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Various concepts of Riesz energy of measures and application to condensers with touching plates

Bent Fuglede and Natalia Zorii

Dedicated to Professor Stephen J. Gardiner
on the occasion of his 60th birthday

Abstract. We develop further the concept of weak α-Riesz energy with $\alpha \in (0, 2]$ of Radon measures μ on $\mathbb{R}^n$, $n \geqslant 3$, introduced in our preceding study and defined by $\int(\kappa_{\alpha/2}\mu)^2\,dm$, m denoting the Lebesgue measure on $\mathbb{R}^n$. Here $\kappa_{\alpha/2}\mu$ is the potential of μ relative to the $\alpha/2$-Riesz kernel $|x-y|^{\alpha/2-n}$. This concept extends that of standard α-Riesz energy, and for μ with $\kappa_{\alpha/2}\mu \in L^2(m)$ it coincides with that of Deny–Schwartz energy defined with the aid of the Fourier transform. We investigate minimum weak α-Riesz energy problems with external fields in both the unconstrained and constrained settings for generalized condensers (A_1, A_2) such that the closures of A_1 and A_2 in $\mathbb{R}^n$ are allowed to intersect one another. (Such problems with the standard α-Riesz energy in place of the weak one would be unsolvable, which justifies the need for the concept of weak energy when dealing with condenser problems.) We obtain sufficient and/or necessary conditions for the existence of minimizers, provide descriptions of their supports and potentials, and single out their characteristic properties. To this end we have discovered an intimate relation between minimum weak α-Riesz energy problems over signed measures associated with (A_1, A_2) and minimum α-Green energy problems over positive measures carried by A_1. Crucial for our analysis of the latter problems is the perfectness of the α-Green kernel, established in our recent paper. As an application of the results obtained, we describe the support of the α-Green equilibrium measure.

1. Introduction

Throughout the paper we fix a natural $n \geqslant 3$ and a real $\alpha \in (0, 2]$. Let $\mathfrak{M}(\mathbb{R}^n)$ stand for the linear space of all real-valued Radon measures μ on $\mathbb{R}^n$, equipped with the *vague* topology, i.e. the topology of pointwise convergence on the class $C_0(\mathbb{R}^n)$ of all (real-valued finite) continuous functions on $\mathbb{R}^n$ with compact support. The *standard* concept of *energy* of a (signed) Radon measure $\mu \in \mathfrak{M}(\mathbb{R}^n)$ relative to the *α-Riesz* kernel $\kappa_\alpha(x, y) := |x-y|^{\alpha-n}$ on $\mathbb{R}^n$, $|x-y|$ being the Euclidean distance between $x, y \in \mathbb{R}^n$, is introduced by

$$E_\alpha(\mu) := E_{\kappa_\alpha}(\mu) := \int \kappa_\alpha(x, y)\, d(\mu \otimes \mu)(x, y) \tag{1.1}$$

whenever $E_\alpha(\mu^+)+E_\alpha(\mu^-)$ or $E_\alpha(\mu^+, \mu^-)$ is finite, and finiteness of $E_\alpha(\mu)$ means that κ_α is $\bigl(|\mu| \otimes |\mu|\bigr)$-integrable, i.e. $E_\alpha(|\mu|) < \infty$. Here μ^+ and μ^- denote the positive and negative parts in the Hahn–Jordan decomposition of a measure $\mu \in \mathfrak{M}(\mathbb{R}^n)$, $|\mu| := \mu^+ + \mu^-$, and

$$E_\alpha(\mu^+, \mu^-) := E_{\kappa_\alpha}(\mu^+, \mu^-) := \int \kappa_\alpha(x, y)\, d(\mu^+ \otimes \mu^-)(x, y)$$

is the (standard) α-Riesz *mutual energy* of μ^+ and μ^-.

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The α-Riesz kernel is *strictly positive definite* in the sense that $E_\alpha(\mu)$, $\mu \in \mathfrak{M}(\mathbb{R}^n)$, is $\geqslant 0$ (whenever defined), and equals 0 only for $\mu = 0$. The set $\mathcal{E}_\alpha(\mathbb{R}^n)$ of all $\mu \in \mathfrak{M}(\mathbb{R}^n)$ with $E_\alpha(\mu) < \infty$ therefore forms a pre-Hilbert space with the (standard) inner product $\langle \mu, \nu \rangle_\alpha := E_\alpha(\mu, \nu)$, $\mu, \nu \in \mathcal{E}_\alpha(\mathbb{R}^n)$, and the (standard energy) norm $\|\mu\|_\alpha := \sqrt{E_\alpha(\mu)}$.

Fix an (open connected) domain D in $\mathbb{R}^n$. An ordered pair $\mathbf{A} = (A_1, A_2)$, where A_1 is a relatively closed subset of D and $A_2 = D^c := \mathbb{R}^n \setminus D$, is said to be a (*generalized*) *condenser* in $\mathbb{R}^n$, while A_1 and A_2 are termed the *positive* and *negative plates* of $\mathbf{A}$. Note that although $A_1 \cap A_2 = \varnothing$, $\mathrm{Cl}_{\mathbb{R}^n} A_1$ and A_2 may have points in common; otherwise we shall call $\mathbf{A} = (A_1, A_2)$ a *standard* condenser. A measure $\mu \in \mathfrak{M}(\mathbb{R}^n)$ is said to be *associated with* a generalized condenser $\mathbf{A}$ if μ^+ and μ^- are carried by A_1 and A_2, respectively. The set $\mathfrak{M}(\mathbf{A})$ consisting of all those μ forms a convex cone in $\mathfrak{M}(\mathbb{R}^n)$, and so does $\mathcal{E}_\alpha(\mathbf{A}) := \mathfrak{M}(\mathbf{A}) \cap \mathcal{E}_\alpha(\mathbb{R}^n)$.

As a preparation for a study of minimum (standard) α-Riesz energy problems over subclasses of $\mathcal{E}_\alpha(\mathbf{A})$, $\mathbf{A}$ being a generalized condenser in $\mathbb{R}^n$, one considered in a recent paper [15] the α-Green kernel $g = g^\alpha_D$ on D, associated with the α-Riesz kernel κ_α [23, Chapter IV, Section 5]. It is claimed in [15, Lemma 2.4] that if a bounded positive Radon measure ν on D has finite α-Green energy $E_g(\nu)$, defined by (1.1) with g in place of κ_α, then the (signed) Radon measure $\nu - \nu'$ on $\mathbb{R}^n$, ν' being the α-Riesz swept measure of ν onto D^c [23, Chapter IV, Section 5], must have finite (standard) α-Riesz energy. Regrettably, the short proof of Lemma 2.4 in [15] was incomplete (a matter of $\infty - \infty$), and actually the lemma fails in general, as seen by the counterexample given in [16, Appendix]. To be precise, the quoted example shows that there is a bounded positive Radon measure ν on D with finite $E_g(\nu)$ such that $E_\alpha(\nu - \nu')$ is not well defined.

Below we argue that this failure is an indication that the standard notion of α-Riesz energy of signed measures is too restrictive when dealing with condenser problems.[1]

The above mentioned error led to the fact that some of the assertions announced in [15] had a gap in their proofs, and actually they fail in general, as will be seen from Example 10.1. Below we show that, nevertheless, these assertions become valid (even in a stronger form) if we replace the standard concept of α-Riesz energy by a weaker concept (e.g. by that of Deny–Schwartz energy of measures treated as tempered distributions).

As shown in [20, Theorem 5.1], [15, Lemma 2.4] quoted above does hold if we replace the standard concept of α-Riesz energy $E_\alpha(\mu)$ of a (signed) Radon measure μ on $\mathbb{R}^n$ by a weaker concept, denoted $\dot{E}_\alpha(\mu)$ and defined essentially (see [20, Definition 4.1]) by

$$\dot{E}_\alpha(\mu) = \int (\kappa_{\alpha/2}\mu)^2 \, dm,$$

$\kappa_{\alpha/2}\mu(\cdot) := \int \kappa_{\alpha/2}(\cdot, y)\, d\mu(y)$ being the potential of μ relative to the $\alpha/2$-Riesz kernel. (Here and in the sequel m denotes the Lebesgue measure on $\mathbb{R}^n$.) Moreover, the concept of weak α-Riesz energy, serving as a main tool of our analysis in [20], enabled us to rectify partly the results announced in [15] (see [20, Section 6], see also Remark 5.4 below for a short survey; compare with a further development of [20, Section 6] achieved in the present study).

