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THE WARING RANK OF BINARY BINOMIAL FORMS

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THE WARING RANK OF BINARY BINOMIAL FORMS

LAURA BRUSTENGA I MONCUSÍ AND SHREEDEVI K. MASUTI

We give an explicit formula for the Waring rank of every binary binomial form with complex coefficients. We give several examples to illustrate this, and compare the Waring rank and the real Waring rank for binary binomial forms.

1. Introduction

This paper concerns symmetric tensor decomposition as a sum of rank one tensors, which is also known as the Waring problem for forms. This topic has a rich history and has recently received huge interest mainly because of its wide applicability in areas as diverse as algebraic statistics, biology, quantum information theory, signal processing, data mining, machine learning; see [Comon et al. 2008; Kolda and Bader 2009; Landsberg 2012].

Let \( \mathbb{k}[x, y] \) be the standard graded polynomial ring with coefficients in the field \( \mathbb{k} \subseteq \mathbb{C} \). For \( d \geq 0 \), we denote by \( \mathbb{k}[x, y]_d \) the \( \mathbb{k} \)-vector space of forms of degree \( d \) in \( \mathbb{k}[x, y] \). For every form \( F \in \mathbb{k}[x, y]_d \), there exist linear forms \( L_1, \ldots, L_r \in \mathbb{k}[x, y]_1 \) and scalars \( a_1, \ldots, a_r \in \mathbb{k} \) with \( r \leq d + 1 \) such that

\[
F = a_1 L_1^d + \cdots + a_r L_r^d
\]

(see [Reznick 2013, Theorem 4.2]). When \( \mathbb{k} = \mathbb{C} \) (resp. \( \mathbb{k} = \mathbb{R} \)), the least of such possible numbers \( r \) is called the Waring rank of \( F \) (resp. the real Waring rank of \( F \)) and we denote it by \( \text{rk}(F) \) (resp. \( \text{rk}_R(F) \)).

The Waring problem is more interesting (and challenging) for coefficient in number fields; see [Reznick 2013]. Because of the direct connection with the real world, there is also a lot of interest in the real Waring rank; see for instance [Boij et al. 2011]. Except for Proposition 3.6 and Examples 3.7, 3.8, and 3.10, we will consider \( \mathbb{k} = \mathbb{C} \).

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J. Alexander and A. Hirschowitz [1995] found the Waring rank of a generic form in any number of variables, which was a longstanding open problem for more than a hundred years. However, the Waring rank for generic forms does not provide information for the Waring rank of a specific form.

There has been an intense research on the Waring rank of binary forms which goes back to the work of J. J. Sylvester. Sylvester [1851a; 1851b] gave an explicit algorithm to compute the Waring rank of a binary form. But in practice, it is unfeasible for real applications. We refer to [Reznick 2013] for an excellent survey on the Waring problem for binary forms.

As an immediate consequence of Sylvester algorithm one can give an explicit formula for the Waring rank of monomials in $\mathbb{C}[x, y]$. Moreover, E. Carlini, M. V. Catalisano and A. V. Geramita recently gave an explicit formula for the Waring rank of monomials in any number of variables; see [Carlini et al. 2012, Proposition 3.1]. This motives us to look beyond monomials for binary forms.

The main result of this paper is an explicit formula for the Waring rank of binomials in $\mathbb{C}[x, y]$ (see Theorem 3.1). In general, it is difficult to describe the Waring rank of $F_1 + \cdots + F_k$ in terms of the Waring ranks of $F_1, \ldots, F_k$ as is evidenced by Strassen’s conjecture; see [Carlini et al. 2012; 2017; 2018; Teitler 2015]. In particular, our formula for binomials is far from being a trivial generalization of the monomial case.

Our main technique to compute the Waring rank of a binomial form $F \in \mathbb{C}[x, y]$ is Sylvester algorithm; see Section 2A. In order to apply this algorithm, we need to find a form of least degree in the apolar ideal $F^\perp$, and check whether it is square-free. For this, we give a nonzero form $g_1$ in $F^\perp$ and, computing the Hilbert function of $F^\perp$ in certain degrees, we are able to conclude that $g_1$ is of least degree in $F^\perp$. Hence, we avoid to compute the entire apolar ideal $F^\perp$. The techniques we use are elementary, but to obtain the correct result is not trivial.

The paper is organized as follows: In Section 2, we fix some notation and gather some preliminary results. In Section 3, we give an explicit formula for the Waring rank of a binomial form. We also give several examples illustrating our result. We finish Section 3 by comparing the Waring and the real Waring rank of binary binomials.

2. Preliminaries

Let $S, T$ denote respectively the standard graded polynomial rings $\mathbb{C}[x, y], \mathbb{C}[X, Y]$. For $d \geq 0$, denote respectively by $S_d, T_d$ the $d$-th graded component of $S, T$.

2A. Apolarity theory and Sylvester algorithm. Consider the apolar action of $T$ on $S$, that is consider $S$ as a $T$-module by means of the differentiation action

$$\circ : T \times S \rightarrow S$$

$$(g, F) \mapsto g \circ F = g(\partial_X, \partial_Y)(F).$$
Definition 2.1. Let $F$ be a form in $S_d$. A form $g \in T_{d'}$ is called apolar to $F$ when
\[ g \circ F \in S_{d-d'} \] is the zero form. The apolar ideal to $F$, denoted as $F^\perp$, is the
homogeneous ideal in $T$ generated by all the forms apolar to $F$, or equivalently
\[ F^\perp = \{ g \in T \mid g \circ F = 0 \} \subseteq T. \]

The so-called Sylvester algorithm below, an algorithm to compute Waring rank of
binary forms, is a consequence of Sylvester’s theorem developed in [Sylvester 1851a; 1851b].
Modern proofs of Sylvester’s theorem may be found in [Reznick 2013, Theorem 2.1] (which is an elementary proof), [Kung and Rota 1984, Section 5], and with further discussion in [Kung 1986; 1987; 1990; Reznick 1996]. Here we state the final version of the algorithm; see [Carlini et al. 2014, Remark 4.16], [Bernardi et al. 2011, Algorithm 2] or [Comas and Seiguer 2011, Section 3]. Recall
that given $F \in S_d$, by the structure theorem (see [Iarrobino and Kanev 1999, Theorem 1.44(iv)]), the apolar ideal $F^\perp$ to $F$ is complete intersection and it can be
generated by forms $g_1, g_2 \in T$ with $\deg(g_1) + \deg(g_2) = d + 2$.

