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ANCIENT SOLUTIONS TO THE CURVE SHORTENING FLOW SPANNING THE HALFPLANE

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Abstract. In this note we construct an infinite family of compact ancient solutions to the Curve Shortening Flow which span the halfplane.

1. Introduction

The Curve Shortening Flow (CSF) is a family of immersed curves which moves along the curvature vector $\kappa$. The CSF is the one dimensional example of the Mean Curvature Flow (MCF), the most studied extrinsic geometric flows.

In this paper, we restrict our attention to ancient solutions to the CSF. A solution $\{\gamma_t\}$ to the CSF is called ancient if $\gamma_t$ is defined in $t \in (-\infty, T_0)$ for some $T_0$.

Besides the straight lines and the shrinking circles, the first non-trivial examples of ancient solutions are given by Abresch and Langer [1], where they give a complete classification of compact self-shrinking solutions to the CSF. This result is generalized in [15], where Halldorsson gives a classification of all self-similar solutions to the CSF. Here self-similar means that the CSF evolves by a combination of rotation, translation and scaling. The first ancient solution which is not self-similar is the paperclip solution, also known as the Angenent oval (introduced in [18], [2]). The Angenent oval is a convex embedded compact ancient solution lying inside a slab, which can be formed by gluing two Grim Reapers moving in opposite directions. More examples are constructed in [4], [21] by gluing finitely many Grim Reapers.

Although ancient solutions to the CSF exist in abundance, there are only four convex embedded one up to ambient isometry and parabolic rescaling. Daskalopoulos, Hamilton, and Sesum show in [12] that the shrinking circle and the Angenent oval are the only compact embedded ancient solutions. Recently, Bourri, Langford and Tinaglia show in [7] that the stationary line and the Grim Reaper are the only noncompact convex embedded example. Thus the classification of ancient embedded convex solutions to the CSF is complete.

On the other hand, from the constructions in [4], [21], [15], it is unlikely to obtain a complete classification of all ancient solutions. One may wonder what geometric conditions can be imposed in order to obtain some classification results.

Recently, Chini and Møller prove in [11] a bi-halfspace property for ancient codimension one MCF: Let $\{M_t : t \in (-\infty, T_0)\}$ be an $n$-dimensional properly immersed ancient solution to the MCF in $\mathbb{R}^{n+1}$. For each $t$, let $\text{Conv}(M_t)$ be the convex hull of $M_t$ in $\mathbb{R}^{n+1}$.
Then up to ambient isometry and parabolic rescaling,
\begin{equation}
\mathcal{M} := \bigcup_{t < T_0} \text{Conv}(M_t)
\end{equation}
is either
\begin{itemize}
  \item the hyperplane \( L = \{ x \in \mathbb{R}^{n+1} : x_{n+1} = 0 \} \),
  \item the slab \( S = \{ x \in \mathbb{R}^{n+1} : 0 < x_{n+1} < 1 \} \),
  \item the halfspace \( H = \{ x \in \mathbb{R}^{n+1} : x_{n+1} > 0 \} \)
  \item or the whole \( \mathbb{R}^{n+1} \).
\end{itemize}

As a result, the set of all properly immersed ancient solutions is divided into four classes, and one might ask if one can classify any of these. The stationary hyperplane is the only one with \( \mathcal{M} = L \). On the other hand, there are lots of ancient solutions with \( \mathcal{M} = \mathbb{R}^{n+1} \), which (e.g.) include all properly immersed self-shrinking solutions \cite{10}.

For the remaining cases, note that the Grim Reaper and the Angenent oval have \( \mathcal{M} = S \), so are the examples in \cite{4}. In higher dimensions, compact convex ancient solutions of the MCF lying inside a slab is constructed in \cite{6}, \cite{20}. Indeed, Wang shows in \cite{20} that for any convex ancient solutions, \( \mathcal{M} \) is either a slab or the whole space.

It is interesting that, to the author’s knowledge (see also p.4 in \cite{11}), there isn’t any example of compact ancient solution which satisfies \( \mathcal{M} = H \). The main goal of this paper is to construct an infinite family of such examples in the case of CSF.

**Theorem 1.1.** There is an infinite family of compact immersed ancient solutions \( \{ \mathcal{C}(t) \}_{t < T_0} \) to the CSF so that
\[ \bigcup_{t < T_0} \text{Conv}(\mathcal{C}(t)) \]
is the halfspace. Moreover, \( \mathcal{C}(t) \) becomes embedded at some time before extinction.

In the above theorem, we consider only compact solutions since non-compact examples can be constructed easily by gluing infinitely many Grim Reapers. This can be done using similar techniques in \cite{21}, although that is not explicitly written down.

The examples constructed in Theorem 1.1 are not embedded. It is not known if there is any embedded compact ancient solution to CSF with \( \mathcal{M} = \mathbb{H} \), or if the Angenent oval is the only compact embedded example inside a slab.

Next we briefly describe the proof of Theorem 1.1. Recall that in \cite{4}, \cite{21}, Angenent and You construct examples of ancient solutions to the CSF by gluing either finitely or infinitely many Grim Reapers along the common asymptotes. Thus their examples either lie in a slab, or is non-compact. In our situation we modify their construction. Roughly speaking, the ancient solutions constructed in Theorem 1.1 have the following property: there is a sequence of compact ancient solutions to the CSF \( \{ \mathcal{C}_n(t) \} \) so that \( \mathcal{C}_0(t) \) is the Angenent oval, and there is a sequence of time
\[ T_1 > \cdots > T_n > \cdots, \quad T_n \to -\infty, \]
so that for each \( n \in \mathbb{N} \) (see Figure 1),
\begin{itemize}
  \item when \( t << T_n \), \( \mathcal{C}_n(t) \) is formed by gluing a figure 8 (modeled by gluing four Grim Reapers) to \( \mathcal{C}_{n-1}(t) \),
\end{itemize}
• when $t \sim T_n$, the figure 8 in $C_n(t)$ disappears, and $C_n(t)$ becomes embedded locally, and
• when $t >> T_n$, $C_n(t)$ is close to $C_{n-1}(t)$.

Then the ancient solutions constructed in Theorem 1.1 is the limit of $\{C_n(t)\}$ as $n \to \infty$.

Figure 1: A figure 8 disappears around time $T_n$.

Next we compare the technical difficulties in proving Theorem 1.1 with those in [4], [21]. In their works, the $C^0$ error at the tip regions are controlled by proving the convexity there, which holds when $-t$ is large enough. In our situation, since we require the figure 8 to unfold and disappear, we need $C^0$ estimates also when $-t$ is not large. Also, the error is harder to control since we need to glue infinitely many Grim Reapers.

To obtain the required $C^0$ estimates, we construct the sequence of ancient solutions $\{C_n(t)\}$ in such a way that for each $t$, the (signed) area bounded by $C_n(t)$ are uniformly bounded. Then we prove (in Lemma 2.6) a $C^0$ estimates directly using the area bound. We remark that some forms of area estimates is also essential in the construction in [4], [21].

In section 2, we describe the construction of the sequence of ancient solutions $\{C_n(t)\}_{n=0}^{\infty}$ and prove some properties about them. In section 3, we derive some $C^0$ estimates between $C_n(t)$, $C_m(t)$ for $m \neq n$. In section 4, we prove Theorem 1.1.

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2. Compact ancient solutions $\mathcal{C}_n(t)$

First we recall some basic facts about Curve Shortening Flow (CSF). For any immersed curve $\gamma : I \to \mathbb{R}^2$, the curvature is given by

$$\kappa = \langle \nabla e, \nu \rangle,$$

where $e = \gamma'/|\gamma'|$ is the unit vector field along $\gamma$ and $\nu$ is the unit normal vector fields. The CSF is a family of immersed curves $\{\gamma_t : I \to \mathbb{R}^2 : t \in [t_1, t_2]\}$ so that

$$\frac{\partial \gamma_t}{\partial t} = \vec{\kappa},$$

where $\vec{\kappa} = \kappa \nu$ is the curvature vector. The CSF is the one dimensional version of the Mean Curvature Flow (MCF).

