ON GENERALIZING DESCARTES’ RULE OF SIGNS TO HYPERSURFACES

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Abstract. We give partial generalizations of the classical Descartes’ rule of signs to multivariate polynomials (with real exponents), in the sense that we provide upper bounds on the number of connected components of the complement of a hypersurface in the positive orthant. In particular, we give conditions based on the geometrical configuration of the exponents and the sign of the coefficients that guarantee that the number of connected components where the polynomial attains a negative value is at most one or two. Our results fully cover the cases where such an upper bound provided by the univariate Descartes’ rule of signs is one. This approach opens a new route to generalize Descartes’ rule of signs to the multivariate case, differing from previous works that aim at counting the number of positive solutions of a system of multivariate polynomial equations.

Keywords: semi-algebraic set; signomial; Newton polytope; connectivity; convex function

1. Introduction

Descartes’ rule of signs, established by René Descartes in his book La Géométrie in 1637, provides an easily computable upper bound for the number of positive real roots of a univariate polynomial with real coefficients. Specifically, it states that the polynomial cannot have more positive real roots than the number of sign changes in its coefficient sequence (excluding zero coefficients). In 1828, Gauss improved the rule by showing that the number of positive real roots, counted with multiplicity, and the number of sign changes in the coefficients sequence, have the same parity [18]. Since then, several different proofs were published e.g. [1, 14, 32], and several generalizations were made in several directions. In 1918, Curtiss gave a proof that works for real exponents and even for some infinite series [14]. In 1999, Grabner showed that Descartes’ bound is sharp, that is, for every given sign sequence, one can always find compatible coefficients such that the polynomial has the maximum possible number of positive roots provided by Descartes’ bound [20]. Generalizations of the Descartes’ rule to other types of functions in one variable are also available [21, 31].

Efforts to generalize Descartes’ rule of signs to the multivariate case have focused on systems of \( n \) multivariate polynomial equations in \( n \) variables, and on bounding the number of solutions in the positive orthant using sign properties of the coefficients of the system. The first conjecture for such a bound was published in 1996 by Itenberg and Roy [22]. They were able to show their conjecture for some special cases. The first non-trivial example supporting the conjecture was presented by Lagarias and Richardson [24] in 1997. Almost at the same time, Li and Wang gave a counterexample to the Itenberg-Roy conjecture [25]. The first generalization was given recently and identifies systems with at most one solution in the positive orthant [27], see also [12]. Afterwards, a sharp upper bound was given for systems of polynomials supported on circuits [8, 9]. In these works, the bound is given in terms of the sign variation of a sequence associated both with the exponents and the coefficients of the system. To the best of our knowledge, these are the only known generalizations of Descartes’ rule of signs to the multivariate case.

Descartes’ rule of signs allows however for a “dual” presentation: it gives an upper bound on the number of connected components of \( \mathbb{R}^n_{>0} \) minus the zero set of the polynomial, and if the sign of the highest degree term is fixed, then it also gives an upper bound on the number of connected components where the polynomial evaluates positively or negatively. Specifically, if we write \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \) with \( a_n \neq 0 \), and let \( \rho \) be the Descartes’ bound on the number of positive roots, then there are at most \( \rho + 1 \) connected components. If \( \rho \) is odd, the
We allow real exponents, restricted to the positive orthant), and consider the sets following sense. Let $f$ evaluates positively or negatively, are shown in red and blue respectively. (a) $8 - 12x + 6x^2 - x^3$. (b) $9 - 15x + 7x^2 - x^3$. (c) $15 - 23x + 9x^2 - x^3$. (d) $3 - 7x + 5x^2 - x^3$.

upper bounds for the number of components where $f$ is positive or negative agree, while if $\rho$ is even, then there are at most $\frac{n}{2} + 1$ connected components where $f$ attains the sign of $a_n$. For example, if after ignoring zero coefficients, the sign sequence of the coefficients is $(+ + - -)$, then there is one connected component where the polynomial evaluates positively and one where it evaluates negatively. If the sequence is $(+ - + -)$, then there are at most two connected components where the polynomial evaluates positively and at most two where it evaluates negatively, see Fig. 1.

With this presentation, Descartes’ rule of signs may be generalized to hypersurfaces in the following sense. Let $f: \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial (a multivariate generalized polynomial, where we allow real exponents, restricted to the positive orthant), and consider the sets

$$(1) \quad V_{>0}(f) := \{ x \in \mathbb{R}^n_{>0} \mid f(x) = 0 \}, \quad V_{\leq 0}(f) := \mathbb{R}^n_{>0} \setminus V_{>0}(f).$$

We aim at bounding the number of connected components of $V_{\leq 0}(f)$ in terms of the relative position of the exponent vectors of each monomial of $f$ in $\mathbb{R}^n$, and the sign of the coefficients. This leads to the formulation of the following problem for the generalization of Descartes’ rule of signs to hypersurfaces.

**Problem 1.1.** Consider a signomial $f: \mathbb{R}^n_{>0} \to \mathbb{R}$ with $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$, and $\sigma(f) \subseteq \mathbb{R}^n$ a finite set. Find a (sharp) upper bound on the number of connected components of $V_{\leq 0}(f)$, where $f$ takes negative (resp. positive) values, based on the sign of the coefficients and the geometry of $\sigma(f)$.

In this paper we address Problem 1.1 for generic $n$ in some scenarios, which, in particular, include the univariate Descartes’ rule of signs when the upper bound on the number of connected components where $f$ is negative is one, that is, when the sign sequence is one of $(+ \cdots + - \cdots -)$, $(- \cdots - + \cdots +)$, or $(+ \cdots + - \cdots - + \cdots +)$.

Specifically, we show that $V_{\leq 0}(f)$ has at most one connected component where $f$ is negative if $f$ has only one negative coefficient (Corollary 3.7). The same holds if there exists a hyperplane separating the exponents with positive coefficients from those with negative coefficients (Theorem 3.9), or if the exponents with negative exponents lie on a simplex such that the exponents with positive coefficient lie outside the simplex in a certain way (Theorem 4.6). A detailed account of our results is given in Section 5. We focus on finding upper bounds for the number of negative connected components, as statements about the number of positive connected components of $V_{\leq 0}(f)$ follows by studying $-f$.

If $f$ is a polynomial, that is, $\sigma(f) \subseteq \mathbb{Z}^n_{>0}$, the set $V_{\leq 0}(f)$ is semi-algebraic and hence it has a finite number of connected components [5, Theorem 5.22]. Computing topological invariants of semi-algebraic sets, such as the number of connected components, has been heavily studied in real algebraic geometry. Upper bounds of the sum of the Betti numbers of a semi-algebraic set in terms of the number of variables, the degree and the number of the defining polynomials can be found for example in [3, Theorem 1], [17, Theorem 6.2], and [7, Theorems 1.8 and 2.7]. For
the number of connected components of a semi-algebraic set, that is, the 0-th Betti number, an upper bound was given in [6, Theorem 1], [2, Theorem 1.1].

There exist several algorithms to compute the number of connected components of a semi-algebraic set. One algorithm is provided by Cylindrical Algebraic Decomposition, but it has double exponential complexity (see [3, Remark 11.19]). A more efficient way to compute connected components is using so-called road maps. In this way, one has an algorithm with single exponential complexity. For more details about this algorithm, see [4, Section 3].

The Descartes’ rule of signs is of special importance in applications where positive solutions to polynomial systems are the object of study. This is the case in models in biology and (bio)chemistry where variables are concentrations or abundances. It is precisely in this setting, namely the theory of biochemical reaction networks, where our motivation to consider Problem 1.1 comes from. In an upcoming paper, we show that the connectivity of the set of parameters that give rise to multistationarity in a reaction network [11, 13, 23] relies on the number of connected components of the complementary of a hypersurface. The hypersurface of interest is large for realistic networks, with many monomials and variables, and hence not manageable by algorithms from semi-algebraic geometry. The advantage of the techniques presented here is that they rely on linear optimization problems, and can handle this application.

The paper is organized as follows. In Section 2, we provide the notation and basic results on signomials. In Section 3, we give bounds answering Problem 1.1 using separating hyperplanes (Theorem 3.5, 3.9, 3.11), while in Section 4 bounds are found by providing conditions that guarantee that the signomial can be transformed into a convex function, while preserving the number of connected components of \( V_{>0}^c(f) \) (Theorem 4.6). In Section 5 we compare the two approaches. Throughout we illustrate our results with examples and figures, worked out using SageMath [30].

**Notation.** \( \mathbb{R}_{\geq 0}, \mathbb{R}_{> 0} \) and \( \mathbb{R}_{< 0} \) refer to the sets of non-negative, positive and negative real numbers respectively. We denote the Euclidean scalar product of two vectors \( v, w \in \mathbb{R}^n \) by \( v \cdot w \). For a set \( \sigma \subseteq \mathbb{R}^m \), a matrix \( M \in \mathbb{R}^{n \times m} \) and a vector \( v \in \mathbb{R}^n \) we write \( M \sigma + v \) for the set \( \{ M s + v \mid s \in \sigma \} \). For two sets \( A, B \subseteq \mathbb{R}^n \), the set \( A + B = \{ a + b \mid a \in A, b \in B \} \) is the Minkowski sum of \( A \) and \( B \). By convention, the maximum over an empty set is \(-\infty\), and the minimum over an empty set is \( \infty \). The symbol \#S denotes the cardinality of a finite set \( S \).

## 2. Preliminaries

The central object of study is a function

\[
(2) \quad f: \mathbb{R}_{>0}^n \to \mathbb{R}, \quad f(x) = \sum_{\mu \in \sigma(f)} c_{\mu} x^{\mu}, \quad \text{with} \ c_{\mu} \in \mathbb{R} \setminus \{0\},
\]

where \( \sigma(f) \subseteq \mathbb{R}^n \) is a finite set, called the support of \( f \). Here \( x^{\mu} \) is the usual short notation for \( x_1^{\mu_1} \ldots x_n^{\mu_n} \). To emphasize that we restrict the domain of \( f \) to the positive orthant, we call \( f \) a signomial. That is, a signomial is a generalized polynomial on the positive orthant. The term signomial was introduced by Duffin and Peterson in the early 1970s [15]. Since then, it is commonly used in geometric programming [10, 29].

Given a signomial \( f \) as in (2) and a set \( S \subseteq \sigma(f) \), we define the restriction of \( f \) to \( S \) by considering the monomials with exponent vectors in \( S \):

\[
(3) \quad f|_S(x) = \sum_{\mu \in S} c_{\mu} x^{\mu}.
\]

With the notation in (1) and by continuity, the signomial \( f \) has constant sign in each connected component of \( V_{>0}^c(f) \).

