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Orbital stability of compact three-planet systems, I: Dependence of system lifetimes on initial orbital separations and longitudes

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ABSTRACT

We explore the orbital dynamics of systems consisting of three planets, each as massive as the Earth, on coplanar, initially circular, orbits about a star of one solar mass. The initial semimajor axes of the planets are equally spaced in terms of their mutual Hill radius, which is equivalent to a geometric progression of orbital periods for small planets of equal mass. Our simulations explore a wide range of spacings of the planets, and were integrated for virtual times of up to 10 billion years or until the orbits of any pair of planets crossed. We find the same general trend of system lifetimes increasing exponentially with separation between orbits seen by previous studies of systems of three or more planets. One focus of this paper is to go beyond the rough trends found by previous numerical studies and quantitatively explore the nature of the scatter in lifetimes and the destabilizing effects of mean motion resonances. In contrast to previous results for five-planet systems, a nontrivial fraction of three-planet systems survive at least several orders of magnitude longer than most other systems with similar initial separation between orbits, with some surviving \(10^{10}\) years at much smaller orbital separations than any found for five-planet systems. Substantial shifts in the initial planetary longitudes cause a scatter of roughly a factor of two in system lifetime, whereas the shift of one planet’s initial position by 100 m along its orbit results in smaller changes in the logarithm of the time to orbit crossing, especially for systems with short lifetimes.

1. Introduction

NASA’s Kepler mission has discovered hundreds of multiple planet systems (Lissauer et al., 2014; Rowe et al., 2014). Many of these multi-planet systems are quite tightly packed, with adjacent planets’ orbits much closer than the orbits of our Solar System’s planets, both in physical distance and dynamically (Lissauer et al., 2011a; Fabrycky et al., 2014). For example, the Kepler-11 system harbors six known planets, five of which have orbital periods between 10 and 47 days (Lissauer et al., 2011b). Studying the orbital stability of such tightly-packed multiple planet systems provides clues to how they form and how long they survive.

The stability of systems of two planets has been characterized analytically (Hill, 1878; Gladman, 1993). In contrast, analytic studies have not provided as comprehensive a solution to delineating stability boundaries for systems with more than two planets, so such systems have been studied extensively using numerical integrations (e.g., Chambers et al. (1996), Marzari and Weidenschilling (2002), Smith and Lissauer (2009), Pu and Wu (2015), Morrison and Kratter (2016), Tamayo et al. (2016), Obertas et al. (2017)). Lissauer (1995) examined the spacing of planets and moons within the Solar System and proposed approximate criteria for global stability of more than two planets based upon resonance overlapping and the Hill/Jacobi exclusion zone.

The present study can be seen as an extension of Smith and Lissauer (2009) (henceforth SL09), who investigated the stability of closely-spaced three-planet systems with the same masses and relative spacings as we use herein. The main goal of this paper is to perform a much more extensive study of the stability timescale of three-planet systems that are uniformly spaced in units of mutual Hill radii to better survey their diversity in lifetime, considering the influence of the initial planetary longitudes as well as (the previously-varied parameter) orbital separation. We compare the lifetimes of the systems that we study with the lifetimes of analogous five planet systems presented by Obertas et al. (2017) (henceforth OVT17) and those of three-planet systems orbiting a Centauri A or B (Quarles and Lissauer, 2018). For this purpose, we have

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integrated ~ 18,000 systems, each consisting of three Earth-like planets orbiting a solar mass star, until a pair of planetary orbits crossed or a pre-selected interval of 10^8 or 10^10 virtual years elapsed. Throughout this work, we quote values of the initial separations between the planets’ orbits in terms of mutual Hill sphere radii (Eq. 1). Nonetheless, since the masses of the planets and the star are identical for all of the simulations presented herein, the listed planetary separations can easily be scaled to values relative to the critical separation for two-planet systems defined by the resonance overlap criterion (Chirikov, 1979; Wisdom, 1986; Deck et al., 2013; Hadden and Lithwick, 2018) and for overlapping of three-body resonances in systems of three planets (Quillen, 2011; Petit et al., 2020).

In Section 2, we present the methodology used to perform our integrations. An analysis of the lifetimes of a set of simulations with initial longitudes of the outer two planets chosen randomly and independently, which we performed to provide a direct comparison of the lifetimes of three-planet systems with those of analogous five-planet systems simulated by OVT17, is given in Section 3. We present exponential fits to the lifetimes of these systems as a function of initial orbital separation in Section 3.1; deviations from the exponential trend are analyzed in Sections 3.2 and 4.1. Section 4 shows the results of the primary set of simulations that we performed, which were begun with the planets approaching conjunction. In Section 5, we explain the displacement of short-lifetime regions inwards of first-order resonances found by OVT17. Lifetimes of planetary systems with the same combination of initial angles used by SL09 are presented and analyzed in Section 6. Section 7 analyzes the effects of very small changes in initial longitudes on lifetimes. We present additional runs testing the effects of large changes in initial longitudes of the planets on system lifetimes in Section 8. We summarize and discuss our results in Section 10 and compare our findings with those of previous numerical studies in Section 11. Section 12 is an epilogue discussing departures from the Newtonian dynamics of point-mass bodies that must be considered when applying our results to planets on very short-period orbits.

2. Methods

We use numerical integrations to study the stability of systems consisting of three planets, each with mass equal to that of the Earth, 1 M⊙, orbiting a one solar mass, 1 M⊙, star. In all cases, the systems are coplanar, the planets orbit the star in the same direction, the orbits of the planets are initially circular, and the inner planet initially orbits at 1 AU from the star. The problem is modeled using purely Newtonian dynamics, so provided time is measured in units of the initial orbital period of the inner planet, the equations and therefore the results would be unchanged if the masses of the star and planets were scaled up or down proportionately, keeping the mass ratios constant, and/or if the planets were all moved closer to or farther from the star, as long as the distance ratios were kept constant and speeds were adjusted to keep the initial orbits circular. Our integrations are performed using heliocentric coordinates; because the planets/star mass ratio is quite small, the difference from Jacobi coordinates is slight.

2.1. Initial semimajor axes of the planets

We measure the initial orbital separations using the unit of mutual Hill radius of neighboring planets, RH,j,k, where j = 1 refers to the inner planet, j = 2 to the middle planet, and j = 3 to the outer planet. The mutual Hill radius is given by:

\[ R_{H,j,k} = \left[ \frac{m_j + m_k}{3M_j} \right] \frac{1}{2} (a_j + a_{j+1}) = \left( \frac{2M_k}{3(M_j + (j-1)M_k)} \right)^{1/3} (a_j + a_{j+1}), \]

where \( a_j \) and \( m_j \) are the semimajor axis and the mass of the \( j \)th planet, respectively, and we define

\[ \tilde{M}_j = M_* + \sum_{k=1}^{j-1} m_k = M_* + (j-1)M_*, \]

where \( M_* \) is the mass of the host star. The first equalities in Eqs. (1) and (2) give the general formulas for any stellar and planetary masses, whereas the second parts are applicable specifically to the systems studied herein. We use a dimensionless number, \( \beta \), to specify the initial spacing of adjacent planetary orbits:

\[ \beta \equiv \frac{a_{j+1} - a_j}{R_{H,j+1}}. \]

Eq. (1) can be rearranged to give:

\[ a_{j+1} = a_j + \beta R_{H,j+1} = a_j + \beta \left[ \frac{m_j + m_{j+1}}{3M_j} \right] \left( \frac{a_j + a_{j+1}}{2} \right)^{1/3}. \]

Solving Eq. (4) for \( a_{j+1} \) yields:

\[ a_{j+1} = \alpha_j + \beta \left( \frac{M_k}{12(M_j + (j-1)M_k)} \right)^{1/3} \left( \frac{M_*}{12(M_j + (j-1)M_k)} \right) \left( \frac{M_*}{3(M_j + (j-1)M_k)} \right)^{1/3} \left( \frac{a_j + a_{j+1}}{2} \right)^{1/3}. \]

The first equality in Eq. (5) gives the general formula for any stellar and planetary masses, whereas the second part is applicable specifically to the systems studied herein. We used Eq. (5) to determine, based upon the spacing parameter \( \beta \) and the Hill radius, the initial semimajor axes of all planets in our simulated systems.

An initially circular and coplanar two-planet system will remain stable forever if the initial Hill separation \( \beta \geq 2\sqrt{3} \approx 3.4641 \). (If, in analogy to the simulations presented in this article, both of the planets have masses equal to that of Earth, the star’s mass is 1 M⊙, and the inner planet orbits at 1 AU, then the critical semimajor axis for the outer planet is \( a_2 \approx 1.04464 \) AU, which implies an initial orbital period of 1.0677 years.) For this reason we do not compute any systems for which \( \beta < 2\sqrt{3} \).

2.2. Initial longitudes of the planets

Following OVT17, we drew the initial longitudes for the middle and outer planets randomly and independently\(^2\) for each of the simulations presented in Section 3. To separate the effects of variations in initial longitudes from those caused by differences in the initial distances between the orbits, the same initial planetary longitudes are used for all of the simulations presented within each of Sections 4, 6, 8 and 9, and a very small range in initial longitudes is examined in Section 7.

For our Primary set of runs, results of which are presented in Section 4, the initial mean anomaly\(^3\) of the \( j \)th planet is \( 2\pi(j-1)\lambda \) degrees, where

\[ \lambda \equiv \frac{1 + \sqrt{\pi}}{2} = 1.618\ldots \]

is Euler’s number (a.k.a. the golden ratio). Simulations presented in

\(^2\) Our Eq. (5) is equivalent to Eq. (4) of SL09. We discovered that the expanded version of this formula given in Eq. (5) of SL09 contains a misprint. The correct form is given in Eq. (5) of Smith and Lissauer (2010). The calculations in both Smith and Lissauer papers used the correct formula.

\(^3\) OVT17 also drew the initial longitude of the inner planet in their study, but since only the relative longitudes of the planets matter, we placed the initial longitude of the inner planet at 0 for simplicity.

\(^4\) As the orbits initially circular, this is also the starting value of the true anomaly, i.e., the initial longitude, \( \theta \).
Section 6 use the same initial longitudes employed by SL09, where the initial mean anomaly of the $j^{th}$ planet is $2\pi(j-1)\beta$ radians (see Fig. 1); SL09 chose this initial longitudinal separation in order to avoid choosing special orientations (i.e., conjunction, opposition, etc.) regardless of the number of planets simulated. Section 7 investigates the effects of chaos by moving the starting position of one planet along its orbit by 100 m relative to the longitudes in our SL09 simulations. In Section 8, we present crossing times from simulations of system with initial planetary longitudes that are three of the vertexes of a regular hexagon and compare them with those presented in Section 4. Results for systems started with all planets at the same longitude (Aligned) are given in Section 9.