When an immersed curve $\gamma$ is given by a graph $u$, the curvature is given by

$$(2.1) \quad \kappa = \frac{u''}{(1 + u'^2)^{3/2}};$$

and when $\gamma_t$ is a family of immersed curves given by a graph $u = u(t, \cdot)$, then up to tangential diffeomorphisms, the CSF equation is equivalent to

$$(2.2) \quad \frac{\partial u}{\partial t} = \frac{u''}{1 + u'^2}.$$

(2.2) has the following essential consequence: if $\gamma_t$ is a family of CSF which is given by a graph in $y$, that is, $\gamma_t(y) = (u(t, y), y)$ for some $u(t, \cdot) : [a(t), b(t)] \to \mathbb{R}$ with $u(t, a(t)) = u(t, b(t)) = 0$, then

$$(2.3) \quad \frac{d}{dt} \int_{a(t)}^{b(t)} u(t, y) dy = \theta(b(t)) - \theta(a(t)),$$

where $\theta(y) = \arctan u'(y)$ is the angle the graph $u$ made with the $y$-axis.

Next we recall the construction of a sequence $\{\mathcal{C}_n(t)\}_{n=1}^{\infty}$ of compact ancient solutions to the CSF in $\mathbb{R}^2$ formed by gluing finitely many Grim Reapers in [4].

Let $\{a_n\}_{n=1}^{\infty}$ be any sequence of positive real numbers so that

- $a_n \geq 1$ for all $n \in \mathbb{N}$,
- $\sum a_n^{-1}$ diverges, and
- $\sum a_n^{-2}$ converges.

For example, $a_n = n$ satisfies the above conditions. Define $h_0 = 0$ and

$$(2.4) \quad h_n = a_1^{-1} + a_2^{-1} + \cdots + a_n^{-1}, \quad \text{for all } n \in \mathbb{N}.$$ 

Let $G : (0, \pi) \to \mathbb{R}$ be given by

$$G(y) = \ln \sin y.$$

Then

$$G(t) : \{x = G(y + \pi) - t\}$$

is a translating solution to the CSF. Any ambient isometry and parabolic rescaling of $G_0(t)$ is called a Grim Reaper.
For each \( m \in \mathbb{N} \), define
\[
G_m^-(y) = \frac{1}{a_m} G \left( a_m(y - 2\pi h_{m-1}) \right), \\
G_m^+(y) = \frac{1}{a_m} G \left( a_m(y - 2\pi h_{m-1} - a_m^{-1}\pi) \right).
\]

and
\[
G_m^-(t) : \{ x = -G_m^-(y) + a_m t \}, \\
G_m^+(t) : \{ x = G_m^+(y) - a_m t \}.
\]

Note that for each \( m \in \mathbb{N} \), \( G_m^-(t) \) and \( G_m^+(t) \) are Grim Reapers lying in the slab \( \{ 2\pi h_{m-1} < y < 2\pi h_{m-1} + a_m^{-1}\pi \} \) and \( \{ 2\pi h_{m-1} + a_m^{-1}\pi < y < 2\pi h_m \} \) respectively.

Let \( n \in \mathbb{N} \) and let \( B_1, \cdots, B_n \) and \( L \) be any positive numbers. We write
\[
C_n = B_1 + \cdots + B_n
\]
and for any \( t < -C_n - 1 \), consider the following (portions of) Grim Reapers:
\[
G_0(t) : x \geq 0 \tag{2.6} , \\
G_m(t + C_m) : x \leq 0, \; m = 1, 2, \cdots, n. \tag{2.7} \\
G_m(t + C_m - L a_m^{-2}), x \geq 0, \; m = 1, 2, \cdots, n. \tag{2.8}
\]

Note that \( G_0(t), G_1(t) \) shares a common asymptotes \( \{ y = 0 \} \) and for \( m = 1, \cdots, n \), \( G_m(t + C_m) \) shares the common asymptotes \( \{ y = 2\pi h_{m-1} \}, \{ y = 2\pi h_{m-1} + \pi a_m^{-1} \} \) with \( G_{m-1}(t + C_{m-1} - L a_m^{-2}) \) and \( G_m(t + C_m - L a_m^{-2}) \) respectively.

In the following we glue the above Grim Reapers along their common asymptotes to form a broken curve as in section 3.2 of [4].

Remark 1. In [4], the gluing is done using smooth functions. However, to simplify our argument (so that maximum principle is easily applicable) we choose the following specific gluing:

(I) The Grim Reaper \([2.6]\) intersects \( \{ x = 1 \} \) at two points \( (1, y_1), (1, y_2) \) with \( y_1 < y_2 \). Join \( (0, -\pi) \) to \( (1, y_1) \) and \( (1, y_2) \) to \( (0, 0) \) respectively by straight lines.

(II) For each \( m = 1, 2, \cdots, n \), the Grim Reaper \([2.7]\) intersects \( \{ x = -a_m^{-1} \} \) at \( (-a_m^{-1}, z_1^-), (-a_m^{-1}, z_2^-) \) with \( z_1^- < z_2^- \). Join \( (0, 2\pi h_{m-1}) \) to \( (-a_m^{-1}, z_1^-) \) and \( (-a_m^{-1}, z_2^-) \) to \( (0, 2\pi h_{m-1} + a_m^{-1}\pi) \) respectively by straight lines.

(III) For each \( m = 1, 2, \cdots, n \), the Grim Reaper \([2.8]\) intersects \( \{ x = a_m^{-1} \} \) at \( (a_m^{-1}, z_1^+), (a_m^{-1}, z_2^+) \) with \( z_1^+ < z_2^+ \). Join \( (0, 2\pi h_{m-1} + a_m^{-1}\pi) \) to \( (a_m^{-1}, z_1^+) \) and \( (a_m^{-1}, z_2^+) \) to \( (0, 2\pi h_m) \) respectively by straight lines.

The resulting curve formed by gluing the Grim Reapers \([2.6], \tag{2.7}, \tag{2.8} \) as in Remark 1 is graphical in \( y \). Hence the curve is given by the graph of a function \( f_n(t, \cdot) : (-\pi, 2\pi h_n) \to \mathbb{R} \).
Figure 2: graph of $\bar{g}_n(t, \cdot)$ when $n = 1$, with the gluing (I), (II), (III) marked in green, yellow and red respectively.

For any $t < -C_n - 1$, the broken solution $\mathcal{C}_n(t)$ is defined as

$$\mathcal{C}_n(t) = \{ x = \bar{g}_n(t, y) : -\pi \leq y \leq 2\pi h_n \}^r,$$

where for any subset $X \subset \mathbb{R}^2$ we write $X^r = X \cup RX$ and $R$ is the reflection along the $y$-axis (see Figure 3). To be precise, $\mathcal{C}_n(t)$ consists of two graphs

$$\{ x = \bar{g}_n(t, y) : -\pi \leq y \leq 2\pi h_n \}, \quad \{ x = -\bar{g}_n(t, y) : -\pi \leq y \leq 2\pi h_n \}$$

and we glue them at the common end points $(0, -\pi)$, $(0, 2\pi h_n)$. The resulting curve $\mathcal{C}_n(t)$ is an image of a continuous mapping $S^1 \to \mathbb{R}^2$ which is immersed away from finitely many points.

Figure 3: the broken curve $\mathcal{C}_n(t)$ when $n = 1$.

For each $\alpha > C_n + 1$, let $\mathcal{C}_n^\alpha(t)$ be the CSF with $\mathcal{C}_n^\alpha(-\alpha) = \overline{\mathcal{C}}_n(-\alpha)$.

**Remark 2.** The CSF starting at a piecewise smooth curve $\gamma : S^1 \to \mathbb{R}^2$ is constructed in [3] as follows: let $\{ \gamma_k : S^1 \to \mathbb{R}^2 \}$ be a sequence of smooth immersions with bounded derivatives which converges in $C^0$ to $\gamma$. For each $k$, we consider the CSF $\gamma_k(t)$ starting at $\gamma_k$. For small $t > 0$, the interior estimates [3, Theorem 3.1] enable us to take the limit of $\gamma_k(t)$ as $k \to \infty$ and construct a CSF $\{ \gamma(t) \}$ starting at $\gamma$. The non-compact situation is done similarly in [13].