Theorem 2.2 (Sylvester algorithm). Let $F$ be a form in $S_d$. Let $g_1, g_2 \in T$ be
generators of the apolar ideal $F^\perp$ with $\deg(g_1) \leq \deg(g_2)$. Then,
\[
\rk(F) = \begin{cases} 
\deg(g_1) & \text{if } g_1 \text{ is square-free,} \\
 d + 2 - \deg(g_1) & \text{otherwise.}
\end{cases}
\]

2B. Hilbert function. Let $I$ be a homogeneous ideal in $T$. The Hilbert function
of $T/I$ is an important numerical invariant associated to $T/I$ defined as
\[
\HF_{T/I}(d) = \dim_{\mathbb{C}} T_d - \dim_{\mathbb{C}} I_d,
\]
where $I_d$ denotes the $\mathbb{C}$-vector space of degree $d$ forms in $I$.

For $F \in S_d$, let $\langle F \rangle$ denote the $T$-submodule of $S$ generated by $F$, that is, the
$\mathbb{C}$-vector space generated by $F$ and by the corresponding derivatives of all orders.
It is well-known that $\langle F \rangle$ determines the Hilbert function of $T/F^\perp$ as follows (see
[Geramita 1996]):
\[\tag{†} \HF_{T/F^\perp}(d) = \dim_{\mathbb{C}} \langle F \rangle_d \]
where $\langle F \rangle_d$ denotes the $\mathbb{C}$-vector space generated by the forms of degree $d$ in $\langle F \rangle$.

3. Waring rank of binary binomial forms

In this section, we give an explicit formula for the Waring rank of every binary bi-
nomial form (Theorem 3.1). We also give several examples. Example 3.3 illustrates
the result itself. Example 3.4 shows that the Waring rank of a trinomial depends
on its coefficients. Examples 3.7–3.10 compare the Waring and the real Waring
rank. Example 3.9 also shows that, in fact, Theorem 3.1 can be used to compute
the Waring rank of every form in the $GL(\mathbb{C}, S_d)$-orbit of the set of binary binomial forms.

Recall that, given a nonzero homogeneous ideal $I$ of $T$, the *initial degree* of $I$ is the least integer $i > 0$ for which $I_i$ is not zero.

Observe that every binary binomial form can be expressed as $F = x^r y^s (ay^\alpha + bx^\alpha)$, with $r, s \geq 0$ and $\alpha \geq 1$. Moreover, since the Waring rank of $F$ is invariant by a linear change of coordinates, $\text{rk}(F)$ does not depend on the coefficients $a$ and $b$. So, the Waring rank of $F$ is determined by the values of $r, s$ and $\alpha$, as Theorem 3.1 below shows.

**Theorem 3.1.** Let $F = ax^r y^s (ay^\alpha + bx^\alpha) \in S_d$ be a binomial form, with $ab \neq 0$, $0 \leq r \leq s$ and $1 \leq \alpha$. Let $q$, $j$ be the unique nonnegative integers such that $r = q \alpha + j$ with $0 \leq j < \alpha$ and set $\delta = r + \alpha - s$. Then the Waring rank of $F$ can be computed using Table 1.

In order to prove Theorem 3.1, we use Sylvester’s algorithm. This requires finding a form $g_1$ of least degree in $F^\perp$ and checking whether it is square-free. We take advantage of the Hilbert function of $T/F^\perp$ to conclude that the $g_1$ in $F^\perp$ that we find is indeed a form of least degree in $F^\perp$. The computation of this form $g_1$ and whether it is square-free depends on $\delta$ and $j$. For this reason, we split the proof in five different cases: Case (1) for $\delta \leq 0$, and Cases (2.i)–(2.iv) for $\delta > 0$: (2.i) $0 \leq j < \lceil(\delta - 1)/2\rceil$; (2.ii) $\delta$ is odd and $j = (\delta - 1)/2$; (2.iii) $\lceil(\delta - 1)/2\rceil < j < \delta - 1$, or $\delta$ is even and $j = (\delta)/2$; and (2.iv) $\delta - 1 \leq j$. For each case, we find $g_1$, which will be square-free or not depending on additional conditions for the exponents of $F$. This is also a reason for getting five different cases in the table of the theorem. Notice that case $\delta \leq 0$ in the table is covered in Case (1) and $\delta > 0$ is covered in Case (2): the first row of $\delta > 0$ in the table is in Case (2.i), the second and third rows of $\delta > 0$ in the table are in Case (2.iv) and the last row of the table is covered in Cases (2.i)–(2.iv).

**Proof of Theorem 3.1.** Since $\text{rk}(F)$ is invariant by a linear change of coordinates, for the sake of simplicity we assume that $a = ((r + \alpha)!)/(r!(s + \alpha))!$ and $b = 1$.
For every pair of integers \( m, n \), we set \( \left[ \frac{m}{n} \right] \) equal to \( \frac{m!}{(m-n)!} \) if \( m \geq n \geq 0 \) and equal to zero otherwise.