2.3. Integrator and duration of the simulations

The Wisdom-Holman method MVS (mixed-variable symplectic) integrator in the Mercury software package (Chambers, 1999), with a timestep of 18 days, was used to perform all of the integrations presented in this manuscript. Integrations presented in Section 7 that study the effects of very small alterations in initial orbital longitudes on system lifetimes use a minor modification to Mercury provided to us by John Chambers (private communication, 2020) that more accurately handles the input of initial orbital longitudes of the planets; this modification does not, to the extent tested, systematically affect lifetimes of systems (Table 7).

We integrate each system until two orbits cross (the apoapsis of the innermost orbit exceeds the periapsis of the middle orbit or the apoapsis of the middle orbit exceeds the periapsis of the outermost orbit) or until a specified amount of time, equal to either $10^8$ or $10^{10}$ times the initial orbital period of the innermost planet, has elapsed. We refer to this orbital crossing time as the lifetime of the system or the time for the system to become unstable, and denote it by $t_c$. We quote values of $t_c$ normalized by the initial period of the innermost planet (i.e., in years).

Our code tests for orbit crossing once every year for nondimensional orbital separation $\beta < 5$ and once every $\sim 10$ years for $\beta \geq 5$. In some systems, orbits may have crossed briefly but returned to nested positions prior to the next check, so very short-lived episodes of orbit crossing may be missed. Excluding very short-lived systems, most of the time in the integration is typically spent waiting for small variations in eccentricity to diffusively build up and allow stronger interactions between the planets, so we expect that this rarely leads to substantial overestimates of system lifetimes, especially if there is an actual close encounter between two planets. However, our procedure does quantize estimated values of $t_c$ by rounding actual crossing times upwards by up to $\sim 10$ years.

3. Results: random initial longitudes

For the integrations presented in this section, we select values of the initial longitudes of the middle and outer planets, $\theta_3$ and $\theta_5$, for each system randomly and independently from a uniform distribution in angle, as was done by Obertas et al. (2017). As in OVT17 and throughout this paper, the same value of $\beta$ (denoted as $\Delta$ by OVT17) was used for the inner and outer pairs of planets.

We simulated systems for all separations $\beta$ that are multiples of 0.0005 within the range $[3.4645, 6.73]$, and multiples of 0.01 within the ranges $(6.73, 9]$, $[9.5, 9.61]$ and $[10.53, 10.67]$. We stopped each integration if two orbits crossed or when the simulation time reached a limit of $10^8$ years ($10^{10}$ years for $\beta < 5.98$).

Fig. 2 shows the crossing time, $t_c$, as a function of initial orbital separations given in mutual Hill radii up to $\beta = 8.5$. The overall trend is that more widely spaced systems tend to be longer-lived, but the data show significant scatter about the general trend. We expect that the choice of relative initial longitudes of the planets, mean motion resonances (MMRs) and chaos all contribute to this scatter; we analyze these three effects in Sections 4, 5 and 7, respectively.

Of the more widely-spaced systems studied, only two in the range $[8.51, 9]$ went unstable during the $10^8$ years simulated – the $\beta$ values of these systems were 8.78 and 8.79, placing them slightly inward of the 13:11 MMR of neighboring planets, and even these systems survived for more than $6 \times 10^7$ years. The intervals simulated for separations $\beta > 9$ were selected to cover the regions near and slightly inwards of the 6:5 MMR of neighboring planets, as well as similar regions in the vicinity of the 11:9 MMR of neighboring planets and the 3:2 MMR of the inner and outer planets, respectively. None of the runs with separations $\beta > 9$ became unstable during the $10^8$ orbits integrated.

Comparing our Fig. 2 with Figs. 2 and 3 of OVT17 confirms similarities in trends of lifetime vs. orbital separation for systems of three and five planets, apart from the tendency of three-planet systems to be longer-lived at a given separation (Chambers et al., 1996, SL09). Although the spread in lifetimes of most three-planet systems is similar to that present in the OVT17 data for five-planet systems, a small but not insignificant fraction of three-planet systems survive for orders of magnitude longer than most systems having similar initial orbital separations. These exceptionally long-lived systems are only found over limited regions in orbital separation, $\beta$, and are discussed further in Sections 3.2, 4.1 and 6–10.

3.1. Lifetime vs. separation: exponential fits

Previous studies have found that log$_t$ of systems of three or more planets increases approximately linearly with the initial separation measured in mutual Hill radii (Chambers et al., 1996; Duncan and Lissauer, 1997; Faber and Quillen, 2007; Smith and Lissauer, 2009; Morrison and Kratter, 2016; Obertas et al., 2017), according to the following relationship:

$$
\log t_c = \log\left(f(\beta) + g(\beta)\right)
$$

where $f(\beta)$ and $g(\beta)$ are functions of the initial separation $\beta$. The term $f(\beta)$ represents the initial linear increase of lifetime with separation, while $g(\beta)$ accounts for the logarithmic decrease in lifetime as the separation increases.
we used the durations of the simulations in place of crossing times for systems that survived; this introduces a downward bias in the fit at high values of $\beta$ and thus reductions in the value of $b$, but by smaller amounts than if the surviving systems were omitted. To reduce the amount of bias, fits were not performed over regions in which integrations of stable systems were halted prior to $10^{10}$ years.

We fit our data over the interval $3.4645 \leq \beta \leq 5.9795$ to the form of Eq. (7) and computed values of the constants to be $b = 1.4220$ and $c = -2.8482$. Extrapolation of this exponential fit to more widely-spaced systems estimates the separation at which $t_c = 10^{10}$ years to be $\beta = 9.0921$. The analogous coefficients for five-planet systems found by OVT17 were $b = 0.951$ and $c = -1.202$ (their fit covered the range $2\sqrt{3} \leq \beta \leq 8.4$). These coefficients imply similar lifetimes for systems of three and five planets with $\beta \approx 2\sqrt{3}$, but that the lifetimes of three-planet systems characteristically increase faster with orbital separation than do those of five-planet systems.

Because the fit data all have $\beta$ values well above 0, the derived value of the constant $c$, which corresponds to the value of the fit line at $\beta = 0$, depends strongly on the value of the slope, $b$. To reduce this correlation, following Quarles and Lissauer (2018), we performed fits to the shifted exponential

$$\log t_c = b' \beta + c',$$

where $b' \equiv \beta - 2\sqrt{3}$.

Fig. 2 shows that system lifetimes exhibit considerable scatter about the general log-linear trend. The departures from the exponential fit line are neither random nor normally distributed. The ensemble of system lifetimes oscillates aperiodically relative to the exponential fit, with typical wavelength and amplitude increasing at larger separations, but remaining $\lesssim 1$ in log, for the fit interval. The primary cause of this oscillation is the dips in lifetimes associated with mean motion resonances among the planets. However, the most extreme deviations from the exponential fits to system lifetime are the upward spikes in lifetime

Table 2 shows results\footnote{Note that these values, as well as all other results presented herein, apply only to system of three planets with equal masses that orbit a star whose mass ratio to each of the planets is $M_1/M_\ast$.} from fitting Eq. (8) to various subsets of our crossing times, including data from Sets of runs with identical starting longitudes (Sections 4 and 6–9) as well as from the independent random draw longitudes Set displayed in Fig. 2. In addition to quoting values of the coefficients $b'$ and $c'$ and the standard deviation of points from the fit line, $\sigma_{\text{exp}}$, we give the value of $\beta$ (not $\beta'$, which is less familiar) at which the exponential fit predicts $t_c = 10^{10}$ years and three other measures of dispersion of the crossing times that are described in Section 3.2.2. This shifted fit returned the coefficients listed in the third numerical row of Table 2 for the same data fit above; coefficients for other ranges in $\beta$ are listed in the first and second rows. Note that the vast majority of the runs used in the fits presented in Table 2 ended in orbit crossing, and all of the exceptions were run for $10^{10}$ years. The number of stopped runs in the data set used for each fit is also provided in this table.

3.2. Deviations from the exponential fits

Fig. 2 shows that system lifetimes exhibit considerable scatter about the general log-linear trend. The departures from the exponential fit line are neither random nor normally distributed. The ensemble of system lifetimes oscillates aperiodically relative to the exponential fit, with typical wavelength and amplitude increasing at larger separations, but remaining $\lesssim 1$ in log, for the fit interval. The primary cause of this oscillation is the dips in lifetimes associated with mean motion resonances among the planets. However, the most extreme deviations from the exponential fits to system lifetime are the upward spikes in lifetime
near $\beta = 5.15, 5.41, 5.7$ and 6, all of which are located far from the strongest MMRs.

3.2.1. “Spikes” of anomalously long-lived systems

Looking more closely at the first spike, 3 systems in the range $[5.147, 5.158]$ had $t_c > 10^5$ years, 2 of which had $t_c > 10^6$ years, but no system survived for $>10^7$ years. Given that we performed 2000 runs per unit increment in $\beta$, 2 stable runs corresponds to an effective width of $\approx 0.001$ in $\beta$; i.e., the integral over the phase space of initial conditions of systems with $\beta = 5.15$ surviving for more than ten million years is roughly equal to the value that it would have if all systems within a range in $\beta$ that is 0.001 wide survived for $>10^7$ years. Analogously, the lack of any long-lived systems with very tight spacing implies that the effective width of systems with $\beta < 5$ and $t_c > 10^5$ years is $\approx 0.0005$.

The numbers of survivors for various time intervals within each of the upward spikes in system lifetimes are listed in Table 3. Systems in the first three spikes for random longitudes were followed for well over an order of magnitude longer than the trend at their separations, and within statistical uncertainties, approximately half of the runs within each of these spike regions went unstable with each increase of one in $\log t$. Fig. 2 displays a wide region of closely-spaced triangles (signifying lower bounds in lifetimes of systems that survived for the entire time interval simulated, $10^8$ years in this case) beginning near $\beta = 6.30$ and ending just above $\beta = 6.70$. We selected an upper boundary for the widest separation spike for random longitudes runs at 6.4095 because the space between it and the next triangle, at 6.444, represented the largest distance between $10^8$ year survivors within that range (and the distance to the subsequent triangle, at 6.4705 is the second largest). This also facilitates comparison of effective widths with the spike having similar boundaries in the SL09 longitudes set of runs.