[1] Except for the case of a standard condenser $\mathbf{A}$ in $\mathbb{R}^n$ with nonzero Euclidean distance between A_1 and A_2, see Remarks 2.6, 5.2 and Section 5.3 below.

The set $\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$ of (signed) Radon measures μ on $\mathbb{R}^n$ with finite $\dot{E}_\alpha(\mu)$, or equivalently with $\kappa_{\alpha/2}\mu \in L^2(m)$, forms a pre-Hilbert space with the (weak) inner product $\langle\mu,\nu\rangle^{\cdot}_\alpha := \langle\kappa_{\alpha/2}\mu, \kappa_{\alpha/2}\nu\rangle_{L^2(m)}$, $\mu,\nu \in \dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$, and the (weak energy) norm $\|\mu\|^{\cdot}_\alpha := \sqrt{\dot{E}_\alpha(\mu)} = \|\kappa_{\alpha/2}\mu\|_{L^2(m)}$ [20, Section 4]. The Riesz composition identity [26] implies that

$$\mathcal{E}^+_\alpha(\mathbb{R}^n) = \dot{\mathcal{E}}^+_\alpha(\mathbb{R}^n) \ \text{ and } \ \mathcal{E}_\alpha(\mathbb{R}^n) \subset \dot{\mathcal{E}}_\alpha(\mathbb{R}^n),$$

where $\dot{\mathcal{E}}^+_\alpha(\mathbb{R}^n) := \dot{\mathcal{E}}_\alpha(\mathbb{R}^n) \cap \mathfrak{M}^+(\mathbb{R}^n)$; and moreover

$$E_\alpha(\nu) = \dot{E}_\alpha(\nu) \ \text{ for any } \nu \in \mathcal{E}_\alpha(\mathbb{R}^n). \tag{1.2}$$

However, as seen from [16, Appendix] and [20, Theorem 5.1], there exists a (signed) measure $\mu \in \dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$ such that $E_\alpha(\mu)$ is not well defined, and hence $\mu \notin \mathcal{E}_\alpha(\mathbb{R}^n)$. Thus $\mathcal{E}_\alpha(\mathbb{R}^n)$ forms a *proper* subset of $\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$, which by [20, Theorem 4.1] is *dense* in $\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$ in the topology determined by the weak energy norm as well as in the (induced) vague topology.

Let S^*_α denote the Hilbert space of all real-valued tempered distributions $T \in S^*$ on $\mathbb{R}^n$ (see [27]) with finite Deny–Schwartz energy

$$\|T\|^2_{S^*_\alpha} := \int_{\mathbb{R}^n} \frac{|\mathcal{F}[T](y)|^2}{|y|^\alpha}\, dm(y),$$

where $\mathcal{F}[T]$ is the Fourier transform of $T \in S^*$. We refer to [11, 12] (see also [23, Chapter VI, Section 1]) for the definition of $\|T\|^2_{S^*_\alpha}$ as well as the properties of the space S^*_α. As shown in [20, Theorem 4.2], every $\nu \in \dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$ can be treated as an element of S^*_α with

$$\|\nu\|^2_{S^*_\alpha} = C_{n,\alpha}\|\nu\|^{\cdot 2}_\alpha,$$

$C_{n,\alpha} \in (0,\infty)$ depending on n and α only, and moreover S^*_α is a completion of the pre-Hilbert space $\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$ in the topology determined by the Deny–Schwartz norm $\|\cdot\|_{S^*_\alpha}$. (This result for $\mathcal{E}_\alpha(\mathbb{R}^n)$ in place of $\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$ goes back to Deny [11].)

Thus the concept of Deny–Schwartz energy coincides (up to a constant factor) with that of weak α-Riesz energy if restricted to measures of the class $\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$. It is however still unknown whether these two concepts are identical if considered over *all* (signed) Radon measures on $\mathbb{R}^n$ (see an open question raised in [12, p. 85] and [20, Section 7], the former work dealing with positive measures only).

In the present paper we proceed further with a study of minimum weak α-Riesz, or equivalently Deny–Schwartz, energy problems for a generalized condenser $\mathbf{A}$, initiated in [20]. Similarly as in [15], the measures are now influenced additionally by external fields f and/or constraints σ (see Section 5 for the precise formulations of the problems in question).

It is shown in Theorem 5.11 below that [15, Lemma 4.2] on the equivalence of [15, Problem 3.1] and [15, Problem 3.2] becomes valid if we require additionally that the plates A_1 and $A_2 = D^c$ of the condenser $\mathbf{A}$ satisfy the separation condition[2]

$$\operatorname{dist}(A_1, D^c) := \inf_{(x,y)\in A_1\times D^c} |x-y| > 0. \tag{1.3}$$

(Note that such $\mathbf{A}$ is certainly standard.) As seen from the quoted theorem, the class of measures admissible in [15, Problem 3.1] can then equivalently be defined as

[2]Because of the error in [15, Lemma 2.4], [15, Lemma 4.2] was unjustified in the stated form. Moreover, it fails in general, as will be seen from Example 10.1.

the set $\widetilde{\mathcal{H}}$ of $\mu - \mu'$ where μ ranges over certain positive measures carried by A_1 and having finite α-Green energy, while μ' is the α-Riesz balayage of μ onto D^c. Omitting now the separation condition (1.3), we see from Theorem 5.12 below that the stated equivalence of [15, Problem 3.1] and [15, Problem 3.2], the former now being applied to $\widetilde{\mathcal{H}}$ serving as a (new) class of admissible measures, remains valid if the standard α-Riesz energy is replaced by the weak energy.

Sufficient and/or necessary conditions for the solvability of the minimum weak α-Riesz energy problems are provided in Theorems 6.1, 6.6 and Corollary 6.7. See also Theorems 8.1, 8.4, 8.6, and 8.10 where such criteria are formulated in terms of variational inequalities for the α-Riesz potentials. An analysis of the supports of the minimizers is given by Theorem 7.1, 8.4, and 8.11. The results obtained are in fact stronger than those announced in [15] since a crucial key to our current proofs is the perfectness of the α-Green kernel, established in our recent study [19]. As an application of the quoted results, we describe the support of the α-Green equilibrium measure (see Theorem 9.2).

Example 10.1 shows that the above mentioned assertions in general fail if the weak α-Riesz energy is replaced by the standard energy. This justifies the need for the concept of weak α-Riesz energy when dealing with condenser problems.

Remark 1.1. The generalized condensers $\mathbf{A} = (A_1, D^c)$ such that the *unconstrained* minimum weak α-Riesz energy problems are solvable differ drastically from those for which the solvability occurs in the *constrained* setting. Indeed, if $f = 0$ and $\sigma = \infty$ (*no* external field and *no* constraint), then the solvability of the problems implies that $c_\alpha(\partial D \cap \mathrm{Cl}_{\mathbb{R}^n} A_1) = 0$, $c_\alpha(\cdot)$ being the α-Riesz capacity.[3] But if the constraint in question is *bounded*, then the problems turn out to be solvable even if $A_1 = D$; see Remark 9.3 below for details.

Remark 1.2. The results announced in [15] have also been rectified in part in a recent work [16]. However, minimum (standard) α-Riesz energy problems for a generalized condenser $\mathbf{A}$ were analyzed in [16] only in the constrained setting, and the constraints were required to be of finite (standard) α-Riesz energy, which was crucial for the proofs in [16]. The concept of weak α-Riesz energy, serving as a main tool of our present study, enables us to rectify the results announced in [15] in both the unconstrained and constrained settings, and the constraints in question are no longer required to satisfy any additional assumptions.

2. Preliminaries

Let X be a locally compact (Hausdorff) space [6, Chapter I, Section 9, n° 7], to be specified below. For the goals of the present study it is enough to assume that X is metrizable and *countable at infinity*, where the latter means that X can be represented as a countable union of compact sets [6, Chapter I, Section 9, n° 9]. Then the vague topology[4] on $\mathfrak{M}(X)$ satisfies the first axiom of countability [17, Remark 2.5], and vague convergence is entirely determined by convergence of sequences. The vague topology on $\mathfrak{M}(X)$ is Hausdorff, and hence a vague limit of any sequence in $\mathfrak{M}(X)$ is *unique* (whenever it exists). We denote by $S^\mu_X = S(\mu)$ the *support* of $\mu \in \mathfrak{M}(X)$. A measure μ is said to be *bounded* if $|\mu|(X) < \infty$. Let $\mathfrak{M}^+(X)$ stand for the (convex, vaguely closed) cone of all positive $\mu \in \mathfrak{M}(X)$.