**Case (1)** Suppose \( \delta \leq 0 \), that is \( s \geq r + \alpha \). Clearly, \( g_1 = X^{r+\alpha+1} \in F_\perp \). We claim that the initial degree of \( F_\perp \) is \( r + \alpha + 1 \). For \( 0 \leq i \leq s \), we have

\[
X^i Y^{s-i} \circ F = \begin{cases} 
\alpha \cdot \left[ \begin{array}{c} r \\ i \\ s-i \end{array} \right] \cdot x^{r-i} y^{\alpha+i} + \left[ \begin{array}{c} r+\alpha \\ i \\ s-i \end{array} \right] \cdot x^{r+\alpha-i} y^i & \text{if } 0 \leq i \leq r, \\
\left[ \begin{array}{c} r+\alpha \\ i \\ s-i \end{array} \right] \cdot x^{r+\alpha-i} y^i & \text{if } r < i \leq r + \alpha, \\
0 & \text{if } r + \alpha < i \leq s.
\end{cases}
\]

It is easy to see that the set \( \{ X^i Y^{s-i} \circ F : 0 \leq i \leq r + \alpha \} \subseteq \span{F}_{r+\alpha} \) is \( \mathbb{C} \)-linearly independent. Therefore by (†),

\[
\HF_{T/F_\perp}(r + \alpha) = \dim_\mathbb{C} \span{F}_{r+\alpha} = r + \alpha + 1.
\]

This implies that the nonzero homogeneous elements of \( F_\perp \) have degree at least \( r + \alpha + 1 \). Since \( g_1 \in F_\perp \), the initial degree of \( F_\perp \) is \( r + \alpha + 1 \).

Therefore \( g_1 \) is part of a minimal generating set of \( F_\perp \) and hence there exists \( g_2 \in T_{s+1} \) such that \( F_\perp = \langle g_1, g_2 \rangle \). As \( s \geq r + \alpha \) and \( g_1 \) is not square free, by Sylvester’s algorithm

\[
\rk(F) = s + 1.
\]

**Case (2)** Assume \( \delta > 0 \), that is \( s < r + \alpha \). We split this case in four cases.

**Case (2.i)** \( 0 \leq j \leq \lceil (\delta - 1)/2 \rceil - 1 \).

First we show that the initial degree of \( F_\perp \) is \( s + j + 2 \). For this it suffices to show that \( \HF_{T/F_\perp}(s + j + 1) = s + j + 2 \) and that there exists a nonzero form of degree \( s + j + 2 \) in \( F_\perp \).

First assume that \( s < \alpha - j - 2 \). Then \( r \leq s \), implies that \( r < \alpha \), and hence \( r = j \).

Therefore

\[
X^{\alpha-1-i} y^i \circ F = \begin{cases} 
\left[ \begin{array}{c} r+\alpha \\ \alpha-1-i \\ i \\ s-i \end{array} \right] \cdot x^{r+i+1} y^{s-i} & \text{if } 0 \leq i \leq s, \\
\left[ \begin{array}{c} r \\ \alpha-1-i \\ i \\ s-i \end{array} \right] \cdot x^{r+1+i-\alpha} y^{s+i-\alpha-i} & \text{if } \alpha - j - 2 < i \leq \alpha - 1.
\end{cases}
\]

Hence \( \span{F} \) contains all the monomials of degree \( s + j + 1 \). Therefore by (†),

\[
\HF_{T/F_\perp}(s + j + 1) = s + j + 2.
\]

Now let \( s \geq \alpha - j - 2 \). We have

(1) \( X^{r+\alpha-j-1-i} Y^i \circ F \)

\[
= \begin{cases} 
\left[ \begin{array}{c} r+\alpha \\ r+\alpha-j-1-i \\ i \\ j+s-i \end{array} \right] \cdot x^{j+i+1} y^{s-i} & \text{if } 0 \leq i \leq \alpha - j - 2, \\
\left[ \begin{array}{c} r \\ r+\alpha-j-1-i \\ i \\ j+s-i \end{array} \right] \cdot a \cdot x^{j+1+i-\alpha} y^{s+i-\alpha-i} + \left[ \begin{array}{c} r+\alpha \\ r+\alpha-j-1-i \\ i \\ j+s-i \end{array} \right] \cdot x^{j+1+i} y^{s-i} & \text{if } \alpha - j - 2 < i \leq s, \\
\left[ \begin{array}{c} r \\ r+\alpha-j-1-i \\ i \\ j+s-i \end{array} \right] \cdot a \cdot x^{j+1+i-\alpha} y^{s+i-\alpha-i} & \text{if } s < i \leq r + \alpha - j - 1.
\end{cases}
\]
Thus taking $i = 0, 1, \ldots, \alpha - j - 2$ in (1) we get
\[
\{x^{j+1}y^s, x^{j+2}y^{s-1}, \ldots, x^{\alpha-1}y^{s+\alpha+j+2}\} \subseteq (F)_{s+j+1}.
\]

Notice that $q\alpha - j - 2 < s$. Therefore taking $i = k\alpha, k\alpha + 1, \ldots, k\alpha + \alpha - j - 2$ for $1 \leq k \leq q - 1$ in (1), we get
\[
\{x^{k\alpha+(j+1)}y^{s-k\alpha}, x^{k\alpha+(j+2)}y^{s-1-k\alpha}, \ldots, x^{k\alpha+\alpha-1}y^{s-\alpha+j+2-k\alpha}\} \subseteq (F)_{s+j+1}
\]
for all $0 \leq k \leq q - 1$. By assumption on $j$, we have $2(j+1) \leq \delta$ and hence
\[
(q+1)\alpha - (j+1) > s.
\]

Therefore taking $i = (q+1)\alpha - (j+1), (q+1)\alpha - j, \ldots, (q+1)\alpha - 1 = r + \alpha - j + 1$ in (1), we get
\[
\{x^{q\alpha}y^{s+j+1-q\alpha}, x^{q\alpha+1}y^{s+j-q\alpha}, \ldots, x^{q\alpha+j}y^{s-q\alpha+1}\} \subseteq (F)_{s+j+1}.
\]

Since $q\alpha - 1 < s$, taking $i = k\alpha - (j+1), k\alpha - j, \ldots, k\alpha - 1$ for $k = q, q-1, \ldots, 1$ in (1), we get
\[
\{x^{k\alpha}y^{s+j+1-k\alpha}, x^{k\alpha+1}y^{s+j-k\alpha}, \ldots, x^{k\alpha+j}y^{s-k\alpha+1}\} \subseteq (F)_{s+j+1}
\]
for all $0 \leq k \leq q$. From equations (2) and (3) we conclude that
\[
\{x^iy^{s+j+1-i} : 0 \leq i \leq r\} \subseteq (F)_{s+j+1}.
\]

Hence taking $i = r - j, r - j + 1, \ldots, s - 1, s$ in (1) we get
\[
\{x^{r+1}y^{s-r+j}, x^{r+2}y^{s-r+j-1}, \ldots, x^{s+j}y, x^{s+j+1}\} \subseteq (F)_{s+j+1}.
\]

Therefore we conclude that all the monomials of degree $s + j + 1$ belong to $(F)_{s+j+1}$.

