A note on the parallel sum

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ABSTRACT

By using a variational principle we find a necessary and sufficient condition for an operator to majorise the parallel sum of two positive definite operators. This result is then used as a vehicle to create new operator inequalities involving the parallel sum.

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1. Introduction

Anderson and Duffin defined the parallel sum $A : B$ of two positive definite operators $A$ and $B$ by setting

$$A : B = \frac{1}{A^{-1} + B^{-1}},$$

and they proved [1, Lemma 18] that for any vector $\xi$ the inner product

$$((A : B)\xi \mid \xi) = \inf_{\eta} \{ (A\eta \mid \eta) + (B(\xi - \eta) \mid \xi - \eta) \}. \tag{1}$$
We begin by giving an intuitive proof of the variational result in (1). The purpose of this note is then to establish that the operator inequality

\[ A : B \leq H \]

is valid, if and only if there exists an operator \( C \) such that

\[ H = C^* AC + (I - C^*)B(I - C). \]

This result then functions as a generator of operator inequalities involving the parallel sum. We refer to [3] for a recent paper on the parallel sum.

2. Preliminaries

We first establish the rule of differentiating an expectation value with respect to a vector,

\[ d_x(Ax \mid x)\xi = 2\text{Re}(Ax \mid \xi). \]

Indeed,

\[
d_x(Ax \mid x)\xi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \left( A(x + \varepsilon \xi) \mid x + \varepsilon \xi \right) - \left( Ax \mid x \right) \right)
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \varepsilon(Ax \mid \xi) + \varepsilon(A\xi \mid x) + \varepsilon^2(A\xi \mid \xi) \right) = 2\text{Re}(Ax \mid \xi).
\]

Let \( A, B \) be positive definite matrices and consider to a given vector \( x \) the vector function

\[ f(\xi) = (A\xi \mid \xi) + (B(x - \xi) \mid x - \xi). \]

It is manifestly convex with derivative

\[ df(\xi)\eta = 2\text{Re}(A\xi \mid \eta) - 2\text{Re}(B(x - \xi) \mid \eta) \]
\[ = 2\text{Re}(A\xi - B(x - \xi) \mid \eta). \]

The derivative vanishes in all \( \eta \) if and only if

\[ A\xi - B(x - \xi) = 0 \quad \text{or} \quad (A + B)\xi = Bx, \]

and this is equivalent to

\[ \xi = (A + B)^{-1}Bx. \tag{2} \]

In addition,
\[ x - \xi = x - (A + B)^{-1}Bx = (A + B)^{-1}(A + B)x - Bx = (A + B)^{-1}Ax. \]

We thus obtain that
\[ (A\xi | \xi) = (A(A + B)^{-1}Bx | (A + B)^{-1}Bx) \]
and
\[ (B(x - \xi) | x - \xi) = (B(A + B)^{-1}Ax | (A + B)^{-1}Ax). \]

Since \( f \) is convex the global minimum of \( f \) is obtained in \( \xi \) with minimum value
\[ f(\xi) = (A\xi | \xi) + (B(x - \xi) | x - \xi). \]

Since
\[ B(A + B)^{-1}A = (A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B, \]
we calculate the global minimum value to be
\[ f(\xi) = ((A^{-1} + B^{-1})^{-1}x | (A + B)^{-1}Bx + (A + B)^{-1}Ax) \]
\[ = ((A^{-1} + B^{-1})^{-1}x | x) = ((A : B)x | x), \]
where \( A : B \) is the parallel sum of \( A \) and \( B \). It is also half of the harmonic mean. In conclusion, we recover (1) and obtain the inequality
\[ ((A : B)x | x) = f(\xi) \leq f(\eta) \]
for any other vector \( \eta \). For an arbitrary operator \( D \) we set \( \eta = D\xi \) and obtain
\[ ((A : B)x | x) \leq f(D\xi) \]
\[ = (AD\xi | D\xi) + (B(x - D\xi) | x - D\xi) \]
\[ = (AD(A + B)^{-1}Bx | D(A + B)^{-1}Bx) \]
\[ + (B(x - D(A + B)^{-1}Bx) | x - D(A + B)^{-1}Bx), \]
where we used (2). Putting \( C = D(A + B)^{-1}B \) this is equivalent to
\[ ((A : B)x | x) \leq (C^*ACx | x) + ((I - C^*)B(I - C)x | x). \]

We have thus proved the following result.
Theorem 2.1. Let $A$ and $B$ be positive definite operators. Then

$$A : B \leq C^* AC + (I - C^*) B (I - C)$$

for an arbitrary operator $C$.

