Scheme for Universal High-Dimensional Quantum Computation with Linear Optics

Paesani, Stefano; Bulmer, Jacob F. F.; Jones, Alex E.; Santagati, Raffaele; Laing, Anthony

Published in:
Physical Review Letters

DOI:
10.1103/PhysRevLett.126.230504

Publication date:
2021

Document version
Publisher's PDF, also known as Version of record

Document license:
CC BY

Citation for published version (APA):
Scheme for Universal High-Dimensional Quantum Computation with Linear Optics

Stefano Paesani,1,2,* Jacob F. F. Bulmer,1 Alex E. Jones,1 Raffaele Santagati,1,3 and Anthony Laing1

1 Quantum Engineering Technology Labs, H. H. Wills Physics Laboratory and Department of Electrical and Electronic Engineering, University of Bristol, Bristol BS8 1FD, United Kingdom

2 Center for Hybrid Quantum Networks (Hy-Q), Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

3 International Iberian Nanotechnology Laboratory (INL), Avenida Mestre José Veiga 4715-330 Braga, Portugal

(Received 25 October 2020; accepted 19 March 2021; published 11 June 2021)

Photons are natural carriers of high-dimensional quantum information, and, in principle, can benefit from higher quantum information capacity and noise resilience. However, schemes to generate the resources required for high-dimensional quantum computing have so far been lacking in linear optics. Here, we show how to generate GHZ states in arbitrary dimensions and numbers of photons using linear optical circuits described by Fourier transform matrices. Combining our results with recent schemes for qudit Bell measurements, we show that universal linear optical quantum computing can be performed in arbitrary dimensions.

DOI: 10.1103/PhysRevLett.126.230504

Photons are natural carriers of high-dimensional quantum information, and, in principle, can benefit from higher quantum information capacity and noise resilience. However, schemes to generate the resources required for high-dimensional quantum computing have so far been lacking in linear optics. Here, we answer this question by showing that all high-dimensional entangled states with fixed numbers of photons can be generated with linear optics and, in principle, with a scalable architecture. That is, we show that universal linear optical quantum computing (LOQC) is possible in arbitrary dimensions. Key to this result are linear optical schemes for the generation of heralded N-photon GHZ entanglement in arbitrary dimension d. Combining these schemes with previous results on Bell measurements with photonic qudits to fuse d-dimensional GHZ states, and techniques from qubit LOQC architectures, we obtain a scheme for universal measurement-based quantum computing with photonic qudits.

Heralded high-dimensional GHZ entanglement.—The GHZ states for N photons, each encoding a qudit of dimension d, are defined as $|\text{GHZ}(N,d)\rangle = \sum_{k=0}^{d-1} |k\rangle^\otimes N / \sqrt{d}$, and represent the initial resource states in an architecture that builds cluster states of qudits for universal quantum computing. Our scheme to generate these states, shown schematically in Fig. 1(a), consists of single photon sources, linear optical elements, and photon-number resolving detectors. Our scheme exploits the zero-transmission law (ZTL) in discrete Fourier transform (DFT) interferometers (previously investigated, e.g., for the verification of boson sampling protocols [19,25,26]). This law is a generalization of the Hong-Ou-Mandel interference that suppresses output coincidences from a balanced beam splitter (two-mode DFT). The ZTL result states that, if m single photons are sent in the m individual input modes of an m-mode DFT, described by the unitary $U_{j,k} = \exp\{ik(2\pi i/m)\}/\sqrt{m}$, then the output configurations $\hat{c}$ with nonzero amplitude must satisfy

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.
The vector $\vec{c}$ has a number of elements given by the number of photons, here $m$. Each element $c_i$ labels the $i$th photon’s output mode, with modes indexed from 0 to $m - 1$. For example, $\vec{c} = (0, 1, 1, 4)$ represents the configuration with one photon in each of modes 0 and 4, two photons in mode 1, and zero photons in all remaining modes. The ZTL condition is valid for any value of $m$.

To describe how the ZTL can be used to generate GHZ states, we introduce a notation for the encoding of individual qudits in the output modes of the DFT. Each qudit of dimension $d$ is encoded in a single photon propagating through $d$ optical modes (see, e.g., Refs. [7,8]). We will denote as $X \subset \{0, \ldots, m - 1\}$ the set of $Nd$ modes used to encode the $N$ qudits, while its complement $\bar{X} = \{0, \ldots, m - 1\} \setminus X$ (the remaining set of $m - Nd$ modes) represents all the auxiliary modes used for heralding the desired state. Each of the $N$ qudits is defined via a set of $d$ distinct modes $Q_i = \{x_{0}^{(i)}, x_{1}^{(i)}, \ldots, x_{d-1}^{(i)}\} \subset X$, with $x_{k}^{(i)} \neq x_{k'}^{(i)}$ if $k \neq k'$, so that all the qudits $\{Q_i\}$ form a partition of $X$. We say that the $i$th qudit is in the logical state $|k_i\rangle$ if a single photon is present in mode $x_{k_i}^{(i)}$, while all other modes in the set $Q_i$ are vacuum.