Comparison with the findings of OV17 for 5-planet systems shows that the most tightly-spaced long-term stable regions occur at smaller $\beta$ and much smaller values of the exponential trend in lifetime for 3-planet systems. Also, the first such regions are far narrower in $\beta$ and occur in regions where they occupy much smaller fractions of phase space than the most tightly-spaced stable zones for 5-planet systems. The first (lowest $\beta$ value) upward spike found for five-planet systems by OV17 started at $\approx 8.6$, but at that separation the trend in system lifetime is already higher, and the orbital spacing is closer to $\beta = 8.3$, and by that separation, the typical time to the subsequent triangle at 6.4705 is the second largest. This also facilitates comparison of effective widths with the spike having similar boundaries in the SL09 longitudes set of runs.

Obertas et al. (2017) did not find any (five-planet) systems whose lifetimes are much more than one order of magnitude above their exponential fit until $\beta > 8.3$, and by that separation, the typical time to orbit crossing is already several million years. This is the beginning of the approach to their first spike, which is located on both sides of the 13:11 MMR that lies at $\beta \approx 9$. Note that in said spike, lifetimes probably don’t exceed $10^9$ years by much, since a large fraction of the systems that survived for 1 Gyr become unstable between $10^9$ and $10^{10}$ years. In contrast, typical systems in their spikes at separations of $\beta \geq 10$ may well persist far longer. Our first spike is at $\beta = 5.15$, where the exponential fit gives $t_c < 10^5 \text{yr}$; although this spike and the next one near $\beta = 5.41$ are very narrow, a broader spike is present starting around $\beta = 5.7$, where the exponential fit for $t_c$ is still well below $10^5 \text{yr}$.

3.2.2. Distribution of deviations from the general trend

Table 2 provides four measures of the dispersion of system lifetime from fits to system lifetime data: The standard deviation of log$t_c$ from the exponential fit is denoted by $\sigma_{\text{exp}}$. The oscillatory pattern produced by the dips in $t_c$ near resonances is responsible for much of the scatter, so to measure the remaining variations we use the standard deviation from the local rolling median, $\sigma_{\text{local}}$, where the value of the local rolling mean was computed by averaging the values of log$t_c$ for the fifth and sixth longest-lived systems among the ten systems with values of $\beta$ nearest to the system being considered (five below and five above). Both of these quantifications of the spread in system lifetimes sum over squares of distances of individual systems from expectations, and thus their values are sensitive to small numbers of outliers, such as anomalously long-lived runs in the spikes; our prescription of stopping integrations at $10^{13}$ years thus also affects these measures.

Fig. 3 shows the cumulative fractional distribution of the ensemble of distances from the log$t_c$ values for individual systems to the best-fit exponential approximation given in the third numerical row of Table 2 and various related distributions. The distribution of deviations (solid black curve in Fig. 3) is analogous to a cumulative gaussian centered at zero with standard deviation of $\sigma_{\text{exp}} = 0.434$ (dashed green curve). Although the departures from the cumulative gaussian approximation (dashed black curve in Fig. 3) are small, they are non-random. There are 34 systems (0.68%) with lifetimes that are more than 3 $\sigma_{\text{exp}}$ above the exponential fit (compared to 0.13% for a gaussian distribution) and 16 systems (0.32%) exceed 6 $\sigma_{\text{exp}}$ (compared to 1 in $10^{-7}$% for a gaussian). All of these exceptionally long-lived systems have separations that place them within the upward spike regions near $\beta = 5.15, 5.41$ and 5.7 or the first portion of the next spike that begins near the upper end of the range in $\beta$ considered here. If one excludes these 34 anomalously long-lived systems, the slopes flattens slightly to $b' = 1.3892$, the “intercept” goes up a little to $c' = 2.1005$, the standard deviation of the points to the exponential fit $\sigma_{\text{exp}}$ drops by 12% to 0.438, but the best fitting cumulative gaussian dispersion value drops by just 1%, to $\sigma_{\text{cum}} = 0.430$. We include values $\sigma_{\text{exp}}$ and the analogous $\sigma_{\text{local}}$ in Table 2 because, unlike the standard measures of dispersion, their values are not sensitive to small numbers of points far from the fit curve. The ratio $\sigma_{\text{cum}}/\sigma_{\text{local}}$ is $\approx 1.6$ for all three intervals in $\beta$ examined for the set of runs with Random longitudes; as deviations add in quadratures and this ratio exceeds $\sqrt{2}$, the variance of the local trend relative to the exponential fit must be larger than the variance of individual points relative to the local trend.

The red curves in Fig. 3 show the cumulative distribution excluding the 34 points in the spikes that survive for more than 3 $\sigma_{\text{cum}}$ longer than the prediction from the exponential fit. The red dashed curve showing the difference between this cumulative distribution and the best-fit gaussian never exceeds $\approx 1.5\%$ of the total number of runs. Nonetheless this difference between the two distributions is statistically significant. For example, there are 95 points in the interval $(-2, 1)$, but only 29 points in the interval (1,2), whereas the gaussian approximation predicts the same number of points within these regions, a difference that greatly exceeds expected stochastic variations. The dashed red difference curve is largely symmetric about zero. But this isn’t the type of symmetry that would be present if the distributions of deviations were the same on the upside as on the downside - that would be antisymmetric, with values above the mean being equal to negative those for the same amount below the mean. These curves show that the narrow resonant dips cause there to be more systems with lifetimes that are roughly $2/\sqrt{\exp}$ below the fit line than expected (producing the rise to the positive peak near $-0.6$ on the abscissa and bringing the fit line below the majority of points), so there are fewer systems than expected below the fit line by $\approx 1.5\sigma_{\text{cum}}$ (thus causing the drop to the most negative value near 0 on the abscissa), much more than expected above the fit line by $\approx 1.5\sigma_{\text{cum}}$, and a deficit of systems (outside of the spikes) that survive $\approx 2\sigma_{\text{cum}}$ above the fit line. The larger deviations of the points below the line is compensated for by the median being above the mean. This is because, excluding the upward spikes, the largest deviations from the fits are the (downward) dips near the strongest resonances.

Much of the deviations of lifetimes from the best fitting exponential can be attributed to the oscillatory pattern caused by dips in lifetimes near strong resonances, although there is also some scatter about the
local average. Fig. 4 is analogous to Fig. 3 (same colour code and same range), but shows the cumulative fractional distribution of the ensemble of distances between log $t_c$, of individual systems and the local running median defined above as the average of log $t_c$ of the fifth and sixth longest-lived systems among the ten systems with most similar spacing. The solid black curve is the distribution of the differences, which has a standard deviation of $\sigma_{\text{local}} = 0.268$, much smaller than the standard deviation of $\sigma_{\text{cum}} = 0.434$ found by comparing with the best-fit exponential. The dashed red curve in Fig. 4, representing the differences of the cumulative distribution from a gaussian with the same standard deviation, is nearly antisymmetric about the vertical axis, implying a symmetric distribution of deviations from the local mean. The wings are higher for than a gaussian, compensated for by an excess of systems with lifetimes very similar to the running median.

We performed a similar analysis of the crossing times of five-planet systems integrated by Obertas et al. (2017). Results for four different upper limits to the range of orbital separations analyzed are listed in Table 4. Five-planet systems have smaller deviations from the log-linear fit over the same range in orbital separation that we examined for three-planet systems with randomly-selected longitudes (Table 2), but the deviations are substantially larger when the entire zone prior to the first regions of long-term stability is considered. The increases from the smallest region to the one extending to $\beta = 8.3$ are primarily caused by the increasing depth of the resonant dips, whereas those for the two largest regions fit are dominated by the first upward spike in the OVT17 data. Regardless of the range examined, 5-planet systems have characteristic variations from the local median lifetime, $\sigma_{\text{local}}$, that are ~25% smaller than those seen with 3-planet systems with Randomly chosen longitudes.

4. Initial longitudes approaching conjunction

Our Primary Set of integrations$^7$ consisted of three-planet systems with the same set of initial longitudes of each of the planets used for all integrations in the Set, as described in Section 2 and enumerated in the second row of Table 1. Each system began with a different value of orbital separation $\beta$, and every system was followed until a pair of orbits crossed or the simulation time reached a pre-determined limit of either 10$^6$ years or 10$^{10}$ years. System lifetimes ranged from <100 years to >10$^{10}$ years, with the longest-lived systems being computationally expensive. Thus, to maximize scientific returns from our computational resources, we explored parameter space with nonuniform resolution in $\beta$. We covered the region 3.465 $\leq \beta \leq$ 6.750 using steps of 0.001 in $\beta$ in most of this interval, but only considering multiples of 0.01 in $\beta$ within the long-lived regions 5.91 $\leq \beta < 5.99$, 6.016 $\leq \beta < 6.08$ and 6.62 $< \beta$. For $\beta > 6.75$, a large fraction of the systems were very long-lived, so we stopped most systems that had not already become unstable at 10$^8$ years, although we followed simulations for all multiples of 0.1 from $\beta = 6.8$ to $\beta = 7.5$, as well as for the separations $\beta = 8$ and $\beta = 8.5$, until instability or 10$^{10}$ years had elapsed.

Fig. 5 shows the crossing time, $t_c$, as a function of initial orbital separations given in mutual Hill radii and as a function of initial period ratios; all runs from our Primary set of integrations are represented. The overall trend is again that more widely spaced systems tend to be more stable, but there remains considerable variation in system lifetimes about this general trend. We expect that mean motion resonances (MMRs) and chaos contribute to this scatter, and we analyze these effects in Section 5 and Section 7.

We fit our data at multiples of 0.01 in $\beta$ over the interval 3.47 $< \beta < 6.75$ to the form of Eq. (7) and computed values of the constants to be $b = 1.5559$ and $c = -3.3186$. Extrapolation of this exponential fit to more widely-spaced systems estimates the separation at which $t_c = 10^8$ years to be $\beta = 8.5602$. Globally, this fit is consistent with the results of SL09, who found $b = 1.496$ and $c = -3.142$ (their fit covered the range $3 \leq \beta \leq 7.2$ at 0.1 resolution, plus points at 7.5 and 8). As with the crossing times of runs with random initial longitudes, the data show significant scatter about the general trend. To isolate the contribution of the initial longitudes to this scatter, in Section 4.1 we compare this scatter to that found in Section 3.2.