[3] Throughout the paper ∂Q denotes the boundary of a set $Q \subset \mathbb{R}^n$ relative to $\mathbb{R}^n$.

[4] We shall tacitly assume to be known this and other notions defined for $X = \mathbb{R}^n$ in the Introduction.

Given a set $Q \subset X$, let $\mathfrak{M}^+(Q;X)$ consist of all $\mu \in \mathfrak{M}^+(X)$ *carried by* Q, which means that $X \setminus Q$ is locally μ-negligible, or equivalently that Q is μ-measurable and $\mu = \mu|_Q$, where $\mu|_Q = 1_Q \cdot \mu$ is the trace (restriction) of μ on Q [7, Chapter V, Section 5, n° 3, Example]. (Here 1_Q denotes the indicator function of Q.) If Q is closed, then μ is carried by Q if and only if it is supported by Q, i.e. $S(\mu) \subset Q$. It follows from the countability of X at infinity that the concept of local μ-negligibility coincides with that of μ-negligibility; and hence $\mu \in \mathfrak{M}^+(Q;X)$ if and only if $\mu^*(X \setminus Q) = 0$, $\mu^*(\cdot)$ being the *outer measure* of a set. Denoting by $\mu_*(\cdot)$ the *inner measure* of a set, for any $\mu \in \mathfrak{M}^+(Q;X)$ we thus get

$$\mu^*(Q) = \mu_*(Q) =: \mu(Q).$$

Write $\mathfrak{M}^+(Q,q;X) := \bigl\{\mu \in \mathfrak{M}^+(Q;X) : \ \mu(Q) = q\bigr\}$, where $q > 0$.

The following well known fact (see e.g. [18, Section 1.1]) will often be used.

Lemma 2.1. *Let ψ be a lower semicontinuous* (l.s.c.) *function on X, nonnegative unless X is compact. The mapping $\mu \mapsto \langle\psi,\mu\rangle := \int \psi\,d\mu$ is then vaguely l.s.c. on $\mathfrak{M}^+(X)$.*

A (function) *kernel* on X is defined as a symmetric l.s.c. function $\kappa : X \times X \to [0,\infty]$. Given $\mu,\nu \in \mathfrak{M}(X)$, we denote by $E_\kappa(\mu,\nu)$ and $\kappa\mu$ the (standard) *mutual energy* and the *potential* relative to the kernel κ. Let $\mathcal{E}_\kappa(X)$ consist of all $\mu \in \mathfrak{M}(X)$ whose (standard) energy $E_\kappa(\mu)$ is finite, which means that $E_\kappa(|\mu|) < \infty$, and let $\mathcal{E}^+_\kappa(X) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(X)$.

In all that follows we assume a kernel κ to be strictly positive definite. Then $\mathcal{E}_\kappa(X)$ forms a pre-Hilbert space with the (standard) inner product $\langle\mu,\nu\rangle_\kappa := E_\kappa(\mu,\nu)$, $\mu,\nu \in \mathcal{E}_\kappa(X)$, and the (standard energy) norm $\|\mu\|_\kappa := \sqrt{E_\kappa(\mu)}$ (see [18]). The (Hausdorff) topology on $\mathcal{E}_\kappa(X)$ determined by the (standard energy) norm $\|\cdot\|_\kappa$ is termed *strong*.

For a set $Q \subset X$ write $\mathcal{E}^+_\kappa(Q;X) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(Q;X)$ and $\mathcal{E}^+_\kappa(Q,q;X) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(Q,q;X)$, where $q \in (0,\infty)$. The (*inner*) *capacity* $c_\kappa(Q)$ of Q relative to the kernel κ is defined by

$$c_\kappa(Q)^{-1} := \inf_{\mu \in \mathcal{E}^+_\kappa(Q,1;X)} \|\mu\|^2_\kappa \tag{2.1}$$

(see e.g. [18, 25]). Then $0 \leqslant c_\kappa(Q) \leqslant \infty$. (Here and in the sequel the infimum over the empty set is taken to be $+\infty$. We also set $1\big/(+\infty) = 0$ and $1\big/0 = +\infty$.)

Because of the strict positive definiteness of the kernel κ, $c_\kappa(K) < \infty$ for every compact set $K \subset X$. Furthermore, by [18, p. 153, Eq. (2)],

$$c_\kappa(Q) = \sup c_\kappa(K) \quad (K \subset Q,\ K \text{ compact}). \tag{2.2}$$

An assertion $\mathcal{U}(x)$ involving a variable point $x \in X$ is said to hold c_κ-*nearly everywhere* (c_κ-*n.e.*) on $Q \subset X$ if $c_\kappa(N) = 0$, where N consists of all $x \in Q$ for which $\mathcal{U}(x)$ fails. It is often used that $c_\kappa(N) = 0$ if and only if $\mu_*(N) = 0$ for every $\mu \in \mathcal{E}^+_\kappa(X)$ [18, Lemma 2.3.1]. We shall sometimes need also the concept of c_κ-*quasi everywhere* (c_κ-*q.e.*) where the exceptional set N is supposed to have *outer* capacity zero. These two concepts of negligibility coincide if the exceptional sets are *capacitable* relative to the kernel κ.

As in [23, p. 134], we call a measure $\mu \in \mathfrak{M}(X)$ c_κ-*absolutely continuous* if $\mu(K) = 0$ for every compact set $K \subset X$ with $c_\kappa(K) = 0$. It follows from (2.2) that for such μ, $|\mu|_*(Q) = 0$ for every $Q \subset X$ with $c_\kappa(Q) = 0$. Hence, every $\mu \in \mathcal{E}_\kappa(X)$ is c_κ-absolutely continuous; but not conversely [23, pp. 134–135].

Definition 2.2. Following [18], we call a (strictly positive definite) kernel κ *perfect* if every strong Cauchy sequence in $\mathcal{E}^+_\kappa(X)$ converges strongly to any of its vague cluster points[5].

Remark 2.3. On $X = \mathbb{R}^n$, $n \geqslant 3$, the α-Riesz kernel $\kappa_\alpha(x, y) = |x - y|^{\alpha-n}$, $\alpha \in (0, n)$, is strictly positive definite and moreover perfect [11, 12]; thus so is the Newtonian kernel $\kappa_2(x, y) = |x-y|^{2-n}$ [9]. Recently it has been shown by the present authors that if X is an open set D in $\mathbb{R}^n$, $n \geqslant 3$, and g^α_D, $\alpha \in (0, 2]$, is the α-Green kernel on D [23, Chapter IV, Section 5], then $\kappa = g^\alpha_D$ likewise is strictly positive definite and moreover perfect [19, Theorems 4.9, 4.11].

Theorem 2.4 (see [18])**.** *If a kernel κ is perfect, then the cone $\mathcal{E}^+_\kappa(X)$ is strongly complete and the strong topology on $\mathcal{E}^+_\kappa(X)$ is finer than the* (*induced*) *vague topology on $\mathcal{E}^+_\kappa(X)$.*

Remark 2.5. In contrast to Theorem 2.4, for a perfect kernel κ the whole pre-Hilbert space $\mathcal{E}_\kappa(X)$ is in general strongly *incomplete*, and this is the case even for the α-Riesz kernel of order $\alpha \in (1, n)$ on $\mathbb{R}^n$, $n \geqslant 3$ (see [9]); compare with the following Remark 2.6.

Remark 2.6. The concept of perfect kernel is an efficient tool in minimum energy problems over classes of *positive* Radon measures with finite energy. Indeed, if $Q \subset X$ is closed and κ is perfect, then problem (2.1) has a (unique) solution $\mu_{Q,\kappa}$ if and only if $0 < c_\kappa(Q) < \infty$ [18, Theorem 4.1]; such $\mu_{Q,\kappa}$ is termed the (*inner*) *κ-capacitary measure* on Q. Later the concept of perfectness has been shown to be efficient also in minimum (standard) energy problems over classes of (*signed*) measures associated with a *standard condenser* (see [31]–[33]; see also the earlier study [29] pertaining to the α-Riesz kernel on $\mathbb{R}^n$). The approach developed in [31]–[33] substantially used *the assumption of the boundedness of the kernel on the product of the oppositely charged plates of a condenser*,[6] which made it possible to extend Cartan's proof [9] of the strong completeness of the cone $\mathcal{E}^+_{\kappa_2}(\mathbb{R}^n)$ of all *positive* measures on $\mathbb{R}^n$ with finite Newtonian energy to an arbitrary perfect kernel κ on a locally compact space X and suitable classes of (*signed*) measures $\mu \in \mathcal{E}_\kappa(X)$. In turn, this strong completeness theorem for metric subspaces of signed $\mu \in \mathcal{E}_\kappa(X)$ made it possible to develop a fairly general theory of standard condensers, actually even with *countably many* plates.

