

(i) Let \( P \) be an orthogonal projection. We obtain the inequality

\[ A : B \leq PAP + (I - P)B(I - P). \]

Setting \( A = B \) it reduces to the familiar inequality

\[ \frac{1}{2}A \leq PAP + (I - P)A(I - P). \]

(ii) The inequality

\[ A : B \leq (A + B)^{-1}(BAB + ABA)(A + B)^{-1} \]

is valid, and it is strict, since for \( A = B \) it reduces to \( \frac{1}{2}A \leq \frac{1}{2}A. \)

(iii) Let \( p \) be a real number. We obtain the inequality

\[ A : B \leq (A^p : B^p)(A^{2p-1} : B^{2p-1})^{-1}(A^p : B^p), \]

and it reduces to equality for \( p = 1 \). The inequality is strict for arbitrary \( p \), since for \( A = B \) it reduces to \( \frac{1}{2}A \leq \frac{1}{2}A. \)

**Proof.** By setting \( C = P \) and applying Theorem 2.1 we obtain (i). By setting \( C = B(A + B)^{-1} \) we obtain \( I - C = A(A + B)^{-1} \) and thus

\[ C^*AC + (I - C)^*B(I - C) \]
\[ = (A + B)^{-1}BAB(A + B)^{-1} + (A + B)^{-1}ABA(A + B)^{-1} \]

from which (ii) follows. Finally, we set \( C = (A^p + B^p)^{-1}B^p \) and since \( I - C = (A^p + B^p)^{-1}A^p \) and \( A^p(A^p + B^p)^{-1}B^p = B^p(A^p + B^p)^{-1}A^p \) we obtain
\[ C^* AC + (I - C)^* B(I - C) \]
\[ = (A^p : B^p)(A^{2p-1} : B^{2p-1})^{-1}(A^p : B^p) \]

as desired. This proves (iii). \(\square\)

By multiplying (iii) in Theorem 3.1 by 2 we obtain the inequality between harmonic means

\[
H_2(A, B) \leq H_2(A^p, B^p)H_2(A^{2p-1}, B^{2p-1})^{-1} H_2(A^p, B^p)
\]  

(3)

for positive definite operators \(A\) and \(B\) and arbitrary \(p \in \mathbb{R}\). If we in particular put \(p = 1/2\) we obtain

\[
H_2(A, B) \leq H_2(A^{1/2}, B^{1/2})^2.
\]  

(4)

This is an improvement of the inequality

\[
H_2(A, B)^{1/2} \leq H_2(A^{1/2}, B^{1/2})
\]

which is plain. Indeed, for \(0 \leq p \leq 1\), we obtain by operator concavity of the function \(t \to t^p\) the inequality

\[
H_2(A, B)^p = \left(\frac{2}{A^{-1} + B^{-1}}\right)^p = \left(\frac{A^{-1} + B^{-1}}{2}\right)^{-p} \leq \frac{2}{A^{-p} + B^{-p}} = H_2(A^p, B^p).
\]  

(5)

The reverse inequality is obtained for \(-1 \leq p \leq 0\) and \(1 \leq p \leq 2\) by operator convexity. It is interesting to note that the inequality

\[
H_2(A, B) \leq H_2(A^p, B^p)^{1/p}
\]  

(6)

is false for \(p = 1/4\) with counter examples in two-by-two matrices. We conjecture that (6) is false for \(0 < p < 1/2\) and true for \(1/2 \leq p \leq 1\).
3.1. The power means

Bhagwat and Subramanian [2, Section 4] introduced for \( p > 0 \) the power mean

\[
M_p(A, B) = \left( \frac{A^p + B^p}{2} \right)^{1/p} \tag{7}
\]

of positive definite operators \( A \) and \( B \). If \( p \geq 1 \) then the function \( t \to t^{1/p} \) is operator concave and thus

\[
M_p(A, B) \geq \frac{A + B}{2} \geq 2(A : B) > A : B.
\]

The parallel sum is thus majorized by the power mean. However, this result can in general not be extended to \( 0 < p < 1 \).

**Example 3.2.** Consider the two-by-two matrices

\[
A = \begin{pmatrix} 0.14623 & -0.07525 \\ -0.07525 & 0.03873 \end{pmatrix}, \quad B = \begin{pmatrix} 0.733 & -0.43 \\ -0.43 & 0.2525 \end{pmatrix}.
\]

\( A \) has approximately eigenvalues \( \{0.184955, 5.00338 \cdot 10^{-6}\} \) and \( B \) has approximately eigenvalues \( \{0.985315, 0.00018522\} \), so they are positive definite. Setting \( p = 1/2 \) the smallest eigenvalue of

\[
\left( \frac{A^{1/2} + B^{1/2}}{2} \right)^2 - (A : B)
\]

is approximately \(-1.57101 \cdot 10^{-6}\).

**Declaration of competing interest**

There is no competing interest.

**References**

