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A force proportional to velocity squared derived from spacetime algebra

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ABSTRACT

The underlying geometry of spacetime algebra allows one to derive a force by contracting the relativistic generalization of angular momentum, \( \mathbf{M} \), with the mass-current, \( m \mathbf{v} \), where \( \mathbf{v} \) is a proper 4-vector velocity. By applying this force to a cosmological object, a repulsive inverse distance-square law is found, which is proportional to the velocity dispersion squared of that structure. It is speculated if this finding may be relevant to the recent suggestion, that such a force may accelerate the expanding universe with no need for a cosmological constant.

Key words: acceleration of particles – cosmology: theory.

1 INTRODUCTION

At high velocities space and time are merged together into Minkowski spacetime, and even though both distances and durations depend on the observers frame, then the 4-vector \( (ct, \mathbf{x}) \) has invariant length under Lorentz transformations. Similarly, the energy–momentum 4-vector, \( (\varepsilon/c, \mathbf{p}) \), is a proper 4-vector in Minkowski space. The classical conservation laws, like energy and momentum conservation arising from symmetries in time and space, thus have related conservation laws in relativistic physics.

However, for other objects, such as the (polar vector) dynamic mass moment, \( \mathbf{L} = \mathbf{p} \times \mathbf{v} \), the corresponding relativistic conservation laws are often not discussed in detail. One reason for this is that there is no way of combining \( \mathbf{p} \) and \( \mathbf{L} \) into a proper 4-vector. Instead, one must combine \( \mathbf{p} \) and \( \mathbf{L} \) into an antisymmetric rank-2 tensor, \( M^{\mu \nu} \) (Landau & Lifshitz 1975).

A very similar issue is well-known from electromagnetism, where the (polar vector) electric field, \( \mathbf{E} \), and the (axial vector) magnetic field, \( \mathbf{B} \), also combine into an antisymmetric rank-2 tensor, \( F^{\mu \nu} \). When learning about electromagnetism this is frustrating to some, since most of our intuition is based on the fields \( \mathbf{E} \) and \( \mathbf{B} \), however, when performing a Lorentz transformation, one most often finds oneself performing the calculations with the physically somewhat less transparent tensor \( F^{\mu \nu} \).

While contemplating the physical meaning of the mathematical space where \( F^{\mu \nu} \) and \( M^{\mu \nu} \) live, one is naturally drawn to spacetime algebra (STA; Hestenes 1966, 2003), which provides a geometric explanation for the connection between tensors like \( F^{\mu \nu} \) and \( M^{\mu \nu} \) and Minkowski space.

In the sufficiently mature scientific field of STA, it is well-known how the electromagnetic bi-vector field, \( \mathbf{F} \), naturally is used to derive both Maxwell’s equations and the Lorentz force. Given the strong mathematical similarities between the ‘electromagnetic’ bi-vector \( \mathbf{F} \) and the ‘mechanics’ bi-vector \( \mathbf{M} \), it appears natural to derive the equations and forces which are dictated by \( \mathbf{M} \), in particular since the structure of STA uniquely defines these equations and forces.

Below, the same methods are applied to the bi-vector field, \( \mathbf{M} \), as have previously been applied to \( \mathbf{F} \) in electromagnetism, and it is shown how new forces appear naturally from the bi-vector field \( \mathbf{M} \). One of these inverse distance-square force-terms depends on the internal velocity dispersion of an object (which for instance could be a distant galaxy). It is speculated to which degree this new force possibly may be related to the force which was recently suggested as an explanation for the observed acceleration of the universe (Loeve, Nielsen & Hansen 2021).