**Definition 2.1.** Let \( f \) be a signomial in \( n \) variables.

- A connected component \( U \) of \( V_{>0}^c(f) \) is said to be positive if \( f(x) > 0 \) for every \( x \in U \).
- We say \( U \) is negative, if \( f(x) < 0 \) for every \( x \in U \).
The convex hull of $\sigma(f)$ is called the Newton polytope of $f$ and denoted by $N(f)$.

The sets of positive exponents and negative exponents are defined respectively as

$$
\sigma_+(f) := \{ \alpha \in \sigma(f) \mid c_\alpha > 0 \} \quad \text{and} \quad \sigma_-(f) := \{ \beta \in \sigma(f) \mid c_\beta < 0 \}.
$$

**Example 2.2.** The support of the signomial

$$
p_1(x_1, x_2) = x_1^{2.5} - 2x_1^{0.5}x_2^2 + x_1^{0.5} - x_1^{2.5}x_2^{-2}
$$

is $\sigma(p_1) = \{(2.5, 0), (0.5, 2), (0.5, 0), (2.5, -2)\}$. The exponents $(2.5, 0)$, $(0.5, 0)$ are positive, the exponents $(0.5, 2), (2.5, -2)$ are negative. The Newton polytope of $p_1$ and the positive and negative connected components of $V^c_{>0}(p_1)$ are displayed in Fig. 2.

In what follows, it will be convenient to consider transformations of the support that do not change the number of negative (resp. positive) connected components. Any invertible matrix $M \in \text{GL}_n(\mathbb{R})$ induces a function

$$
h_M : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}, \quad x \mapsto x^M := (x^{M_1}, \ldots, x^{M_n})
$$

where $M_1, \ldots, M_n$ denote the columns of $M$. The function $h_M$ is called a monomial change of variables and it is a homeomorphism.

**Lemma 2.3.** For $M \in \text{GL}_n(\mathbb{R})$, $v \in \mathbb{R}^n$, and a signomial $f$ on $\mathbb{R}^n_{>0}$, define the signomial

$$
F_{M,v,f} : \mathbb{R}^n_{>0} \to \mathbb{R}, \quad F_{M,v,f}(x) = x^v f(h_M(x)).
$$

There is a homeomorphism between the positive (resp. negative) connected components of $V^c_{>0}(f)$ and $V^c_{>0}(F_{M,v,f})$. Furthermore,

$$
\sigma_+(F_{M,v,f}) = M\sigma_+(f) + v \quad \text{and} \quad \sigma_-(F_{M,v,f}) = M\sigma_-(f) + v.
$$

**Proof.** If $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$, we have

$$
F_{M,v,f}(x) = x^v f(h_M(x)) = \sum_{\mu \in \sigma(f)} c_\mu x^v (x^M)^\mu = \sum_{\mu \in \sigma(f)} c_\mu x^{M \mu + v}.
$$

From this, the second part of the lemma follows.

For the first part, clearly, the identity map induces a sign-preserving homeomorphism between $V^c_{>0}(F_{M,v,f})$ and $V^c_{>0}(f \circ h_M)$, and the map $h_M$ induces a homeomorphism between $V^c_{>0}(f \circ h_M)$ and $V^c_{>0}(f)$, which also preserves the sign of each connected component.

In view of Lemma 2.3, we could for example assume that all exponent vectors belong to $\mathbb{Q}^n$. Moreover, if $\sigma(f) \subseteq \mathbb{Q}^n$, then $f$ can be replaced by a polynomial without changing the number of negative (resp. positive) connected components of $V^c_{>0}(f)$.
Example 2.4. The matrix \( M = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0 \end{pmatrix} \) and the vector \( v = (-0.25, -0.25) \) transform the signomial \( p_1 \) from Example 2.2 to the polynomial \( F_{M,v,p_1}(x_1, x_2) = x_1 x_2 - 2x_2 + 1 - x_1 \).

3. Paths on logarithmic scale

In this section, we provide the first results towards Problem 1.1. The idea behind the proofs in this section relies on reducing the multivariate signomial to a univariate signomial, and applying Descartes’ rule of signs. To this end, given \( v \in \mathbb{R}^n \) and \( x \in \mathbb{R}^n_{>0} \), we consider continuous paths

\[
\gamma_{v,x} : [1, \infty) \to \mathbb{R}^n_{>0}, \quad t \mapsto (t^{v_1} x_1, \ldots, t^{v_n} x_n).
\]

In logarithmic scale, applying the coordinate-wise natural logarithm map

\[
\log : \mathbb{R}^n_{>0} \to \mathbb{R}^n, \quad (x_1, \ldots, x_n) \mapsto (\log(x_1), \ldots, \log(x_n)),
\]

each path \( \gamma_{v,x} \) is transformed into a half-line \([0, \infty) \to \mathbb{R}^n, s \mapsto s v + \log(x)\), with start point \( \log(x) \) and direction vector \( v \). Since the logarithm map \( \log \) is a homeomorphism, the topological properties of \( f^{-1}(\mathbb{R}_{<0}) \) and of its image under \( \log \) are the same. This observation gives us an easy geometric way to think about paths \( \gamma_{v,x} \).

Given a signomial \( f \), each \( v \in \mathbb{R}^n \) and \( x \in \mathbb{R}^n_{>0} \) induce a signomial function in one variable:

\[
f_{v,x} : \mathbb{R}_{>0} \to \mathbb{R}, \quad t \mapsto \sum_{\mu \in \sigma(f)} (c_{\mu} x^\mu) t^{\mu}.
\]

Note that \( f_{v,x}(1) = f(x) \). Since the restriction of \( f_{v,x} \) to \([1, \infty)\) is the composition \( f \circ \gamma_{v,x} \), understanding the properties of \( f_{v,x} \) allows us to determine whether the path \( \gamma_{v,x} \) is in the pre-image of the negative real line under \( f \). This motivates the study of signomials in one variable. The following lemma will be used repeatedly in what follows. Its proof is a direct application of the Descartes’ rule of signs.

Lemma 3.1. Let \( g : \mathbb{R}_{>0} \to \mathbb{R}, g(t) = \sum_{\nu \in \sigma(g)} a_{\nu} t^{\nu} \) be a signomial in one variable such that \( g(1) < 0 \).

(i) If the sign sequence of the coefficients of \( g \) has at most two sign changes, and the leading coefficient is positive, then there is unique \( \rho \in (1, \infty) \) such that \( g(\rho) = 0 \), and it holds \( g(t) < 0 \) for all \( t \in [1, \rho) \) and \( g(t) > 0 \) for all \( t \in (\rho, \infty) \).

(ii) If the sign sequence of the coefficients of \( g \) has at most one sign change, and the leading coefficient is negative, then \( g(t) < 0 \) for all \( t \in [1, \infty) \).

Every \( v \in \mathbb{R}^n \) and \( a \in \mathbb{R} \), define a hyperplane \( \mathcal{H}_{v,a} := \{ \mu \in \mathbb{R}^n \mid v \cdot \mu = a \} \), which in turn defines two half-spaces

\[
\mathcal{H}_{v,a}^+ := \{ \mu \in \mathbb{R}^n \mid v \cdot \mu \geq a \} \quad \text{and} \quad \mathcal{H}_{v,a}^- := \{ \mu \in \mathbb{R}^n \mid v \cdot \mu \leq a \}.
\]

We let \( \mathcal{H}_{v,a}^+ \) denote the interior of \( \mathcal{H}_{v,a}^+ \), and \( \mathcal{H}_{v,a}^- \) respectively. Although \( \mathcal{H}_{v,a} = \mathcal{H}_{-v,-a} \), the choice of sign determines which half-space is positive and which one is negative.

As we will see in Lemma 3.3 the relative position of a hyperplane \( \mathcal{H}_{v,a} \) and the points in \( \sigma(f) \) gives valuable information about the behavior of the function \( f_{v,x} \) in (6). To this end, we introduce the following types of vectors \( v \).

Definition 3.2. Let \( v \in \mathbb{R}^n \).

(i) We say that \( v \) is a separating vector of \( \sigma(f) \) if for some \( a \in \mathbb{R} \) it holds

\[
\sigma_-(f) \subseteq \mathcal{H}_{v,a}^+, \quad \sigma_+(f) \subseteq \mathcal{H}_{v,a}^-.
\]

The separating vector \( v \) is strict if \( \sigma_-(f) \cap \mathcal{H}_{v,a}^+ \neq \emptyset \), and very strict if additionally \( \sigma_-(f) \cap \mathcal{H}_{v,a} = \emptyset \) for some \( a \in \mathbb{R} \). We \( S_v^-(f) \) denote the set of separating vectors of \( \sigma(f) \).
Let \( \sigma + (f) \) be a signomial and positive coefficients are grouped. This has the following consequences.

Proof. For each partition \( \sigma + (f) = A_1 \cup A_2 \) into disjoint subsets, we define

\[
E_{A_1, A_2} := \left\{ v \in \mathbb{R}^n \mid \max_{\alpha \in A_1} v \cdot \alpha \leq \min_{\beta \in \sigma_-(f)} v \cdot \beta \quad \text{and} \quad \max_{\beta \in \sigma_-(f)} v \cdot \beta \leq \min_{\alpha \in A_2} v \cdot \alpha \right\}.
\]

Then \( E_-(f) = \bigcup_{A_1, A_2} E_{A_1, A_2} \). For every \( v, w \in E_{A_1, A_2} \), \( \lambda, \mu \in \mathbb{R}_{>0} \), and \( \beta \in \sigma_-(f) \), it holds that

\[
(\lambda v + \mu w) \cdot \alpha \leq (\lambda v + \mu w) \cdot \beta \quad \text{for all } \alpha \in A_1 \quad \text{and} \quad (\lambda v + \mu w) \cdot \beta \leq (\lambda v + \mu w) \cdot \alpha \quad \text{for all } \alpha \in A_2.
\]

Thus, the sets \( E_{A_1, A_2} \) are convex cones and \( E_-(f) \) a union of convex cones. It follows directly from the definition that if \( v \in E_{A_1, A_2} \), then \( \lambda v \in E_{A_1, A_2} \) if \( \lambda \geq 0 \) and \( \lambda v \in E_{A_2, A_1} \) if \( \lambda \leq 0 \).

The fact that \( S^-(f) \) is a convex cone follows by setting \( A_1 = \sigma_+(f) \) and \( A_2 = \emptyset \).

Enclosing and separating vectors order the exponents of \( f_{v,x} \) in \([2]\), such that the negative and positive coefficients are grouped. This has the following consequences.

Lemma 3.4. Let \( f : \mathbb{R}^n_{>0} \rightarrow \mathbb{R} \) be a signomial and \( x \in \mathbb{R}^n_{>0} \).

(i) If \( v \in E_-(f) \), then there are at most two sign changes in the coefficient sign sequence of the signomial \( f_{v,x} \). If \( v \) is additionally strict, then both the leading coefficient and the coefficient of smallest degree of \( f_{v,x} \) are positive.