![Figure 4](image-url)

**Fig. 4.** Comparison of cumulative distribution of deviations in lifetimes of systems with randomly-drawn initial longitudes in the range 3.4645 $\leq \beta \leq$ 5.9795 from log-linear fits with local median. The four curves associated with the left axis labels are indicated in the key near the top of the figure: The solid black line shows the cumulative portion of the ensemble of runs with lifetimes (log $t_c$) less than the amount given by the horizontal coordinate “above” the local median. The green dashed curve represents a cumulative gaussian distribution with dispersion of 0.268. The solid red and dotted blue curves are analogous representations with the 34 systems with lifetimes $> 3 \sigma_{\text{exp}}$ above the exponential fit removed. The two curves associated with the right axis are listed in the lower key: The dashed black (red) curve shows the difference between the solid black (red) “observed” cumulative distribution and the dashed green (dotted blue) cumulative gaussian approximation thereto. The red and blue curves omit exceptionally long-lived runs in the upward spikes of system lifetimes, whereas the black and green curves include all data within the specified range. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Initial planetary longitudes.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set(s) of runs</td>
<td>Section(s)</td>
</tr>
<tr>
<td>Random</td>
<td>3</td>
</tr>
<tr>
<td>Primary</td>
<td>4, 5</td>
</tr>
<tr>
<td>SL09, SL09$^\dagger$, Chaos</td>
<td>6, 7</td>
</tr>
<tr>
<td>Hexagonal</td>
<td>8</td>
</tr>
<tr>
<td>Aligned</td>
<td>9</td>
</tr>
</tbody>
</table>

Initial longitudes of middle and outer planets ($\theta_2$ and $\theta_3$, respectively) relative to that of the inner planet ($\theta_1$), measured along the direction of the orbits, for each set of runs presented herein. Values presented in this table have been rounded to the nearest 0.01”. The initial longitudes of the SL09, SL09$^\dagger$ and Chaos sets differ by <0.01”.

$^7$ We initiated our study using these initial planetary longitudes and we integrated most of the systems in this set prior to realizing how much lifetimes at a given orbital separation could depend on initial longitudes. Once we became aware of these dependences, we decided to consider other starting longitudes, but allowed fewer of those integrations to proceed beyond 10$^8$ years because of computational costs.
Table 2

<table>
<thead>
<tr>
<th>Set of runs</th>
<th>Range</th>
<th>Reso n</th>
<th>Slope, b'</th>
<th>Intercept, c'</th>
<th>( \beta ) at ( \log t_c - 10 )</th>
<th>( t_c &gt; 10^{10} )</th>
<th>( \sigma_{exp} )</th>
<th>( \sigma_{local} )</th>
<th>( \sigma_{exp}' )</th>
<th>( \sigma_{local}' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random</td>
<td>3.4645-5.1870</td>
<td>0.0005</td>
<td>1.474 ± 0.014</td>
<td>2.046 ± 0.014</td>
<td>8.862 ± 0.044</td>
<td>0</td>
<td>0.420</td>
<td>0.288</td>
<td>0.414</td>
<td>0.261</td>
</tr>
<tr>
<td>Random</td>
<td>3.4645-5.9100</td>
<td>0.0005</td>
<td>1.396 ± 0.010</td>
<td>2.099 ± 0.014</td>
<td>9.122 ± 0.031</td>
<td>3</td>
<td>0.484</td>
<td>0.352</td>
<td>0.436</td>
<td>0.267</td>
</tr>
<tr>
<td>Random</td>
<td>3.4645-6.9795</td>
<td>0.0005</td>
<td>1.422 ± 0.010</td>
<td>2.078 ± 0.014</td>
<td>9.035 ± 0.029</td>
<td>6</td>
<td>0.497</td>
<td>0.368</td>
<td>0.434</td>
<td>0.268</td>
</tr>
<tr>
<td>Primary</td>
<td>3.465-5.187</td>
<td>0.001</td>
<td>1.468 ± 0.019</td>
<td>2.108 ± 0.019</td>
<td>8.840±0.059</td>
<td>0</td>
<td>0.420</td>
<td>0.280</td>
<td>0.393</td>
<td>0.225</td>
</tr>
<tr>
<td>Primary</td>
<td>3.465-6.910</td>
<td>0.001</td>
<td>1.399 ± 0.014</td>
<td>2.163 ± 0.020</td>
<td>9.066 ± 0.044</td>
<td>1</td>
<td>0.507</td>
<td>0.349</td>
<td>0.416</td>
<td>0.234</td>
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<tr>
<td>Primary</td>
<td>3.47-6.30</td>
<td>0.01</td>
<td>1.603 ± 0.070</td>
<td>2.026 ± 0.114</td>
<td>8.439 ± 0.152</td>
<td>8</td>
<td>0.955</td>
<td>0.664</td>
<td>0.549</td>
<td>0.319</td>
</tr>
<tr>
<td>Primary</td>
<td>3.47-6.75</td>
<td>0.01</td>
<td>1.556 ± 0.054</td>
<td>2.071 ± 0.103</td>
<td>8.568 ± 0.117</td>
<td>9</td>
<td>0.925</td>
<td>0.629</td>
<td>0.526</td>
<td>0.324</td>
</tr>
<tr>
<td>Primary</td>
<td>3.5-7.5</td>
<td>0.1</td>
<td>1.680 ± 0.128</td>
<td>1.799 ± 0.300</td>
<td>8.346 ± 0.207</td>
<td>5</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>SL09</td>
<td>3.465-5.187</td>
<td>0.001</td>
<td>1.495 ± 0.020</td>
<td>1.968 ± 0.020</td>
<td>8.838 ± 0.060</td>
<td>0</td>
<td>0.414</td>
<td>0.246</td>
<td>0.394</td>
<td>0.236</td>
</tr>
<tr>
<td>SL09</td>
<td>3.465-6.910</td>
<td>0.001</td>
<td>1.492 ± 0.015</td>
<td>1.967 ± 0.021</td>
<td>8.847 ± 0.042</td>
<td>0</td>
<td>0.528</td>
<td>0.297</td>
<td>0.434</td>
<td>0.254</td>
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<tr>
<td>SL09</td>
<td>3.465-6.300</td>
<td>0.001</td>
<td>1.383 ± 0.012</td>
<td>2.064 ± 0.020</td>
<td>8.921 ± 0.038</td>
<td>0</td>
<td>0.540</td>
<td>0.300</td>
<td>0.447</td>
<td>0.258</td>
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<tr>
<td>SL09</td>
<td>3.47-7.15</td>
<td>0.01</td>
<td>1.478 ± 0.038</td>
<td>1.977 ± 0.081</td>
<td>8.892 ± 0.090</td>
<td>5</td>
<td>0.733</td>
<td>0.375</td>
<td>0.574</td>
<td>0.306</td>
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<tr>
<td>SL09</td>
<td>3.5-7.5</td>
<td>0.05</td>
<td>1.621 ± 0.076</td>
<td>1.808 ± 0.179</td>
<td>8.512 ± 0.136</td>
<td>3</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>SL09</td>
<td>3.47-6.32</td>
<td>0.01</td>
<td>1.398 ± 0.037</td>
<td>2.027 ± 0.062</td>
<td>9.166 ± 0.113</td>
<td>0</td>
<td>0.519</td>
<td>0.286</td>
<td>0.460</td>
<td>0.257</td>
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<tr>
<td>Chaos</td>
<td>3.47-6.32</td>
<td>0.01</td>
<td>1.406 ± 0.037</td>
<td>2.028 ± 0.060</td>
<td>9.132 ± 0.108</td>
<td>0</td>
<td>0.507</td>
<td>0.325</td>
<td>0.438</td>
<td>0.273</td>
</tr>
<tr>
<td>Hexagonal</td>
<td>3.465-5.187</td>
<td>0.001</td>
<td>1.405 ± 0.019</td>
<td>2.106 ± 0.019</td>
<td>9.084 ± 0.065</td>
<td>0</td>
<td>0.401</td>
<td>0.240</td>
<td>0.395</td>
<td>0.224</td>
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<tr>
<td>Hexagonal</td>
<td>3.465-6.910</td>
<td>0.001</td>
<td>1.344 ± 0.012</td>
<td>2.150 ± 0.017</td>
<td>9.310 ± 0.133</td>
<td>0</td>
<td>0.418</td>
<td>0.254</td>
<td>0.412</td>
<td>0.235</td>
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<tr>
<td>Hexagonal</td>
<td>3.47-5.94</td>
<td>0.01</td>
<td>1.337 ± 0.038</td>
<td>2.166 ± 0.054</td>
<td>9.323 ± 0.129</td>
<td>0</td>
<td>0.421</td>
<td>0.277</td>
<td>0.411</td>
<td>0.253</td>
</tr>
<tr>
<td>Aligned</td>
<td>3.465-5.187</td>
<td>0.001</td>
<td>1.461 ± 0.024</td>
<td>2.348 ± 0.024</td>
<td>7.802 ± 0.072</td>
<td>2</td>
<td>0.497</td>
<td>0.325</td>
<td>0.410</td>
<td>0.229</td>
</tr>
</tbody>
</table>

Fig. 5. Lifetime, \( t_c \), of three-planet systems as a function of the initial separation of the orbits of neighboring planets in units of mutual Hill sphere radii, \( \beta \) (Eq. 3), for our Primary set of runs. The open triangles represent systems that existed up to 10\(^{10}\) years because statistics are poor for both measures and \( \sigma_{local} \) extends well beyond the local region.

Table 2 also provides fits parameters for systems with larger orbital separation, where the resolution in \( \beta \) is 0.01. The local median can change nontrivially over distances of 0.05 in \( \beta \), leading to systematic increases in \( \sigma_{local} \) for fits to data at resolution of 0.01. Thus, we caution the reader against overinterpreting these statistical measures. We don’t include fits for separations at which only 0.1 resolution data are available up to 10\(^{10}\) years because statistics are poor for both measures and \( \sigma_{local} \) extends well beyond the local region.

As was the case for systems with randomly-selected initial planetary longitudes, the departures from the exponential trend in system lifetimes are neither random nor normally distributed. For the set of systems with random, independently drawn, initial longitudes, the ensemble of system lifetimes oscillates aperiodically relative to the exponential fit, and the most extreme deviations from the exponential fits to system lifetime occur in upward spikes. The key difference is that there are fewer upward spikes for this fixed set of initial longitudes, but those that are present have more members (larger effective width).