2 SPACETIME ALGEBRA

STA starts with Minkowski space, \( \mathcal{M}_{1,3} \), with the metric signature \( (+, -, -, -) \), and a chosen basis \( \{ \gamma_{\mu} \}_{\mu=0}^{3} \) of \( \mathcal{M}_{1,3} \). These 4 orthonormal vectors are the basis for 1-blades. The 2-blade elements are the six antisymmetric products \( \gamma_{\mu \nu} \equiv \gamma_{\mu} \gamma_{\nu} \). The product is here given by the sum of the dot and wedge product: \( ab = a \cdot b + a \wedge b \) (Doran & Lasenby 2007; Hestenes 2015). The wedge operator, \( \wedge \), is the 4D generalization of the 3D cross-product. Continuing over 3-blades, \( \gamma_{\mu \nu \lambda} \), one finally reaches the highest grade, the pseudo-scalar \( I \equiv \gamma_{0} \gamma_{1} \gamma_{2} \gamma_{3} \), which represents the unit 4-volume in any basis. Interestingly, one has \( I = -1 \).

For the discussion below, the bi-vectors are important: these are oriented plane segments, and examples include the electromagnetic field \( \mathbf{F} = \mathbf{E} + \mathbf{B} I \), and the angular momentum \( \mathbf{M} = x \wedge p \), where \( x \) and \( p \) are proper 4-vectors (Doran & Lasenby 2007; Hestenes 2015). From a notational point of view, vector-arrows are used above spatial 3-vectors like \( \mathbf{E} \) or \( \mathbf{p} \), no-vector-arrows are used for proper 4-vectors like \( x \) and \( w \), and boldface is used for bi-vectors like \( \mathbf{F} \) and \( \mathbf{M} \).

3 ELECTROMAGNETISM

The case of electromagnetism in STA is well described in the literature (Dressel et al. 2015; Hestenes 2015), and serves as a starting point here. The 4 Maxwell’s equations can be written

\[
\nabla \mathbf{F} = j,
\]

(1)
where the complex current may contain both electric (vector) and magnetic (trivector) parts, \( j_\alpha + j_\omega I \). The bi-vector is given by \( \mathbf{F} = \tilde{E} + \tilde{B} I \), and the derivative \( \nabla \mathbf{F} = \nabla \cdot \mathbf{F} + \nabla \times \mathbf{F} \) produces both a vector and a trivector field.

Using the time-direction, \( \gamma_0 \), one can decompose the derivative along a direction parallel to and perpendicular to \( \gamma_0 \), \( \nabla = \left( \delta_0 - \mathbf{V} \right) \gamma_0 \), where \( \mathbf{V} \) is the frame-dependent relative 3-vector derivative. It is now straightforward to expand equation (1) to the 4 Maxwell’s equation (Dressel et al. 2015; Hestenes 2015).

It is important to stress, that equation (1) is not only a matter of compact notation, it is indeed the only logical extension beyond the most trivial equation in STA, \( \nabla \mathbf{F} = 0 \). The only thing missing is to connect the bi-vector field to observables: this is done through observations, which also establish the units of \( \tilde{E} \) and \( \tilde{B} \).

### 3.1 The Lorentz force

The classical Lorentz force is given by

\[
\frac{d\mathbf{p}}{dt} = q \left( c \tilde{E} + \tilde{v} \times \tilde{B} \right),
\]

where effectively arose as a clever guess to explain observations. The cross, \( \times \), refers to the normal 3D cross-product. In the standard Euler–Lagrange formalism, the Lorentz force appears when adding a term to the Lagrangian, \( q \mathbf{v} \rho A^\mu \), where \( A \) is the 4-vector potential.

In STA, the Lorentz force appears when one contracts the bi-vector field \( \mathbf{F} \) with a proper 4-vector velocity \( \mathbf{w} \). Using that the 4-vector \( \mathbf{w} \) is connected with the para-vector, \( \mathbf{w}_0 + \mathbf{w} \), via \( \gamma_0 \) (Doran & Lasenby 2007), namely \( \mathbf{w} = (\mathbf{w}_0 + \mathbf{w}) \), one gets

\[
(\mathbf{F} \cdot \mathbf{w}) \frac{d\gamma_0}{d\tau} = q \tilde{E} \cdot \tilde{v} + q \left( c \tilde{E} + \tilde{v} \times \tilde{B} \right),
\]

where the first term on the r.h.s. is the rate of work, \( ds/d(\tau) \), we use \( \tilde{w} = \gamma \tilde{v} \), and the last parenthesis on the r.h.s is exactly the Lorentz force in equation (2).