Note that $C_n^\alpha(t)$ are invariant under the reflection $R$. We write

$$C_n^\alpha(t) = \{ x = g_n^\alpha(t, y) : y_{0,n}^{\alpha,-}(t) \leq y \leq y_{0,n}^{\alpha,+}(t) \}^r.$$  

(2.9)

Here $g_n^\alpha$ satisfies the graphical CSF equation (2.2) and $g_n^\alpha(-\alpha, \cdot) = \bar{g}_n(-\alpha, \cdot)$. Note that $C_n^\alpha(t)$ is defined as least when $-\alpha < t < -C_n - 1$ and intersects the $y$-axis at $2n + 2$ points

$$-\pi < y_{0,n}^{\alpha,-}(t) < y_{0,n}^{\alpha,+}(t) < \cdots < y_{n,n}^{\alpha,-}(t) < y_{n,n}^{\alpha,+}(t) < 2\pi h_n.$$  

(2.10)
2n of them (besides \((0, y_{0,n}^-(t)), (0, y_{0,n}^+(t))\)) are also self-intersection points of \(C_n^\alpha(t)\).

The following theorem is proved in section 4 of [4].

**Theorem 2.1.** For each \(n \in \mathbb{N}\), the CSFs \(C_n^\alpha(t)\) converges, as \(\alpha \to +\infty\), to an ancient solution \(C_n(t)\) to the CSF.

Just like \(C_n^\alpha(t)\), the immersion \(C_n(t)\) is defined when \(t < -C_n - 1\) and intersects the \(y\)-axis at \(2n + 2\) points

\[-\pi < y_{0,n}^-(t) < y_{0,n}^+(t) < \cdots < y_{n,n}^-(t) < y_{n,n}^+(t) < 2\pi h_n.\]

2n of them (besides \((0, y_{0,n}^-(t)), (0, y_{0,n}^+(t))\)) are also self-intersection points of \(C_n(t)\).

For each \(n \in \mathbb{N}\) we write

\[C_n(t) = \{x = g_n(t, y) : y_{0,n}^-(t) \leq y \leq y_{n,n}^+(t)\},\]

where

\[g_n = \lim_{\alpha \to \infty} g_n^\alpha.\]

**Lemma 2.1.** If \(m > n\), then \(g_m(t, y) > g_n(t, y)\) for all \((t, y)\) so that both \(g_n, g_m\) are defined.

**Proof.** For any \(t < -C_m - 1\), choose \(\alpha > -t\). Then

\[C_n^\alpha(-\alpha) = \overline{C}_n(-\alpha),\quad C_m^\alpha(-\alpha) = \overline{C}_m(-\alpha).\]

Also, by Remark 1, \(\overline{C}_m(-\alpha, y) = \overline{g}_n(-\alpha, y)\) whenever \(\overline{g}_n(-\alpha, y)\) is defined. As in Remark 2 we choose a sequence of function \(f_k : [a_k, b_k] \to \mathbb{R}\) with the following properties:

- \(-\pi < a_k < b_k < 2\pi h_n,
- a_k \to -\pi\) and \(b_k \to 2\pi h_n\) as \(k \to \infty\),
- \(f_k(a_k) = f_k(b_k) = 0\),
- \(f_k(y) < \overline{g}_n(-\alpha, y)\) for all \(y \in [a_k, b_k]\),
- for each \(k\),

\[\{x = f_k(y) : y \in [a_k, b_k]\}\]

is a smooth immersion, and

- \(f_k(\cdot) \to \overline{g}_n(-\alpha, \cdot)\) in \(C^0\) as \(k \to \infty\).

For each \(k\), let

\[\{x = f_k(t, y) : a_k(t) \leq y \leq b_k(t)\}\]

be the CSF starting at \((2.12)\) with \(f_k(-\alpha, \cdot) = f_k(\cdot)\). Then \(f_k(t, y)\) satisfies the graphical CSF equation \((2.2)\). Note that \(\overline{g}_m(-\alpha, \cdot) > f_k(\cdot)\). For each \(k\) and for all \(t > -\alpha\), we claim that

\[g_m^\alpha(t, y) > f_k(t, y)\]

for all \(y \in [a_k(t), b_k(t)]\). The inequality is obvious when \(t\) is close to \(-\alpha\). In general we argue by contradiction: assume that \((2.14)\) fails at the first instance \(t_0\). Since \((2.14)\) holds for all \(t < t_0\), the graph of \(f_k(t_0, \cdot)\) touches the graph of \(g_m^\alpha(t_0, \cdot)\) from below. By [3 Theorem 1.1], these two graphs can touch at only finitely many points. Since \(f_k\),
\( g_m^\alpha \) both satisfy (2.2), the strong maximum principle implies that the graph of \( f_k(t_0, \cdot) \) and \( g_m^\alpha(t_0, \cdot) \) cannot touch in the interior \((a_k(t_0), b_k(t_0))\). Similarly, represent \( f_k \) (resp. \( g_m^\alpha \)) locally around \((0, a_k(t_0))\) (resp. \((0, y_{0,m}^-((t)))\)) as a graph of \( x \), the strong maximum principle implies that the two graphs cannot touch at \((0, a_k(t_0)), (0, y_{0,m}^-((t)))\). Lastly, if the graphs touch at \((0, b(t_0))\), then either
\[
\lim_{y \to b(t_0)^-} (g_m^\alpha)'(t_0, y) = -\infty
\]
when \( b_k(t_0) = g_{k,m}^{\alpha,+}(t_0) \) or
\[
(g_m^\alpha)'(t_0, b_k(t_0)) \leq 0,
\]
when \( b_k(t_0) = g_{k,m}^{\alpha,+}(t_0) \) for some \( k = n, \ldots, m - 1 \). Representing \( f_k \), \( g_m^\alpha \) locally around \((0, b_k(t_0))\) as graphs of \( x \), one sees that the first case is impossible by the strong maximum principle. For the second case, using also
\[
\lim_{y \to b_k(t_0)^-} f_k'(t_0, y) = -\infty,
\]
there is \( y < b_k(t_0) \) so that \( g(t_0, y) < f_k(t_0, y) \), which is again impossible. Thus the graph of \( g_m^\alpha(t_0, \cdot) \) and \( f_k(t_0, \cdot) \) do not touch and this contradicts to the choice of \( t_0 \). This finishes the proof of (2.14).

Taking \( k \to \infty \), we have
\[
g_m^\alpha(t, y) \geq g_n^\alpha(t, y)
\]
for all \( \alpha > -t \). By (2.11), we have \( g_m(t, y) \geq g_n(t, y) \). An argument using \[17\] Theorem 1.1] and strong maximum principle again imply \( g_m(t, y) \geq g_n(t, y) \) whenever \( g_n(t, y) \), \( g_m(t, y) \) are defined.

**Lemma 2.2.** Let \( n \in \mathbb{N} \) and \( m \geq n \). If \( t < -C_n - 1 \) and \( g_m(t, y) \) is defined at \( t \), then
\[
g_m(t, y) < -G_n^-(y) + a_n(t + C_n)
\]
for all \( 2\pi h_{n-1} \leq y \leq 2\pi h_{n-1} + \pi a_n^{-1} \).

**Proof.** Let \( -\alpha < t \). By definition,
\[
g_m(-\alpha, y) \leq -G_n^-(y) + a_n(t + C_n)
\]
for all \( y \in (2\pi h_{n-1}, 2\pi h_{n-1} + a_n^{-1} \pi) \). Since \(-G_n^-(y) + a_n(t + C_n)\) converges to \(+\infty\) as \( y \to 2\pi h_{n-1}, 2\pi h_{n-1} + a_n^{-1} \pi, g_m^\alpha(t, y) \) is bounded and both \( g_m^\alpha(t, y) \), \(-G_n^-(y) + a_n(t + C_n)\) satisfy (2.2), the comparison principle \[17\] Theorem 4.3] implies that
\[
g_m^\alpha(t, y) \leq -G_n^-(y) + a_n(t + C_n)
\]
for all \(-\alpha < t < -C_n - 1\). Taking \( \alpha \to \infty \), we have \( g_m(t, y) \leq G_n^+(y) - a_n(t + C_n) \) and the strictly inequality follows from strong maximum principle.