(ii) If \( v \in S^-(f) \), then there is at most one sign change in the coefficient sign sequence of the signomial \( f_{v,x} \). If \( v \) is strict, then the leading coefficient of \( f_{v,x} \) is negative.

Additionally if \( f(x) < 0 \), then the following statements hold:

(i') If \( v \in E_-(f) \), then there is a unique \( \rho \in (1, \infty] \) such that \( f_{v,x}(t) < 0 \) for all \( t \in [1, \rho) \) and \( f_{v,x}(t) > 0 \) for all \( t > \rho \) (note that \( \rho \) might be infinite).

(ii'') If \( v \in S^-(f) \), then \( f_{v,x}(t) < 0 \) for all \( t \in [1, \infty) \).

Proof. (i) and (i'). For \( v \in E_-(f) \), \( v \) orders the exponents \( v \cdot \mu \) such that the sign sequence is \((\cdots - + \cdots - + \cdots +)\), with potentially one or more of the three blocks of repeated signs not present. The positive blocks are present if \( v \) is strict by definition, showing (i).
For $f(x) < 0$, if the leading coefficient of $f_{v,x}$ is positive, then Lemma 3.1(i) gives the existence of a unique $\rho \in (1, \infty)$ satisfying (i)’ in the statement. If the leading coefficient of $f_{v,x}$ is negative, then $v \in S^-(f)$ and this case is covered next, and gives $\rho = \infty$.

(ii) and (ii’). From $v \in S^-(f)$, it follows that the signomial $f_{v,x}$ has at most one sign change in its coefficient sequence, as $\max_{\alpha \in \sigma_+(f)} v \cdot \alpha \leq \min_{\beta \in \sigma_-(f)} v \cdot \beta$. If $v$ is strict, then for at least one $\beta_0 \in \sigma_-(f)$ we have $\max_{\alpha \in \sigma_+(f)} v \cdot \alpha < v \cdot \beta_0$, and hence the leading term is negative, showing (ii). If $f_{v,x}(1) = f(x) < 0$, $f_{v,x}$ must have some negative coefficient. Using $v \in S^-(f)$, we conclude that the leading coefficient is negative and $v$ is strict. Lemma 3.1(ii) gives now statement (ii’).

For a set $U \subseteq \mathbb{R}^n$, we let $\text{Lin}(U)$ denote the vector subspace of $\mathbb{R}^n$ generated by $U$.

**Theorem 3.5.** Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. If

$$\text{Lin}(S^-(f)) + E^-(f) = \mathbb{R}^n,$$

then $f^{-1}(\mathbb{R}_{<0})$ is path connected. In particular, $V^c_{>0}(f)$ has at most one negative connected component.

**Proof.** The empty set is path connected, so we assume $f^{-1}(\mathbb{R}_{<0}) \neq \emptyset$. The proof is constructive, and we give a continuous path between every pair of points $x, y \in f^{-1}(\mathbb{R}_{<0})$. Specifically, we find vectors $v, u \in S^-(f)$ and $w \in E^-(f)$ such that

$$\gamma_{w,\gamma_{v,x}(e)}(e) = \gamma_{u,y}(e),$$

where $e$ denotes Euler’s number. This means that a path from $x$ to $y$ is obtained by concatenating the path $\gamma_{v,x}$ in $[1, e]$ from $x$ to $\gamma_{v,x}(e)$, with the path $\gamma_{w,\gamma_{v,x}(e)}$ in $[1, e]$ from $\gamma_{v,x}(e)$ to $\gamma_{w,\gamma_{v,x}(e)}(e)$, and finally with the reverse of the path $\gamma_{u,y}$ in $[1, e]$ from $y$ to $\gamma_{w,\gamma_{v,x}(e)}(e)$. With this in place, all we need is to ensure that these paths are in the pre-image of $f^{-1}(\mathbb{R}_{<0})$ as well.

Equality (9) is equivalent to $e^{w_i} e^{v_i} x_i = e^{u_i} y_i$ for all $i = 1, \ldots, n$, which in turn is equivalent to

$$v - u + w = \lambda, \quad \text{with} \quad \lambda = (\log(\frac{w_1}{x_1}), \ldots, \log(\frac{w_n}{x_n})).$$

By assumption, there exist $z_1, \ldots, z_m \in S^-(f)$, $\nu_1, \ldots, \nu_m \in \mathbb{R}$ and $w \in E^-(f)$ such that

$$\lambda = \sum_{i=1}^m \nu_i z_i + w.$$

Since $S^-(f)$ is a convex cone by Lemma 3.3,

$$v := \sum_{\nu_i > 0} \nu_i z_i, \quad u := \sum_{\nu_i < 0} (-\nu_i) z_i,$$

both belong to $S^-(f)$. Thus, we have found $v, u \in S^-(f)$ and $w \in E^-(f)$ such that $\lambda = v - u + w$ and hence equality (9) holds.

Since $v, u \in S^-(f)$ and on $[1, \infty)$ it holds $f \circ \gamma_{v,x} = f_{v,x}$ and $f \circ \gamma_{u,y} = f_{u,y}$, the paths $\gamma_{v,x}$ and $\gamma_{u,y}$ are included in $f^{-1}(\mathbb{R}_{<0})$ by Lemma 3.4(ii’).

For $z = \gamma_{v,x}(e)$, we have $f_{w,z}(1) = f(z) = f(\gamma_{v,x}(e)) < 0$ and $f_{w,z}(e) = f(\gamma_{u,y}(e)) = f(\gamma_{u,y}(e)) < 0$. Lemma 3.4(ii) implies that $f_{w,z}(t) < 0$ for all $t \in [1, e]$. Thus, the path segment $\gamma_{w,z}$ on $[1, e]$ is contained in $f^{-1}(\mathbb{R}_{<0})$. This concludes the proof.

In low dimensional cases, the assumption of Theorem 3.5 can be seen directly from visual inspection of the support of the signomial. Otherwise, computer algebra systems such as SageMath can be employed.

**Example 3.6.** Consider the signomial

$$p_2(x_1, x_2) = x_1^3 x_2^2 + x_1^2 x_2^3 - 5 x_1^2 x_2^3 - 3 x_1^2 x_2^2 + x_1 x_2 + x_2^2,$$

with $\sigma(p_2)$ depicted in Fig. 3(a). The vector $s = (1, -1)$ is a separating vector of $\sigma(p_2)$ that spans $\text{Lin}(S^-(p_2))$. The vector $z = (1, 1)$ is an enclosing vector of $\sigma(p_2)$, see Fig. 3(a). By Lemma 3.3, the subspace spanned by $z$ is contained in $E^-(p_2)$. 

\[ \text{□} \]
As $v$ varies, the path segments given by $\gamma_{v,x}$, $\gamma_{w,x(e)}$ and $\gamma_{u,y}$ restricted to $[1,e]$ are shown: $x = (3,1.25)$, $y = (0.75,1)$, where $v \approx 0.29s$, $u \approx 0.87s$, $w \approx -0.80z$.

In conclusion, $\text{Lin}(S^-(p_2)) + E^-(p_2) = \mathbb{R}^2$. By Theorem 3.5, $V_{S_0}^- (p_2)$ has at most one negative connected component. In Fig. 3(b) we show the path between two points where $p_2$ is negative as constructed in the proof of Theorem 3.5.

**Corollary 3.7.** Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. If at most one coefficient of $f$ is negative, then $E^-(f) = \mathbb{R}^n$. In particular, $f^{-1}(R_{<0})$ is path connected, and $V_{S_0}^-(f)$ has at most one negative connected component.

**Proof.** If $\sigma_-(f) = \emptyset$, then $V_{S_0}^-(f) = \emptyset$. If $f$ has one negative exponent $\beta$, then for any $v \in \mathbb{R}^n$, the choice $a = b = v \cdot \beta$ satisfies the conditions for $v$ being an enclosing vector from Definition 3.2(ii). Hence $v \in E^-(f)$. The last statement follows from Theorem 3.5.

Corollary 3.7 covers the case of Theorem 3.5 where $E^-(f) = \mathbb{R}^n$. As we show next, the case $\text{Lin}(S^-(f)) = \mathbb{R}^n$ holds when there exists a very strict separating vector.

**Proposition 3.8.** Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial and $v \in \mathbb{R}^n$ a very strict separating vector of $\sigma(f)$. Then there exists a basis $\{w_1, \ldots, w_n\}$ of $\mathbb{R}^n$, consisting of very strict separating vectors, and a constant $c \in \mathbb{R}$ such that for every $i \in \{1, \ldots, n\}$ it holds

$$\sigma_-(f) \subseteq \mathcal{H}_{w_i,c}^+ \quad \sigma_+(f) \subseteq \mathcal{H}_{w_i,c}^-.$$  

In particular, $\text{Lin}(S^-(f)) = \mathbb{R}^n$ and $f^{-1}(\mathbb{R}_{<0})$ is path connected.

**Proof.** Define $a := \max_{\alpha \in \sigma_+(f)} v \cdot \alpha$, $b := \min_{\beta \in \sigma_-(f)} v \cdot \beta$, $c := \frac{a+b}{2}$.

As $v \in S^-(f)$, $\sigma_-(f) \subseteq \mathcal{H}_{w_i,c}^+$ and $\sigma_+(f) \subseteq \mathcal{H}_{w_i,c}^-$ by (7). Since $v$ is very strict, we have $b > c > a$. Choose $v_2, \ldots, v_n \in \mathbb{R}^n$ such that $v, v_2, \ldots, v_n$ are linearly independent and define $K := \min_{i=2, \ldots, n} \min_{\mu \in \sigma(f)} v_i \cdot \mu$, $L := \max_{i=2, \ldots, n} \max_{\mu \in \sigma(f)} v_i \cdot \mu$.