Therefore by (\dagger),
\[
HF_{T/F^\perp}(s+j+1) = s+j+2.
\]

Hence the nonzero homogeneous elements of $F^\perp$ have degree at least $s+j+2$.

We claim that
\[
g_1 = \sum_{i=0}^{q} (-1)^i X^{r+1-i\alpha} Y^{i\alpha+s-r+j+1} \in (F^\perp)_{s+j+2}.
\]
If \( q = 0 \), then \( r = j \). Therefore \( g_1 = X^{r+1}Y^{s+1} \) which clearly belongs to \( F^\perp \). Hence assume that \( q > 0 \). Then

\[
g_1 \circ F = 0 + \left[ \frac{r + \alpha}{r + 1} \right] \cdot \left[ \frac{s}{s - r + j + 1} \right] \cdot x^{\alpha-1} y^{r-j-1} + \sum_{i=1}^{q-1} (-1)^i \left[ a \cdot \left[ \frac{r}{r + 1 - i\alpha} \right] \cdot \left[ \frac{s + \alpha}{i\alpha + s - r + j + 1} \right] \cdot x^{i\alpha-1} y^{(1-i)\alpha+r-j-1} + \right. \\
\left. \left[ \frac{r + \alpha}{r + 1 - i\alpha} \right] \cdot \left[ \frac{s}{i\alpha + s - r + j + 1} \right] \cdot x^{(i+1)\alpha-1} y^{-i\alpha+r-j-1} \right. \\
\left. + (-1)^q a \cdot \left[ \frac{r}{r + 1 - q\alpha} \right] \cdot \left[ \frac{s + \alpha}{q\alpha + s - r + j + 1} \right] \cdot x^{q\alpha-1} y^{(1-q)\alpha+r-j-1} + 0. \right.
\]

Observe that, since \( a = ((r + \alpha)!s!)/(r!(s + \alpha))! \), for \( i = 1, \ldots, q \),

\[
a \cdot \left[ \frac{r}{r + 1 - i\alpha} \right] \cdot \left[ \frac{s + \alpha}{i\alpha + s - r + j + 1} \right] = \left[ \frac{r + \alpha}{r + 1 - (i - 1)\alpha} \right] \cdot \left[ \frac{s}{(i - 1)\alpha + s - r + j + 1} \right].
\]

Hence, the previous sum is telescopic and \( g_1 \circ F = 0 \). Thus the initial degree of \( F^\perp \) is \( s + j + 2 \). Therefore

\[
F^\perp = (g_1, g_2)
\]

for some \( 0 \neq g_2 \in T_{r+\alpha-j} \). Moreover,

\[
g_1 = \begin{cases} 
Y^2 \left( \sum_{i=0}^{q} (-1)^i X^{r+1-i\alpha} Y^{i\alpha+s-r+j-1} \right) & \text{if } j \geq 1 \text{ or } s - r > 0, \\
Y \left( \sum_{i=0}^{q} (-1)^i X^{r+1-i\alpha} Y^{i\alpha} \right) & \text{if } j = 0 \text{ and } r = s.
\end{cases}
\]

Suppose that \( j = 0 \) and \( r = s \). Then

\[
\left( \sum_{i=0}^{q} (-1)^i X^{r+1-i\alpha} Y^{i\alpha} \right) (X^\alpha + Y^\alpha) = X^{r+\alpha+1} + (-1)^q XY^{r+\alpha}.
\]

Since \( X^{r+\alpha+1} + (-1)^q XY^{r+\alpha} \) is square-free, \( g_1 \) is square-free if \( j = 0 \) and \( r = s \). Clearly, if \( j \geq 1 \), or \( s - r > 0 \), then \( g_1 \) is not square-free. Therefore by Sylvester algorithm

\[
\text{rk}(F) = \begin{cases} 
\rho + \alpha - j & \text{if } j \geq 1 \text{ or } s - r > 0, \\
\rho + 2 & \text{if } j = 0 \text{ and } r = s.
\end{cases}
\]
Case (2.ii) $\delta$ is odd and $j = (\delta - 1)/2$.

First we prove that the initial degree of $F^\perp$ is $s + j + 1$. For $0 \leq i \leq r + \alpha - j$, we have

\[
X^{r+\alpha-j-i} Y^i \circ F
\]

\[
= \begin{cases} 
[r+\alpha]_{i} \cdot s \cdot x^{i+j} y^{s-i} & \text{if } 0 \leq i \leq \alpha - j - 1, \\
[r+\alpha-j-i] \cdot [s+\alpha]_{i} \cdot a \cdot x^{i+j} y^{s+i} & \text{if } \alpha - j - 1 < i \leq s, \\
[r+\alpha-j-i] \cdot [s+\alpha]_{i} \cdot a \cdot x^{i+j} y^{s+i} & \text{if } s < i \leq r + \alpha - j.
\end{cases}
\]

Substituting $i = 1, \ldots, \alpha - j - 1$ in (4), we get

\[
\{x^{j+1} y^{s-1}, x^{j+2} y^{s-2}, \ldots, x^{\alpha-1} y^{s+\alpha-j+1}\} \subseteq \langle F \rangle_{s+j}.
\]