Next we define a sequence of ancient solutions \( O_n(t) \). For each \( n \geq 0 \), let
\[
(2.15) \quad \overline{O}_n(t) = \{ x = \bar{g}_n(t, y) : 2\pi h_{n-1} + a_n^{-1} \pi \leq y \leq 2\pi h_n \}^{re}.
\]
As in the construction of \( C_n(t) \), we let \( \alpha > 0 \) and let \( O_n^\alpha(t) \) be the CSF with \( O_n^\alpha(-\alpha) = \overline{O}_n(-\alpha) \). Write
\[
(2.16) \quad O_n^\alpha(t) = \{ x = o_n^\alpha(t, y) : o_n^\alpha^-(t) \leq y \leq o_n^\alpha^+(t) \}^{re},
\]
Lemma 2.4. There is $o_n^a(t, \cdot) \geq 0$. Take $\alpha \to \infty$, we obtain the ancient solution $O_n(t)$. Note that $O_0(t) = C_0(t)$ by construction. Write

\begin{equation}
O_n(t) = \{x = o_n(t, y) : o_n^-(t) \leq y \leq o_n^+(y)\}^{re},
\end{equation}

where $o_n(t, y) \geq 0$. For $n \geq 1$, the strong maximum principle implies that

\begin{equation}
o_n(t, y) < g_n(t, y)
\end{equation}

whenever $t > -C_n - 1$.

Next we estimate the “area” bounded between

- $o_n$ and $g_n$,
- $g_m$ and $-G^-(\cdot) + a_n(t + C_n)$, and
- $g_m$ and $g_m$

respectively. To be precise, let $f_i : I_i \to \mathbb{R}$, $i = 1, 2$ be two functions defined on two intervals with $I_1 \subset I_2$, $f_1 \leq f_2$ and $f_i(\partial I_i) = 0$ for $i = 1, 2$. The signed area bounded between $f_1$ and $f_2$ is defined as

\[
\int_{I_2} f_2 - f_1,
\]

where we extend the domain of $f_1$ to $I_2$ by setting $f_1(x) = 0$ for all $x \in I_2 \setminus I_1$. If $f_2 \geq 0$ in $I_2 \setminus I_1$, then the signed area is just the area under the graph $|f_2 - f_1|$, and in this case we simply use the word area.

**Lemma 2.3.** For each $n \geq 1$, there is $A_n > 0$ depending on $\{a_n\}$ so that for all $t < -C_n - 1$, the area bounded between $o_n(t, y)$ and $g_n(t, y)$ in $[y_{n,n}^-(t), y_{n,n}^+(t)]$ is smaller than $A_n$.

**Proof.** For any $\alpha > C_n + 1$ and $t \in [-\alpha, -C_n - 1]$, Let $A^\alpha(t)$ be the area bounded between $o_n^\alpha(t, \cdot)$ and $g_n^\alpha(t, \cdot)$ in $[y_{n,n}^-(t), y_{n,n}^+(t)]$. Note that

\begin{equation}
A^\alpha(-\alpha) = 0,
\end{equation}

Since $o^\alpha(-\alpha, y) = g_n^\alpha(-\alpha, y) = g_n(-\alpha, y)$ for all $y \in [2\pi h_{n-1} + a_n^{-1}\pi, 2\pi h_n]$. Using (2.3), we have

\begin{equation}
\frac{d}{dt} A^\alpha(t) = \theta(t),
\end{equation}

here $\theta(t)$ is the angle $\{x = g_n^\alpha(t, y)\}$ made with the $x$-axis at $(0, y_{n,n}^-(t))$. Arguing as in the proof of Lemma 3.5 in [4], we have $|\theta(t)| \leq M e^{\delta(t + C_n)}$, where $M, \delta$ depends only on $a_n$. Together with (2.19) the lemma is proved by integrating (2.20) and take $\alpha \to \infty$. \□

The following Lemma is proved similarly as in Lemma 3.5 and 3.6 in [4].

**Lemma 2.4.** There is $A'_n > 0$ depending on $\{a_n\}$ so that for all $t < -C_n - 1$, the area bounded between $\max\{G_n^-(y) - a_n^{-1}(t + C_n), 0\}$ and $|g_n(t, y)|$ in $[y_{n-1,n}^+(t), y_{n,n}^-(t)]$ is smaller than $A'_n$. 
Lemma 2.5. For any \( m, n \in \mathbb{N} \) with \( m > n \), the signed area bounded between \( g_n \) and \( g_m \) is
\[
L(a_{n+1}^{-2} + \cdots + a_m^{-2})
\]
whenever both \( g_n, g_m \) are defined.

Proof. It suffices to show the case when \( m = n + 1 \). We first calculate the area bounded between \( \tilde{g}_\alpha(t, \cdot) \) and \( \tilde{g}_{\alpha+1}(t, \cdot) \) for \( t > -\alpha \) and then take \( \alpha \to \infty \). Since \( \tilde{g}_\alpha, \tilde{g}_{\alpha+1} \) both satisfy the CSF equation, the area bounded between \( \tilde{g}_\alpha(t, \cdot) \) and \( \tilde{g}_{\alpha+1}(t, \cdot) \) is independent of \( t \) by (2.3). Thus it suffices to calculate the area \( A_\alpha \) bounded between \( \tilde{g}_n(\alpha, \cdot) \) and \( \tilde{g}_{n+1}(\alpha, \cdot) \). From the construction described in Remark 1, since \( \tilde{g}_n(\alpha, \cdot) = \tilde{g}_{n+1}(\alpha, \cdot) \) on \([-\pi, 2\pi h_{n-1}]\), we have
\[
A_\alpha = La_{n+1}^{-2} + o(1),
\]
as \( \alpha \to \infty \). Hence the lemma is proved by taking \( \alpha \to \infty \). \( \square \)

Before we move to the next section, we prove a lemma which provides a \( C^0 \) estimates using the area bound. In the following, we assume that either \( r(t, \cdot) = G(y) - t \) or \( r(t, \cdot) = o_0(t, \cdot) \).

Lemma 2.6. Given any \( A > 0 \), there are \( E, M > 0 \) depending only on \( A \) such that the following holds: For any \( M \geq M \) and \( T \in \mathbb{R} \) (when \( r(t, \cdot) = o_0(t, \cdot) \), we also assume \( T \leq T^* \), so that \( \max_y o_0(T^*, y) > 1 \)). Let
\[
\{ C(t) : t \in [T - M, T] \}
\]
be a CSF of immersed curves in \( \mathbb{R}^2 \). Assume that for any \( t \in [T - M, T] \), \( C(t) \) is represented as a graph
\[
\{ x = g_c(t, y) : y^-(t) \leq y \leq y^+(t) \}
\]
inside the region (here \( R_T = \max_y r(T, y) \))
\[
\mathcal{R} := [R_T - 1, \infty) \times I
\]
with \( g_c(t, y^+(t)) = R_T - 1 \) and \( I \) is an interval with length \( \leq 4\pi \). Also,
\[
1 \) for each \( t \), \( g_c(t, y) \) has exactly one local maximum and no local minimum (except at the boundary points \( y^+(t) \)),
\[
2 \) \( g_c(t, y) > r(t, y) \) for all \( t \in [T - M, T] \), and
\[
3 \) the area bounded between the two graphs \( g_c(t, y), r(t, y) \) in \( \mathcal{R} \) is smaller than \( A \).

Then \( \max_y g_c(T, y) - R_T < E \).
Proof. At time $t = T$, the graph $\{x = r(t, y)\}$ intersects with the line $\{x = R_T - 1\}$ at two points $(R_T - 1, c), (R_T - 1, d)$ with $c < d$. Note that $d - c \geq \ell$ for some positive constant $\ell$ (if $r(t, y) = G(y) - t, d - c$ is constant). Define

\[
K = \frac{2(A + 5\pi)}{\ell}
\]  

and

\[
E = \frac{20\pi K}{\ell}.
\]  

Note that $K, E$ depend only on $A$.

We argue by contradiction. Assume that $\max_y g_C(T, y) - R_T > E$. From the CSF equation (2.2), the maximum value of $g_C$ is non-increasing in $t$. As a result,

$$\max_y g_C(t, y) - R_T > E$$

for all $t \in [T - M, T]$ and in the region

$$[R_T - 1, R_T + E] \times I,$$

the curve $C(t)$ is a union of two graphs in $x$ with uniform $C^0$ bound by assumption (1). By the gradient estimates for graphical MCF [14, Corollary 5.3], assuming that $M \geq 1$, the gradient of these two graphs are also bounded uniformly. Let $v_1(x), v_2(x)$ be those two graphs at time $t = T$ with $v_1 < v_2$.