A set $Q \subset X$ is said to be *locally closed* in X if for every $x \in Q$ there is a neighborhood V of x in X such that $V \cap Q$ is a closed subset of the subspace Q [6, Chapter I, Section 3, Definition 2], or equivalently if Q is the intersection of an open and a closed subset of X [6, Chapter I, Section 3, Proposition 5]. The latter implies that this Q is universally measurable, and hence $\mathfrak{M}^+(Q; X)$ consists of all the restrictions $\mu|_Q$ where μ ranges over $\mathfrak{M}^+(X)$. On the other hand, by [6, Chapter I, Section 9, Proposition 13] a locally closed set Q itself can be thought of as a locally compact subspace of X. Thus $\mathfrak{M}^+(Q; X)$ consists, in fact, of all those $\nu \in \mathfrak{M}^+(Q)$ for each of which there is $\hat{\nu} \in \mathfrak{M}^+(X)$ with the property

$$\hat{\nu}(\varphi) = \int \varphi|_Q \, d\nu \quad \text{for every } \varphi \in C_0(X). \tag{2.3}$$

[5] It follows from Theorem 2.4 that for a perfect kernel such a vague cluster point exists and is unique.

[6] For any classical kernel κ on $\mathbb{R}^n$ the quoted assumption of the boundedness of κ on the product of the oppositely charged plates is equivalent to the separation condition (1.3).

We say that such $\hat{\nu}$ *extends* $\nu \in \mathfrak{M}^+(Q)$ by 0 off Q to all of X. A sufficient condition for (2.3) to hold is that ν be bounded.

3. α-Riesz balayage and α-Green kernel

In the rest of the paper fix $n \geqslant 3$, $\alpha \in (0, 2]$ and a domain $D \subset \mathbb{R}^n$ with $c_{\kappa_\alpha}(D^c) > 0$, and assume that either $\kappa = \kappa_\alpha$ is the *α-Riesz kernel* on $X = \mathbb{R}^n$, or $\kappa = g^\alpha_D$ is the *α-Green kernel* on $X = D$. We simply write α instead of κ_α if κ_α serves as an index, and we use the short form 'n.e.' instead of 'c_α-n.e.' if this will not cause any misunderstanding.

When speaking of a positive Radon measure μ on $\mathbb{R}^n$, we always tacitly assume that for the given α, $\kappa_\alpha\mu$ is not identically infinite. This implies that

$$\int_{|y|>1} \frac{d\mu(y)}{|y|^{n-\alpha}} < \infty \tag{3.1}$$

(see [23, Eq. (1.3.10)]), and consequently that $\kappa_\alpha\mu$ is finite (c_α-)n.e. on $\mathbb{R}^n$ [23, Chapter III, Section 1]; these two implications can actually be reversed.

Definition 3.1. A (signed) measure $\nu \in \mathfrak{M}(D)$ is termed *extendible* if there are $\widehat{\nu^+}$ and $\widehat{\nu^-}$ extending ν^+ and ν^-, respectively, by 0 off D to all of $\mathbb{R}^n$, see (2.3), and if these $\widehat{\nu^+}$ and $\widehat{\nu^-}$ satisfy the general convention (3.1). We identify this $\nu \in \mathfrak{M}(D)$ with its extension $\hat{\nu} := \widehat{\nu^+} - \widehat{\nu^-}$, and we therefore write $\hat{\nu} = \nu$.

Every bounded measure $\nu \in \mathfrak{M}(D)$ is extendible. The converse holds if D is bounded, but not in general (e.g. not if D^c is compact). The set of all extendible measures consists of all the restrictions $\mu|_D$ where μ ranges over $\mathfrak{M}(\mathbb{R}^n)$, see the end of Section 2. Also note that for any extendible measure $\nu \in \mathfrak{M}(D)$, $\kappa_\alpha\nu$ is finite n.e. on $\mathbb{R}^n$, for $\kappa_\alpha\nu^\pm$ is so.

The *α-Green kernel* $g = g^\alpha_D$ on D is defined by

$$g^\alpha_D(x, y) := \kappa_\alpha\varepsilon_y(x) - \kappa_\alpha\varepsilon_y^{D^c}(x) \quad \text{for all } x, y \in D,$$

where ε_y denotes the unit Dirac measure at a point y and $\varepsilon_y^{D^c}$ its *α-Riesz balayage* (sweeping) onto the (closed) set D^c, determined uniquely in the frame of the classical approach by [19, Theorem 3.6] pertaining to positive Radon measures on $\mathbb{R}^n$. See also the book by Bliedtner and Hansen [5] where balayage is studied in the setting of balayage spaces.

We shall simply write μ' instead of μ^{D^c} when speaking of the α-Riesz balayage of $\mu \in \mathfrak{M}^+(D; \mathbb{R}^n)$ onto D^c. According to [19, Corollaries 3.19, 3.20], for any $\mu \in \mathfrak{M}^+(D; \mathbb{R}^n)$ *the balayage μ' is c_α-absolutely continuous and it is determined uniquely by the relation*

$$\kappa_\alpha\mu' = \kappa_\alpha\mu \quad \textit{n.e. on } D^c \tag{3.2}$$

among the c_α-absolutely continuous measures supported by D^c. Furthermore, there holds the integral representation [19, Theorem 3.17][7]

$$\mu' = \int \varepsilon'_y \, d\mu(y). \tag{3.3}$$

[7]In the literature the integral representation (3.3) seems to have been more or less taken for granted, though it has been pointed out in [7, Chapter V, Section 3, n° 1] that it requires that the family $(\varepsilon'_y)_{y\in D}$ be *μ-adequate* in the sense of [7, Chapter V, Section 3, Definition 1]; see also counterexamples (without μ-adequacy) in Exercises 1 and 2 at the end of that section. A proof of this adequacy has therefore been given in [19, Lemma 3.16].

If moreover $\mu \in \mathcal{E}^+_\alpha(D;\mathbb{R}^n)$, then the balayage μ' is in fact *the orthogonal projection of μ onto the convex cone $\mathcal{E}^+_\alpha(D^c;\mathbb{R}^n)$* [19, Theorem 3.1], i.e. $\mu' \in \mathcal{E}^+_\alpha(D^c;\mathbb{R}^n)$ and

$$\|\mu-\theta\|_\alpha > \|\mu-\mu'\|_\alpha \quad \text{for all} \quad \theta \in \mathcal{E}^+_\alpha(D^c;\mathbb{R}^n),\ \theta \neq \mu'. \tag{3.4}$$

If now $\nu \in \mathfrak{M}(D)$ is an extendible (signed) measure, then $\nu' := \nu^{D^c} := (\nu^+)' - (\nu^-)'$ is said to be a *balayage* of ν onto D^c. It follows from [23, p. 178, Remark] that the balayage ν' is determined uniquely by (3.2) with ν in place of μ among the c_α-absolutely continuous (signed) measures supported by D^c.

Definition 3.2 (see [8, Theorem VII.13])**.** A closed set $Q \subset \mathbb{R}^n$ is said to be *α-thin at infinity* if either Q is compact, or the inverse of Q relative to $S(0,1) := \bigl\{x \in \mathbb{R}^n : |x| = 1\bigr\}$ has $x = 0$ as an α-irregular boundary point (cf. [23, Theorem 5.10]).

Remark 3.3. Any closed set Q that is not α-thin at infinity is of infinite capacity $c_\alpha(Q)$. Indeed, by the Wiener criterion of α-regularity, Q is not α-thin at infinity if and only if

$$\sum_{k\in\mathbb{N}} \frac{c_\alpha(Q_k)}{q^{k(n-\alpha)}} = \infty,$$

where $q > 1$ and $Q_k := Q \cap \bigl\{x \in \mathbb{R}^n : q^k \leqslant |x| < q^{k+1}\bigr\}$, while by [23, Lemma 5.5] $c_\alpha(Q) < \infty$ is equivalent to the relation

$$\sum_{k\in\mathbb{N}} c_\alpha(Q_k) < \infty.$$

These observations also imply that the converse is not true, i.e. there is Q with $c_\alpha(Q) = \infty$, but α-thin at infinity (see also [10, pp. 276–277]).