As $j + 1 = \delta - j = r + \alpha - s - j$, we have $j + s + 1 - \alpha = r - j = q \alpha$. Therefore taking $i = k \alpha + 1, k \alpha + 2, \ldots, k \alpha + \alpha - j - 1$ for $1 \leq k \leq q - 1$ in (4), we get

\[
\{x^{k \alpha+j+1} y^{s-1-k \alpha}, x^{k \alpha+j+2} y^{s-k \alpha-2}, \ldots, x^{k \alpha+\alpha-1} y^{s-(k+1)\alpha+j+1}\} \subseteq \langle F \rangle_{s+j}
\]

for all $0 \leq k \leq q - 1$. Taking $i = s + 1, s + 2, \ldots, r + \alpha - j$ in (4) we get

\[
\{x^{q \alpha} y^{s-q \alpha+j}, x^{q \alpha+1} y^{s-q \alpha+j-1}, \ldots, x^{q \alpha+j} y^{s-q \alpha}\} \subseteq \langle F \rangle_{s+j}.
\]

Since $q \alpha \leq s$, substituting $i = k \alpha + \alpha - j, k \alpha + \alpha - j + 1, \ldots, k \alpha + \alpha$ for $k = q - 1, q - 2, \ldots, 1, 0$ in (4) we get

\[
\{x^{k \alpha} y^{s-k \alpha+j}, x^{k \alpha+1} y^{s-k \alpha+j-1}, \ldots, x^{k \alpha+j} y^{s-k \alpha}\} \subseteq \langle F \rangle_{s+j}.
\]

for all $0 \leq k \leq q$. From equations (5) and (6) we conclude that

\[
\{x^i y^{s+j-i} : 0 \leq i \leq r\} \subseteq \langle F \rangle_{s+j}.
\]

Hence taking $i = r - j + 1, r - j + 2, \ldots, s - 1, s$ in (4) we get

\[
\{x^{r+1} y^{s-r+j-1}, x^{r+2} y^{s-r+j-2}, \ldots, x^{s+j-1} y, x^{s+j}\} \subseteq \langle F \rangle_{s+j}.
\]

Therefore we conclude that all the monomials of degree $s + j$ are in $\langle F \rangle_{s+j}$. Hence by (†), $HF_{F^\perp} (s + j) = s + j + 1$. Therefore the nonzero homogeneous elements of $F^\perp$ have degree at least $s + j + 1$.

We claim that

\[
g_1 = \sum_{i=0}^{a+1} (-1)^i X^{s+j+1-i\alpha} Y^{i\alpha} \in (F^\perp)_{s+j+1}.
\]
Indeed,

\[
g_1 \circ F = 0 + \sum_{i=1}^{q} (-1)^i \left( a \cdot \left[ \begin{array}{c} r \\ s + j + 1 - i \alpha \end{array} \right] \cdot \left[ \begin{array}{c} r + \alpha \\ i \alpha \end{array} \right] \cdot x^{r+i \alpha-s-j-1} y^s \right) + (-1)^{q+1} \alpha \cdot \left[ \begin{array}{c} r \\ 0 \\ (q+1) \alpha \end{array} \right] \cdot x^r y^s - q + 1 = 0.
\]

Observe that, since \( a = ((r + \alpha)!s!)/(r!(s + \alpha)!), \) for all \( i = 1, \ldots, q \)

\[
a \cdot \left[ \begin{array}{c} r \\ s + j + 1 - i \alpha \end{array} \right] \cdot \left[ \begin{array}{c} s + \alpha \\ i \alpha \end{array} \right] = \left[ \begin{array}{c} r + \alpha \\ s + j + 1 - (i-1) \alpha \end{array} \right] \cdot \left[ \begin{array}{c} s \\ (i-1) \alpha \end{array} \right].
\]

Hence, the previous sum is telescopic and \( g_1 \circ F = 0. \) Therefore there exists \( 0 \neq g_2 \in T_{r+\alpha-j+1} \) such that \( F^\perp = (g_1, g_2). \) As

\[
g_1(X^\alpha + Y^\alpha) = X^{s+j+1+\alpha} + (-1)^{q+1} Y^{s+j+1+\alpha},
\]

and \( X^{s+j+1+\alpha} + (-1)^{q+1} Y^{s+j+1+\alpha} \) is square-free, \( g_1 \) is also square-free. Hence by Sylvester algorithm \( \text{rk}(F) = s + j + 1 = r + \alpha - j. \)

**Case (2.iii) [\((\delta-1)/2]\lt j \lt \delta-1 \text{ or } \delta \text{ is even and } j = \delta/2.**

Let \( k = \delta - j. \) We show that \( \text{rk}(F) = r + \alpha - j = s + k. \) For this it suffices to show that \( \text{HF}_{T/F^\perp}(s + k - 1) = s + k \) and that there exists a square-free polynomial of degree \( s + k \) in \( F^\perp. \) For \( 0 \leq i \leq r + \alpha - k + 1, \) we have

\[
X^{r+\alpha-k+1-i} Y^i \circ F
\]

\[
= \begin{cases} 
X^{k-1+i} \cdot x^{k-1+i} y^{s-i} & \text{if } 0 \leq i \leq \alpha-k, \\
X^r \cdot a \cdot x^{k-1+i} \cdot y^{s+i} & \text{if } \alpha-k < i \leq s, \\
X^r \cdot a \cdot x^{k-1+i} \cdot y^{s+i} & \text{if } s < i \leq r+\alpha-k+1.
\end{cases}
\]

Substituting \( i = 0, 1, \ldots, \alpha - k \) in (7) we get

\[
\{ x^{k-1} y^s, x^k y^{s-1}, \ldots, x^{\alpha-1} y^{s-\alpha+k} \} \subseteq \langle F \rangle_{s+k-1}.
\]

Notice that \( r = q \alpha + j = q \alpha + (\delta - k). \) This implies that \( (q+1)\alpha - k = s. \) Hence taking \( i = m \alpha, m \alpha + 1, \ldots, m \alpha + \alpha - k \) for \( m = 1, 2, \ldots, q \) in (7) we get

\[
\{ x^{m \alpha+(k-1)} y^{s-m \alpha}, x^{m \alpha+k} y^{s-1-m \alpha}, \ldots, x^{m \alpha+\alpha-1} y^{s-\alpha+k-m \alpha} \} \subseteq \langle F \rangle_{s+k-1}
\]
for all $0 \leq m \leq q$. Now taking $i = (q + 1)\alpha - k + 1, (q + 1)\alpha - k + 2, \ldots, (q + 1)\alpha - 1$ in (7) we get

$$\{x^{q\alpha} y^{s-q\alpha+k-1}, x^{q\alpha+1} y^{s-q\alpha+k-2}, \ldots, x^{q\alpha+k-2} y^{s-q\alpha+1}\} \subseteq \langle F \rangle_{s+k-1}.$$