If the current was complex there could be another force term allowed (Dressel et al. 2015), namely

\[
\mathbf{F} \cdot (\mathbf{w} I) \gamma_0 = \left( (\mathbf{F} \wedge \mathbf{w}) I \right) \gamma_0 = \tilde{B} \cdot \tilde{v} + \left( c \tilde{B} - \tilde{v} \times \tilde{E} \right).
\]

To summarize, the full 4 Maxwell’s equations appear naturally from the geometric structure of STA, through the equation \( \nabla \mathbf{F} = J \). The Lorentz force also appears naturally in STA when contracting the bi-vector field, \( \mathbf{F} \) with the 4-vector current, \( dp/d\tau = \mathbf{F} \cdot (\mathbf{wo}) \), where \( p \) is the proper energy–momentum 4-vector.

### 4 Angular Momentum

In order to generalize the 3D angular momentum, \( \tilde{L} = \tilde{x} \times \tilde{p} \), one uses the proper 4D \( x = (ct + \tilde{x}) \gamma_0 \) and \( p = (\epsilon/c + \tilde{p}) \gamma_0 \), to create the bi-vector \( \mathbf{M} \) (Landau & Lifshitz 1975; Dressel et al. 2015)

\[
\mathbf{M} = x \wedge p = \frac{\epsilon \tilde{x}}{c} - ct \tilde{p} - \tilde{x} \times \tilde{p} I = -\tilde{N} - \tilde{L} I.
\]

Only \( \mathbf{M} \) is a proper geometric object, and the split into dynamic mass moment and angular momentum requires that one specifies \( \gamma_0 \), in exactly the same way that \( \mathbf{F} \) is the proper geometric object of electromagnetism, and the separation into \( \tilde{E} \) and \( \tilde{B} \) fields requires specification of a frame by the choice of \( \gamma_0 \).

It is now clear how everything can be repeated from the case of electromagnetism: where one had a bi-vector \( \mathbf{F} \) and relative 3-vectors \( \tilde{E} \) (polar) and \( \tilde{B} \) (axial), then one now has a bi-vector \( \mathbf{M} \) and relative 3-vectors \( -\tilde{N} \) (polar) and \( -\tilde{L} \) (axial). The signs could have been defined away, but are kept to agree with the standard notation in the literature (Landau & Lifshitz 1975). When deriving the Lorentz force for electromagnetism, by dotting the bi-vector field \( \mathbf{F} \) with a charge-current, \( q \omega_0 \), one needs experimental data to get the units right (\( \epsilon_0 \) and \( \mu_0 \) for the E- and B-fields, respectively) (Hestenes 2015).

In a similar fashion, experimental data are needed to get the units for a force defined by dotting the field \( \mathbf{M} \) with a ‘mass-current’, \( m \omega_0 \).

The simplest possible equation describing the evolution of the bi-vector field is specified by the structure of STA, namely

\[
\nabla \mathbf{M} = j_\mathbf{M}.
\]

This letter is not focusing on the details of the source on the r.h.s. (which could be zero), however, for the sake of generality, it is allowed to contain both a vector and a trivector term \( j_\mathbf{M} = j_1 + j_3 I \). The resulting equations split into two equations for the relative scalars

\[
-\mathbf{V} \cdot \tilde{N} = \rho_1,
\]

\[
-\mathbf{V} \cdot \tilde{L} = \rho_3,
\]

(where \( \rho_1 \) refer to the 0-component of the sources) and two equations for the relative 3-vectors, just like Maxwell’s equations did.

\[
\partial_0 \tilde{N} + \mathbf{V} \times \tilde{L} = -\tilde{J}_1,
\]

\[
\partial_0 \tilde{L} + \mathbf{V} \times \tilde{N} = \tilde{J}_3,
\]

where \( \tilde{J}_1 \) refer to the three spatial components of the sources. The details of these four equations will be discussed elsewhere.