Example 3.4 (see [31, Example 5.3])**.** Let $n = 3$ and $\alpha = 2$. Define the rotation body

$$Q_\varrho := \bigl\{x \in \mathbb{R}^3 :\ 0 \leqslant x_1 < \infty,\ x_2^2 + x_3^2 \leqslant \varrho^2(x_1)\bigr\}, \tag{3.5}$$

where ϱ is given by one of the following three formulae:

$$\varrho(x_1) = x_1^{-s} \quad \text{with} \quad s \in [0,\infty), \tag{3.6}$$

$$\varrho(x_1) = \exp(-x_1^s) \quad \text{with} \quad s \in (0,1], \tag{3.7}$$

$$\varrho(x_1) = \exp(-x_1^s) \quad \text{with} \quad s \in (1,\infty). \tag{3.8}$$

Then Q_ϱ is not 2-thin at infinity if ϱ is defined by (3.6), Q_ϱ is 2-thin at infinity but has infinite Newtonian capacity if ϱ is given by (3.7), and finally $c_2(Q_\varrho) < \infty$ if (3.8) holds.

Theorem 3.5 (see [19, Theorem 3.22])**.** *The set D^c is not α-thin at infinity if and only if for every bounded measure $\mu \in \mathfrak{M}^+(D)$ we have $\mu'(\mathbb{R}^n) = \mu(\mathbb{R}^n)$.*[8]

Theorem 3.6 (see [19, Theorem 4.12])**.** *For any relatively closed subset F of D with $c_g(F) < \infty$ there exists a unique α-Green equilibrium measure on F, i.e. a measure $\gamma_{F,g} = \gamma_F \in \mathcal{E}^+_g(F;D)$ such that $\gamma_F(D) = \|\gamma_F\|^2_g = c_g(F)$ and*[9]

$$\begin{aligned} g\gamma_F &= 1 \quad (c_\alpha\text{-})\text{n.e. on } F, \\ g\gamma_F &\leqslant 1 \quad \text{on } D. \end{aligned} \tag{3.9}$$

[8] In general, $\nu^{D^c}(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n)$ for every $\nu \in \mathfrak{M}^+(\mathbb{R}^n)$ [19, Theorem 3.11].

[9] If N is a given subset of D, then $c_g(N) = 0$ if and only if $c_\alpha(N) = 0$ [15, Lemma 2.6]. Thus any assertion involving a variable point holds (c_α-)n.e. on $Q \subset D$ if and only if it holds c_g-n.e. on Q.

This γ_F is characterized uniquely within $\mathcal{E}^+_g(F;D)$ by (3.9), and it is the (unique) solution to the problem of minimizing $E_g(\nu)$ over the (convex) class Γ_F of all (signed) $\nu\in\mathcal{E}_g(D)$ with the property $g\nu\geqslant 1$ n.e. on F. That is,

$$c_g(F)=\|\gamma_F\|^2_g=\min_{\nu\in\Gamma_F}\|\nu\|^2_g. \tag{3.10}$$

If $I_{F,\alpha}$ consists of all α-irregular points of F, then (3.9) can be specified as follows:

$$g\gamma_F=1 \quad\text{on}\quad F\setminus I_{F,\alpha}. \tag{3.11}$$

Remark 3.7. If F is a relatively closed subset of D with $0<c_g(F)<\infty$, then

$$\gamma_{F,g}=c_g(F)\mu_{F,g},$$

where $\mu_{F,g}$ is the (unique) g-capacitary measure on F (which exists, see Remarks 2.3 and 2.6).

Corollary 3.8. *If F is a relatively closed subset of D with $c_g(F)<\infty$, then*

$$c_\alpha\bigl(\partial D\cap \mathrm{Cl}_{\mathbb{R}^n}F\bigr)=0.$$

Proof. According to Theorem 3.6, there is the α-Green equilibrium measure $\gamma=\gamma_F$ on F. By Lusin's type theorem [23, Theorem 3.6] applied to each of $\kappa_\alpha\gamma$ and $\kappa_\alpha\gamma'$, there exists for any $\varepsilon>0$ an open set $\Omega\subset\mathbb{R}^n$ with $c_\alpha(\Omega)<\varepsilon$ such that $\kappa_\alpha\gamma$ and $\kappa_\alpha\gamma'$ are both continuous relative to $\mathbb{R}^n\setminus\Omega$. Since there is no loss of generality in assuming $I_{F,\alpha}\subset\Omega$, we thus get from Lemma 3.9 below and (3.11)

$$\kappa_\alpha\gamma=\kappa_\alpha\gamma'+1 \quad\text{on}\quad (\mathrm{Cl}_{\mathbb{R}^n}F)\setminus\Omega.$$

As ε is arbitrary, $\kappa_\alpha\gamma=\kappa_\alpha\gamma'+1$ c_α-q.e. on $\partial D\cap\mathrm{Cl}_{\mathbb{R}^n}F$. But $\kappa_\alpha\gamma=\kappa_\alpha\gamma'$ holds c_α-n.e. on D^c by (3.2), hence c_α-q.e. because $\bigl\{\kappa_\alpha\gamma\neq\kappa_\alpha\gamma'\bigr\}$ is a Borel set, and the corollary follows. □

The following three known assertions establish relations between potentials and standard energies relative to the kernels κ_α and $g=g^\alpha_D$.

Lemma 3.9 (see [16, Lemma 3.4]). *For any extendible (signed) measure $\mu\in\mathfrak{M}(D)$ the α-Green potential $g\mu$ is finite (c_α-)n.e. on D and given by $g\mu=\kappa_\alpha\mu-\kappa_\alpha\mu'$.*

Lemma 3.10 (see [16, Lemma 3.5]). *Suppose that $\mu\in\mathfrak{M}(D)$ is extendible and the extension belongs to $\mathcal{E}_\alpha(\mathbb{R}^n)$. Then $\mu\in\mathcal{E}_g(D)$, $\mu-\mu'\in\mathcal{E}_\alpha(\mathbb{R}^n)$ and moreover*

$$\|\mu\|^2_g=\|\mu-\mu'\|^2_\alpha=\|\mu\|^2_\alpha-\|\mu'\|^2_\alpha. \tag{3.12}$$

Lemma 3.11 (see [20, Lemma 3.4]). *Let A_1 be a relatively closed subset of D with the separation property (1.3). Then a bounded measure $\mu\in\mathfrak{M}^+(A_1;D)$ has finite $E_g(\mu)$ if and only if its extension has finite standard α-Riesz energy, and in the affirmative case (3.12) holds. Furthermore, $c_g(A_1)<\infty$ if and only if $c_\alpha(A_1)<\infty$.*

4. Auxiliary results

In all that follows fix a generalized condenser $\mathbf{A}=(A_1,A_2)$ (see the Introduction). Write

$$\mathfrak{M}(\mathbf{A},\mathbf{1}):=\bigl\{\mu\in\mathfrak{M}(\mathbf{A}):\ \mu^+(A_1)=\mu^-(A_2)=1\bigr\},$$

where $\mathbf{1}:=(1,1)$. *To avoid trivialities, throughout the paper we assume that*

$$c_\alpha(A_i)>0 \quad\text{for } i=1,2. \tag{4.1}$$

Then $\mathcal{E}_\alpha(\mathbf{A},\mathbf{1}):=\mathcal{E}_\alpha(\mathbb{R}^n)\cap\mathfrak{M}(\mathbf{A},\mathbf{1})$ is nonempty in accordance with [18, Lemma 2.3.1], and hence so is $\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1}):=\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)\cap\mathfrak{M}(\mathbf{A},\mathbf{1})$, see (1.2). Note that in general $\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})$

is a *proper* subset of $\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})$, which is seen from the counterexample given in [16, Appendix] and Theorems 3.5, 4.2; compare with the following Lemma 4.1.