We next investigate the range of the operator function

$$F(C) = C^* AC + (I - C^*) B (I - C)$$

to given positive definite operators $A$ and $B$. We consider the operator equation $F(C) = H$ and rewrite the equation as

$$C^* (A + B) C + B - C^* B - BC = H.$$ 

By multiplying with $(A + B)^{-1/2}$ from the left and from the right the equation is equivalent to

$$\begin{align*}
(A + B)^{-1/2} C^* (A + B) C (A + B)^{-1/2} &+ (A + B)^{-1/2} B (A + B)^{-1/2} \\
- (A + B)^{-1/2} C^* B (A + B)^{-1/2} - (A + B)^{-1/2} BC (A + B)^{-1/2} \\
&= (A + B)^{-1/2} H (A + B)^{-1/2}.
\end{align*}$$

We now set

$$X = (A + B)^{1/2} C (A + B)^{-1/2} \quad \text{and} \quad Y = (A + B)^{-1/2} B (A + B)^{-1/2}$$

and rewrite the equation as

$$X^* X + Y - X^* Y - YX = (A + B)^{-1/2} H (A + B)^{-1/2},$$

which again may be written as

$$(X - Y)^* (X - Y) - Y^2 + Y = (A + B)^{-1/2} H (A + B)^{-1/2}$$

or

$$(X - Y)^* (X - Y) = (A + B)^{-1/2} H (A + B)^{-1/2} + Y^2 - Y$$

$$= (A + B)^{-1/2} (H - B + B (A + B)^{-1} B) (A + B)^{-1/2}$$

$$= (A + B)^{-1/2} (H - B (A + B)^{-1} (A + B - B)) (A + B)^{-1/2}$$

$$= (A + B)^{-1/2} (H - (A : B)) (A + B)^{-1/2}.$$ 

The equation can thus be solved if and only if
\[ H \geq A : B. \]

Under this condition we may find positive definite solutions in \( X \) given by

\[ X = Y + \left( (A + B)^{-1/2}(H - (A : B))(A + B)^{-1/2} \right)^{1/2} \]

and then obtain

\[
C = (A + B)^{-1/2}X(A + B)^{1/2} = (A + B)^{-1/2}Y(A + B)^{1/2} \\
+ (A + B)^{-1/2}\left( (A + B)^{-1/2}(H - (A : B))(A + B)^{-1/2} \right)^{1/2}(A + B)^{1/2} \\
= (A + B)^{-1}B + \left( (A + B)^{-1}(H - (A : B)) \right)^{1/2}.
\]

Note that the operator appearing inside the square root in the last formula line may not be self-adjoint. It is however similar to a positive semi-definite operator and therefore has a unique square root with positive spectrum. We have obtained.

**Theorem 2.2.** Let \( A, B \) and \( H \) be positive definite operators. The operator equation

\[ F(C) = C^*AC + (I - C^*)B(I - C) = H \]

has solutions in \( C \) if and only if \( H \geq A : B. \) One of the solutions is then given by

\[ C = (A + B)^{-1}B + \left( (A + B)^{-1}(H - (A : B)) \right)^{1/2}. \]

3. **Generating operator inequalities**

   Theorem 2.1 may serve as a generator for operator inequalities by suitably choosing the operator \( C \). For \( C = \lambda I \), where \( 0 \leq \lambda \leq 1 \), we obtain

   \[ A : B \leq \lambda^2 A + (1 - \lambda)^2 B. \]

   By setting \( \lambda = 0 \), \( \lambda = 1/2 \) or \( \lambda = 1 \) we obtain the well-known inequalities

   \[ A : B \leq B, \quad A : B \leq \frac{A + B}{4}, \quad A : B \leq A. \]

   Setting \( C = (A + B)^{-1}B \) we obtain equality

   \[ A : B = F(C). \]

   Indeed, we note that
\[ I - C = I - (A + B)^{-1}B = (A + B)^{-1}(A + B - B) = (A + B)^{-1}A. \]

Therefore,
\[
F(C) = B(A + B)^{-1}A(A + B)^{-1}B + A(A + B)^{-1}B(A + B)^{-1}A
\]
\[ = B(A + B)^{-1}A(A + B)^{-1}B + A(A + B)^{-1}A(A + B)^{-1}B \]
\[ = (B^{-1} + A^{-1})^{-1} = A : B. \]

We next use Theorem 2.1 to obtain new operator inequalities.