In the following, we show that it is possible to choose $\epsilon > 0$ such that the vectors $w_i := v + \epsilon v_i$, $i = 2, \ldots, n$ with the given $c$ satisfy (11). For $\beta \in \sigma_-(f)$ and $i \in \{2, \ldots, n\}$, using that $v_i \cdot \beta \geq K$ and $v \cdot \beta \geq b$, it holds that

$$w_i \cdot \beta = v \cdot \beta + \epsilon (v_i \cdot \beta) \geq b + \epsilon K \begin{cases} \geq b > c & \text{if } K \geq 0 \text{ and for } \epsilon > 0, \\ > b + \frac{a+b}{2K} K = c & \text{if } K < 0 \text{ and for } 0 < \epsilon < \frac{a-b}{2K}. \end{cases}$$

\[\text{Figure 3. Illustration of Example 3.6. (a) Shows } \sigma(p_2) \text{ with blue indicating negative exponents and red positive exponents. The vector } s = (1,-1) \text{ is a separating vector, } z = (1,1) \text{ is an enclosing vector. (b) The negative connected component of } V_{S_0}^- (p_2) \text{ is shown in blue. The path segments given by } \gamma_{v,x}, \gamma_{w,x(e)} \text{ and } \gamma_{u,y} \text{ restricted to } [1,e] \text{ are shown: } x = (3,1.25), y = (0.75,1), \text{ where } v \approx 0.29s, u \approx 0.87s, w \approx -0.80z.\]
Similarly, for every $\alpha \in \sigma_+(f)$ and $i \in \{2, \ldots, n\}$, it follows that

$$w_i \cdot \alpha = v \cdot \alpha + \epsilon (v_i \cdot \alpha) \leq a + \epsilon L \left\{ \begin{array}{ll} \leq a < c & \text{if } L \leq 0 \text{ and for } \epsilon > 0, \\ < a + \frac{b-a}{2L} L = c & \text{if } L > 0 \text{ and for } 0 < \epsilon < \frac{b-a}{2L}. \end{array} \right.$$  \hspace{1cm} (12)

Therefore, one can choose $\epsilon > 0$ such that $w_2, \ldots, w_n$ satisfy \((11)\) and \((12)\). Therefore, $v, w_2, \ldots, w_n$ are very strict separating vectors, and since $\epsilon \neq 0$, they form a basis of $\mathbb{R}^n$. Hence $\text{Lin}(\mathcal{S}^-(f)) = \mathbb{R}^n$. The last statement follows from Theorem 3.5. \hfill \Box

As we will see below in Proposition 3.16, the existence of a very strict separating vector characterizes when $\text{Lin}(\mathcal{S}^-(f)) = \mathbb{R}^n$ holds. However, the existence of a strict separating vector is enough for connectivity of $f^{-1}(\mathbb{R}_{<0})$.

**Theorem 3.9.** Let $f: \mathbb{R}_{\geq 0}^n \to \mathbb{R}$ be a signomial. If there exists a strict separating vector of $\sigma(f)$, then

(i) $f^{-1}(\mathbb{R}_{<0})$ is path connected and non-empty.

(ii) The closure of $f^{-1}(\mathbb{R}_{<0})$ equals $f^{-1}(\mathbb{R}_{\leq 0})$.

**Proof.** Let $v \in \mathcal{S}^- (f)$ be a strict separating vector. First note that for any $x \in \mathbb{R}_{\geq 0}^n$, the leading coefficient of $f_{v,x}$ is negative by Lemma 3.4(ii), and hence $f^{-1}(\mathbb{R}_{<0}) \neq \emptyset$. Define

$$a := \max_{\alpha \in \sigma_+(f)} v \cdot \alpha, \quad \text{and} \quad M := \{ \beta \in \sigma_-(f) \mid v \cdot \beta = a \} = \sigma_-(f) \cap \mathcal{H}_{v,a}.$$  

Since $v$ is a strict separating vector, $\sigma_-(f) \setminus M \neq \emptyset$. Consider the restriction of $f$ to $\sigma(f) \setminus M$, c.f. (3):

$$\tilde{f} := f|_{\sigma(f)\setminus M}.$$  

As $\tilde{f}$ is obtained from $f$ only by removing monomials with negative coefficients, $f(x) \leq \tilde{f}(x)$ for all $x \in \mathbb{R}_{\geq 0}^n$ and hence $f^{-1}(\mathbb{R}_{<0}) \subseteq \tilde{f}^{-1}(\mathbb{R}_{\leq 0})$. By construction $\sigma_-(f) \neq \emptyset$, and $v$ is also a strict separating vector of $\sigma(\tilde{f})$, which additionally satisfies

$$\max_{\alpha \in \sigma_+(f)} v \cdot \alpha < \min_{\beta \in \sigma_-(f)} v \cdot \beta.$$  

Hence, $v$ is a very strict separating vector of $\sigma(\tilde{f})$. By Proposition 3.8, $\tilde{f}^{-1}(\mathbb{R}_{<0})$ is path connected and non-empty. Therefore, it is enough to show that for every $x \in f^{-1}(\mathbb{R}_{<0})$, there exists a path from $x$ to a point in $\tilde{f}^{-1}(\mathbb{R}_{<0})$ and contained in $f^{-1}(\mathbb{R}_{<0})$.

Let $x \in f^{-1}(\mathbb{R}_{<0})$. Since $v \in \mathcal{S}^-(f)$ and $v \in \mathcal{S}^-(f)$, and strict in both cases, Lemma 3.4(iii) gives that $f(\gamma_{v,x}(t)) < 0$ for all $t \geq 1$ and the univariate signomial $f_{v,x}$ has negative leading term. This means that for some $t_0$ large enough, $\tilde{f}_{v,x}(t_0) < 0$. Hence, the path $\gamma_{v,x}$ from $x$ to $\gamma_{v,x}(t_0)$ is contained in $f^{-1}(\mathbb{R}_{<0})$ and ends in $f^{-1}(\mathbb{R}_{<0})$, as desired. This concludes the proof of (i).

Finally, we show statement (ii). Let $x \in f^{-1}(\{0\})$. Since $v \in \mathcal{S}^-(f)$ and strict, Lemma 3.4(ii) gives that $f_{v,x}(t) < 0$ for all $t > 1$. Thus the sequence $(\gamma_{v,x}(1 + \frac{1}{n}))_{n \in \mathbb{N}}$ belongs to $f^{-1}(\mathbb{R}_{<0})$. As $\gamma_{v,x}$ is continuous and $\gamma_{v,x}(1) = x$, the sequence $(\gamma_{v,x}(1 + \frac{1}{n}))_{n \in \mathbb{N}}$ converges to $x$. So each $x \in f^{-1}(\mathbb{R}_{<0})$ is the limit of a convergent sequence in $f^{-1}(\mathbb{R}_{<0})$. Hence $f^{-1}(\mathbb{R}_{\leq 0}) \subseteq f^{-1}(\mathbb{R}_{<0})$. The other inclusion is clear by the continuity of $f$. \hfill \Box

**Example 3.10.** Consider the signomial

$$p_3(x_1, x_2) = -x_1^3 x_2 + 3x_1^2 x_2^2 - x_1^2 x_2^2 - x_1 x_2^3 + x_1 x_2^2 - 3x_2 x_2 + 2.$$  

Then $v = (1, -1) \in \mathcal{S}^-(p_3)$ is strict, see Fig. 4(a), and by Theorem 3.9, $V_{>0}(p_3)$ has one negative connected component.

Note that the proof of Theorem 3.9 is constructive. Consider the signomial obtained by removing the negative monomials on the separating hyperplane $\mathcal{H}_{v,-1}$ from Fig. 4(a):

$$\tilde{p}_3(x_1, x_2) = 3x_1^3 x_2 - x_1^3 x_2^2 + x_1 x_2^2 - 3x_1 x_2 + 2.$$
By construction, see (7), Consider the restriction of \( f \) and the negative exponents of \( f \) to \( \sigma^\ast(f) \). With analogous techniques, the existence of strict enclosing vectors of \( f \) is strict, the following sets are non-empty:

\[
\gamma_v, x, \gamma_v, y, \gamma_v, x, \gamma_v, y.
\]

As \( v \) is strict, the following sets are non-empty:

\[
M := \{ \beta \in \sigma_-(f) \mid \max_{a \in \sigma_+(f)} v \cdot a < v \cdot \beta \}, \quad N := \{ \beta \in \sigma_+(f) \mid v \cdot \beta < \min_{a \in \sigma_+(f)} v \cdot a \}.
\]

Consider the restriction of \( f \) to the sets \( M \cup \sigma_+(f) \) and \( N \cup \sigma_+(f) \):

\[
\tilde{f}_M := f_{|M \cup \sigma_+(f)} \quad \tilde{f}_N := f_{|N \cup \sigma_+(f)}.
\]

By construction, \( v \) and \( -v \) are strict separating vectors of \( \sigma(\tilde{f}_M) \) and \( \sigma(\tilde{f}_N) \) respectively. Hence \( \tilde{f}_M^{-1}(\mathbb{R}_{<0}) \) and \( \tilde{f}_N^{-1}(\mathbb{R}_{<0}) \) are path connected by Theorem 3.9. Additionally, as the set of negative exponents of \( \tilde{f}_M \) or \( \tilde{f}_N \) is included in that of \( f \), it holds \( \tilde{f}(x) \leq \tilde{f}_N(x) \) and \( f(x) \leq \tilde{f}_M(x) \) for all \( x \in \mathbb{R}_{>0} \) and hence

\[
\tilde{f}_M^{-1}(\mathbb{R}_{<0}) \subseteq f^{-1}(\mathbb{R}_{<0}), \quad \tilde{f}_N^{-1}(\mathbb{R}_{<0}) \subseteq f^{-1}(\mathbb{R}_{<0}).
\]

With this in place, if we show that for every \( x \in f^{-1}(\mathbb{R}_{<0}) \) there is a continuous path to a point in \( \tilde{f}_M^{-1}(\mathbb{R}_{<0}) \) or to a point in \( \tilde{f}_N^{-1}(\mathbb{R}_{<0}) \) and this path is contained in \( f^{-1}(\mathbb{R}_{<0}) \), then the number of connected components of \( f^{-1}(\mathbb{R}_{<0}) \) is at most 2.

Fix \( x \in f^{-1}(\mathbb{R}_{<0}) \). As \( v \) is a strict separating vector of \( \sigma(\tilde{f}_M) \) and \( -v \) of \( \sigma(\tilde{f}_N) \), there exist \( t_x, d_x > 1 \) such that \( \gamma_{v, x}(x) \in \tilde{f}_M^{-1}(\mathbb{R}_{<0}) \) and \( \gamma_{-v, x}(d_x) \in \tilde{f}_N^{-1}(\mathbb{R}_{<0}) \) by Lemma 3.4(ii).
Consider the signomial

\[ \gamma = (0.15, 0.95) \]

The vector \( v = (1, -1) \) is a strict enclosing vector for \( -p_4 \), see Fig. 3a). Hence, the number of negative connected components of \( V_{\geq 0}(p_4) \) is at most two by Theorem 3.11.

In Fig. 5(b), the idea of the proof of Theorem 3.11 is illustrated. The two following signomials are considered

\[ \tilde{p}_{4,M}(x_1, x_2) = x_1^5 x_2^5 + x_1^4 x_2^5 + x_1^3 x_2^3 - x_1^5 + 3x_1^3 x_2^3 + x_1 x_2, \]

\[ \tilde{p}_{4,N}(x_1, x_2) = x_1^5 x_2^5 - x_1^4 x_2^5 + x_1^3 x_2^3 + x_1 x_2^3 - x_1 x_2^3 + 3x_1^3 x_2^3 + x_1 x_2. \]

For each of these signomials, the pre-image of \( \mathbb{R}_{<0} \) is path connected and contained in \( p_4^{-1}(\mathbb{R}_{<0}) \).

Using the paths \( \gamma_{v, x} \) or \( \gamma_{-v, x} \), any point \( x \in p_4^{-1}(\mathbb{R}_{<0}) \) is connected to one of these two connected sets.