For simplicity, we will focus on heralding configurations where all the $m - N$ auxiliary photons are detected in the zeroth mode, and vacuum in the remaining heralding modes $\bar{X} \setminus \{0\}$, as shown in Fig. 1(a). In this way, all auxiliary photons give zero contributions to the sum in Eq. (1), and only contributions from the $N$ encoding photons remain. Under this condition, we can write the set of all possible remaining $N$-photon output states induced by the heralding on $\bar{X}$ and the ZTL as

$$B_{\bar{X}} = \left\{ \bar{b}_k = (y_{k,1}, y_{k,2}, \ldots, y_{k,N}) \mid \sum_{i=1}^{N} y_{k,i} \mod m = 0, y_{k,i} \in X \right\}. \quad (2)$$

Here $\bar{b}_k$ are the output configurations allowed by the ZTL, indexed by $k$, and $y_{k,i}$ represent each photon’s output port for that configuration, as in Eq. (1).

We will now show that the following conditions for the set $B_{\bar{X}}$ are sufficient (but not necessary) to obtain a GHZ state at the output: (1) $B_{\bar{X}}$ contains exactly $d$ configurations, i.e., $|B_{\bar{X}}| = d$. (2) $B_{\bar{X}}$ forms a partition of $\bar{X}$. (3) For all $\bar{b}_k \in B_{\bar{X}}$, $\sum_{i=1}^{N} y_{k,i} = m$. This is a slightly more restrictive form of the ZTL. The first step is to show that, if these three conditions are satisfied, we can specify $N$ well-defined photonic qudits, given by

\[
\sum_{i=1}^{m} c_i \mod m = 0. \quad (1)
\]
success probability for the state generation is given by generated with linear optics. Supplemental Material 2 \[27\]. While this particular solutions for any \(m\) can easily be found by taking \(m = 2d + 1\) and \(\chi = \{1,2,\ldots,2d\}\). This gives \(B_X = \{(1,2d),(2,2d-1),\ldots,(d,d+1)\}\) and qudit modes \(Q_1 = \{1,2,\ldots,d\}\), \(Q_2 = \{2d,2d-1,\ldots,d+1\}\), as shown in Fig. 1(b). The success probability for the state generation is given by \(d(2d-1)!/(2d+1)^{2d-1}\) (see Supplemental Material 1 and 6 for additional schemes for qudit Bell states \[27\]). For small values of \(N\) and \(d\), optimized solutions for our heralded GHZ generation scheme can be found numerically. For example, in Fig. 1(c) we report the optimized solution for the case of heralded GHZ generation for \((N,d) = (3,3)\), which requires \(m = 25\). The list of modes used for the encoding in this case is \(\chi = \{1,2,3,4,5,9,13,16,22\}\), and the only three-element combinations (including repetitions) that sum up to multiples of \(m\), given by \(B_X = \{(1,2,22), (3,9,13), (4,5,16)\}\), satisfy conditions 1-3. This therefore provides heralded GHZ generation for the three qudits defined in the modes \(Q_1 = \{1,3,4\}\), \(Q_2 = \{2,9,5\}\), \(Q_3 = \{22,13,16\}\). Note that, at the output of the DFT, adjacent modes associated to different qudits can be interleaved; in such cases, a network of swaps is required to separate the different qudits and address them individually. However, without further optimization, the three-dimensional GHZ generation success probability remains very small, approximately \(10^{-10}\).

The main reason for low success probabilities is the use of a single heralding pattern among exponentially many possible outcomes. While the choice of a single heralding pattern was done to simplify the treatment in the general case of arbitrary \(N\) and \(d\), for a given \(N\) and \(d\) many more heralding patterns are likely to generate GHZ entanglement. We show in Supplemental Material 3 \[27\] that these (combinatorially many) valid heralding patterns can be used, in conjunction with feed-forward operations and balancing circuitry, to significantly improve the success probability. For example, Monte Carlo simulations of the \((N,d) = (3,3)\) GHZ generation scheme of Fig. 1(c) show that the success probability is boosted to approximately \(10^{-4}\). This indicates that many orders of magnitude improvements can be found through solution-specific optimizations. We estimate that, with this optimization, heralded \(3,3\) GHZ generation could be achieved at 0.7 kHz rates with state-of-the-art quantum photonic hardware. Furthermore, in Supplemental Material 2 \[27\] we report an algorithm to estimate optimized solutions for larger values of \(N\) and \(d\).