Lemma 4.1. *If* $\mathbf{A}$ *is a (standard) condenser with the separation property* (1.3), *then*

$$\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})=\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1}).$$

Proof. Fix $\mu\in\mathfrak{M}(\mathbf{A},\mathbf{1})$. By the Riesz composition identity and Fubini's theorem,

$$\begin{aligned}\int\kappa_{\alpha/2}\mu^+\kappa_{\alpha/2}\mu^-\,dm&=\int\Bigl(\int\kappa_{\alpha/2}(x,y)\,d\mu^+(y)\Bigr)\Bigl(\int\kappa_{\alpha/2}(x,z)\,d\mu^-(z)\Bigr)\,dm(x)\\&=\int\Bigl(\int|x-y|^{\alpha/2-n}|x-z|^{\alpha/2-n}\,dm(x)\Bigr)\,d(\mu^+\otimes\mu^-)(y,z)\\&=\int\Bigl(\int|x-y-z|^{\alpha/2-n}|x|^{\alpha/2-n}\,dm(x)\Bigr)\,d(\mu^+\otimes\mu^-)(y,z)\\&=\int\kappa_\alpha(y,z)\,d(\mu^+\otimes\mu^-)(y,z)\leqslant\bigl[\operatorname{dist}(A_1,D^c)\bigr]^{\alpha-n}<\infty.\end{aligned}$$

Assuming now $\mu\in\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})$, we thus obtain

$$\bigl(\kappa_{\alpha/2}\mu^+\bigr)^2+\bigl(\kappa_{\alpha/2}\mu^-\bigr)^2=\bigl(\kappa_{\alpha/2}\mu\bigr)^2+2\kappa_{\alpha/2}\mu^+\,\kappa_{\alpha/2}\mu^-\in L^1(m),$$

which yields $\kappa_{\alpha/2}\mu^+,\kappa_{\alpha/2}\mu^-\in L^2(m)$. This means that $\mu^+,\mu^-\in\dot{\mathcal{E}}^+_\alpha(\mathbb{R}^n)$ $\bigl(=\mathcal{E}^+_\alpha(\mathbb{R}^n)\bigr)$, and hence $\mu\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})$. Since $\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})\subset\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})$ by (1.2), the lemma follows. □

We shall also need the following two assertions, the former being known.

Theorem 4.2 (see [20, Theorem 5.1]). *Let* μ *be an extendible (signed) Radon measure on* D *with* $E_g(\mu)<\infty$. *Then* $\mu-\mu'$ *has finite weak* α*-Riesz energy* $\dot{E}_\alpha(\mu-\mu')$, *and moreover*

$$E_g(\mu)=\dot{E}_\alpha(\mu-\mu'). \tag{4.2}$$

Theorem 4.3. *Assume that* D^c *is not* α*-thin at infinity. For any* $\mu\in\mathcal{E}^+_g(A_1,1;D)$ *there is a sequence* $\{\mu_j\}_{j\in\mathbb{N}}\in\mathcal{E}^+_\alpha(A_1,1;\mathbb{R}^n)$, *each* μ_j *being compactly supported in* D, *which with the notations* $\nu_j:=\mu_j-\mu_j'$ *and* $\nu:=\mu-\mu'$ *possesses the following properties:*

(a) $\nu\in\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})$ *and* $\{\nu_j\}_{j\in\mathbb{N}}\subset\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})$;
(b) $\|\nu_j-\nu\|^{\cdot}_\alpha\to0$ *as* $j\to\infty$;
(c) $\nu_j^\pm\to\nu^\pm$ *vaguely in* $\mathfrak{M}(\mathbb{R}^n)$ *as* $j\to\infty$.

Proof. Applying Theorems 3.5 and 4.2 to the (bounded, and hence extendible) measure μ, we obtain $\nu:=\mu-\mu'\in\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})$, which is the former relation in (a).

Choose an increasing sequence $\{K_j\}_{j\in\mathbb{N}}$ of compact sets with the union D and write $\tilde\mu_j:=\mu|_{K_j}$, where $\mu|_{K_j}$ is the trace of μ on K_j. It follows from the definition of $\tilde\mu_j$ that $\kappa_\alpha\tilde\mu_j\uparrow\kappa_\alpha\mu$ pointwise on $\mathbb{R}^n$ and also that the increasing sequence $\{\tilde\mu_j\}_{j\in\mathbb{N}}$ converges to μ vaguely in $\mathfrak{M}(\mathbb{R}^n)$. We therefore see from the proof of [19, Theorem 3.6] that $\{\tilde\mu_j'\}_{j\in\mathbb{N}}$ likewise is increasing and converges vaguely to μ'. Also write $\mu_j:=\tilde\mu_j/\tilde\mu_j(K_j)$, $j\in\mathbb{N}$. Since $\tilde\mu_j(K_j)\uparrow\mu(A_1)=1$, we infer that $\mu_j\to\mu$ and $\mu_j'\to\mu'$ vaguely in $\mathfrak{M}(\mathbb{R}^n)$, which is (c).

Furthermore, $g\tilde\mu_j\uparrow g\mu$ pointwise on D, and also

$$\lim_{j\to\infty}\|\tilde\mu_j\|_g=\|\mu\|_g<\infty. \tag{4.3}$$

The former relation implies that $\langle\tilde\mu_j,\tilde\mu_p\rangle_g\geqslant\|\tilde\mu_p\|_g^2$ for all $j\geqslant p$, and hence

$$\|\tilde\mu_j-\tilde\mu_p\|_g^2\leqslant\|\tilde\mu_j\|_g^2-\|\tilde\mu_p\|_g^2,$$

which together with (4.3) proves that $\{\tilde\mu_j\}_{j\in\mathbb{N}}$ is a strong Cauchy sequence in the pre-Hilbert space $\mathcal{E}_g(D)$. Since the kernel g is perfect [19, Theorem 4.11], we thus see by Definition 2.2 that $\tilde\mu_j\to\mu$ in $\mathcal{E}_g(D)$ strongly, and consequently

$$\lim_{j\to\infty}\|\mu_j-\mu\|_g=0. \tag{4.4}$$

Since $S_D^{\mu_j}$ is compact and $E_g(\mu_j)<\infty$, it follows from Theorem 3.5 and Lemma 3.11 that $\nu_j:=\mu_j-\mu_j'\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})$, which is the latter relation in (a). Furthermore, we get from Theorem 4.2

$$\|\nu_j-\nu\|^{\cdot}_\alpha=\|(\mu_j-\mu)-(\mu_j'-\mu')\|^{\cdot}_\alpha=\|\mu_j-\mu\|_g,$$

which establishes (b) when combined with (4.4). $\square$

5. Statements of the problems

5.1. **Notations, permanent assumptions, historical remarks.** Let $\mathfrak{C}(A_1)$ consist of all $\xi\in\mathfrak{M}^+(A_1;\mathbb{R}^n)$ with $\xi(A_1)>1$; such ξ will serve as (upper) *constraints* for measures of the class $\mathfrak{M}^+(A_1,1;\mathbb{R}^n)$. For any given $\xi\in\mathfrak{C}(A_1)$ write

$$\dot{\mathcal{E}}^\xi_\alpha(\mathbf{A},\mathbf{1}):=\bigl\{\mu\in\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1}):\ \mu^+\leqslant\xi\bigr\},$$

where $\mu^+\leqslant\xi$ means that $\xi-\mu^+\geqslant 0$. To combine (whenever this is possible) formulations related to minimum weak α-Riesz problems in both the unconstrained and constrained settings, write $\dot{\mathcal{E}}^\sigma_\alpha(\mathbf{A},\mathbf{1})$, $\sigma\in\mathfrak{C}(A_1)\cup\{\infty\}$, where the formal notation $\sigma=\infty$ means that *no* upper constraint is imposed on the positive parts of $\mu\in\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})$, i.e. $\dot{\mathcal{E}}^\infty_\alpha(\mathbf{A},\mathbf{1}):=\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})$.

Fix $f:\mathbb{R}^n\to[-\infty,\infty]$, to be treated as an *external field*, and let $\dot{\mathcal{E}}^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1})$ consist of all $\nu\in\dot{\mathcal{E}}^\sigma_\alpha(\mathbf{A},\mathbf{1})$ such that f is $|\nu|$-integrable. For such ν write $\langle f,\nu\rangle:=\int f\,d\nu$ and

$$\dot G_{\alpha,f}(\nu):=\|\nu\|^{\cdot\,2}_\alpha+2\langle f,\nu\rangle. \tag{5.1}$$

Note that if $\dot{\mathcal{E}}^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1})$ is nonempty (sufficient conditions for this to hold will be provided below), then it forms a convex subcone of the convex cone $\dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})$.