Using assumption (2) and the Mean Value Theorem, there are $x_1, x_2 \in (R_T - 1, R_T + E)$ so that

$$|v_i'(x_i)| \leq 4\pi/E.$$

Then for all $x \in [R_T - 1, R_T - 1 + K]$,

$$|v_i'(x)| \leq |v_i'(x_i)| + \sup |v_i''||x - x_i|
\leq 4\pi/E + \sup |v_i''|E.$$

Next we use $|v_i'| \leq C$, (2.1) for $i = 1, 2$ to obtain

\[
|v_i'(x)| \leq 4\pi/E + C \sup |\kappa_i|E.
\]

Using the interior estimates [13, Corollary 3.2 (i)], we have

\[
\sup |\kappa_i| \leq \frac{C}{\sqrt{M}}
\]

for some universal constant $C$. Now we choose

\[
M = CE^4,
\]

here $C$ is a suitable universal constant so that when $M \geq M$, (2.25) and (2.26) imply

$$|v_i'(x)| \leq 5\pi/E, \quad \forall x \in [R_T - 1, R_T - 1 + K].$$

Hence we have

$$v_2 - v_1 \geq d - c - 10\pi K/E \geq \frac{\ell}{2}.$$
for all $x \in [R_T - 1, R_T - 1 + K]$ by \((2.24)\). By \((2.23)\), we have
\[
\int_{R_T-1}^{R_T-1+K} (v_2 - v_1)dx \geq \frac{\ell}{2}K = A + 5\pi.
\]
But this is impossible, since by assumption (3) and that $v_2 - v_1 \leq 4\pi$,
\[
A \geq \int_{R_T}^{R_T-1+K} (v_2 - v_1)dx \geq \int_{-R_T-1}^{R_T-1+K} (v_2 - v_1) - 4\pi.
\]
Thus we have arrived at a contradiction and finished the proof of the Lemma. \(\square\)

3. $C^0$-estimates between $C_n(t)$, $C_m(t)$

In this section we compare $C_n(t)$ with $C_m(t)$ for some particular time $t$. First we construct a sequence of ancient solutions to the CSF $\{B_n(t)\}$, which are used to show the embeddedness of $C_n(t)$.

For any $L > 0$ and $t < -1$, let $\overline{b}(t, \cdot) : (0, 2\pi) \rightarrow \mathbb{R}$ be formed by gluing two Grim Reapers similar as in Remark $\Box$. That is,

- $\overline{b}(t, y) = -G(y) + t$ if $\arcsin e^{1+t} \leq y \leq \pi - \arcsin e^{1+t}$,
- $\overline{b}(t, y) = G(y - \pi) - (t - L)$ if $\pi + \arcsin e^{1+t-L} \leq y \leq 2\pi - \arcsin e^{1+t-L}$, and
- $\overline{b}(t, y)$ is linear in the intervals $(0, \arcsin e^{1+t})$, $(\pi - \arcsin e^{1+t}, \pi)$, $(\pi, \pi + \arcsin e^{1+t-L})$, and $(2\pi - \arcsin e^{1+t-L}, 2\pi)$, joining
  - $(0, 0)$ to $(-1, \arcsin e^{1+t})$,
  - $(-1, \pi - \arcsin e^{1+t})$ to $(0, \pi)$,
  - $(0, \pi)$ to $(1, \pi + \arcsin e^{1+t-L})$, and
  - $(1, 2\pi - \arcsin e^{1+t-L})$ to $(0, 2\pi)$.

Let $\overline{B}(t)$ be the curve
\[
\overline{B}(t) = \{ (\overline{b}(t, y) : 0 \leq y \leq 2\pi \} \cup \{(x, 0) : x \geq 0 \}\}^r.
\]

For any $\alpha > 1$, let $B^\alpha(t)$ be the CSF with $B^\alpha(-\alpha) = \overline{B}(-\alpha)$. For each $t > -\alpha$, $B^\alpha(t)$ is a union of two graphs in $y$. Hence we write
\[
B^\alpha(t) = \{ x = b^\alpha(t, y) : 0 < y \leq b^\alpha(t) \}\}^r,
\]
where $b^\alpha(t, \cdot)$ satisfies the CSF \((2.22)\) and $b^\alpha(-\alpha, y) = \overline{b}(-\alpha, y)$ for all $y > 0$.

As in \([4]\), one can show that $B^\alpha(t)$ converges to an ancient solution to the CSF $B(t)$ as $\alpha \rightarrow \infty$. We write
\[
B(t) = \{ x = b(t, y) : 0 < y \leq b(t) \}\}^r,
\]
where
\[
b(t, y) = \lim_{\alpha \rightarrow \infty} b^\alpha(t, y).
\]

Figure 5: the barrier $B(t)$. 

---

\[ \int_{R_T-1}^{R_T-1+K} (v_2 - v_1)dx \geq \frac{\ell}{2}K = A + 5\pi. \]

Thus we have arrived at a contradiction and finished the proof of the Lemma. \(\square\)
Arguing as in Lemma 2.5 one has

\[ \int_0^{b(t)} b(t, y) dy = L \]

for all \( t \) so that \( b(t, \cdot) \) is defined.

**Lemma 3.1.** There is \( L > 0 \) so that whenever \( L > \mathcal{L} \), \( B(t) \) becomes embedded at some time (and thus is defined for all \( t \)).

**Proof.** Observe that \( B(t) \) becomes embedded if \( b(t, y) > 0 \) for all \( y \in (0, b(t)) \). Also, if \( b_1(t, y), b_2(t, y) \) are constructed from above using \( L = L_1, L_2 \) respectively with \( L_1 < L_2 \), then

\[ b_1(t, y) \leq b_2(t, y) \]

by maximum principle. Thus it suffices to show the embeddedness for \( L \) large enough. To this end, let \( b_{\min} \) be the global minimum of \( b(-1, y) \), where \( b(t, y) \) is defined using a fixed \( L_0 \). We assume that \( b_{\min} < 0 \), or there is nothing to prove. Let \( G(t) \) be a Grim Reaper so that

- \( G(t) \) is symmetric about \( \{ y = \pi/2 \} \) and is moving in the positive \( x \) direction, and
- has width larger than \( \pi \),

and if we represent \( G(t) \) as a graph of the function \( \{ x = g_t(y) \} \),

- the minimum of \( g_{-1}(y) \) is smaller than \( b_{\min} \), and
- the graphs of \( g_{-1}(y) \) and \( b(-1, y) \) intersect at only one point, which is in the region \( \{ y > 3\pi/2 \} \).

Using (3.5), one sees that the same conditions are satisfied by \( G(t) \) if \( b(t, y) \) is defined using any \( L \geq L_0 \). Being a translator, \( G(t) \) leaves the region \( \{ x \leq 0 \} \) at some time \( T > -1 \). Now we choose \( L \) large enough (depending on \( T \)) so that the following is true: there is an Angenent oval \( \gamma \) in \( \mathbb{R}^2 \) so that

\[ \gamma \subset \{ (x, y) \in \mathbb{R}^2 : x \in (\pi, 3\pi/2), \max\{g_{-1}(y), 1\} < y < b(-1, y) \} \]

\( \gamma \) is symmetric about \( \{ y = y_0 \} \) for some \( y_0 \in (\pi, 3\pi/2) \) and the area enclosed by \( \gamma \) is larger then \( (T + 1)/2\pi \).

![Figure 6: The Angenent oval \( \gamma \) (in red), \( G(-1) \) (in purple) and \( b(-1, y) \) (in green).](image)
Let $\gamma(t)$ be the CSF with $\gamma(-1) = \gamma$. Since $\{\gamma(t)\}$ shrinks to a point and the area enclosed by $\gamma(t)$ decreases with a constant rate $2\pi$ [8, Appendix B, Proposition 1], $\gamma(t)$ is defined at least up to time $T$. On the other hand, since $\gamma$ is disjoint from $G(-1)$ and $B(-1)$, $\gamma(t)$ is also disjoint to $G(t)$, $B(t)$ for all $t \in [-1, T]$ [17, Theorem 5.8]. Then $g_t(y_0) < b(t, y_0)$ for all $t \in [-1, T]$ and in particular $b(T, y_0) > 0$ by the choice of $T$. This implies that $b(T, y) > 0$ for all $y \in [y_0, b(T))$.

It remains to show that $b(t, y) > 0$ for all $y \in (0, y_0)$. Since $b(t, y) \to +\infty$ as $y \to 0^+$, there is $y^0 > 0$ so that $b(t, y) > g_t(y)$ for all $t \in [-1, T]$ and $y \in (0, y^0]$. In particular, we have $b(t, y^0) > g_t(y^0)$ for all $t \in [-1, T]$. Thus $b \geq g$ on the parabolic boundary of $[-1, T] \times [y^0, y_0]$. By the comparison principle [17, Theorem 4.3], $b \geq g$ on $[-1, T] \times [y^0, y_0]$. This finishes the proof since $g(T, y)$ is positive for all $y$.