In the region \([3.47, 6.75]\) at 0.01 resolution in \( \beta \), the median difference in absolute value of \( \log t_c \), of neighboring points (lifetimes of systems having the closest values of \( \beta \), excluding pairs of points that were both stopped after 10\(^{10}\) years) is 0.442. The signs of the differences in lifetime of neighboring systems from the general trend line are positively correlated, with 68% of runs having \( t_c \) on the same side of the log-linear fit line as the run with \( \beta \) value 0.01 smaller (compared with 50% if no correlation existed).

Fig. 6 shows the cumulative fractional distribution of the ensemble of distances between \( \log t_c \) and the best-fit exponential approximations given in the fifth, seventh and eighth columns of Table 2. The distribution up to 5.910 (black curve in Fig. 5) is analogous to a cumulative gaussian distribution centered at zero with standard deviation \( \sigma_{cum} = 0.416 \) (dotted green curve), only \( \approx 5\% \) smaller than the dispersion over a similar range in planetary separation for systems with randomly-drawn initial longitudes. As for systems with randomly-drawn longitudes, the gaussian fit is a fairly good approximation apart from the excess of unusually long lived systems: 0.82% (20 systems) have lifetimes that are \( >3 \sigma_{cum} \) above the exponential fit (compared to 0.13% for a gaussian distribution) and 0.45% (11 systems) exceed 6 times the
standard deviation (compared to \(1 \times 10^{-7}\)\(\beta_6\) for a gaussian). Most of these exceptionally long-lived systems have separations that place them within the upward spike region near \(\beta = 5.38\); the exceptions are the system with \(\beta = 3.744\), which has a lifetime \(3.4\sigma_{cum}\) above the fit line (the system with \(\beta = 3.691\) has a lifetime \(2.9\sigma_{cum}\) above the fit line), and the systems with the two largest values of \(\beta\) within this interval, 5.909 and 5.91, which are \(3.8\sigma_{cum}\) and \(3.1\sigma_{cum}\) above the fit line, respectively. The other curves deviate more visibly from normal distributions, because the percentage of their points within spikes is much larger, leading to a displacement of the median (where the curves on the figure reach 50%) from 0 to balance the significant depressions of the curves leading to a displacement of the median (where the curves on the figure reach 50%).

5. Resonances and displacement of resonant dips

SLO9 identified a drop in the lifetime of systems associated with a first-order mean motion resonance (where the ratio of the orbital periods of a pair of planets can be written in the form \(N/(N-1)\), with \(N\) being an integer) between neighboring planets. First-order resonances of neighboring planets have the strongest effects on lifetimes. They consist of resonance triplets—both the inner and outer pairs are near resonance (these resonances are slightly offset, by a fractional difference of order \(M_j/M_\star\), but this offset is small compared to the resonance widths), and the planets are near a (zeroth-order) 3-body resonance of the form \(N_j - (2N_1 + 1)n_{j+1} + (N_1 + 1)n_{j+2} = 0\), where \(n_j\) is the angular velocity of the \(j\)th planet.

Obertas et al. (2017) showed dips in lifetime at several first-order and second-order (period ratio \(N/(N-2)\)) resonances, and they discussed resonances at greater length. They found that the dips near first-order MMRs were centered at slightly smaller values of \(\beta\) than the resonance locations, but those at second-order MMRs were centered at

These departures from normality would be even larger if we had the computer resources to integrate those systems that survived for \(10^{10}\) years for even longer times. The spikes in system lifetimes seen for randomly-selected initial longitudes near \(\beta = 5.15, 5.7\) and 6.35 are absent in Fig. 5, but the very narrow spike centered at \(\beta \approx 5.385\), the wider long-lived region around \(\beta = 6\), and the even wider stable region starting at \(\beta = 6.75\) remain. Looking more closely at the first spike, 18 systems in the region [5.377, 5.409] had \(t_\varepsilon > 10^7\) years, 13 of which had \(t_\varepsilon > 10^8\) years, including 7 that had \(t_\varepsilon > 10^9\) years, 2 with \(t_\varepsilon > 10^10\) years, and 1 with \(t_\varepsilon > 10^{11}\) years. As with the spikes seen in the Random longitude runs with \(\beta < 5.8\), within statistical uncertainties, approximately half of the runs within the spike region went unstable with each unit increment in log-\(\varepsilon\). The numbers of surviving systems imply an effective width in \(\beta\) of \(0.018\) for survival time of \(10^7\) years, dropping to \(-0.001\) for survival time of \(10^{10}\) years. However, lifetimes are distributed differently within the second spike, with 9 of the 13 runs at multiples of 0.01 in \(\beta\) that survived for more than 10 Myr surviving for the full 10 Gyr integrations (Table 3). Both types of dropoff are present in some of the spikes detected at other initial longitudes; see Section 10 for details.

### Table 3

Heights and widths of spikes of long-lived closely-spaced systems.

<table>
<thead>
<tr>
<th>Set of runs</th>
<th>Location</th>
<th>(t_\varepsilon &gt; 10^6) yr</th>
<th>(t_\varepsilon &gt; 10^7) yr</th>
<th>(t_\varepsilon &gt; 10^8) yr</th>
<th>(t_\varepsilon &gt; 10^9) yr</th>
<th>(t_\varepsilon &gt; 10^{10}) yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random</td>
<td>[5.147, 5.1585]</td>
<td>3; 0.0015</td>
<td>2; 0.001</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Random</td>
<td>[5.402, 5.4165]</td>
<td>7; 0.0025</td>
<td>4; 0.002</td>
<td>2; 0.001</td>
<td>1; 0.0005</td>
<td>1; 0.0005</td>
</tr>
<tr>
<td>Random</td>
<td>[5.677, 5.773]</td>
<td>–</td>
<td>14; 0.007</td>
<td>9; 0.0045</td>
<td>3; 0.0015</td>
<td>3; 0.0015</td>
</tr>
<tr>
<td>Random</td>
<td>[5.902, 5.9730]</td>
<td>–</td>
<td>9; 0.0045</td>
<td>3; 0.0015</td>
<td>3; 0.0015</td>
<td>3; 0.0015</td>
</tr>
<tr>
<td>Random</td>
<td>[5.98, 6.0840]</td>
<td>–</td>
<td>18; 0.009</td>
<td>13; 0.0065</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Random</td>
<td>[6.3055, 6.4095]</td>
<td>–</td>
<td>–</td>
<td>37; 0.0185</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Primary</td>
<td>[5.577, 5.409]</td>
<td>18; 0.018</td>
<td>13; 0.013</td>
<td>7; 0.007</td>
<td>2; 0.002</td>
<td>1; 0.001</td>
</tr>
<tr>
<td>Primary</td>
<td>[5.92, 6.07]</td>
<td>–</td>
<td>13; 0.13</td>
<td>12; 0.12</td>
<td>10; 0.1</td>
<td>9; 0.09</td>
</tr>
<tr>
<td>Primary</td>
<td>[6.8, 6.9]</td>
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<td>–</td>
<td>–</td>
<td>2; 0.2</td>
<td>1; 0.1</td>
</tr>
<tr>
<td>Primary</td>
<td>[7.2, 7.4]</td>
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<td>–</td>
<td>–</td>
<td>3; 0.3</td>
<td>2; 0.2</td>
</tr>
<tr>
<td>SLO9</td>
<td>[5.675, 5.748]</td>
<td>–</td>
<td>30; 0.030</td>
<td>8; 0.008</td>
<td>2; 0.002</td>
<td>0</td>
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<tr>
<td>SLO9</td>
<td>[6.33, 6.41]</td>
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<td>7; 0.07</td>
<td>5; 0.05</td>
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<td>5; 0.25</td>
<td>4; 0.2</td>
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<tr>
<td>Hexagonal</td>
<td>[5.95, 6.09]</td>
<td>–</td>
<td>11; 0.11</td>
<td>10; 0.10</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Aligned</td>
<td>[5.131, 5.187]</td>
<td>18; 0.018</td>
<td>7; 0.007</td>
<td>5; 0.005</td>
<td>4; 0.004</td>
<td>2; 0.002</td>
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<td>40; 0.040</td>
<td>33; 0.033</td>
<td>28; 0.028</td>
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<td>–</td>
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<tr>
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<td>25; 0.05</td>
<td>26; 0.029</td>
<td>27; 0.027</td>
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<td>–</td>
</tr>
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<td>16; 0.16</td>
<td>14; 0.14</td>
<td>–</td>
<td>–</td>
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</tbody>
</table>

Locations (interval from the first system within the spike through that of the last such system), followed by number of runs surviving for the indicated times and effective width in \(\beta\) of the ensemble of survivors for that duration. Spikes are defined by a minimum duration (always chosen to be an integer power of ten) to be considered anomalous, no numbers are quoted for survival times not considered anomalously high for their location, and data are also missing for durations longer than those at which we capped our integrations. Simulations in the spike for Random longitudes around \(\beta = 6\) and for Aligned systems around \(\beta = 5.2\) were allowed to run for different maximum times depending on the value of orbital separation; in order to represent all results available, the data for each of these spikes are split over two rows. The region in the Primary Set near \(\beta = 6\) was simulated with non-uniform resolution, so only systems with \(\beta\) values that are multiples of 0.01 (the lower resolution) are considered here.
Errors may be present elsewhere in the table, but these are likely to be far less than the statistical uncertainties resulting from the small sizes of the samples.

Comparison of the survival times of integrations for the chaos test (innermost planet 100 m ahead) with those using SL09 longitudes at the same value of \( \beta \) at log\( _e \) \( \sim \) 10, \( \tau_t > 10^{10} \) years, the number of runs that survived for the entire 10\(^9\) years simulated, and the four measures of dispersions of the data listed for three-planet systems in Table 2.

Comparison of system lifetimes: SL09 vs. chaos longitudes. Table 6 Comparison of system lifetimes: SL09 vs. chaos longitudes.