Now substituting $i = m\alpha - k + 1, m\alpha - k + 2, \ldots, m\alpha - 1$ for $m = q, q - 1, \ldots, 1$ in (7) we get

$$(9) \quad \{x^{m\alpha} y^{s-m\alpha+k-1}, x^{m\alpha+1} y^{s-m\alpha+k-2}, \ldots, x^{m\alpha+k-2} y^{s-m\alpha-1}\} \subseteq \langle F \rangle_{s+k-1}$$

for all $0 \leq m \leq q$. From equations (8) and (9) we conclude that

$$\{x^{m\alpha} y^{s+k-1-m\alpha}, x^{m\alpha+1} y^{s+k-2-m\alpha}, \ldots, x^{m\alpha+\alpha-1} y^{s+k-m\alpha-\alpha} : 0 \leq m \leq q\} \subseteq \langle F \rangle_{s+k-1}.$$

Notice that $(q + 1)\alpha - 1 = s + k - 1$. Therefore all the monomials of degree $s + k - 1$ are in $\langle F \rangle$, and thus $\langle F \rangle_{s+k-1} = S_{s+k-1}$. Hence by (†) HF$_{T/F\perp}(s + k - 1) = s + k$.

Next we claim that

$$g_1 = \sum_{i=0}^{q+1} (-1)^i X^{s+k-i\alpha} Y^{i\alpha} \in (F\perp)_{s+k}.$$

We have

$$g_1 \circ F = 0 + \left[ \begin{array}{c} r + \alpha \\ s + k \\ 0 \end{array} \right] \cdot \left[ \begin{array}{c} s \\ 0 \end{array} \right] \cdot x^{r+\alpha-s-k} y^s$$

$$+ \sum_{i=1}^{q} (-1)^i \left( a \cdot \left[ \begin{array}{c} r \\ s + k - i\alpha \end{array} \right] \cdot \left[ \begin{array}{c} s + \alpha \\ i\alpha \end{array} \right] \cdot x^{r-s-k+i\alpha} y^{s+(1-i)\alpha} + \left[ \begin{array}{c} r + \alpha \\ s + k - i\alpha \\ i\alpha \end{array} \right] \cdot \left[ \begin{array}{c} s \\ 0 \\ (q + 1)\alpha \end{array} \right] \cdot x^r y^{s-q\alpha} + 0. \right)$$

Observe that, since $a = ((r + \alpha)! s!)/((s + \alpha)!)$, for all $i = 1, \ldots, q$

$$a \cdot \left[ \begin{array}{c} r \\ s + k - i\alpha \end{array} \right] \cdot \left[ \begin{array}{c} s + \alpha \\ i\alpha \end{array} \right] = \left[ \begin{array}{c} r + \alpha \\ s + k - (i - 1)\alpha \end{array} \right] \cdot \left[ \begin{array}{c} s \\ (i - 1)\alpha \end{array} \right].$$

Hence, the previous sum is telescopic and $g_1 \circ F = 0$. Also

$$g_1(X^\alpha + Y^\alpha) = X^{s+k+\alpha} + (-1)^{q+1} Y^{s+k+\alpha}.$$

As $X^{s+k+\alpha} + (-1)^{q+1} Y^{s+k+\alpha}$ is square-free, $g_1$ is also square-free. Therefore by Sylvester algorithm $\text{rk}(F) = s + k = s + \delta - j = r + \alpha - j.$
Case (2.iv) \( \delta - 1 \leq j \leq \alpha - 1 \).

First we show that the initial degree of \( F^\perp \) is \( s + 1 \). For this it suffices to show that \( \text{HF}_{T/F^\perp}(s) = s + 1 \) and that there exists a nonzero form of degree \( s + 1 \) in \( F^\perp \). For \( 0 \leq i \leq r + \alpha \), we have

\[
X^{r+\alpha-i}Y^i \circ F
\]

\[
\begin{cases}
[r+r+\alpha-i].\begin{bmatrix} r+\alpha \\ r \end{bmatrix} \cdot \begin{bmatrix} s+\alpha \\ i \end{bmatrix} \cdot x^iy^{s-i} & \text{if } 0 \leq i < \alpha, \\
[r+r+\alpha-i].\begin{bmatrix} r+\alpha \\ r \end{bmatrix} \cdot \begin{bmatrix} s+\alpha \\ i \end{bmatrix} \cdot a \cdot x^{i-\alpha}y^{s+\alpha-i} + [r+r+\alpha-i].\begin{bmatrix} s \\ i \end{bmatrix} \cdot x^iy^{s-i} & \text{if } \alpha \leq i \leq s, \\
[r+r+\alpha-i].\begin{bmatrix} r+\alpha \\ r \end{bmatrix} \cdot \begin{bmatrix} s+\alpha \\ i \end{bmatrix} \cdot a \cdot x^{i-\alpha}y^{s+\alpha-i} & \text{if } s < i \leq r + \alpha.
\end{cases}
\]

Therefore

\[
\{x^iy^{s-i} : 0 \leq i < \alpha \}
\]

\[
\cup \left\{ \begin{bmatrix} r \\ r+\alpha-i \end{bmatrix} \cdot \begin{bmatrix} s+\alpha \\ i \end{bmatrix} \cdot a \cdot x^{i-\alpha}y^{s+\alpha-i} + [r+r+\alpha-i].\begin{bmatrix} s \\ i \end{bmatrix} \cdot x^iy^{s-i} : \alpha \leq i \leq s \right\}
\subseteq \langle F \rangle_s.
\]

This implies that \( \langle F \rangle_s \) contains all the monomials of degree \( s \). Hence by (†),

\[
\text{HF}_{T/F^\perp}(s) = s + 1.
\]