From now on, we fix $L > \bar{L}$. Let $T^e$ be the time where $B(t)$ becomes embedded.

Next we define $B_n(t)$ by parabolic rescaling and space time translations: Given any sequence $\{B_n\}_{n=1}^\infty$ of positive numbers, define

\begin{equation}
B_n(t) = \frac{1}{a_n} B(a_n^2 t + C_n) + (0, 2\pi h_{n-1}),
\end{equation}

Write

\begin{equation}
B_n(t) = \{ x = b_n(t, y) : 2\pi h_{n-1} < y \leq b_n(t) \}^{re}.
\end{equation}

Then

\begin{equation}
\int_{2\pi h_{n-1}}^{b_n(t)} b_n(t, y) dy = L a_n^{-2}
\end{equation}

and $B_n(t)$ becomes embedded when $a_n^2 t + C_n = T^e$, or

\begin{equation}
t = T^e_n := -C_n + T^e / a_n^2.
\end{equation}

**Lemma 3.2.** For any $n \in \mathbb{N}$, $b_n(t, \cdot) > g_n(t, \cdot)$.

**Proof.** Let

\begin{equation}
\overline{B}_n(t) = \frac{1}{a_n} \overline{B}(a_n^2 t + C_n) + (0, 2\pi h_{n-1}).
\end{equation}

From the construction of $B(t)$, one sees that $B_n(t)$ is the limit of $B^\alpha_n(t)$ as $\alpha \to +\infty$, where for any $\alpha > C_n + 1 / a_n^2$, $B^\alpha_n(t)$ is the CSF with

$B^\alpha_n(-\alpha) = \overline{B}_n(-\alpha)$.

Also, from the construction of $B(t)$ and $\overline{C}_n(t)$ in Remark[11] $\overline{B}_n(t) = \overline{C}_n(t)$ in $\{ y > 2\pi h_{n-1} \}$.

Arguing similarly as in the proof of Lemma [2.1] and Remark [2] $b_n^\alpha \geq g_n^\alpha$ in the region $y > 2\pi h_{n-1}$. Hence $b_n \geq g_n$ by taking $\alpha \to \infty$. This implies $b_n > g_n$ by the strong maximum principle. \qed

Note that we cannot argue directly by maximum principle and Lemma [3.2] that $C_n(t)$ becomes embedded in the region $\{ y > 2\pi h_{n-1} \}$. To show the embeddedness, we need to control the area bounded between $b_n$ and $g_n$ in that region.
Proposition 3.1. Let $n$ be fixed. For all $\delta > 0$, there is $B^e_n > 0$ depending only on $a_{n-1}, a_n, \delta$ so that for all $B_n \geq B^e_n$, the immersed curve $C_n(T^e_n + \delta)$ is embedded in the region $\{ y \geq 2\pi h_{n-1} \}$.

Proof. Let $M > 0$. For any time $t < T^e_n$ and $\alpha > -t$, the graph $\{ x = b_n^\alpha(t, y) \}$ intersects $\{ x = M \}$ at either one or three points (depending on $t$). Let $y = b_{n,M}^\alpha(t)$ be the smallest $y$-coordinates of those points. Similarly,

$$\{ x = g_n^\alpha(t, y) : y_{n-1,n}^- \leq y \leq y_{n-1,n}^+(t) \}$$

intersect $\{ x = M \}$ at two points and we let $y_{n,M}^\alpha(t)$ be the larger $y$-coordinates among those two. Then for any $\epsilon > 0$, as in the proof of [4, Lemma 3.5], one can find $M, B^e_n$ large so that if $B_n \geq B^e_n$, the region (see Figure 5) bounded by

- $\{ (M, y) : y_{n,M}^- \leq y \leq b_{n,M}^\alpha(t) \}$,
- $\{ x = b_n^\alpha(t, y) : b_{n,M}^- \leq y \leq b_n^\alpha(t) \}$,
- $\{ (0, y) : y_{n,n}^+ \leq y \leq b_n^\alpha(t) \}$, and
- $\{ x = g_n^\alpha(t, y) : y_{n,M}^- \leq y \leq y_{n,n}^+ \}$.

has area less than $\epsilon$. Taking $\alpha \rightarrow +\infty$, one sees that the area of the region bounded by

- $\{ (M, y) : y_{n,M}^- \leq y \leq b_{n,M}^\alpha(t) \}$,
- $\{ x = b_n^\alpha(t, y) : b_{n,M}^- \leq y \leq b_n^\alpha(t) \}$,
- $\{ (0, y) : y_{n,n}^+ \leq y \leq b_n^\alpha(t) \}$, and
- $\{ x = g_n^\alpha(t, y) : y_{n,M}^- \leq y \leq y_{n,n}^+ \}$

is less than $\epsilon$ at all time $t$, where

$$y_{n,M}(t) := \lim_{\alpha \rightarrow +\infty} y_{n,M}^\alpha(t), \ b_{n,M}(t) := \lim_{\alpha \rightarrow +\infty} b_{n,M}^\alpha(t).$$

Figure 7: the area is less than $\epsilon$.

By Lemma 3.1 and (3.6), $B_n(t)$ becomes embedded at some time. Then it is proved in [19] (see also [16, Theroem 2.5]) that $B_n(t)$ are asympototic and converge to the straight line $\{ y = 0 \}$ as $t \rightarrow +\infty$. Thus the curvature of $B_n(t)$ are uniformly bounded (say, by $C = C_B_n$). Let $\epsilon < \pi/C^2$, Then for each $t \leq T^e_n$, the ball of radius $\sqrt{\epsilon/\pi}$ touching the local maximum of $b_n(t, y)$ from below has area $\epsilon$. Thus $g_n(t, y)$ intersects this ball. In particular, for any $\delta > 0$, one can choose $\epsilon$ small enough so that whenever $B_n \geq B^e_n$, the curve $C_n(t)$ is defined at least up to $T^e_n + \delta$.

At time $t = T^e_n$, the function $b_n(T^e_n(t, y))$ intersects $\{ x = 0 \}$ at two points $b_n(T^e_n(t))$ and $b^e$ with $b^e < b_n(T^e_n(t))$. Let $\delta_1$ be a positive number so that

$$\delta_1 < \min\{ b^e - 2\pi h_{n-1}, b_n(T^e_n) - b^e \}.$$
By choosing a smaller $\epsilon$ (thus a larger $B^e_n$) if necessary, we assume
\[
\min \left\{ \int_{b^e-\delta}^{b^e} b_n(T_n^e, y)dy, \int_{b^e}^{b^e+\delta} b_n(T_n^e, y)dy \right\} > \epsilon.
\]
The area estimates between $g_n$ and $b_n$ implies that $g_n(T_n^e, y)$ satisfies
\[
b^e - \delta_1 < y_{n-1}^e(T_n^e) < b^e < y_{n,n}^e(T_n^e) < b^e + \delta_1,
\]
that is, $g_n(T_n^e, \cdot)$ is positive outside $(b^e - \delta_1, b^e + \delta_1)$ in the region $\{y \geq 2\pi h_{n-1} \}$. Thus it suffices to show that $g_n$ is positive inside $(b^e - \delta_1, b^e + \delta_1)$ at time $t = T_n^e + \delta$.

To see this, let $b_\delta$ be the local minimum of $b_n(T_n^e + \delta, y)$ in $(b^e - \delta_1, b^e + \delta_1)$ (we assume that $\delta > 0$ is small so that $b_\delta$ exists). Note that in $(b^e - \delta_1, b^e + \delta_1)$, by the gradient estimates [14, Corollary 5.3], we have
\[
|g_n'(T_n^e + \delta, \cdot)| \leq C,
\]
for some universal constant $C$. Now we argue by contradiction that $g_n(T_n^e + \delta, \cdot)$ is positive in $(b^e - \delta_1, b^e + \delta_1)$; if not, then $g_n(T_n^e + \delta, y_0) \leq 0$ for some $y_0 \in (b^e - \delta_1, b^e + \delta_1)$. Then $g_n(T_n^e + \delta, y) \leq b_\delta/2$ for all $y \in (b^e - \delta_1, b^e + \delta_1)\text{ with } |y - y_0| < b_\delta/2C$ by (3.10). Thus
\[
\int_I (b_n(t,y) - g_n(t,y))dy \geq b_\delta^2/8C.
\]
for some interval $I \subset (b^e - \delta_1, b^e + \delta_1)$. But this is impossible if $\epsilon < b_\delta^2/8C$. Thus $g_n$ is positive at time $t = T_n^e + \delta$ for small enough $\epsilon$ in the region $\{y \geq 2\pi h_{n-1} \}$.