Instead, the force which appears from the contraction with a mass-current, \( m \omega_0 \), where \( \omega \) again is a proper 4-velocity, and \( m \) is the inertia of the test particle, will now be calculated. From the term \( \mathbf{M} \cdot \mathbf{w} \) one gets

\[
(\mathbf{M} \wedge \mathbf{w}) \frac{d\gamma_0}{d\tau} = -\tilde{N} \cdot \frac{\tilde{v}}{c} + \left( -\tilde{N} - \frac{\tilde{v}}{c} \times \tilde{L} \right).
\]

The first term on the r.h.s. is similar to a rate of work. However, the last parenthesis of equation (11) contains the new forces of interest here, and will be discussed in Section 5 below. One could also have considered a force arising from \( (\mathbf{M} \wedge \mathbf{w}) I \), which looks like

\[
(\mathbf{M} \wedge \mathbf{w}) I \frac{d\gamma_0}{d\tau} = -\tilde{L} \cdot \frac{\tilde{v}}{c} + \left( -\tilde{L} + \frac{\tilde{v}}{c} \times \tilde{N} \right),
\]

however, it is left for a future analysis to study this.

### 5 The New Force Terms

Let us consider a collection of particles at a large distance, \( \tilde{r}_0 \). The particles may have different inertia, \( m_i \), but move collectively with an average velocity, \( \tilde{V} \). If one considers particles in a cosmological setting, then the velocity is a combination of the Hubble expansion and peculiar velocity, \( \tilde{V} = H \tilde{r}_0 + \tilde{v}_p \), and at large distances the peculiar velocity is subdominant. The collection of particles may have internal motion, which is simplified with an internal velocity dispersion, \( \sigma^2 \).

Practically, when calculating the velocity dispersion there will be terms including both the Hubble expansion, \( v_H = H \tilde{r}_0 \), and also the
background density of both matter and the cosmological constant, however, these terms happen to exactly cancel each other (Falco et al. 2013), and one can therefore calculate $\sigma^2$ as if the structure is alone in a non-expanding universe.

The dynamic mass moment is given by

$$\vec{N} = \sum \left( ct \vec{p}_i - \vec{e}_i \vec{r}_i \right),$$

(13)

where the sum is over all particles involved (Landau & Lifshitz 1975). If one divides both terms by the total energy, $\varepsilon_{\text{tot}} = \sum \varepsilon_i$, then one gets

$$\frac{\vec{N}}{\varepsilon_{\text{tot}}} = \left( \frac{ct \sum \vec{p}_i}{\varepsilon_{\text{tot}}} - \frac{\sum \vec{e}_i \vec{r}_i}{c} \right),$$

(14)

The first term is just $ct$ times the average velocity. At small velocities one has $\varepsilon \approx m c^2$ and hence the last term describes the relativistic centre of inertia, $\vec{R}_m = \sum (m_i \vec{r}_i) / \sum m_i$. If the centre of inertia moves at constant velocity (now ignoring sums over particles), then one has $\vec{r} = \vec{r}_0 + V t$, and hence

$$\vec{N} = -m c \vec{r}_0.$$

(15)

When considering the Lorentz force in equation (3) one needs to get the units right to get $\vec{E} = q \vec{E} / (4 \pi \varepsilon_0 |r|^3)$, which includes the observable vacuum permittivity, $\varepsilon_0$, and also Coulomb’s inverse distance-square law (resulting from Gauss’ and Faradays’ laws combined). Effectively, this means dividing by $\varepsilon_0 |r|^2$.

The new force terms in the parenthesis of equation (11) will now be considered. Since the mass-current is related to gravity, one should multiply by Newton’s gravitational constant, $G = 6.67 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$. To get a well-behaved field one divides by distance to power 3, which will lead to an inverse distance-square force: this comes from the integral over equation (8), using the sphericity from equation (10) with $J_1 = 0$ and the expression in equation (15). This last point is easily recognized by considering the change of notation, $\vec{N} \to \vec{g} / (4 \pi G)$, and $\rho_1 \to \rho_m$ which is the mass density, which means that equation (8) is written as $\vec{\nabla} \cdot \vec{g} = -4 \pi G \rho_m$. This equation is clearly recognized as leading to Newton’s gravitational law. Finally to get the units right, it is divided by $c$.