**Remark 3.13.** One might be tempted to believe that in the situation of Theorem 3.11, \( V_{\geq 0}(f) \) has at most one positive connected component. However, Example 2.2 shows already a counter example, as \( V_{\geq 0}(p_1) \) has two positive connected components, and the vector \( v = (0, 1) \) satisfies the hypotheses of Theorem 3.11, see Fig. 2.

A direct consequence of Theorems 3.9 and 3.11 applies to the case where the positive exponents of \( f \) belong to a hyperplane that does not contain all the negative exponents of \( f \).

**Corollary 3.14.** Let \( f : \mathbb{R}_{>0}^n \to \mathbb{R} \) be a signomial. If for some \( v \in \mathbb{R}^n \) and \( a \in \mathbb{R} \)

\[ \sigma_+(f) \subseteq \mathcal{H}_{v,a} \quad \text{and} \quad \sigma_-(f) \not\subseteq \mathcal{H}_{v,a}, \]

then \( V_{>0}(f) \) has at most two negative connected components.
Proof. The conditions imply that either \( v \) is a strict enclosing vector of \( \sigma(-f) \), or either \( v \) or \(-v\) is a strict separating vector of \( \sigma(-f) \). The statement then follows from Theorem 3.11 or Theorem 3.9.

**Corollary 3.15.** Let \( f : \mathbb{R}^n_{>0} \to \mathbb{R} \) be a signomial. If

\[ \#\sigma_+(f) \leq \dim N(f), \]

then \( V_{c_0}(f) \) has at most two negative connected components.

Proof. Since \( \#\sigma_+(f) \leq \dim N(f) \leq n \), the points \( \sigma_+(f) \) lie on an affine subspace of dimension at most \( \dim N(f) - 1 \). Necessarily, this subspace cannot contain all negative exponents. Hence, there exists an affine hyperplane \( \mathcal{H}_{c_0} \) containing \( \sigma_+(f) \) and not containing \( \sigma_-(f) \). Now, the statement follows from Corollary 3.14.

We note that Corollary 3.7 and Proposition 3.8 give conditions that guarantee that either \( \mathcal{E}^-(f) = \mathbb{R}^n \) or \( \text{Lin}(\mathcal{S}^-(f)) = \mathbb{R}^n \) and hence Theorem 3.5 applies. We conclude this section by showing that the given conditions cover essentially all cases where this holds.

**Proposition 3.16.** Let \( f : \mathbb{R}^n_{>0} \to \mathbb{R} \) be a signomial.

(i) \( \text{Lin}(\mathcal{S}^-(f)) = \mathbb{R}^n \) if and only if the support of \( f \) has a very strict separating vector.

(ii) \( \mathcal{E}^-(f) = \mathbb{R}^n \) if and only if either \( f \) has at most one coefficient negative, or \( \sigma_-(f) = \sigma(f) \), or \( \dim N(f) = 1 \) and \( \text{Conv}(\sigma_-(f)) \cap \sigma_+(f) = \emptyset \).

Proof. (i) The reverse implication is Proposition 3.8. To show the forward implication, assume \( \text{Lin}(\mathcal{S}^-(f)) = \mathbb{R}^n \). Let \( w_1, \ldots, w_n \in \mathbb{R}^n \) be linearly independent separating vectors and let \( a_1, \ldots, a_n \in \mathbb{R} \) such that

\[ \sigma_-(f) \subseteq \mathcal{H}_{w_i,a_i}^+, \quad \sigma_+(f) \subseteq \mathcal{H}_{w_i,a_i}^-, \quad \text{for } i = 1, \ldots, n. \]

Write \( \{z\} = \bigcap_{i=1}^n \mathcal{H}_{w_i,a_i} \), and define

\[ w_0 := \sum_{i=1}^n w_i, \quad a_0 := w_0 \cdot z = \sum_{i=1}^n a_i. \]

We show that \( w_0 \) is a very strict separating vector. For any \( \mu \in \sigma(f) \), using (13) we have

\[ w_0 : \mu = \sum_{i=1}^n w_i : \mu \begin{cases} < \sum_{i=1}^n a_i = a_0 & \text{if } \mu \in \sigma_+(f), \mu \neq z, \\ > \sum_{i=1}^n a_i = a_0 & \text{if } \mu \in \sigma_-(f), \mu \neq z, \\ = \sum_{i=1}^n a_i = a_0 & \text{if } \mu = z. \end{cases} \]

As both \( z \in \sigma_+(f) \) and \( z \in \sigma_-(f) \) cannot hold, we conclude that \( w_0 : \alpha \leq a_0 \leq w_0 : \beta \) with at least one inequality strict, for any \( \alpha \in \sigma_+(f) \) and \( \beta \in \sigma_-(f) \). This proves that \( w_0 \) is a very strict separating vector.

(ii) The reverse implication follows directly as every vector is an enclosing vector under the given hypotheses. For the forward implication, assume that \( \sigma_-(f) \neq \sigma(f) \) has at least two elements and \( \dim(N(f)) > 1 \). It is then possible to choose affinely independent \( \beta_0, \beta_1 \in \sigma_-(f) \) and \( \alpha \in \sigma_+(f) \). So there exist hyperplanes \( \mathcal{H}_{w_0,a_0}, \mathcal{H}_{w_1,a_1} \) such that \( \alpha, \beta_0 \in \mathcal{H}_{w_0,a_0}, \beta_1 \in \mathcal{H}_{w_1,a_1} \), and \( \alpha, \beta_1 \in \mathcal{H}_{w_1,a_1} \). From this follows that

\[ (w_0 + w_1) : \beta_1 < a_0 + a_1 = (w_0 + w_1) : \alpha < (w_0 + w_1) : \beta_0. \]

Therefore, \( w_0 + w_1 \notin \mathcal{E}^-(f) \) and \( \mathcal{E}^-(f) \neq \mathbb{R}^n \).

Assume now that \( \sigma_-(f) \neq \sigma(f) \) has at least two elements, \( \dim N(f) = 1 \), and \( \text{Conv}(\sigma_-(f)) \cap \sigma_+(f) \neq \emptyset \). Then it is possible to choose \( \beta_0 < \alpha < \beta_1 \) with \( \beta_0, \beta_1 \in \sigma_-(f) \) and \( \alpha \in \sigma_+(f) \). Then the vector 1 is not an enclosing vector.

**Remark 3.17.** The techniques used in this section rely on the logarithmic paths given by a vector \( v \) and a fixed point \( x \). If \( \sigma(f) \subseteq \mathbb{Z} \), one can consider \( f \) as a Laurent polynomial with complex coefficients. The image of \( V_{c_0}(f) \) under the logarithm map is contained in the amoeba of \( f \) introduced by Gelfand et al. [19] which is the image of the set \( \{z \in (\mathbb{C}^*)^n \mid f(z) = 0\} \) under
the map \((C^*)^n \rightarrow \mathbb{R}^n, (z_1, \ldots, z_n) \mapsto (\log(|z_1|), \ldots, \log(|z_n|))\). Forsberg et al. [16] provided results regarding the number and structure of the connected components of the complement of the amoeba. The statements in [16] seem to be related to Problem 1.1, however, we have not found any direct connection between the two questions.

4. Convexification of signomials

In Section 3, we used continuous paths (5), which are half-lines on logarithmic scale, to derive bounds for the number of negative connected components of \(V^c_{<0}(f)\), where \(f\) is a signomial function. In this section, we take a different approach to bound the number of negative connected components of \(V^c_{>0}(f)\). We use the almost trivial observation that every sublevel set of a convex function is a convex set (see e.g. [28, Theorem 4.6.]). Therefore, \(V^c_{>0}(f)\) has at most one negative connected component, if \(f\) is a convex function. With this in mind, we investigate what signomials can be transformed into a convex function using Lemma 2.3.

From [26] Theorem 7, one can easily derive a sufficient condition for convexity of signomials.

**Lemma 4.1.** A signomial \(f: \mathbb{R}^n_{>0} \rightarrow \mathbb{R}\) is a convex function if the following holds:

(a) For each \(\alpha \in \sigma_+(f)\), it holds that
(i) \(\alpha_i \leq 0\) for all \(i = 1, \ldots, n\), or
(ii) there exists \(j \in \{1, \ldots, n\}\) such that \(\alpha_i = 0\) for all \(i \neq j\) and \((1, \ldots, 1) \cdot \alpha \geq 1\),
(b) For each \(\beta \in \sigma_-(f)\), it holds that \(\beta_i \geq 0\) for all \(i = 1, \ldots, n\) and \((1, \ldots, 1) \cdot \beta \leq 1\).

**Proof.** By [26] Theorem 7, hypotheses (a) and (b) imply that each term \(c_\alpha x^\alpha, \alpha \in \sigma_+(f)\) and \(c_\beta x^\beta, \beta \in \sigma_-(f)\) is convex. The result follows from the fact that the sum of convex functions is convex.

We proceed to interpret the conditions in Lemma 4.1 geometrically.

**Definition 4.2.** Given an \(n\)-simplex \(P \subseteq \mathbb{R}^n\) with vertices \(\{\mu_0, \ldots, \mu_n\}\), define the outer cone at the vertex \(\mu_k\) as

\[P^{out,k} := \left\{ \sum_{i=0}^n \lambda_i \mu_i \mid \sum_{i=0}^n \lambda_i = 1, \lambda_i \leq 0 \text{ for all } i \neq k \right\}, \quad k = 0, \ldots, n.\]

Note that it follows that \(\lambda_k > 0\). We write \(P^{out}\) for the union of the outer cones \(P^{out,0}, \ldots, P^{out,n}\).

Every \(n\)-simplex \(P \subseteq \mathbb{R}^n\) has \(n + 1\) facets, each face \(F\) is supported on a hyperplane \(H_{\mu_F, a_F}\), and then \(P = \bigcap_{F \subseteq P} \text{facet } H_{\mu_F, a_F}\). Fig. 6(a) shows an example in the plane. The next proposition gives a geometric interpretation of \(P^{out}\).

**Proposition 4.3.** Let \(P = \text{Conv}(\{\mu_0, \ldots, \mu_n\}) \subseteq \mathbb{R}^n\) be an \(n\)-simplex. A point \(\alpha \in \mathbb{R}^n\) belongs to \(P^{out,k}\) for \(k \in \{0, \ldots, n\}\), if and only if \(\alpha \in H_{\mu_F, a_F}^+\) for all facets \(F\) of \(P\) containing \(\mu_k\). In that case, it holds \(\alpha \in H_{\mu_F, a_F}^-\) for the facet \(F\) not containing \(\mu_k\).