From now on we assume that in the constructions of $\{C_n(t)\}$ we have chosen $B_n \geq B^e_n$ for all $n \in \mathbb{N}$, where $B^e_n$ is chosen as in Proposition 3.1 with $\delta = 1$.

**Corollary 3.1.** For each $n \in \mathbb{N}$, assume that $B_m \geq B^e_n$ for all $m = 1, 2, \cdots, n$. Then $C_n(t)$ becomes embedded and shrinks to a point. $C_n(t)$ are all defined up to time $T_0$.

**Proof.** From the choice of $B^e_n$, by Proposition 3.1 we have $g_n(t,y) > 0$ in the region $y > 2\pi h_{n-1}$ after time $T_n^e + 1$. Since $g_n(t,y) > g_k(t,y)$ when $n > k$, an inductive argument shows that $g_n$ becomes positive after some time and thus $C_n(t)$ becomes embedded. Since $g_n > g_0$, this also shows that $C_n(t)$ is defined up to time $T_0$.

**Definition 3.1.** For any $t < -C_n - 1$, we say that an immersed curve $C$ is close to $C_n(t)$ if the followings hold:

1. $C$ can be written as
   \[
   C = \{x = g_C(y) : y_0^- \leq y \leq y_0^+\}_{t = 0}^{t = \tau_C}
   \]
   for some function $g_C : [y_0^-, y_0^+] \to \mathbb{R}$ with $g_C(y_0^-) = g_C(y_0^+) = 0$.
2. $2\pi h_n + \pi a_{n+1}^{-1} > y_0^+ > y_0^- > -\pi$;
3. $g_C(y) \geq g_n(t, y)$ and the area bounded between $g_C(y)$ and $g_n(t, y)$ is less than $L \sum_{k=n}^{\infty} a_k^{-2}$;
4. $g_C(y) < G_m(y) + a_m(t + C_m)$ for $m = 1, 2, \cdots, n$.
5. $g_C$ has exactly $n+1$ local maximum and $n$ local minimum.
Proposition 3.2. Let $n \in \mathbb{N}$ be fixed. Then there are $\overline{M}_n, M_n > 0$ depending only on \( \{a_n\} \) such that the following holds: For all $M_n \geq \overline{M}_n$ and $B_n \geq \max\{\overline{M}_n, B_n^\ast\}$, let $C$ be an immersed curve that is close to $C_n(-C_n - M_n)$. Then $C(-C_n - 1 + t)$ is close to $C_{n-1}(-C_n - 1 + t)$ whenever $t \in (\overline{M}_n, B_n)$. Here $C(t)$ is the CSF with $C(-C_n - M_n) = C$.

Proof. Since $g_C(\cdot) > g_n(-C_n - M_n, \cdot)$, $C(t)$ is defined at least up to time $T_0$ by Corollary 3.1. We write

\[(3.11) \quad C(t) = \{ x = g_C(t, y) : y_-(t) \leq y \leq y_+(t) \}^{re}, \]

where $g_C(-C_n - M_n, \cdot) = g_C(\cdot)$ and $g_C(t, y)$ satisfies (2.2).

To show that $C(t)$ is close to $C_{n-1}(t)$ for some $t$, we need only to show (2) and (5) in Definition 3.1. (1) is true for all $t$, (3) follows from Lemma 2.1 and Lemma 2.5, while (4) can be proved similarly as in the proof of Lemma 2.2.

By (3) and (4) we have

\[g_n(-C_n - M_n, y) \leq g_C(y) \leq G_m^-(y) + a_n M_n.\]

Arguing as in the proof of Lemma 2.1 and Lemma 2.2, we have

\[(3.12) \quad g_n(t, y) \leq g_C(t, y) \leq G_m^-(y) + a_m (t + C_m)\]

for all $-C_n - M_n < t < -C_n - 1$ and $m = 1, 2, \ldots, n$. Just like $g_n(t, y)$, for all $t < -C_n - 1$, $g_C(t, y)$ intersects \( \{ x = 0 \} \) at

\[y_{C,0}(t) < y_{C,1}(t) < \cdots < y_{C,n-1}(t) < y_{C,n-1}(t) < y_{C,n}(t) < y_{C,n}(t).\]

Using Lemma 2.4 and Lemma 2.6 (with suitable parabolic rescalings the reflection $R$), there are $E_n^-, M_n^- > 0$ depending only on \( \{a_n\} \) so that

\[g_n(-C_n - 1, y) + a_n \geq -E_n^-, \quad \forall y \in (y_{n-1,n}^-(C_n - 1), y_{n,n}^-(C_n - 1))\]

whenever $M_n \geq M_n^-$. By (3.12), we have

\[(3.13) \quad g_C(-C_n - 1, y) \geq -E_n^- - a_n, \quad \forall y \in (y_{C,n-1}^-(C_n - 1), y_{C,n}^-(-C_n - 1)).\]

Similarly, using Lemma 2.3 and Lemma 2.6 (with suitable parabolic rescalings and $A = L \sum a_n^2 + A_n$), there are $E_n^+, M_n^+ > 0$ such that if $M_n \geq M_n^+$, then

\[(3.14) \quad g_C(-C_n - 1, y) - a_n(1 + L A_n^{-1}) \leq E_n^+, \quad \forall y \geq y_{C,n}^-(C_n - 1).\]

Choose

\[M_n = \max\{M_n^-, M_n^+\}, \quad E_n = \max\{E_n^-, a_n, E_n^+, a_n + L A_n^{-1}\}.\]

Then whenever $M_n \geq M_n$, $|x| \leq E_n$ for all $(x, y) \in C(-C_n - 1)$ and $y \geq y_{C,n}^-(C_n - 1)$.

Next we construct another barrier $P_n$. At time $t = -C_n - 1$, $G_n^-(C_n - 1)$ intersect $x = E_n$ at two points $p_n < p^n$. Define

\[P_n = X^{re},\]

where

\[X = \{ G_n^-(C_n - 1) : y \leq p_n \} \cup \{ E_n \} \times [p_n, 2\pi h_n + \pi a_{n+1}^{-1}] \cup [0, E_n] \times \{ 2\pi h_n + \pi a_{n+1}^{-1} \}.\]
Let $\mathcal{P}_n(t)$ be the CSF with $\mathcal{P}_n(0) = \mathcal{P}_n$. For all $t > 0$, $\mathcal{P}_n(t)$ is graphical in $y$ when $x > 0$, and $\mathcal{P}_n(t)$ is given by

$$
(3.15) \quad \mathcal{P}_n(t) = \{x = p_n(t, y) : 2\pi h_{n-1} < y \leq p^n(t)\}^r.
$$

Also $p_n(t, \cdot) > g_C(-C_n - 1 + t, \cdot)$ for all $t > 0$ by maximum principle. Note that $\mathcal{P}_n(t)$ is an entire graph in $x$ for $t > 0$.

From [19], [16], the CSF $\mathcal{P}_n(t)$ tends to $\{y = 2\pi h_{n-1}\}$ uniformly as $t \to \infty$. Thus there is $T^n_p > 0$ so that $\mathcal{P}_n(t)$ lies between $\{y = 2\pi h_{n-1}\}$ and $\{y = 2\pi h_{n-1} + \pi a_n^{-1}\}$ for all $t > T^n_p$. The maximum principle implies that $\mathcal{C}(t)$ also lies in $\{y < 2\pi h_{n-1} + \pi a_n^{-1}\}$ for all $t > -C_n - 1 + T^n_p$.

On the other hand, we have

**Claim:** There is $T^2_n > 0$ independent of $B_n$ so that $\mathcal{C}(-C_n - 1 + T^2_n)$ has exactly $n$ local maximum and $n - 1$ local minimum.