<table>
<thead>
<tr>
<th>Interval in ( \beta )</th>
<th>[3.465, 3.999]</th>
<th>[4.000, 4.999]</th>
<th>[5.000, 5.910]</th>
<th>[3.465, 5.910]</th>
<th>[5.92, 6.75]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of runs in the range</td>
<td>535</td>
<td>1000</td>
<td>911</td>
<td>2446</td>
<td>84</td>
</tr>
<tr>
<td>( &lt;\log \tau_t(\text{Primary}) – \log \tau_t(\text{SL09})&gt; )</td>
<td>0.134</td>
<td>0.130</td>
<td>–</td>
<td>0.003</td>
<td>0.083</td>
</tr>
<tr>
<td>( &lt;\log \tau_t(\text{choas}) – \log \tau_t(\text{SL09})&gt; )</td>
<td>0.313</td>
<td>0.337</td>
<td>0.470</td>
<td>0.381</td>
<td>1.289</td>
</tr>
<tr>
<td># ( \tau_t(\text{SL09}) &lt; 0.5\tau_t(\text{Primary}) )</td>
<td>169 (31.59%)</td>
<td>326 (32.6%)</td>
<td>216 (23.7%)</td>
<td>711 (29.1%)</td>
<td>33 (39.3%)</td>
</tr>
<tr>
<td># ( 0.5\tau_t(\text{SL09}) &lt; \tau_t(\text{Primary}) &lt; 2\tau_t(\text{SL09}) )</td>
<td>293 (54.48%)</td>
<td>533 (53.3%)</td>
<td>475 (52.1%)</td>
<td>1301 (53.2%)</td>
<td>27 (32.1%)</td>
</tr>
<tr>
<td># ( 2\tau_t(\text{Primary}) &lt; \tau_t(\text{SL09}) )</td>
<td>73 (13.65%)</td>
<td>141 (14.1%)</td>
<td>220 (24.1%)</td>
<td>434 (17.7%)</td>
<td>24 (28.6%)</td>
</tr>
<tr>
<td># within 10% of the Primary runs</td>
<td>40 (7.48%)</td>
<td>71 (7.10%)</td>
<td>60 (6.59%)</td>
<td>171 (6.99%)</td>
<td>8 (9.52%)</td>
</tr>
<tr>
<td># within 1% of the Primary runs</td>
<td>3 (0.56%)</td>
<td>7 (0.70%)</td>
<td>2 (0.22%)</td>
<td>12 (0.49%)</td>
<td>0 (0.00%)</td>
</tr>
</tbody>
</table>

Comparison of the survival times of integrations using the SL09 initial longitudes runs with those at the same value of \( \beta \) from our Primary integrations. Note that because we check for orbital crossing once per year for \( \beta < 5 \) and once every ten years for more widely-spaced systems, the listed data are unreliable for runs with short lifetimes. Thus, the data denoted with * are likely to be somewhat in error, but are nonetheless useful for comparison with the values presented in Tables 6-9. Small errors may be present elsewhere in the table, but these are likely to be far less than the statistical uncertainties resulting from the small sizes of the samples.
of the TRAPPIST-1 planetary system, which contains a resonances chain of seven planets (Gillon et al., 2017; Luger et al., 2017; Tamayo et al., 2017), have shown that even for far more widely-spaced systems than those considered herein the stable resonant regions in high multiplicity planetary systems occupy a small volume of parameter space. This smallness is consistent with the apparent absence of systems stabilized by resonant locks among the many thousands of systems that we (and others) have considered. Some tightly-spaced systems far from first-order MMRs (spikes) in system lifetimes are seen near \( \beta = 8 \). No analogous long-lived region exterior to a first-order resonance between neighboring planets and interior to the neighboring higher order resonances \( \approx 5.7, 6.4, 7.3 \). In order to reduce computational costs, most integrations for wider separations (larger values of \( \beta \)) were halted at 10 years if no orbits had crossed; however, simulations were followed for as long as 10 years for \( \beta \) values that are multiples of 0.05 up to \( \beta = 7.5 \). No SL09 longitudes run with \( \beta > 8.18 \) went unstable during the 10 years integration.

Fig. 8 shows that, as for systems starting at our Primary Set of longitudes, the “anomalous peaks” (spikes) in system lifetimes are seen near \( \beta \approx 5.7, 6.4, 7.3 \). The most closely-spaced three-planet system that SL09, who used far lower resolution in \( \beta \), found to be long-lived began with a separation of \( \beta = 6.4 \). In contrast to the lowest \( \beta \) peaks for our Primary runs, all of these spikes are located midway between a first-order resonance between neighboring planets on the inside and a second-order resonance at higher \( \beta \). No analogous long-lived region exterior to a first-order resonance between neighboring planets and interior to the neighboring high-

6. SL09 longitudes

Following SL09, we started the planets in this Set of simulations at widely-separated initial longitudes, as described in Section 2 and enumerated in the third numerical row of Table 1. Our study covered the range 3.465–6.300 with resolution of 0.001 in \( \beta \) and 6.31–8.5, as well as the near-resonance regions 8.75–8.82, 9.56–9.63 and 10.52–10.59, with resolution of 0.01 in \( \beta \). Our integrations were allowed to continue up to 10 years for \( \beta \leq 7.15 \). In order to reduce computational costs, most integrations for wider separations (larger values of \( \beta \)) were halted at 10 years if no orbits had crossed; however, simulations were followed for as long as 10 years for \( \beta \) values that are multiples of 0.05 up to \( \beta = 7.5 \). No SL09 longitudes run with \( \beta > 8.18 \) went unstable during the 10 years integration.

Comparison of the survival times of integrations using Hexagonal longitudes with those at the same value of \( \beta \) from our Primary integrations. The resolution is 0.001 over the interval [3.465, 5.910] and 0.01 within the intervals [6.10, 6.72] and [8.01, 8.19]. The * symbols are explained in the caption to Table 5.
because some of those runs survived the first 10,000 years of evolution (1000 years for initial period ratios of planets (black dots) and the outer pair (green dots) of planets averaged over the range of this article.)

Weak peaks in system lifetime are present near the planetary separations ($\beta$ values) of the two most tightly-spaced spikes in lifetimes seen for our random initial longitudes runs (Fig. 2), with the second peak being substantially less prominent at this combination of longitudes than for random longitudes. The effective width of the spike located near $\beta = 5.7$ is 4.3 times as large as that computed for random orientations for lifetimes exceeding $10^7$ years, but survival rates of the remaining systems are lower for later times, leading to similar equivalent widths on gigayear time scales (Table 5).

We perform statistical studies by comparing lifetime of the SL09 longitudes runs with those from our Primary set in different $\beta$ ranges. For this purpose we consider all multiples of 0.001 in $\beta$ up to 5.910, where we have high-resolution data for both sets of runs, and multiples of 0.01 in $\beta$ within the interval [5.92, 6.75], where all systems in both sets were run for $10^{10}$ years if they did not become unstable prior to that time. Results are shown in Table 5. The first two rows of results show the averages of the absolute and relative difference in log$\tau$, between the Primary runs and our SL09 runs; subsequent rows detail the fractions of runs within various logarithmic distances of their counterpart at the same value of $\beta$ but different initial longitudes. We find that runs in the Primary Set, which were performed with initial longitudes approaching conjunction, are on average slightly longer-lived than the SL09 longitudes Set, wherein the planets begin at widely-separated initial longitudes. The bottom row gives the results of a Kolmogorov-Smirnov test of the hypothesis that the ensemble of system lifetimes within the specified range of $\beta$ for the two Sets were drawn from the same distribution; p-values $<0.05$ are considered evidence that they were not drawn from the same distribution. Note that the comparison is made without considering correlations between $t_c$ and $\beta$ within the specified range of orbital separation.

7. Chaos

We examined the effects of very small changes in initial longitudes by performing additional integrations with initial conditions identical to our SL09 Set of calculations (Section 6) apart from moving one of the planets along its orbit by 100 m ($\approx 3.8 \times 10^{-8}$ degrees). As discussed in Section 2.3, the manner in which the Mercury package reads in longitudes is not accurate enough to make a change this small, so we used the modification generously provided to us by John Chambers that is specified in Footnote 5 for both re-running some of the systems with the nominal SL09 longitudes and performing the slightly different “Chaos” integrations. We used resolution of 0.01 in $\beta$ and considered the regions $5.47 \leq \beta \leq 6.32$ and 6.42 $\leq \beta \leq 7.12$. These ranges in $\beta$ were selected because results presented in the previous section suggested that system lifetimes would be short enough that we could run all simulations to orbit crossing, and this proved to be the case. To distinguish the two sets of runs that were performed using this modified code from those presented in Section 6, we refer to them as the SL09′ Set and the Chaos Set. Results of our chaos test comparing crossing times of systems with one planet differing in initial longitudes by 100 m are displayed in Fig. 9. Statistical comparisons with lifetimes of systems in SL09′ Set with those of the Chaos systems and the SL09 systems at the same separations are presented in Tables 6 and 7, respectively. Differences caused by the change in the code are of comparable magnitude to those caused by moving one planet by 100 m along its orbit. Most of the systems that live for less than 1000 orbits have virtually the same crossing time in all three Sets of integrations. Chaos has more substantial effects (in log$\tau$) on the lifetimes of longer-lived systems, but except in the $6.00 \leq \beta \leq 6.32$ range, which includes few runs and thus poor statistics, these tables
show a greater fraction of systems with survival times differing by less than a factor of two than for the substantially different longitude combinations compared in Tables 5, 8 and 9. As expected, the Kolmorogov-Smirnov tests that we performed all were fully consistent with the compared data sets having been drawn from the same populations. Note that here and elsewhere in this study, we measured the time at which a pair of orbits first crossed. Had we continued our simulations until an actual collision occurred (which would have required the use of a different integration algorithm), then larger differences would be seen for many of the systems where the crossing times are nearly identical, although these differences do not increase in proportion to the lifetime of the runs (Bartram et al., 2021, in preparation).

8. Hexagonal longitudes

In order to provide additional data on the sensitivity of the systems’ lifetimes to initial longitudes, we simulated a set of systems in which the middle planet started 180° behind the inner planet and the outer planet started 60° ahead of the inner planet; we refer to these as Hexagonal longitudes systems. These simulations were performed for values of β that were multiples of 0.001 over the interval [3.465, 5.91] and multiples of 0.01 within the intervals [5.92, 6.72] and [8.01, 8.2]. All Hexagonal longitudes integrations were stopped if no orbits had crossed after 10^8 years; the most tightly-spaced system to survive for 10^8 years had β = 5.95. Lifetimes of the systems with these initial longitudes are shown in Fig. 10.