Constructing universal cluster states of qudits.—The DFT-based schemes described above can be generalized to generate a large class of \(N\)-photon states with dimensionality \(d\), but at the cost of exponentially large resources when increasing \(N\) (see Appendix 4 in the Supplemental Material \[27\]). However, due to the heralded nature of the generated high-dimensional entanglement, we can use these schemes to adapt qubit-based LOQC techniques (e.g., multiplexing and fusion operations \[5,46\]) to construct a high-dimensional architecture that is scalable. Measurement-based quantum computing (MBQC) \[47\] in linear optics typically proceeds by connecting small entangled resource states using probabilistic fusion gates \[24\], to build large cluster states. Similarly, high-dimensional photonic GHZ states can be used as building blocks to construct large
universal for high-dimensional MBQC\cite{49,50}. Figure 2(a) shows a module for the architecture, where multiple GHZ states are fused together to link a single computational state complexity. Nevertheless, for the scheme shown in Fig. 1(b) with $N = 2$ and low values of $d$, the simulation remains tractable on a standard laptop.

For various levels of distinguishability, we show in Fig. 3(a) the fidelity of the generated state to the ideal qudit Bell state, and in Fig. 3(b) the logarithmic negativity as a figure of merit to assess the generated entanglement.

Robustness against photon distinguishability.—While requiring additional resources compared to qubit-based approaches, qudits can provide improved robustness to noise, with potential benefits for quantum communication and fault-tolerant quantum technologies\cite{57–60}. Here, we numerically analyze how distinguishability, an important source of noise in quantum photonics, affects the multi-photon high-dimensional entanglement generated using the schemes proposed in Fig 1. We assume that all photons have pure internal states such that they all share the same value of pairwise indistinguishability $|\langle \psi_i | \psi_j \rangle |^2$. Thus all pairs of photons would exhibit the same Hong-Ou-Mandel interference visibility. To determine the effect of distinguishability, we numerically reconstruct the simulated output heralded state within the qudit space for different values of indistinguishability $|\langle \psi_i | \psi_j \rangle |^2$. The noisy state generation is simulated using an approach by Tichy describing multiphoton interference of partially distinguishable photons [61] (see Supplemental Material 7 for details [27]). Because the number of photons involved in the scheme increases rapidly with $N$ and $d$, this simulation quickly becomes intractable when increasing the heralded state complexity. Nevertheless, for the scheme shown in Fig. 1(b) with $N = 2$ and low values of $d$, the simulation remains tractable on a standard laptop.

For various levels of distinguishability, we show in Fig. 3(a) the fidelity of the generated state to the ideal qudit Bell state, and in Fig. 3(b) the logarithmic negativity as a figure of merit to assess the generated entanglement.

The noisy state generation is simulated using an approach by Tichy describing multiphoton interference of partially distinguishable photons [61] (see Supplemental Material 7 for details [27]). Because the number of photons involved in the scheme increases rapidly with $N$ and $d$, this simulation quickly becomes intractable when increasing the heralded state complexity. Nevertheless, for the scheme shown in Fig. 1(b) with $N = 2$ and low values of $d$, the simulation remains tractable on a standard laptop.

For various levels of distinguishability, we show in Fig. 3(a) the fidelity of the generated state to the ideal qudit Bell state, and in Fig. 3(b) the logarithmic negativity as a figure of merit to assess the generated entanglement.
The logarithmic negativity is an entanglement monotone that upper bounds the distillable entanglement, which in turn quantifies the amount of pure state entanglement that can be extracted under local operations and classical communication [62, 63]. Distinguishability weakens the interference governing the ZTL, and therefore both the state fidelity and generated entanglement decrease when the indistinguishability is reduced. However, while the fidelity is always lower for higher dimensional states, the negativity reaches zero at larger values of photon distinguishability for higher dimensions, as highlighted in the inset of Fig. 3(b). This indicates that for increased dimensionality, the generated entanglement can endure higher levels of photon distinguishability, even if the number of imperfect input photons is larger.

**Discussion.**—The large resource overheads of high-dimensional LOQC could be compensated to some extent by quantum error correction protocols that are more efficient due to robustness to noise that increases with qudit dimension. Furthermore, and similarly to the improvements made in efficiency for qubit-based LOQC [23, 24, 46], we expect our results to provide a first and important step for developing high-dimensional LOQC architectures closer to midterm technological capabilities. For example, as detailed in Supplemental Material 3 [27], by considering all valid heralding events in our GHZ generation scheme for the particular case with \(d = 3\) [see Fig. 1(c)], the resource overheads for multiplexing are reduced from \(\mathcal{O}(10^{10})\) to \(\mathcal{O}(10^4)\), showing that enormous improvements are possible via solution-specific optimization. Crucially, this work proves that such solutions can exist for any dimension.

We thank C. Vigliar, D. Bacco, M. Erhard and M. Krenn for useful discussions. We acknowledge support from the Engineering and Physical Sciences Research Council (EPSRC) Hub in Quantum Computing and Simulation (EP/T001062/1) and Networked Quantum Information Technologies (EP/N509711/1). Fellowship support from EPSRC is acknowledged by A. L. (EP/N003470/1).

---

*stefano.paesani@nbi.ku.dk