**Proof of Claim:** Let $\mathcal{G}(t)$ be a Grim Reaper which

(i) is symmetric about the $y$-axis with width larger than $2E_n$,

(ii) moves along the negative $y$ direction, and

(iii) at $t = 0$, $\mathcal{G}(0)$ intersects $\mathcal{C}(-C_n - 1)$ at two points $(\pm x_0, y_0)$ in the region $\{y \geq 2\pi h_{n-1}\}$ with $|x_0| > E_n$.

Note $\mathcal{G}(t)$ can be chosen using only $E_n, h_n, a_n$ and is independent of the curve $\mathcal{C}(-C_n - 1)$. Since $\mathcal{G}(t)$ moves at constant speed in the negative $y$ direction, there is $T^2_n > 0$ so that $\mathcal{G}(T^2_n)$ do not intersect the region $\{y \geq 2\pi h_{n-1}\}$. We argue that the function $g_C(-C_n - 1 + T^2_n, y)$ has no local maximum and local minimum in the region $\{y \geq 2\pi h_{n-1}\}$, except at the boundary.

We argue by contradiction. If $g_C(-C_n - 1 + T^2_n, \cdot)$ does has a local minimum in $\{y \geq 2\pi h_{n-1}\}$, then the same is true for the function $g_C(-C_n - 1 + t, \cdot)$ for all $t \in [0, T^2_n]$ by maximum principle. For any $t$, let the local maximum and local minimum of $g_C(-C_n - 1 + t, \cdot)$ be occurred at $a(t), b(t)$ respectively, with $y^n(-C_n - 1 + t) > a(t) > b(t)$. Let $e(t) \in \mathbb{R}$ so that $\mathcal{G}(t)$ intersects $\mathcal{C}(-C_n - 1 + t)$ at $(\pm x_t, e(t))$, and let $t_{\max}$ be the time when $\mathcal{G}(t_{\max})$ and $\mathcal{C}(-C_n - 1 + t_{\max})$ intersect at only one point (by symmetry the point of intersection is $(0, y^n(-C_n - 1 + t_{\max}))$).

Note that when $t$ is small, we have $b(t) > e(t)$ and at $t = t_{\max}$,

$$
e(t_{\max}) = y^n(-C_n - 1 + t_{\max}) > b(t_{\max}).$$

By continuity, there is a time $t \in (0, t_{\max})$ so that $e(t) = b(t)$. But this is impossible: since $\mathcal{G}(t)$ is convex, if $\mathcal{G}(t)$ and $\mathcal{C}(-C_n - 1 + t)$ does intersect at the local minimum, it must also intersect at yet another point. This is impossible since the number of intersections along two CSF is non-increasing by maximum principle. This finishes the proof of the claim.

With the above claim we finish the proof of Proposition. Let $\overline{M}_n = \max\{T_{p_n}, T^2_n\}$ and assume that $B_n \geq \max\{\overline{M}_n, B_n^c\}$. Then for all $t \in (\overline{M}_n, B_n)$, $-C_n - 1 + t < -C_n - 1 + B_n = -C_{n-1} - 1$. By the choice of $\overline{M}_n$, we see that whenever $t \geq \overline{M}_n$, (2) and (5) are also satisfied for $\mathcal{C}(-C_n - 1 + t)$, and thus $\mathcal{C}(-C_n - 1 + t)$ is close to $C_{n-1}(-C_{n-1} - 1 + t)$. \(\square\)
4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

**Theorem 4.1.** One can choose $B_n > B_n^c$, $N_n > 0$ and $t_n \in (-C_{n+1}, -C_n - 1)$ so that for all $m \geq n$, $C_m(t_n)$ is close to $C_n(t_n)$ and the curvatures of $C_m(t_n)$ are uniformly bounded by $N_n$.

**Proof.** Let $M_n$ and $\overline{M}_n$ be chosen as in Proposition 3.2. Note that the choice of $\overline{M}_n$, $M_n$ depends only on the sequence $\{a_n\}$.

Now we choose $\{B_n\}$ so that for all $n \in \mathbb{N}$, $B_n \supseteq \{B_n^c, M_{n-1} + \overline{M}_n\}$. Then we have

$$-C_1 + M_n < -C_{n-1} - \overline{M}_{n-1}, \ \forall n \in \mathbb{N}.$$  

Let $t_n = -C_{n+1} + \overline{M}_{n+1}$. Let $m, n \in \mathbb{N}$ with $m > n$. By definition, $C_m(t_n)$ is close to $C_m(t_n)$. Using (4.1) and proposition 3.2, $C_m(t_{n-1})$ is close to $C_{n-1}(t_{m-1})$. After finitely many steps we see that $C_m(t_n)$ is close to $C_m(t_n)$.

Lastly, we show that there are $N_1, \ldots, N_n, \ldots$ so that the curvatures of $C_m(t_n)$ are bounded by $N_n$ for all $m \geq n$. Since $C_m(t_n)$ is close to $C_n(t_n)$, the function $g_m(t_n, y)$ has $n+1$ local maximum and $n$ local minimum. Recall $g_m(t_n, y)$ intersects the $y$-axis at

$$y_{0,m}(t_n) < y_{0,n+1}(t_n) < \cdots < y_{0,m}(t_n) < y_{0,n+1}(t_n).$$

For each $k = 0, \ldots, n$ and $t \leq t_n$, let $T_k(t)$ be the $k$-th tip region

$$T_k(t) := \{(x, y) \in C_m(t) : y \in (y_{k-1,m}^+(t), y_{k,m}^+(t)), |x| > -a_k t_n - 1\}.$$ 

Note that in the tip regions, one can apply Lemma 2.6 to obtain uniform $C^0$ distance estimates to the Grim Reapers. The gradient estimates [14 Corollary 5.3] implies that $|g_m'(t, y)| \leq C(n)$ in the tip regions. Since $g_m$ satisfies the graphical CSF equation (2.2), standard parabolic estimates implies uniform control on $g_m''(t_n, y)$ in the tip regions, which gives a uniform bound for the curvature there.

Away from the tip regions, $C_m(t)$ can be written as a union of $2n$-graphs in $x$ with uniform $C^0$ bounds, at least when $t \in [t_n - 1, t_n]$. Then one can similarly obtain gradient estimates and thus uniform bounds on curvatures. 

**Proof of Theorem 1.1.** From Theorem 4.1 one can find a sequence of bounded open sets $U_n$ so that $\overline{U}_n \subset U_{n+1}$, and $C_m(t_n) \subset U_n$ for all $m \geq n$. The curvatures of $C_m(t_n)$ are uniformly bounded for $m \geq n$. By choosing a slightly larger time $t_n$ for each $n$ and using the CSF equation, one can assume that all derivatives of curvatures of $C_m(t_n)$ are uniformly bounded for $m \geq n$ [15 Section 3]. Thus by [9 Theorem 1.3], we can pick a diagonal subsequence $\{n_j\}_{j=1}^\infty$ so that for all $n \in \mathbb{N}$, $\{C_{n_j}(t_{n_j})\}_{j=1}^\infty$ converges smoothly to a compact immersion $\mathcal{C}^n$. Let $\mathcal{C}^n(t)$ be the CSF with $\mathcal{C}^n(t_0) = \mathcal{C}^n$. Since $C_m(t_n)$ converges smoothly to $\mathcal{C}^n$, the continuous dependence of CSF implies that the sequence of CSF’s $\{C_m(t) : t_n \leq t \leq T_0\}$ converges smoothly to $\{\mathcal{C}^n(t) : t_n \leq t \leq T_0\}$. In particular, for all $m > n$,

$$C^m(t) = C^n(t), \ \forall t \geq t_n.$$ 

Thus we can define, for each $t < T_0$, $\mathcal{C}(t) = C^m(t)$, where $m \in \mathbb{N}$ and $t_m < t$. Then $\{C(t) : t < T_0\}$ is a CSF so that $\mathcal{C}(t_n) = \mathcal{C}^n$ for all $n \in \mathbb{N}$. Since $C(t_n)$ is the limit of
\[ C_m(t_n) \text{ as } m \to \infty, \]
\[ \bigcup_{t<T_0} \text{Conv } C(t) = \{(x, y) : y > -\pi\} \]

is a halfspace. Lastly, in the construction we fix a choice of sequence \( \{a_n\} \). Since different sequences \( \{a_n\} \) lead to ancient solutions with different limit as \( t \to -\infty \), the above construction gives an infinite family of ancient solutions. \( \square \)

**References**

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