We claim that

\[
g_1 = \sum_{i=0}^{q+1} (-1)^i X^{s-(j-\delta)-i\alpha}Y^{i\alpha+(j-\delta)+1} \in (F^\perp)_{s+1}.
\]

Notice that \( (q + 1)\alpha = r - j + \alpha = s + \delta - j \). Therefore we have

\[
g_1 \circ F = 0 + \begin{bmatrix} r+\alpha \\ s-j+\delta \end{bmatrix} \cdot \begin{bmatrix} s \\ j-\delta+1 \end{bmatrix} \cdot x^{r+\alpha-s+j-\delta}y^{s-j+\delta-1}
\]

\[
+ \sum_{i=1}^{q} (-1)^i \left( a \cdot \begin{bmatrix} r \\ s-j+\delta-i\alpha \end{bmatrix} \cdot \begin{bmatrix} s+\alpha \\ i\alpha+j-\delta+1 \end{bmatrix} \right) \cdot x^{r-s+j-\delta+i\alpha}y^{s-j+\delta+(1-i)\alpha-1}
\]

\[
+ \left( a \cdot \begin{bmatrix} r \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s+\alpha \\ (q+1)\alpha+(j-\delta+1) \end{bmatrix} \right) \cdot x^{r}y^{s-q\alpha-(j-\delta)+1} + 0.
\]

Observe that, since \( a = ((r+\alpha)!s!)/(r!(s+\alpha)!) \), for all \( i = 1, \ldots, q \),

\[
a \cdot \begin{bmatrix} r \\ s-j+\delta-i\alpha \end{bmatrix} \cdot \begin{bmatrix} s+\alpha \\ i\alpha+j-\delta+1 \end{bmatrix} \]

\[
= \begin{bmatrix} r+\alpha \\ s-j+\delta-(i-1)\alpha \end{bmatrix} \cdot \begin{bmatrix} s \\ (i-1)\alpha+j-\delta+1 \end{bmatrix}.
\]
Hence, the previous sum is telescopic and \( g_1 \circ F = 0 \). Hence \( g_1 \in (F^\perp)_{s+1} \). Therefore the initial degree of \( F^\perp \) is \( s+1 \). Hence \( g_1 \) is part of a minimal generating set of \( F^\perp \). Therefore there exists \( 0 \neq g_2 \in T_{r+\alpha+1} \) such that \( F^\perp = (g_1, g_2) \). Clearly, if \( j \geq \delta + 1 \), then \( g_1 \) is not square-free. If \( \delta - 1 \leq j \leq \delta \), then

\[
g_1(X^\alpha + Y^\alpha) = \begin{cases} X^{s+\alpha+1} + (-1)^{q+1} Y^{s+\alpha+1} & \text{if } j = \delta - 1, \\ Y(X^{s+\alpha} + (-1)^{q+1} Y^{s+\alpha}) & \text{if } j = \delta, \end{cases}
\]

which implies that \( g_1 \) is square-free. Hence by Sylvester algorithm

\[
\mathrm{rk}(F) = \begin{cases} s + 1 & \text{if } \delta - 1 \leq j \leq \delta, \\ r + \alpha + 1 & \text{if } j \geq \delta + 1. \end{cases}
\]

\[\blacksquare\]

**Remark 3.2.** The generic rank of a form of degree \( r + s + \alpha \) is

\[\left\lceil \frac{r + s + \alpha + 1}{2} \right\rceil;\]

see [Alexander and Hirschowitz 1995; Carlini et al. 2014]. Theorem 3.1 illustrates that the Waring rank of a binomial form behaves as weirdly as possible compared to the generic rank. In particular, the Waring rank of a binomial form can be smaller or larger than the generic rank.

We illustrate Theorem 3.1 in the following example.

**Example 3.3.** Let the notation be as in Theorem 3.1:

(0) \( (r = s = 0 \text{ and } \alpha \geq 1) \) Let \( F = x^\alpha + y^\alpha \). Clearly,

\[
\mathrm{rk}(F) = \begin{cases} 1 & \text{if } \alpha = 1, \\ 2 & \text{if } \alpha = 2. \end{cases}
\]

On the other hand, since \( r = s = 0 \), we have \( \delta = \alpha \geq 1 \) and \( j = 0 \). Therefore \( \mathrm{rk}(F) \) coincides with the Waring rank stated in Theorem 3.1.

(1) \( (r = s \text{ and } \alpha = 1) \) Let \( F = x^r y^{r+1} + x^{r+1} y^r \). In this case \( F^\perp = (g_1, g_2) \) where

\[
g_1 = X^{r+1} - X^r Y + \cdots + (-1)^i X^{r+1-i} Y^i + (-1)^{r+1} Y^{r+1} \quad \text{and} \quad g_2 = X^{r+2}.
\]

Since \( g_1 \) is square-free, by Sylvester algorithm \( \mathrm{rk}(F) = r + 1 \).

On the other hand, since \( r = s \), we have \( \delta = r + \alpha - s = \alpha = 1 \) and hence \( j = 0 \) for every nonnegative integer \( r \). Therefore \( \mathrm{rk}(F) = r + 1 \) by Theorem 3.1 also.

We remark that, in [Reznick and Tokcan 2017, Theorem 3.1], the authors proved that if we consider \( F \in k[x, y] \) for different fields \( k \subseteq \mathbb{C} \), then \( F \) has at least three different relative Waring ranks. Their proof in fact shows that \( \mathrm{rk}(F) = r + 1 \).
(2) \((r = 0, \ s > 0 \ \text{and} \ \alpha \geq 1)\) Let \(F = ay^{s+\alpha} + x^\alpha y^s\) where \(a = (\alpha!s!)/(s+\alpha)!\). In this case \(\delta = r + \alpha - s = \alpha - s\) and \(j = 0\). Hence, by Theorem 3.1,

\[
\text{rk}(F) = \begin{cases} 
\ s + 1 & \text{if } \alpha \leq s, \\
\alpha & \text{if } \alpha > s.
\end{cases}
\]

This can be verified directly (without using Theorem 3.1) as follows: We have

\[
F^\perp = \begin{cases} 
(X^{\alpha+1}, Y^{s+1} - X^\alpha Y^{s-\alpha+1}) & \text{if } \alpha \leq s, \\
(XY^{s+1}, X^\alpha - Y^\alpha) & \text{if } \alpha > s,
\end{cases}
\]

and hence by Sylvester algorithm \(\text{rk}(F)\) is as required.