**Proof.** Denote by \(F_i\) the facet of \(P\) that does not contain \(\mu_i\) and \(H_{\mu_i, a_i}\) a supporting hyperplane. In particular it holds that

\[v_j \cdot \mu_i = a_j \text{ for } i \neq j \quad \text{and} \quad v_i \cdot \mu_i < a_i, \quad \text{for } i = 0, \ldots, n.\]

The condition in the statement is equivalent to the existence of \(k \in \{0, \ldots, n\}\) such that

\[v_i \cdot \alpha \geq a_i \quad \text{for } i \neq k.\]

Write \(\alpha = \sum_{j=0}^n \lambda_j \mu_j\) for \(\lambda_0, \ldots, \lambda_n \in \mathbb{R}\) such that \(\sum_{j=0}^n \lambda_j = 1\). Then

\[v_i \cdot \alpha = \sum_{j=0}^n \lambda_j (v_i \cdot \mu_j) = \lambda_i (v_i \cdot \mu_i) + \sum_{j=0, j \neq i}^n \lambda_j a_i = \lambda_i (v_i \cdot \mu_i) + (1 - \lambda_i) a_i = \lambda_i (v_i \cdot \mu_i - a_i).\]
Using this, condition (15) holds for some $k$ if and only if
\[ \lambda_i(v_i \cdot \mu_i - a_i) \geq 0 \quad \text{for} \quad i \neq k. \]
By (14), this holds if and only if $\lambda_i \leq 0$ for $i \neq k$, that is, if and only if $\alpha \in P^{out,k} \subseteq P^{out}$. As then, $\lambda_k \geq 0$, (16) gives that $v_i \cdot \alpha < a_k$ giving $\alpha \in \mathcal{H}_{v_k,a_k}$. \hfill $\square$

We write $\Delta_n := \text{Conv}\{e_0, e_1, \ldots, e_n\}$ for the standard $n$-simplex in $\mathbb{R}^n$, where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$ and $e_0$ denotes the zero vector.

**Lemma 4.4.** Let $f : \mathbb{R}^n_{\geq 0} \rightarrow \mathbb{R}$ be a signomial. If $\sigma_-(f) \subseteq \Delta_n$ and $\sigma_+(f) \subseteq \Delta_n^{out}$, then $f$ is a convex function.

**Proof.** We show that the conditions in Lemma 4.1 are equivalent to $\sigma_-(f) \subseteq \Delta_n$ and $\sigma_+(f) \subseteq \Delta_n^{out}$. For $\beta \in \mathbb{R}^n$, find the unique $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=0}^n \lambda_i e_i = \beta$ and $\sum_{i=0}^n \lambda_i = 1$. Note that $(1, \ldots, 1) \cdot \beta = \sum_{i=1}^n \lambda_i = 1 - \lambda_0$, which is at most 1 if and only if $\lambda_0 \geq 0$.

Lemma 4.1(b) holds if and only if $\lambda_i \geq 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n \lambda_i \leq 1$. Equivalently, $\lambda_i \geq 0$ for all $i = 1, \ldots, n$ and $\lambda_0 \geq 0$, that is, $\beta \in \Delta_n$. We show now that $\beta \in \Delta_n^{out}$ if and only if Lemma 4.1(a) holds. By definition, $\beta \in \Delta_n^{out}$ if and only if for some $k$,
\[ \lambda_i \leq 0 \quad \text{for} \quad i \neq k. \]

For $k = 0$, (17) holds if and only if $\beta_i \leq 0$ for all $i$, thus Lemma 4.1(a,i) holds. For $k > 0$, (17) holds, if and only if all but the $k$-th coordinate of $\beta$ are non-positive, and $\lambda_0 \leq 0$, equivalently $(1, \ldots, 1) \cdot \beta \geq 1$, which is Lemma 4.1(a,ii). This concludes the proof. \hfill $\square$

We next look into what signomials can be transformed into a convex signomial using the transformations from Lemma 2.3. The next lemma is particularly useful.

**Lemma 4.5.** Let $P, Q \subseteq \mathbb{R}^n$ be $n$-simplices. For every $B \subseteq P$ and $A \subseteq P^{out}$, there exist an invertible matrix $M \in \text{GL}_n(\mathbb{R})$ and a vector $v \in \mathbb{R}^n$ such that $MB + v \subseteq Q$ and $MA + v \subseteq Q^{out}$.

**Proof.** Denote by $\{p_0, \ldots, p_n\}$ and $\{q_0, \ldots, q_n\}$ the vertex sets of $P$ and $Q$ respectively. Since $P$ and $Q$ are simplices, there is an invertible matrix $M \in \text{GL}_n(\mathbb{R})$ such that $M(p_i - p_0) = q_i - q_0$ for $i = 1, \ldots, n$. Define $v := -Mp_0 + q_0$. By construction, it holds that $Mp_i + v = q_i$ for every $i = 0, \ldots, n$.

For each $\mu \in \mathbb{R}^n$, write $\mu = \sum_{i=0}^n \lambda_i p_i$ with $\sum_{i=0}^n \lambda_i = 1$. It holds that
\[ M\mu + v = \sum_{i=0}^n \lambda_i Mp_i + \sum_{i=0}^n \lambda_i v = \sum_{i=0}^n \lambda_i (Mp_i + v) = \sum_{i=0}^n \lambda_i q_i. \]
That is, the coordinates of $\mu$ according to $P$ and those of $M\mu + v$ according to $Q$ are the same.

From this the statement follows. \hfill $\square$

**Theorem 4.6.** Let $f : \mathbb{R}^n_{\geq 0} \rightarrow \mathbb{R}$ be a signomial. If there exists an $n$-simplex $P$ such that
\[ \sigma_-(f) \subseteq P, \quad \text{and} \quad \sigma_+(f) \subseteq P^{out}, \]
then $f^{-1}(\mathbb{R}_{<0})$ is path connected.

**Proof.** By Lemma 4.5 with $B = \sigma_-(f)$ and $A = \sigma_+(f)$, there exists $M \in \text{GL}_n(\mathbb{R})$ and $v \in \mathbb{R}^n$ such that $M\sigma_-(f) + v \subseteq \Delta_n$ and $M\sigma_+(f) + v \subseteq \Delta_n^{out}$. By Lemma 2.3 $\sigma_+(F_{M,v,f}) = M\sigma_+(f) + v$ and $\sigma_-(F_{M,v,f}) = M\sigma_-(f) + v$. Hence by Lemma 4.4 $F_{M,v,f}$ is a convex function and thus $F_{M,v,f}^{-1}(\mathbb{R}_{<0})$ is path connected. By Lemma 2.3 again, $f^{-1}(\mathbb{R}_{<0})$ is also path connected. \hfill $\square$

In view of Theorem 4.6 understanding $P^{out}$ for a simplex $P$ allows us to determine whether $f$ can be transformed to a convex function.

**Example 4.7.** Consider the signomial
\[ p_5(x_1, x_2) = x_1^2 x_2^2 + x_1 x_2^2 - 2 x_1^2 x_2^2 - 3 x_1^2 x_2^2 + x_1 x_2^3 + x_2^4 - x_1 x_2 + 1 \]
and the simplex $P = \text{Conv}((1,1),(4,2),(1,3))$. We have $\sigma_- (p_5) \subseteq P$ and $\sigma_+ (p_5) \subseteq P^{out}$, see Fig. 6. By Theorem 4.6 the set $p_5^{-1}(\mathbb{R}_{<0})$ is path connected.
A direct consequence of Theorem 4.6 states that if all positive exponents are vertices of the Newton polytope and this is a simplex, then \( f^{-1}(\mathbb{R}^-) \) is path connected. Let \( \text{Vert}(N(f)) \) denote the set of vertices of \( N(f) \).

**Corollary 4.8.** Let \( f : \mathbb{R}^n_+ \to \mathbb{R} \) be a signomial. If \( \sigma_+(f) \subseteq \text{Vert}(N(f)) \) and \( N(f) \) is a simplex, then \( f^{-1}(\mathbb{R}^-) \) is path connected.

**Proof.** Let \( d := \dim N(f) \) and denote by \( e_1, \ldots, e_d \) the first \( d \) standard basis vectors of \( \mathbb{R}^n \). Without loss of generality, we can assume that \( \sigma(f) \subseteq \text{Lin}(e_1, \ldots, e_d) \subseteq \mathbb{R}^n \), as this can be achieved via a change of variables as in Lemma 2.3. Hence \( f \) depends only on the variables \( x_1, \ldots, x_d \), and can be seen as a signomial in \( \mathbb{R}^d_+ \) with full dimensional Newton polytope. Viewing \( V^>(c) \) in \( \mathbb{R}^d_+ \), the statement of the corollary follows from Theorem 4.6, since \( \sigma_+(f) \subseteq \text{Vert}(N(f)) \subseteq N(f)^{\text{out}} \) and \( \sigma_-(f) \subseteq N(f) \).

The proof is completed noticing that the pre-image of a connected subset of \( \mathbb{R}^d_+ \) under the projection map \( (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_d) \) is connected in \( \mathbb{R}^n_+ \). \( \square \)

### 5. Comparing the Different Approaches

Theorems 3.5, 3.9, 3.11, 4.6 cover some cases of a generalization of Descartes’ rule of signs to hypersurfaces. In particular, we have shown that \( f^{-1}(\mathbb{R}^-) \) is path connected in the following relevant cases:

- \( f \) has at most one negative exponent.
- There exists a strict separating vector.
- There exists a simplex \( P \) such that negative exponents belong to \( P \) and positive exponents to \( P^{\text{out}} \); in particular if all positive exponents are vertices of the Newton polytope and this is a simplex.

The techniques to study the case where \( f^{-1}(\mathbb{R}^-) \) is path connected could also be used to derive a condition for \( f^{-1}(\mathbb{R}^-) \) having at most two connected components:

- There exists a strict enclosing vector of \( \sigma(-f) \); in particular if the positive exponents belong to a hyperplane that does not contain all negative exponents, or if the number of positive exponents is smaller than \( \dim N(f) \).

Theorem 3.5 and Theorem 4.6 cover all the cases where the classical Descartes’ rule guarantee that the number of negative connected components of \( V^>(c) \) is at most one. These are the cases when the coefficients of the one-variable signomial \( f \) has one of the following sign patterns:

\((-\cdots-+\cdots+)\quad (+\cdots-\cdots-\cdots+)\quad (+\cdots+\cdots-\cdots+)\).
In fact, the conditions in Theorem 3.5 and Theorem 4.6 are equivalent for \( n = 1 \). One should also note that in the one-dimensional case, a separating vector is automatically a strict separating vector. Thus, it seems that Theorem 3.5, 3.9, 4.6 are strongly related, if the signomial has only one variable.

Although Theorems 3.5, 3.9, and 4.6 build on apparently different techniques, we show in this section that their respective conditions are equivalent in several situations. Computationally, checking whether Theorem 3.9 applies is less demanding than to verifying that the conditions of Theorem 4.6 hold.