This implies that one has a force-term that looks like

$$- \kappa \frac{G m_m \rho_0}{|r|^3},$$

(16)

where $\kappa$ is an unknown, dimensionless number, which must be determined from observations, and $m_m$ is from the mass-current, $m_\nu$. In the case of $\kappa = 1$, this is just Newton’s gravitational force. In the above picture, it thus appears that the Newtonian gravity may be interpreted as a gravitational analogue to the Coulomb force from electromagnetism. Since the masses are always positive, the gravitational force is always attractive.

When the structure under consideration contains a dynamical term proportional to the velocity dispersion, $\sigma^2$, which for instance can arise in a dwarf galaxy where the stars and dark matter particles are orbiting in the local gravitational potential, then the potential will be minus 2 times the kinetic energy according to the virial theorem (Binney & Tremaine 2008), $2T + U = 0$, and hence one writes the energy as

$$\varepsilon_i = m_i c^2 - \frac{1}{2} m_i \sigma^2.$$ 

(17)

In this case one ends up with a new force of the form

$$\frac{\kappa}{2} \frac{\sigma^2}{c^2} \frac{m_m G \rho_0}{\|r_0\|^3},$$

(18)

where the dispersion has been normalized to $c$. This force is always repulsive. In the case of $\kappa = 1$, this is just a minor correction to the normal gravitational attraction, e.g. for galaxy clusters with velocity dispersions of 1000 km s$^{-1}$ this is a $10^{-5}$ correction, and for dwarf galaxies much less. Possibly for motion near very compact objects, this correction may eventually be observable.

Velocity-dependent forces are well known, including the Coriolis-force and the Lorentz-force; however, this, is to our knowledge, the first derived long-distance force depending on velocity squared. In an attempt to make kinetic energy depend on relative velocities (rather than absolute velocities) similarly to how potential energy depends on relative position, Schrödinger suggested a new gravitational force proportional to velocity squared (Schrödinger 1925). His force has essentially the same form as the force derived above, however, its existence was postulated on rather philosophical grounds.

A recent paper demonstrated that a universe which contains no dark energy, but instead includes a new repulsive inverse distance-square force proportional to internal velocity dispersion squared, just like equation (18), could have an accelerated expansion which fairly closely mimics the accelerated expansion induced by the cosmological constant (Loeve et al. 2021). In that paper, it was implicitly suggested that such a force might conceivably exist among the DM particles. What has been shown in this letter is, that such a force indeed may exist, and that it is not specifically related to the DM particle, but instead related to gravity in general. One of the concerns with the suggestion discussed in Loeve et al. (2021) is the potential instability of cosmological structures, however, from the derivation above it is clear that the new force derived here comes from the internal dispersion (as opposed to relative velocities), and hence there is no instability concern.

The magnitude of the dimensionless $\kappa$ in equation (18) is unknown in the present derivation, however, it should logically be unity. The new force of the paper (Loeve et al. 2021) should have a numerical value of $\kappa \sim 10^6 - 10^8$. From the derivation above there is no indication where such a large factor should come from.

### 6 Conclusion

It was recently suggested that if a force which depends on velocity squared exists in Nature, then it may induce an effect on cosmological scales which mimics the accelerated expansion of the standard cosmological constant (Loeve et al. 2021). This letter demonstrates one such concrete possibility. The derivation here is framed in the geometric structure of STA (Hestenes 2015), and takes as starting point the relativistic generalization of angular momentum which includes the dynamic mass moment, $\vec{N} = ct \vec{p} - \vec{e} \vec{r} / c$. Since the energy in this term, $\varepsilon$, contains the velocity dispersion of a distant cosmological object, then an inverse distance-square force naturally appears, which is proportional to $\sigma^2$. Such a force may lead to a slightly reduced gravitational force for extremely compact objects. The magnitude of the force derived here is significantly smaller than needed to explain the present day accelerating universe (Loeve et al. 2021).

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