The following example illustrates that unlike the binomial case, the Waring rank of a trinomial may depend on its coefficients.

**Example 3.4.** Consider the quadratic form \(F = x^2 + xy + y^2\). Then

\[
F = [x \ y] A_F [x \ y]^T \quad \text{where} \quad \begin{pmatrix} 1 & 1/2 \\ 1 & 1 \end{pmatrix}.
\]

It is a standard fact from linear algebra that \(\text{rk}(F) = \text{rank}(A_F)\). Hence \(\text{rk}(F) = 2\). On the other hand, if \(G = x^2 + 2xy + y^2 = (x+y)^2\), then \(\text{rk}(G) = 1\). This shows that unlike the binomial case, the Waring rank of a trinomial may depend on its coefficients.

Now, we compare the Waring and the real Waring rank of a binomial form. For this purpose we recall Theorem 3.5 below due to Sylvester [1865]; for a modern proof see [Reznick 2013, Section 3].

**Theorem 3.5** (Sylvester). Let \(F\) be a form in \(\mathbb{R}[x, y]_d\) with \(d \geq 3\) which is not a \(d\)-th power of a linear form. If \(F\) splits into linear factors over \(\mathbb{R}\), then \(\text{rk}_{\mathbb{R}}(F) = d\).

First we note the following result which shows that, unlike in the complex case, there can be at most two possible values of the real rank of a binomial form depending on its coefficients.

**Proposition 3.6.** Consider a real binomial \(F = x^r y^s (ay^\alpha + bx^\alpha)\) with \(ab \neq 0\). For \(\alpha\) odd, the real Waring rank of \(F\) does not depend on the coefficients \(a, b\). For \(\alpha\) even, there are at most two different real Waring ranks for \(F\), depending on the sign of \(ab\).

**Proof.** When \(\alpha\) is odd, there are always real \(\alpha\)-th roots of \(a\) and \(b\) regardless of their sign. So, via a real linear change of coordinates, we may reduce \(F\) to \(x^r y^s (y^\alpha + x^\alpha)\), and its Waring rank does not depend on \(a, b\). Instead, when \(\alpha\) is even, we may reduce \(F\) to either \(x^r y^s (y^\alpha + x^\alpha)\), if \(ab > 0\), or \(x^r y^s (y^\alpha - x^\alpha)\), if \(ab < 0\). \(\square\)
We remark that, for every even $\alpha > 0$, there are binomials $F$ for which $x^r y^s (y^\alpha + x^\alpha)$ and $x^r y^s (y^\alpha - x^\alpha)$ have the same real rank. For instance,

$$\text{rk}_R(x^{2k} + y^{2k}) = \text{rk}_R(x^{2k} - y^{2k}) = 2$$

for every $k \geq 1$. In what follows we give explicit examples of real binomial forms with two distinct ranks depending on their coefficients (Examples 3.8, 3.9 and 3.10). First we note the following example which compares the Waring and the real Waring rank of a binomial form. Also note that, since $\alpha$ is odd in the following example, the real Waring rank of $F$ is independent of its coefficients by Proposition 3.6.

**Example 3.7.** Let $F = x^r y^r (x \pm y)$ where $r \geq 1$. By Theorem 3.1 $\text{rk}(F) = r + 1$. Whereas, since $F$ splits completely into linear factors in $\mathbb{R}$, $\text{rk}_R(F) = 2r + 1$ by Theorem 3.5.

**Example 3.8.** By Theorem 3.5 $\text{rk}_R(x^3 - x y^2) = 3$ whereas $\text{rk}_R(x^3 + x y^2) = 2$ by [Tokcan 2017, Corollary 2.3]. But $\text{rk}(x^3 \pm x y^2) = \text{rk}(y^3 \pm x^2 y) = 2$ by Theorem 3.1 (take $r = 0$, $s = 1$ and $\alpha = 2$ in Theorem 3.1).

The following example is a generalization of Example 3.8. In fact, this example shows that Theorem 3.1 covers a much larger class of binary forms than binomial, since some forms can be transformed to a binomial form under a suitable linear change of coordinates.

**Example 3.9.** Let $F = \ell(x, y)k(x^2 \pm y^2)$ where $\ell(x, y) = Ax + By$ is a linear form. To begin with, $F$ is not a binomial form in $x$ and $y$, but $F$ can be transformed to a binomial form by a linear change of coordinates. Namely, let $x_1 = Bx \mp Ay$ and $y_1 = Ax + By$. Then $F(x_1, y_1) = (1/B^2 \pm A^2) y_1^k (x_1^2 \pm y_1^2)$, which is a binomial form in $x_1$ and $y_1$. Now by Theorem 3.1 (take $r = 0$, $s = k$ and $\alpha = 2$) $\text{rk}(F) = k + 1$. On the other hand, when $A$ and $B$ are real, we have $\text{rk}_R(\ell(x, y)k(x^2 - y^2)) = k + 2$ by Theorem 3.5 and $\text{rk}_R(\ell(x, y)k(x^2 + y^2)) = k + 1$ by [Tokcan 2017, Corollary 2.3].

**Example 3.10.** Let $F = x^r y^s (x^2 - y^2)$. Then by Theorem 3.5 $\text{rk}_R(F) = r + s + 2$. On the other hand, $\text{rk}_R(x^r y^s (x^2 + y^2)) < r + s + 2$ by [Blekherman and Sinn 2016, Theorem 2.2]. But

$$\text{rk}(F) = \begin{cases} s + 2 & \text{if } r = s \text{ and } r \text{ is even,} \\ s + 1 & \text{otherwise.} \end{cases}$$

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