For \( n > 1 \), a first inspection of the examples considered so far gives an idea of what relations not to expect. The existence of a strict separating vector does not guarantee that Theorem 3.5 applies for the signomial \( P \). Particularly, the existence of a strict separating vector does not imply the existence of an \( n \)-simplex as in Theorem 4.6 implies the existence of a separating vector.

**Proposition 5.1.** Let \( f : \mathbb{R}_+^n \to \mathbb{R} \) be a signomial and let \( P \subseteq \mathbb{R}^n \) be an \( n \)-simplex such that \( \sigma_+(f) \subseteq P \) and \( \sigma_-(f) \subseteq \text{pout} \). If there exists \( k \in \{1, \ldots, n\} \) such that \( P_{\text{out},k} \cap \sigma_+(f) = \emptyset \), then \( \sigma(f) \) has a separating vector. Moreover, there is a strict separating vector if there is a negative exponent in \( P \setminus F_k \), where \( F_k \) denotes the facet of \( P \) opposite to \( P_{\text{out},k} \).

**Proof.** Let \( H_{w_0,a_k} \) be a supporting hyperplane for the facet \( F_k \). By hypothesis and from Proposition 4.3 we obtain \( \sigma_+(f) \subseteq H_{w_0,a_k}^\circ \). By hypothesis we also have that \( \sigma_-(f) \subseteq P \subseteq H_{w_0,a_k}^{-}\). Therefore, \( -v_k \) is a separating vector of \( \sigma(f) \). If there is a negative exponent \( \beta_0 \notin F_k \), then \( v_k \cdot \beta_0 < a_k \) giving that \( -v_k \) is strict.

We inspect now whether or when Theorem 3.9 follows from Theorem 4.6, in which case we state that in the one-dimensional case, a separating vector is automatically a strict separating vector. Thus, it seems that Theorem 3.5, 3.9, 4.6 are strongly related, if the signomial has only one variable.

**Lemma 5.2.** Let \( \{H_{w_0,a_0}, \ldots, H_{w_n,a_n}\} \) be a set of hyperplanes of \( \mathbb{R}^n \) such that:

(i) Every proper subset of \( \{w_0, \ldots, w_n\} \) is linearly independent.

(ii) For every \( i \in \{0, \ldots, n\} \) it holds that \( \bigcap_{j=0, j \neq i}^n H_{w_j,a_j} \subseteq H_{w_i,a_i}^{-}\).

Then \( \bigcap_{j=0}^n H_{w_j,a_j} \) is an \( n \)-simplex.

**Proof.** First, note that (ii) implies

\[(ii') \quad \bigcap_{j=0}^n H_{w_j,a_j} = \emptyset.\]

As a finite intersection of closed half-spaces, \( P := \bigcap_{j=0}^n H_{w_j,a_j} \) is a convex polyhedron. Each face of \( P \) has the form

\[P_I = P \cap H_I, \quad H_I = \{x \in \mathbb{R}^n \mid \forall i \in I : \ w_i \cdot x = a_i\},\]

for some non-empty subset \( I \subseteq \{0, \ldots, n\} \). By (i) and (ii'), \( H_I \) is zero dimensional if and only if \( I \) has \( n \) elements. By (ii), for \( I = \{0, \ldots, n\} \setminus \{i\} \), \( P_I \neq \emptyset \) and hence \( P_I \) is a vertex of \( P \), denoted by \( u_i \). Furthermore, the points \( u_0, \ldots, u_n \) are affinely independent. This follows from (ii'), as for each \( k, \mu_i \in H_{w_k,a_k}^{-}\) for \( i \neq k \) and \( \mu_k \notin H_{w_k,a_k}^{-}\). Hence Conv(\( u_0, \ldots, u_n \)) is an \( n \)-simplex. Finally, \( P = \text{Conv}(u_0, \ldots, u_n) \) as \( H_{w_i,a_i}^{-}\) contains a vertex for each \( k \).

The existence of a strict separating vector does not imply the existence of an \( n \)-simplex satisfying the condition in Theorem 4.6. To see this, we consider the signomial \( p_3 \) in Example 3.10. The positive exponent \( (3, 4) \) lies in the convex hull of the negative exponents, and is not a vertex. Therefore, there is no \( n \)-simplex \( P \) such that \( \sigma_-(p_3) \subseteq P \) and \( (3, 4) \in \text{pout} \).
Hence, this subcase of Theorem 3.5 follows from Theorem 4.6.

Proposition 5.3. Let \( f: \mathbb{R}_0^n \to \mathbb{R} \) be a signomial. If there is a very strict separating vector \( v \in \mathbb{R}^n \) of \( \sigma(f) \), then there exists an \( n \)-simplex \( P \) such that \( \sigma_-(f) \subseteq P \) and \( \sigma_+(f) \subseteq \text{Pout} \).

Proof. By Proposition 3.8 there exist \( n \) linearly independent very strict separating vectors \( -w_1, \ldots, -w_n \), and \( c \in \mathbb{R}^n \) such that

\[
\sigma_-(f) \subseteq \bigcap_{i=1}^{n} \mathcal{H}_{w_i, c}^- \quad \text{and} \quad \sigma_+(f) \subseteq \bigcap_{i=1}^{n} \mathcal{H}_{w_i, c}^+.
\]

We consider minus the basis in Proposition 3.8 as separating vectors leave the negative exponents on the positive side of the hyperplane, while the simplex \( P \) leaves them on the negative side of the defining hyperplanes.

We define \( w_0 := -\sum_{i=1}^{n} w_i \), choose \( a_0 \in \mathbb{R} \) such that \( a_0 > \max_{\mu \in \sigma(f)} w_0 \cdot \mu \) and define \( P := \mathcal{H}_{w_0, a_0}^+ \cap \bigcap_{i=1}^{n} \mathcal{H}_{w_i, c}^- \). It then holds that \( \sigma_-(f) \) and \( \sigma_+(f) \) belong to \( \mathcal{H}_{w_0, a_0}^- \). Thus, \( \sigma_-(f) \subseteq P \) and \( \sigma_+(f) \subseteq \text{Pout} \) by Proposition 4.3.

All that is left is to show that \( P \) is an \( n \)-simplex. To this end, we apply Lemma 5.2. It is clear that every subset of \( \{w_0, \ldots, w_n\} \) with \( n \) elements is linearly independent, so Lemma 5.2(i) holds. From (18) follows that

\[
n(-c) \leq \max_{\beta \in \sigma_-(f)} \sum_{i=1}^{n} -w_i \cdot \beta = \max_{\beta \in \sigma_-(f)} w_0 \cdot \beta \leq \max_{\mu \in \sigma(f)} w_0 \cdot \mu < a_0.
\]

For \( x \in \bigcap_{j=1}^{n} \mathcal{H}_{w_j, c}^- \), we obtain \( w_0 \cdot x = -nc < a_0 \), so \( x \in \mathcal{H}_{w_0, a_0}^- \). If \( x \in \mathcal{H}_{w_0, a_0}^- \cap \bigcap_{j=1, j \neq i}^{n} \mathcal{H}_{w_j, c}^- \), again by (19) we have that

\[
w_i \cdot x = -w_0 \cdot x - \sum_{j=1, j \neq i}^{n} w_j \cdot x = -a_0 - (n-1)c < nc - (n-1)c = c.
\]

Hence \( x \in \mathcal{H}_{w_i, c}^- \) for each \( i \in \{1, \ldots, n\} \). We conclude that Lemma 5.2(ii) holds, so \( P \) is an \( n \)-simplex and this completes the proof.

Propositions 3.16(i) and 5.3 show that if \( \text{Lin}(\mathcal{S}^-) = \mathbb{R}^n \), then Theorem 4.6 also applies. Hence, this subcase of Theorem 3.5 follows from Theorem 4.6.
Example 5.4. Let \( f: \mathbb{R}^2_{\gamma_0} \to \mathbb{R} \) be a signomial with only one negative exponent \( \beta_0 \). If \( \sigma_+(f) \) is equal to the vertex set of a regular \( m \)-gon for some \( m \geq 7 \) with circumcenter \( \beta_0 \), then there does not exist a simplex \( P \) such that \( \sigma_-(f) \subseteq P \) and \( \sigma_+(f) \subseteq P_{\text{out}} \).

To see this, assume that such a simplex exists and write \( P = H_{w_0,b_0}^{-} \cap H_{w_1,b_1}^{-} \cap H_{w_2,b_2}^{-} \), with \( w_0, w_1, w_2 \in \mathbb{R}^2 \), and \( b_0, b_1, b_2 \in \mathbb{R} \). For \( a_i := w_i \cdot \beta_0, i = 0, 1, 2 \), the three lines \( H_{w_0,a_0}, H_{w_1,a_1} \), and \( H_{w_2,a_2} \), intersect each other at \( \beta_0 \) and divide the circumsphere of the \( m \)-gon into 6 regions.

Let \( \gamma_0, \gamma_1, \gamma_2 \in [0, \pi] \) be the angles of the regions cut out by \( H_{w_0,a_0} \), \( H_{w_1,a_1} \), and \( H_{w_2,a_2} \). From this follows that \( \gamma_0 + \gamma_1 + \gamma_2 = \pi \). Since \( \sigma_+(f) \subseteq P_{\text{out}} \), the positive points are in alternating regions. Therefore one of the two regions cut out by \( H_{w_0,a_0} \) and \( H_{w_1,a_1} \) with angle \( \gamma_0 \) cannot contain any positive point. Since \( \sigma_+(f) \) is the vertex set of a regular \( m \)-gon, for each pair of consecutive positive point \( \alpha_i, \alpha_{i+1} \) (counted counterclockwise), the angle \( \angle \alpha_i \beta_0 \alpha_{i+1} \) equals \( \frac{2\pi}{m} \). From this follows that \( \gamma_0 \leq \frac{2\pi}{m} \). A similar argument shows that \( \gamma_1 \leq \frac{2\pi}{m} \) and \( \gamma_2 \leq \frac{2\pi}{m} \). We conclude that \( \gamma_0 + \gamma_1 + \gamma_2 \leq \frac{6\pi}{m} \). Since \( m \geq 7 \), this contradicts \( \gamma_0 + \gamma_1 + \gamma_2 = \pi \). Therefore, such a simplex \( P \) does not exist.

Since the set of enclosing vectors \( \mathcal{E}^-(f) \) is a union of lines but not a vector space, the only other case where Theorem 3.5 is informative, is when \( \dim \text{Lin} (S^-(f)) = n - 1 \) and \( \mathcal{E}^-(f) \) contains a one dimensional subspace complementing \( \text{Lin} (S^-(f)) \) to \( \mathbb{R}^n \). Example 3.6 illustrates this situation.

If \( \text{Lin} (S^-(f)) \) contains a strict separating vector, then we can invoke Theorem 3.9. If that is not the case, then necessarily the negative points lie on a line. The existence of an enclosing vector then guarantees that the segment defined by the negative points does not contain a positive point. We show in the last proposition of this work that, in this scenario, a simplex satisfying the assumptions of Theorem 4.6 exists. Fig. 8 illustrates this scenario. Therefore, Theorem 3.5 follows from both Theorem 3.9 and Theorem 4.6 except when \( f \) has at most one negative point.