Results of statistical studies comparing lifetimes of the systems in the Primary and Hexagonal Sets over various ranges in β are shown in Table 8. Systems in our Primary set of runs, which start with the planets approaching conjunction, tend on average to be slightly longer-lived than those in the set with Hexagonal initial longitudes, with statistically significant differences between the two sets of lifetimes seen over the region 4 ≤ β < 5 and differences of lesser significance seen for more widely-spaced orbits.

![Fig. 9. Stability time, t_c, of systems of three 1 M⊕ planets orbiting a 1 M⊙ star, displayed as a function of the initial dynamical separation β. Black points are from SL09 initial conditions. Red points represent systems that were started with the innermost planet displaced by 100 m ahead in initial longitude. Both sets shown here used the modified code to more accurately input initial longitudes of the planets. The solid red line corresponds to the fit to the Chaos runs and the solid gray line shows the fit to the SL09 runs, both over the range 3.47 ≤ β ≤ 6.32. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)](image1)

![Fig. 10. Survival times of integrations using Hexagonal longitudes plotted against orbital separation, β. The black points denote systems with Hexagonal longitudes. The solid gray line corresponds to the fit to lifetimes of systems with Hexagonal longitudes over the range 3.465–5.91 at 0.001 resolution.](image2)

![Fig. 11. Lifetime, t_c, of three-planet systems with initially Aligned longitudes as a function of the separation in units of Hill sphere radii β (Eq. 3). The solid gray line corresponds to the fit over the whole range over which simulations were allowed to proceed for 10^{10} virtual years unless they went unstable, [3.465, 5.187].](image3)

9. Initially aligned longitudes

We perform integrations for systems with the three planets initially aligned (i.e., at conjunction) in the range [3.465, 5.800] at resolution 0.001 in β and covered the range [5.81, 7.15] at resolution 0.01. Results are displayed in Fig. 11. We find 60 systems with β ≤ 5.8 that survived at least 10^9 years, all having β within the range [5,155, 5.354], including 5 within [5,155, 5,187], 10 within [5,188, 5,202], 18 in [5,215, 5,234] and 27 in the interval [5,317, 5,354]. We only integrated the five longest-lived systems within the most closely-spaced of these ranges for >10^10 years. The most tightly-packed system that survived more than 10^10 years (it lasted for 6×10^9 years) has a β value of 5.155, the next one is 5.160 with a crossing time of 1.25 billion years. The two systems that survived for the entire 10^{10} years integrated have dynamical separations of 5.170 and 5.187.
Unlike the other sets for fixed initial longitudes that we studied, where the first upward spikes were located in alternating zones between first and second-order resonances, the first spikes in lifetime for the set of runs with initially aligned longitudes were located in adjacent zones of stability between first and second-order resonances.

A statistical comparison of the lifetimes of systems more tightly spaced than $\beta = 5.188$ (all of which were integrated until two orbits had crossed or $10^{15}$ years had elapsed) with those in our Primary Set of runs is presented in Table 9. The large magnitudes and negative signs of the numbers representing the mean difference between the logarithms of lifetimes given in the first results row of Table 9 (compared to the analogous data in Tables 5–8) imply that initially aligned systems are typically longer-lived than the systems at the other longitude combinations analyzed herein. The Kolmogorov-Smirnov test shows that the computed ensembles of lifetimes are not consistent from being drawn from the same distribution for any of the regions that were compared.

10. Summary and discussion

We integrated the trajectories of bodies in more than 18,000 planar 4-body systems, each consisting of three 1 $M_\star$ planets initially on nested circular orbits about a 1 $M_\odot$ star, until two orbits crossed or a maximum time of $10^8$ or $10^{15}$ years had elapsed. We find the same general trend of system lifetime increasing exponentially with orbital separation apart from dips in lifetimes produced by low-order mean-motion resonances found in previous studies, and present parameters of these fits to this trend and measures of characteristic deviations from these fits for various subsets of our integrations in Table 2.

The orbital crossing times of the major planets that we integrated are shown in Fig. 12, together with a curve representing the theoretical prediction of Petit et al. (2020) for lifetimes of systems with the parameters that we used. The theoretical estimates of stability times are quite good for most of the systems. However, we found regions of long-term stability for some surprisingly closely-spaced three-planet systems, and destabilization by low-order mean motion resonances disrupts some systems of more widely-spaced planets.

The most closely-spaced system to survive for the full $10^{15}$ years integrated has initial separation between neighboring orbits of $\beta = 5.17$ mutual Hill radii which is less than 50% wider than the critical separation required for Hill stability of analogous two-planet systems and 25% closer than the separation beyond which systems of this type are expected to be very long-lived based on the widths and spacings of three-body resonances (Petit et al., 2020). The regions of stability near 5.17 and 5.4 mutual Hill radii (Section 4) are narrow and “porous” (wide swings in lifetimes with small changes in $\beta$ and/or for differing initial planetary longitudes), implying small volumes in the phase space of initial conditions. But the stable regions beginning at $\beta \approx 5.7$ are far broader, albeit restricted in longitude and not continuous for $\beta < 7.15$. We present a more focussed and intensive analysis of very closely-spaced and long-lived three-planet systems in Gavino & Lissauer (in preparation).

Two distributions of instability times within spikes are evident in our data (Table 3). The number of survivors in the spikes for Random longitudes as well as for the most tightly-packed spike regions for Primary, SL09 and Aligned longitudes show drops by a factor of $\sim 2$ per order of magnitude in time. The spikes at greater orbital separation in Primary and Aligned Sets and the one spike seen in the Hexagonal Set have a much larger fraction of long-term survivors. (The statistics are insufficient to assess the SL09 spikes for larger orbital separations.) Perhaps there is something different about the processes driving destabilization within different spikes or different longitudes within a spike; alternatively (or additionally), lifetimes of some of the systems within the wider separation spikes may be orders of magnitude greater than $10^{15}$ years, and the decay rates may become roughly logarithmic at much later times than considered herein. Note that the first two spikes in the Aligned runs are parts of the same spike, divided into two for inclusion in Table 3 because only the inner portion of the region was simulated beyond $10^8$ yr. Also, for the spike surrounding $\beta = 6$, survivors for Random longitudes give a drop off quickly, whereas Primary, Hexagonal and Aligned longitudes show much more gradual loss of systems with time. Thus, the different rates of dropout are more likely to represent end members of a continuum rather than distinct distributions.

Petit et al. (2020)’s estimate for the lifetimes of 3-planet systems with the Earth/Sun mass ratio turns up sharply near period ratio 1.138 (Fig. 12), beyond which lifetimes of many systems are expected to be much longer because three-body resonances no longer overlap (Quillen, 2011; Petit et al., 2020). This period ratio corresponds to $\beta \approx 6.8$ in our Hill sphere radius normalized units, placing it within the last longitude-limited upward spike in lifetimes (moving a little farther out is the dip in lifetimes associated with the 8:7 MMR of neighboring planets, followed by the first upward spike present at all longitude combinations examined).

The broad extents in initial orbital separation and limited ranges of starting planetary longitudes for the spikes in system lifetime observed for $5.5 < \beta < 7$ provide strong evidence that the long survival times of these systems are not the result of planets librating within protective resonances. These spikes extend for large enough ranges in orbital separation that the differences in the relative longitudes of the planets at various $\beta$ values within a spike circulate many times within the number of orbits given by the general exponential trend in $t_s(\beta)$ near the spike.

Although we quote planetary separations in terms of Hill radii for convenience, our results are specific to 1 $M_\star$ planets orbiting a 1 $M_\odot$ star (or other combinations with the same planets/star mass ratio). As planetary mass, $M_p$, is varied, single-encounter dynamics scale approximately with Hill sphere size, $(M_p/M_\star)^{2/3}$, whereas two-planet resonance overlap scales as $(M_p/M_\star)^{1/3}$, three-planet resonance overlap scales as $(M_p/M_\star)^{1/4}$, and locations of individual resonances are independent of mass ratio.
This suggests that the phasing of the early conjunctions of the planets can, in a nontrivial fraction of the cases, have profound effects on the lifetimes of three-planet systems.

In our formalism, all planets begin on circular orbits; we do not try to minimize quantities analogous to proper eccentricities. Fig. 13 compares the early evolution of six of our planetary systems with \( \beta = 6.02 \); three of these systems, those with Primary, Hexagonal and Aligned initial longitudes, are very long lived (\( t_c > 10^5 \) years), whereas the systems beginning with the SL09, Chaos and Random longitudes each go unstable in less than \( 10^4 \) years. For our Primary longitudes, where the planets begin shortly before a three-planet (near) conjunction, eccentricities \( e \approx 0.0014 \) are imparted to the inner and outer planets during the initial planetary conjunction occurring in the first orbit. But the first encounter of the middle planet with its neighbors in the system starting at SL09 longitudes, which occurs in the eighth orbit, together with subsequent planetary encounters during the following two decades, boost the planetary eccentricities. Therefore, the angular momentum deficit (AMD) of this system, which for planar orbits is given by the formula

\[
AMD = m\sqrt{GM_0\left(1 - \sqrt{1 - e^2}\right)}.
\] (9)

grows quickly to values far higher than those obtained by the planets started at the Primary longitudes. The initial conjunctions are different for our systems beginning at our Hexagonal and Aligned longitudes, but the low AMD of the system compared to the values for the Chaos and Random longitudes systems is also established within the first few synodic periods.

Table 2 presents four measures of the scatter of system lifetimes within each range in orbital separation and Set of runs for which we computed log-linear fits on data with resolution of 0.01 in \( \beta \) or better. In most cases, both measure of dispersion relative to the fit lines, \( \sigma_{exp}^{\text{cum}} \) and \( \sigma_{exp}^{\text{cum}} \), grow larger as the upper limit to the range of orbital separations increases. Also, for a given range in orbit separation, both measures of scatter about the local trend, \( \sigma_{local}^{\text{cum}} \) and \( \sigma_{local}^{\text{cum}} \), are typically larger for the Set of runs with randomly-chosen initial longitudes than for Sets of systems with the same combination of initial longitudes, indicating a significant dependence of lifetimes on the starting longitudes of the planets.

Deterministic chaos accounts for some of the scatter in lifetime seen among the systems with similar initial conditions that we integrated, but its effects are typically small, especially for relatively short-lived systems (Fig. 9). Note that we only followed systems until two planetary orbits had crossed. Much larger chaotic variations, especially in fractional changes of the lifetimes of short-lived systems, are found if the orbits are followed to physical collisions of planets with parameters analogous to those of Earth (Bartram et al., 2021).