**Theorem 3.1.** Let \( A, B \) be positive definite operators.

(i) Let \( P \) be an orthogonal projection. We obtain the inequality
\[
A : B \leq PAP + (I - P)B(I - P).
\]

Setting \( A = B \) it reduces to the familiar inequality
\[
\frac{1}{2}A \leq PAP + (I - P)A(I - P).
\]

(ii) The inequality
\[
A : B \leq (A + B)^{-1}(BAB + ABA)(A + B)^{-1}
\]

is valid, and it is strict, since for \( A = B \) it reduces to \( \frac{1}{2}A \leq \frac{1}{2}A \).

(iii) Let \( p \) be a real number. We obtain the inequality
\[
\]

and it reduces to equality for \( p = 1 \). The inequality is strict for arbitrary \( p \), since for \( A = B \) it reduces to \( \frac{1}{2}A \leq \frac{1}{2}A \).

**Proof.** By setting \( C = P \) and applying Theorem 2.1 we obtain (i). By setting \( C = B(A + B)^{-1} \) we obtain \( I - C = A(A + B)^{-1} \) and thus
\[
C^*AC + (I - C)^*B(I - C)
= (A + B)^{-1}BAB(A + B)^{-1} + (A + B)^{-1}ABA(A + B)^{-1}
\]
from which (ii) follows. Finally, we set \( C = (A^p + B^p)^{-1}B^p \) and since \( I - C = (A^p + B^p)^{-1}A^p \) and \( A^p(A^p + B^p)^{-1}B^p = B^p(A^p + B^p)^{-1}A^p \) we obtain
\[ C^*AC + (I - C)^*B(I - C) \]
\[ = (A^p : B^p)(A^{2p-1} : B^{2p-1})^{-1}(A^p : B^p) \]

as desired. This proves \((iii)\). \(\square\)

By multiplying \((iii)\) in Theorem 3.1 by 2 we obtain the inequality between harmonic means

\[ H_2(A, B) \leq H_2(A^p, B^p)H_2(A^{2p-1}, B^{2p-1})^{-1}H_2(A^p, B^p) \] (3)

for positive definite operators \(A\) and \(B\) and arbitrary \(p \in \mathbb{R}\). If we in particular put \(p = 1/2\) we obtain

\[ H_2(A, B) \leq H_2(A^{1/2}, B^{1/2})^2. \] (4)

This is an improvement of the inequality

\[ H_2(A, B)^{1/2} \leq H_2(A^{1/2}, B^{1/2}) \]

which is plain. Indeed, for \(0 \leq p \leq 1\), we obtain by operator concavity of the function \(t \to t^p\) the inequality

\[ H_2(A, B)^p = \left(\frac{2}{A^{-1} + B^{-1}}\right)^p = \left(\frac{A^{-1} + B^{-1}}{2}\right)^{-p} \]
\[ \leq \frac{2}{A^{-p} + B^{-p}} = H_2(A^p, B^p). \] (5)

The reverse inequality is obtained for \(-1 \leq p \leq 0\) and \(1 \leq p \leq 2\) by operator convexity. It is interesting to note that the inequality

\[ H_2(A, B) \leq H_2(A^p, B^p)^{1/p} \] (6)

is false for \(p = 1/4\) with counter examples in two-by-two matrices. We conjecture that (6) is false for \(0 < p < 1/2\) and true for \(1/2 \leq p \leq 1\).
3.1. The power means

Bhagwat and Subramanian [2, Section 4] introduced for $p > 0$ the power mean

$$M_p(A, B) = \left(\frac{A^p + B^p}{2}\right)^{1/p}$$

(7)

of positive definite operators $A$ and $B$. If $p \geq 1$ then the function $t \to t^{1/p}$ is operator concave and thus

$$M_p(A, B) \geq \frac{A + B}{2} \geq 2(A : B) > A : B.$$

The parallel sum is thus majorized by the power mean. However, this result can in general not be extended to $0 < p < 1$.

**Example 3.2.** Consider the two-by-two matrices

$$A = \begin{pmatrix}
0.14623 & -0.07525 \\
-0.07525 & 0.03873
\end{pmatrix}, \quad B = \begin{pmatrix}
0.733 & -0.43 \\
-0.43 & 0.2525
\end{pmatrix}.$$

$A$ has approximately eigenvalues $\{0.184955, 5.00338 \cdot 10^{-6}\}$ and $B$ has approximately eigenvalues $\{0.985315, 0.00018522\}$, so they are positive definite. Setting $p = 1/2$ the smallest eigenvalue of

$$\left(\frac{A^{1/2} + B^{1/2}}{2}\right)^2 - (A : B)$$

is approximately $-1.57101 \cdot 10^{-6}$.

**Declaration of competing interest**

There is no competing interest.

**References**