Proposition 3.16(ii) characterizes when \( \mathcal{E}^-(f) = \mathbb{R}^n \). In particular, this holds when \( f \) has exactly one negative exponent. This example cannot be covered by Theorem 4.6 as the following example illustrates.

**Example 5.4.** Let \( f: \mathbb{R}^2_{\gamma_0} \to \mathbb{R} \) be a signomial with only one negative exponent \( \beta_0 \). If \( \sigma_+(f) \) is equal to the vertex set of a regular \( m \)-gon for some \( m \geq 7 \) with circumcenter \( \beta_0 \), then there does not exist a simplex \( P \) such that \( \sigma_-(f) \subseteq P \) and \( \sigma_+(f) \subseteq P_{\text{out}} \).

To see this, assume that such a simplex exists and write \( P = H_{w_0,b_0}^{-} \cap H_{w_1,b_1}^{-} \cap H_{w_2,b_2}^{-} \), with \( w_0, w_1, w_2 \in \mathbb{R}^2 \), and \( b_0, b_1, b_2 \in \mathbb{R} \). For \( a_i := w_i \cdot \beta_0, i = 0, 1, 2 \), the three lines \( H_{w_0,a_0}, H_{w_1,a_1} \), and \( H_{w_2,a_2} \), intersect each other at \( \beta_0 \) and divide the circumsphere of the \( m \)-gon into 6 regions.

Let \( \gamma_0, \gamma_1, \gamma_2 \in [0, \pi] \) be the angles of the regions cut out by \( H_{w_0,a_0} \), \( H_{w_1,a_1} \), and \( H_{w_2,a_2} \), respectively. Note that \( \gamma_0 + \gamma_1 + \gamma_2 = \pi \). Since \( \sigma_+(f) \subseteq P_{\text{out}} \), the positive points are in alternating regions. Therefore one of the two regions cut out by \( H_{w_0,a_0} \) and \( H_{w_1,a_1} \) with angle \( \gamma_0 \) cannot contain any positive point. Since \( \sigma_+(f) \) is the vertex set of a regular \( m \)-gon, for each pair of consecutive positive point \( \alpha_i, \alpha_{i+1} \) (counted counterclockwise), the angle \( \angle \alpha_i \beta_0 \alpha_{i+1} \) equals \( \frac{2\pi}{m} \). From this follows that \( \gamma_0 \leq \frac{2\pi}{m} \). A similar argument shows that \( \gamma_1 \leq \frac{2\pi}{m} \) and \( \gamma_2 \leq \frac{2\pi}{m} \). We conclude that \( \gamma_0 + \gamma_1 + \gamma_2 \leq \frac{6\pi}{m} \). Since \( m \geq 7 \), this contradicts \( \gamma_0 + \gamma_1 + \gamma_2 = \pi \). Therefore, such a simplex \( P \) does not exist.

Since the set of enclosing vectors \( \mathcal{E}^-(f) \) is a union of lines but not a vector space, the only other case where Theorem 3.5 is informative, is when \( \dim \text{Lin} (S^-(f)) = n - 1 \) and \( \mathcal{E}^-(f) \) contains a one dimensional subspace complementing \( \text{Lin} (S^-(f)) \) to \( \mathbb{R}^n \). Example 3.6 illustrates this situation.

If \( \text{Lin} (S^-(f)) \) contains a strict separating vector, then we can invoke Theorem 3.9. If that is not the case, then necessarily the negative points lie on a line. The existence of an enclosing vector then guarantees that the segment defined by the negative points does not contain a positive point. We show in the last proposition of this work that, in this scenario, a simplex satisfying the assumptions of Theorem 4.6 exists. Fig. 8 illustrates this scenario. Therefore, Theorem 3.5 follows from both Theorem 3.9 and Theorem 4.6 except when \( f \) has at most one negative point.
Proof. Let $w_1, \ldots, w_{n-1}$ be linearly independent non-strict separating vectors. Then with $a_i := \max\{w_i \cdot \alpha \mid \alpha \in \sigma_+(f)\}$, it holds

$$\sigma_+(f) \subseteq \bigcap_{i=1}^{n-1} \mathcal{H}_{w_i,a_i} \quad \text{and} \quad \sigma_-(f) \subseteq L \quad \text{with} \quad L := \bigcap_{i=1}^{n-1} \mathcal{H}_{w_i,a_i}. \tag{20}$$

If $L$ contains all positive exponents, then any simplex $P$ having as edge Conv$(\sigma_-(f))$, satisfies the statement. Hence, we assume that this is not the case. We prove the proposition by applying Lemma 5.2. We introduce the following:

$$v := \sum_{i=1}^{n-1} w_i \in \mathbb{R}^{n-1}, \quad d := \sum_{i=1}^{n-1} a_i \in \mathbb{R}, \quad K := \max\{v \cdot \alpha \mid \alpha \in \sigma_+(f), \; v \cdot \alpha \neq d\} \in \mathbb{R}. \tag{21}$$

By assumption, $\epsilon := d - K > 0$ and we have $\sigma_-(f) \subseteq \mathcal{H}_{v,d}$. Let $z \in \mathbb{R}^n$ such that $z, w_1, \ldots, w_{n-1}$ are linearly independent, and denote by $\beta_0, \beta_1$ the vertices of Conv$(\sigma_-(f))$ where the linear form induced by $z$ attains its minimum and its maximum respectively. These vertices are different, otherwise each $\beta \in \text{Conv}(\sigma_-(f))$ would be the unique solution of $z \cdot \beta = z \cdot \beta_0, w_i \cdot \beta = a_i, i = 1, \ldots, n-1$. This would be a contradiction, since Conv$(\sigma_-(f))$ contains at least two elements.

We let $M := \max\{z \cdot \alpha \mid \alpha \in \sigma_+(f)\}$, choose $\lambda > \mu$ positive real numbers such that

$$\lambda(M - z \cdot \beta_0) \leq \epsilon = d - K, \quad \mu(M - z \cdot \beta_1) \leq \epsilon = d - K, \tag{22}$$

and define $w_0 := v + \lambda z, w_n := -v - \mu z, a_0 := d + \lambda(z \cdot \beta_0), a_n := -d - \mu(z \cdot \beta_1)$. By construction, $\beta_0 \in \mathcal{H}_{-w_0,-a_0}$ and $\beta_1 \in \mathcal{H}_{-w_n,-a_n}$.

We show that $P := \bigcap_{j=1}^{n-1} \mathcal{H}_{-w_j,-a_j}$ is an n-simplex using Lemma 5.2 and satisfies the hypotheses of the statement. Lemma 5.2(ii) holds by construction. To show Lemma 5.2(ii), we consider first $i \in \{0, n\}$. As

$$\bigcap_{j=0}^{n-1} \mathcal{H}_{-w_j,-a_j} = \{\beta_0\}, \quad \bigcap_{j=1}^{n} \mathcal{H}_{-w_j,-a_j} = \{\beta_1\}, \tag{23}$$

it suffices to show that $\beta_0 \in \mathcal{H}_{-w_n,-a_n}$ and $\beta_1 \in \mathcal{H}_{-w_0,-a_0}$. For each $\beta \in \sigma_-(f)$, it holds that

$$w_n \cdot \beta = -v \cdot \beta - \mu(z \cdot \beta) \geq -d - \mu(z \cdot \beta_1) = a_n, \quad \text{and} \quad w_0 \cdot \beta = v \cdot \beta + \lambda(z \cdot \beta) \geq d + \lambda(z \cdot \beta_0) = a_0 \tag{24}$$

as $z$ attains its minimum resp. its maximum on Conv$(\sigma_-(f))$ at $\beta_0$ resp. at $\beta_1$ and $\lambda, \mu > 0$. From these we get that $\beta_0 \in \mathcal{H}_{-w_n,-a_n}$ and $\beta_1 \in \mathcal{H}_{-w_0,-a_0}$ since $z \cdot \beta_1 > z \cdot \beta_0$ and hence the inequalities in (23) and (24) are strict.

Consider now $i \in \{1, \ldots, n-1\}$ and $x \in \bigcap_{j=0,j \neq i}^{n} \mathcal{H}_{-w_i,-a_i}$. In particular, $x \in \mathcal{H}_{w_0,a_0} \cap \mathcal{H}_{w_n,a_n}$. Solving the linear system $w_0 \cdot x = v \cdot x + \lambda(z \cdot x) = a_0$ and $w_n \cdot x = -v \cdot x - \mu(z \cdot x) = a_n$ for $v \cdot x$ and $z \cdot x$ and using the definition of $a_0, a_n$, we obtain

$$z \cdot x = \frac{a_n + a_0}{\lambda - \mu}, \quad v \cdot x = a_0 - \lambda \beta_0 = d + \frac{\lambda}{\lambda - \mu} \beta_1 > d, \tag{25}$$

as $\lambda, \mu, \lambda - \mu, z \cdot \beta_1 - z \cdot \beta_0 > 0$. Hence

$$\sum_{j=1}^{n-1} w_j \cdot x = v \cdot x > d = \sum_{j=1}^{n} a_j. \tag{26}$$

From this follows that $w_j \cdot x > a_j$, since $w_j \cdot x = a_j$ for $j \neq i$. Therefore $x \in \mathcal{H}_{-w_i,-a_i}$ and Lemma 5.2(ii) holds. We conclude that $P$ is an n-simplex.

Finally, we show that $\sigma_-(f) \subseteq P$ and $\sigma_+(f) \subseteq P^{\text{out}}$. The inclusion $\sigma_-(f) \subseteq P$ follows from (20), (23) and (24).

Let $\alpha \in \sigma_+(f)$ and assume that $v \cdot \alpha < d$. By (21),

$$w_0 \cdot \alpha = v \cdot \alpha + \lambda(z \cdot \alpha) \leq K + \lambda M = d - \epsilon + \lambda M \leq d + \lambda(z \cdot \beta_0) = a_0, \tag{27}$$

which implies $\alpha \in \mathcal{H}_{-w_0,-a_0}$. This together with (20) imply that $\alpha \in P^{\text{out}}$ by Proposition 4.3.
Now, consider the case $v \cdot \alpha = d$. In this case, \[ (20) \] implies that $w_i \cdot \alpha = a_i$ for each $i = 1, \ldots, n - 1$. Thus, $\alpha \in L$ and recall $\alpha \notin \text{Conv}(\sigma_-(f))$. Hence $\alpha \in L \setminus \text{Conv}(\sigma_-(f)) \subseteq P_{\text{out}}$, where the last inclusion follows from the fact that the supporting hyperplanes of each cone $P_{\text{out},k}$ are supporting hyperplanes as $P$. \qed

**References**


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