11. Comparison with previous numerical studies

Tamayo et al. (2016) studied a grid of nonuniformly-spaced systems of three 5 M\( \oplus \) planets; the spacings of the inner pair, \( \beta_i \), and the outer pair, \( \beta_o \), were selected randomly and independently within the range [5, 9]; eccentricities and inclinations were drawn randomly over [0, 0.02] and [0, 1], respectively. They did not find any system of (\( \approx 5000/16 \approx 310 \) simulated) with both \( \beta_i < 6 \) and \( \beta_o < 6 \) that was stable for 10^7 years, and only 5 of the \( \approx 625 \) with \( \beta_i + \beta_o < 12 \) endured for the entire 10^7 initial inner planet orbits that they allowed their integrations to run (Daniel Tamayo, private communication, 2020). They use more massive planets, a difference that is expected to increase stability for the majority of \( \beta \) values since resonance overlap (which varies as the 2/7 power of the planets/star mass ratio, Wisdom (1980)) and 3-body resonance density (proportional to the 1/4 power of the planets/star mass ratio, Quillen (2011)) both scale with smaller powers of mass than the Hill sphere size (1/3 power of the planets/star mass ratio). Their inclusion of nonzero \( e \) acts in the opposite sense (Gratia and Lissauer, 2021); the small mutual inclinations of the planets that they integrate probably have little effect on average. Petit et al. (2020) did not find any systems of three \( \approx 3 \) M\( \oplus \) planets more tightly-spaced than both of their analytic stability estimates that survived for the full 10^7 years that they simulated; they did, however, find a few closely-spaced systems of roughly Mercury-mass planets that survived for the entire 1 Gyr.

Obertas et al. (2017) (OVT17) integrated systems that included five 1 M\( \oplus \) planets with randomly and independently drawn values of \( \beta \) (corresponding to an average resolution of 0.0005 for \( \beta < 10 \)) and initial longitudes. The most closely spaced system to survive for the entire 10^10 years that they simulated had \( \beta = 8.6397 \) (Alysa Obertas, private communication, 2020), corresponding to 2.4941 times the critical separation for Hill-Jacobi stability of two-planet systems.

Our results for three-planet systems are, in many respects, qualitatively similar to those found for analogous five-planet systems analyzed by OVT17. Lifetimes of closely-packed, uniformly-spaced, three-planet systems and five-planet systems each typically increase exponentially with distance between the orbits. In most of the regions of phase space that we examined, system lifetimes depend primarily on orbital separation, with starting angular separations between the planets being far greater.
less important, in agreement with the findings of OV17 for five-planet systems.

But there are significant differences between the dynamics of three-planet systems and of five-planet systems. Three-planet systems are generally longer-lived, with the magnitude of the difference increasing at wider separations, in agreement with the findings of Chambers and Mignolli (1997), Smith and Lissauer (2009) (SL09), and Funk et al. (2010). A more qualitative difference that was not previously recognized is that closely-spaced three-planet systems have narrow regions of parameter space that allow for stability for orders of magnitude longer than most systems with similar orbital separations. These regions are located far from strong mean motion resonances. This likely results from three-planet systems having much smaller numbers of combinations of two planets (3 vs. 10) and of three planets (1 vs. 10), resulting in fewer strong two-planet and three-planet mean motion resonances.

For separations $\beta < 7.1$ (slightly more than twice the separation required for Hill stability of two planets), there are no regions in $\beta$ where spikes are present in both of the initial planetary longitude combinations that we studied most extensively (Sections 4 and 6). This is qualitatively different from the spikes present in OV17’s data for 5-planet systems (all with $\beta > 8.5$), where system lifetimes have little dependence on initial planetary longitudes, which were selected randomly and independently for each of the systems that they integrated. Thus, the dynamics of systems of three planets are profoundly different from those of systems with five (or, presumably, more) planets.

Resonances can destabilize planetary systems, but they can also enhance stability. However, in the systems studied herein, as well as those integrated by SL09 and OV17, the only clear signature of resonances in the lifetime plots is destabilization. Regions of stable resonance trappings for three or more planets in known systems such as TRAPPIST-1 are small (Tamayo et al., 2017). In resonance chains, superperiods\(^9\) are equal, and for small $e$ cases such as Kepler-80 and TRAPPIST-1, superperiods are short enough that period ratios for pairs near the same first-order resonance increase nontrivially as one moves outwards (a few parts per thousand from one planet pair to the next), whereas in our formalism, the superperiods for the inner and outer pairs are nearly equal (compare the black and green points in Fig. 7). Thus, not only is the stable region of parameter space small, for small planetary eccentricities it is also somewhat displaced from the region in which all of these studies have searched.

The zones of stability for $\beta < 6.75$, and especially for $\beta < 5.65$, occupy small fractions of parameter space (Table 3). However, these islands of stability are large enough to be found by a survey of parameter space, as opposed to the minuscule stable fractions of parameter space that are occupied by some actual planetary systems with planets in resonance-lock librations (Tamayo et al., 2017), which have only been reproduced numerically by dissipating systems into resonances.

Values of our measure of deviations of system lifetimes relative to the median lifetime of systems with similar orbital separations, $\sigma_{\text{cum}}$ listed in Table 2 for cases where the resolution in orbital separation is 0.001 or better are similar to or slightly larger than dispersion measures found for shadow trajectories of planar 5-planet systems by Hussain and Tamayo (2020) and those for nearly planar 4-planet systems given by Rice et al. (2018). This suggests that much of the local dispersion is the result of chaos, with lesser contributions of systematic variations in lifetimes caused by very small changes (differences of $<0.005$ mutual Hill radii) in initial orbital separation.

Chambers and Lissauer (2018) performed stability studies of systems of 2–6 planets (each 1 M$_{\odot}$) orbiting in the habitable zones of α Centauri A and α Centauri B, with perturbations of the companion star being taken into account. For a given magnitude of characteristic system lifetime, their figures show substantially more scatter in lifetime for systems of two planets (and thus four massive bodies, like our systems of three planets orbiting a single star) than for systems of three or more planets. However, their calculations were not at sufficiently high resolution in orbital separation to determine whether or not spikes of extremely large increases in lifetime are present for two-planet systems around either of the stars in this nearby stellar binary.

12. Epilogue: relativity, tides, and planets on short-period orbits

The integrations reported upon in this work were run until either a pair of planetary orbits crossed or the elapsed time reached a pre-determined value of either $10^{6}$ or $10^{10}$ times the initial orbital period of the inner planet ($10^{6}$ or $10^{10}$ years for the inner orbit located 1 AU from a 1 M$_{\odot}$ star). Ten billion year integrations are sufficient for 1 M$_{\odot}$ planets in the habitable zone of a sunlike star, as the Universe is only slightly older than this, and moreover the main sequence lifetime of a 1 M$_{\odot}$ star is $\sim$10$^{10}$ years. However, the dynamical age (number of orbital periods of the inner planet) of most Kepler multiple planet systems is $10^{11}$ – $10^{12}$ orbits, so one may ask why have we (and almost all previous studies of this type) limited our simulations to dynamical ages of at most $10^{10}$ inner planet orbits, and are the results of such time-limited simulations relevant for the stability of Kepler’s multi-planet systems?

As noted in Section 2, our integrations used purely Newtonian dynamics of point mass particles, neglecting general relativistic deviations (GR) as well as tidal forces, both of which are more important for planets orbiting very close to their stars. Neglecting GR and tides makes the problem scale-invariant, allowing our results to be applied to planetary systems orbiting closer to their stars and to lower (higher)-mass planets orbiting lower (higher)-mass stars provided orbits and masses were varied by the same factor for all bodies in the system\(^10\). In contrast, including GR and/or tides would specialize the study to specific planetary (and stellar) masses and orbital sizes, voiding the scalings discussed at the beginning of Section 2 and expanding the dimensionality of the parameter space of the (already computationally-demanding) problem. Incorporating tidal damping would require even more specialization than accounting for GR because of the great diversity of tidal response characteristics that terrestrial planets may possess.

Although the first-order post-Newtonian deviations from classical dynamics caused by general relativity are important in stabilizing our Solar System, this is because GR moves the locations of resonances rather than making fundamental changes to the dynamics (Batygin and Laughlin, 2008; Laskar, 2008). Therefore, GR may well not be important to statistical samples appropriate for understanding general scalings for planetary systems except in the regime where it substantially alters orbital precession rates or some other key variable.

Tides are more fundamental to the problem than are the principal post-Newtonian terms in GR, as they induce dissipative processes that fundamentally alter the dynamics of the system\(^11\), e.g., by damping eccentricities that have been excited by perturbations among the planets. Tidal damping is likely to be strong enough to prevent the applicability of dissipation-free simulations to planets with orbital periods of a few weeks or less for timescales of billions of years, but tidal damping drops off rapidly with increasing orbital distance. Thus, given the age of the Universe, non-dissipative integrations (the type presented herein) lasting for more than $10^{11}$ orbits would be of minimal applicability to exoplanet systems. Integrations for $10^{10}$ orbits require wall clock times of several days using modern cpu cores, so integrating for longer than this would severely reduce the number of systems that it would be

\(^9\) The superperiod of two planets near but not in a first-order MMR is equal to the periodicity of the near-resonant argument. Its value is given by $|2\pi(N_2 - (N - 1)N_1)| \cdot \frac{1}{P_1}$.

\(^10\) Analogous scalings applicable to planetary accretion simulations that include physical collisions are given in Appendix C of Quintana and Lissauer (2018).

\(^11\) Gravitational radiation also removes mechanical energy from the system, but the amount lost is exceeding small for all known multi-planet systems.
feasible to integrate. Although our longest integrations are only one-tenth of the desired duration of $10^{15}$ orbits, instabilities tend to occur logarithmically with time, i.e., the range in $\beta$ for which orbits cross increases logarithmically with time. The fundamental timescale for instabilities is the synodic period of the planets, which is of order ten orbits for the closely-spaced systems considered herein. Thus, in a logarithmic sense, results for integrations for $10^{16}$ orbits would be expected to reveal roughly 90% of the physics that occurs within $10^{15}$ orbits. Eventually, as computing costs continue to drop, it will be feasible to test this hypothesis with a statistically-significant number of $10^{11}$ orbit integrations.

Declaration of Competing Interest

None.

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References