Fix a nonempty convex cone $\mathcal{H}\subset\dot{\mathcal{E}}^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1})$, to be specified below. Then

$$\dot G_{\alpha,f}(\mathcal{H}):=\inf_{\nu\in\mathcal{H}}\dot G_{\alpha,f}(\nu)<\infty,$$

and hence the following minimum weak α-Riesz energy problem, to be referred to as the *$\mathcal{H}$-problem*, makes sense: *does there exist $\lambda_{\mathcal{H}}\in\mathcal{H}$ with $\dot G_{\alpha,f}(\lambda_{\mathcal{H}})=\dot G_{\alpha,f}(\mathcal{H})$?*

Lemma 5.1. *A solution $\lambda_{\mathcal{H}}$ to the $\mathcal{H}$-problem is unique* (*whenever it exists*).

Proof. This can be established by standard methods based on the convexity of the class $\mathcal{H}$ and the pre-Hilbert structure on the space $\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$. Indeed, if λ and $\breve\lambda$ are two solutions to the $\mathcal{H}$-problem, then we obtain from (5.1)

$$4\dot G_{\alpha,f}(\mathcal{H})\leqslant 4\dot G_{\alpha,f}\Bigl(\frac{\lambda+\breve\lambda}{2}\Bigr)=\|\lambda+\breve\lambda\|^{\cdot\,2}_\alpha+4\langle f,\lambda+\breve\lambda\rangle.$$

On the other hand, applying the parallelogram identity in $\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)$ to λ and $\breve{\lambda}$ and then adding and subtracting $4\langle f, \lambda+\breve{\lambda}\rangle$ we get

$$\|\lambda-\breve{\lambda}\|^{\cdot 2}_\alpha = -\|\lambda+\breve{\lambda}\|^{\cdot 2}_\alpha - 4\langle f,\lambda+\breve{\lambda}\rangle + 2\dot{G}_{\alpha,f}(\lambda) + 2\dot{G}_{\alpha,f}(\breve{\lambda}).$$

When combined with the preceding relation, this yields

$$0 \leqslant \|\lambda-\breve{\lambda}\|^{\cdot 2}_\alpha \leqslant -4\dot{G}_{\alpha,f}(\mathcal{H}) + 2\dot{G}_{\alpha,f}(\lambda) + 2\dot{G}_{\alpha,f}(\breve{\lambda}) = 0,$$

which establishes the lemma because $\|\cdot\|^{\cdot}_\alpha$ is a norm. □

Remark 5.2. Let $f=0$, $\sigma=\infty$ (no external field and no constraint), and let $\mathcal{H}=\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})$. In view of (1.2), the $\mathcal{H}$-problem is then the problem on the existence of $\lambda\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})$ with

$$\|\lambda\|^2_\alpha = \inf_{\nu\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})} \|\nu\|^2_\alpha =: w_\alpha(\mathbf{A},\mathbf{1}), \tag{5.2}$$

$1/w_\alpha(\mathbf{A},\mathbf{1})$ being known as the (standard) α-Riesz *capacity* of the (generalized) condenser $\mathbf{A}$. To avoid trivialities, assume that $c_\alpha(A_1)<\infty$.[10] If moreover the separation condition (1.3) holds, then problem (5.2) is solvable if and only if either $c_\alpha(A_2)<\infty$, or A_2 is *not* α-thin at infinity (see [29, Theorem 5]).

Thus, if A_2 is α-thin at infinity, but $c_\alpha(A_2)=\infty$ (such A_2 exists according to Remark 3.3), then $\|\nu\|^2_\alpha > w_\alpha(\mathbf{A},\mathbf{1})$ for any $\nu\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})$. It can however be shown that any (minimizing) sequence $\{\nu_j\}_{j\in\mathbb{N}}\subset\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})$ with $\lim_{j\to\infty}\|\nu_j\|^2_\alpha = w_\alpha(\mathbf{A},\mathbf{1})$ then converges strongly and vaguely to a (unique) measure $\theta\in\mathcal{E}_\alpha(\mathbf{A})$ such that $\theta^+(A_1)=1$, but $\theta^-(A_2)<1$ (see [29]). Of course, $\|\theta\|^2_\alpha = w_\alpha(\mathbf{A},\mathbf{1})$. Using the electrostatic interpretation, which is possible for the Coulomb kernel $|x-y|^{-1}$ on $\mathbb{R}^3$, we say that the described pair (A_1,A_2) of oppositely charged conductors achieves its equilibrium state only provided that a nonzero part (with mass $1-\theta^-(A_2)>0$) of charge carried by A_2 vanishes at the point at infinity.

This phenomenon, discovered first for $\alpha=2$ in [28], is actually a characteristic feature of *space* condensers; compare with Bagby's study [2] where it has been proven that the infimum of the logarithmic energy over $\mathfrak{M}(\mathbf{A},\mathbf{1})$, $\mathbf{A}=(A_1,A_2)$ being a condenser in $\mathbb{R}^2$ such that $\mathrm{Cl}_{\overline{\mathbb{R}^2}}A_1\cap\mathrm{Cl}_{\overline{\mathbb{R}^2}}A_2=\varnothing$, is *always* an actual minimum. (Here $\overline{\mathbb{R}^2}$ is the one-point compactification of $\mathbb{R}^2$.) Such a drastic difference between the theories of space and plane condensers is caused by the fact that the logarithmic capacity of a plane condenser is invariant with respect to the Möbius transformations (more generally, conformal mappings) of $\overline{\mathbb{R}^2}$, while the Riesz capacity of a space condenser is not so.

Example 5.3. Let $n=3$, $\alpha=2$ and $A_2=D^c=Q_\varrho$, where Q_ϱ is given by (3.5), and let A_1 be a closed set in $\mathbb{R}^3$ with $c_2(A_1)<\infty$ possessing the separation property (1.3). According to [29, Theorem 5], a solution to problem (5.2) (with $\alpha=2$) does exist if ϱ is given by either (3.6) or (3.8), while the problem has no solution if ϱ is defined by (3.7). These (theoretical) results have been illustrated in [21, 24] by means of numerical experiments.

In all that follows we shall always suppose that D^c is not α-thin at infinity.

Remark 5.4. Assume that $f=0$ and $\sigma=\infty$, and define

$$\ddot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1}) := \dot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})\cap\mathrm{Cl}_{\dot{\mathcal{E}}_\alpha(\mathbb{R}^n)}\mathcal{E}_\alpha(\mathbf{A},\mathbf{1}).$$

[10] If $c_\alpha(A_i)=\infty$ for $i=1,2$, then $w_\alpha(\mathbf{A},\mathbf{1})=0$, and hence this infimum cannot be an actual minimum because $0\notin\mathcal{E}_\alpha(\mathbf{A},\mathbf{1})$.

The $\mathcal{H}$-problem with $\mathcal{H} = \ddot{\mathcal{E}}_\alpha(\mathbf{A}, \mathbf{1})$ is in fact [20, Problem 6.2], solved by [20, Theorems 6.1, 6.2]. To be precise, it has been proven in [20, Theorem 6.1] that

$$\inf_{\mu \in \ddot{\mathcal{E}}_\alpha(\mathbf{A},\mathbf{1})} \|\mu\|^{\cdot 2}_\alpha = \inf_{\nu \in \mathcal{E}_\alpha(\mathbf{A},\mathbf{1})} \|\nu\|^2_\alpha \quad \bigl(= w_\alpha(\mathbf{A}, \mathbf{1})\bigr).$$

Furthermore, the $\ddot{\mathcal{E}}_\alpha(\mathbf{A}, \mathbf{1})$-problem is shown to be (uniquely) solvable if and only if

$$c_g(A_1) < \infty, \tag{5.3}$$

and the solution $\lambda_{\mathcal{H}}$ is then given by

$$\lambda_{\mathcal{H}} = \mu_{A_1,g} - \mu'_{A_1,g}, \tag{5.4}$$

where $\mu_{A_1,g}$ is the g-capacitary measure on A_1 (which exists,[11] see Remark 3.7 for $F = A_1$). If moreover the separation condition (1.3) holds, then assumption (5.3) (which by Lemma 3.11 is now equivalent to $c_\alpha(A_1) < \infty$) is also necessary and sufficient for the solvability of problem (5.2), and its solution is again given by (5.4) [20, Theorem 6.2].

Combining Theorems 3.5, 4.2, and 4.3 implies that the quoted results on the solvability of the $\ddot{\mathcal{E}}_\alpha(\mathbf{A}, \mathbf{1})$-problem remain valid if $\ddot{\mathcal{E}}_\alpha(\mathbf{A}, \mathbf{1})$ is replaced by the class of all $\mu \in \dot{\mathcal{E}}_\alpha(\mathbf{A}, \mathbf{1})$ such that $\mu^+ \in \mathcal{E}^+_g(A_1, 1; D)$ and $\mu^- = (\mu^+)'$. This observation may be viewed as a motivation to the study of the $\widetilde{\mathcal{H}}$-problem, $\widetilde{\mathcal{H}}$ being defined by (5.20) below.

In all that follows we assume that either Case I *or Case* II *holds, where:*

I. $f \geqslant 0$ *is l.s.c. on* $\mathbb{R}^n$ *and*

$$f = 0 \quad \text{n.e. on } D^c; \tag{5.5}$$

II. $f = \kappa_\alpha(\zeta - \zeta')$ *where* ζ *is an extendible* (*signed*) *Radon measure on* D *with* $E_g(\zeta) < \infty$ *and* ζ' *is the* α*-Riesz swept measure of* ζ *onto* D^c.

Note that in Case II the external field f is finite n.e. on $\mathbb{R}^n$ according to the general convention (3.1), and it also satisfies (5.5) by (3.2). In fact, then (see Lemma 3.9)

$$f = g\zeta \quad \text{n.e. on } D. \tag{5.6}$$

Also observe that in Case I we actually have $f = 0$ on D^c, for $f \geqslant 0$ is l.s.c.

5.2. **An auxiliary minimum α-Green energy problem.** Being c_g-absolutely continuous, any $\mu \in \mathcal{E}^+_g(A_1, 1; D)$ is c_α-*absolutely continuous* (see footnote 9), which will be used permanently throughout the paper.

Let f and $\sigma \in \mathfrak{C}(A_1) \cup \{\infty\}$ be as indicated in Section 5.1, and let $\mathcal{E}^\sigma_{g,f}(A_1, 1; D)$ consist of all $\mu \in \mathcal{E}^+_g(A_1, 1; D)$ such that $\mu \leqslant \sigma$ and f is μ-integrable. For any $\mu \in \mathcal{E}^\sigma_{g,f}(A_1, 1; D)$ write

$$G_{g,f}(\mu) := \|\mu\|^2_g + 2\langle f, \mu\rangle = |\mu\|^2_g + 2\langle f|_D, \mu\rangle, \tag{5.7}$$

the latter equality being valid since $\mu^*(D^c) = 0$ for any $\mu \in \mathfrak{M}^+(D; \mathbb{R}^n)$, see Section 2. If the class $\mathcal{E}^\sigma_{g,f}(A_1, 1; D)$ is nonempty, or equivalently

$$G^\sigma_{g,f}(A_1, 1; D) := \inf_{\mu \in \mathcal{E}^\sigma_{g,f}(A_1,1;D)} G_{g,f}(\mu) < \infty, \tag{5.8}$$

then the following (auxiliary) minimum α-Green energy problem, which is actually [15, Problem 3.2], makes sense.

[11] It has been used here that $c_g(A_1) > 0$, which is clear from the permanent assumption (4.1) and footnote 9.

Problem 5.5. *Does there exist* $\lambda_{A_1,g} \in \mathcal{E}^\sigma_{g,f}(A_1,1;D)$ *with*

$$G_{g,f}(\lambda_{A_1,g}) = G^\sigma_{g,f}(A_1,1;D)?$$

Lemma 5.6. $G^\sigma_{g,f}(A_1,1;D) > -\infty$.

Proof. This is obvious in Case I since, by the strict positive definiteness of the kernel g,

$$G_{g,f}(\mu) = \|\mu\|^2_g + 2\langle f,\mu\rangle > 0 \quad \text{for every } \mu \in \mathcal{E}^\sigma_{g,f}(A_1,1;D).$$

If Case II takes place, then by (5.6) the equality $f = g\zeta$ holds n.e. on D, and hence μ-a.e. for any (c_α-absolutely continuous) measure $\mu \in \mathcal{E}^\sigma_{g,f}(A_1,1;D)$. Integrating $f = g\zeta$ with respect to μ gives $2\langle f,\mu\rangle = 2E_g(\zeta,\mu) = \|\mu+\zeta\|^2_g - \|\mu\|^2_g - \|\zeta\|^2_g$, and therefore

$$G_{g,f}(\mu) = \|\mu+\zeta\|^2_g - \|\zeta\|^2_g \geqslant -\|\zeta\|^2_g > -\infty. \tag{5.9}$$

This completes the proof by letting here μ range over $\mathcal{E}^\sigma_{g,f}(A_1,1;D)$. □

In all that follows we assume that

$$c_g(A_1^\circ) > 0, \ \textit{where } A_1^\circ := \bigl\{x \in A_1 : \ |f(x)| < \infty\bigr\}. \tag{5.10}$$

Note that in Case II this holds automatically, which is clear from $c_g(A_1) > 0$ (see footnote 11) and the fact that in Case II the external field is finite n.e. on $\mathbb{R}^n$.

Lemma 5.7. $G^\sigma_{g,f}(A_1,1;D)$ *is finite if either* $\sigma = \infty$, *or otherwise if the following two requirements hold:* $\sigma(A_1^\circ) > 1$ *and* $E_g(\sigma|_K) < \infty$ *for every compact* $K \subset A_1^\circ$.

Proof. As seen from Lemma 5.6, it is enough to show that under the stated assumptions $\mathcal{E}^\sigma_{g,f}(A_1,1;D)$ is nonempty. Write $E_k := \{x \in A_1 : \ |f(x)| \leqslant k\}$. As E_k, $k \in \mathbb{N}$, are universally measurable and $E_k \uparrow A_1^\circ$, we get from [18, Lemma 2.3.3]

$$c_g(A_1^\circ) = \lim_{k\to\infty} c_g(E_k). \tag{5.11}$$

Assume first that $\sigma = \infty$. Since $c_g(A_1^\circ) > 0$ by assumption, in view of (5.11) and (2.2) one can choose $k_0 \in \mathbb{N}$ and a compact set $K_0 \subset E_{k_0}$ with $c_g(K_0) > 0$, and hence a measure $\mu \in \mathcal{E}^+_g(K_0,1;D)$. Noting that $|\langle f,\mu\rangle| \leqslant k_0$, we actually have $\mu \in \mathcal{E}^\sigma_{g,f}(A_1,1;D)$.[12]

Consider next the case where $\sigma \in \mathfrak{C}(A_1)$. Then $\sigma(A_1^\circ) > 1$ by assumption, and hence there exist $k \in \mathbb{N}$ and a compact set $K \subset E_k$ such that $\sigma(K) > 1$. Since $E_g(\sigma|_K) < \infty$ and $|f| \leqslant k$ on K, the measure $\sigma|_K/\sigma(K)$ belongs to $\mathcal{E}^\sigma_{g,f}(A_1,1;D)$. □

Unless explicitly stated otherwise, in all that follows $G^\sigma_{g,f}(A_1,1;D)$ *is required to be finite.* Sufficient conditions for this to hold have been provided in Lemma 5.7.

Lemma 5.8 (see [15, Lemma 4.1])**.** *A solution* $\lambda_{A_1,g}$ *to Problem* 5.5 *is unique* (*if it exists*).

Remark 5.9. If $f = 0$ and $\sigma = \infty$, then Problem 5.5 reduces to problem (2.1) with $Q = A_1$ and $\kappa = g$. It therefore has the (unique) solution $\lambda_{A_1,g}$ if and only if condition (5.3) holds, and then (see Remark 3.7)

$$\lambda_{A_1,g} = \mu_{A_1,g} = \gamma_{A_1,g}/c_g(A_1), \tag{5.12}$$

where $\mu_{A_1,g}$, resp. $\gamma_{A_1,g}$, is the g-capacitary, resp. g-equilibrium, measure on A_1.

[12] These arguments can actually be reversed, which proves that in the unconstrained case ($\sigma = \infty$) the permanent assumption (5.10) is necessary and sufficient for the finiteness of $G^\sigma_{g,f}(A_1,1;D)$.

5.3. **A minimum weak α-Riesz energy problem for a condenser with separated plates.** Throughout Section 5.3, $\mathbf{A}$ is a (standard) condenser possessing the separation property (1.3). Denoting by $\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ the class of all $\nu\in\mathcal{E}_{\alpha}(\mathbf{A},\mathbf{1})$ such that $\nu^+\leqslant\sigma$ and f is $|\nu|$-integrable, we then obtain from Lemma 4.1

$$\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})=\dot{\mathcal{E}}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}). \tag{5.13}$$

(In general, $\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ is a *proper* subset of $\dot{\mathcal{E}}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$.) For any $\nu\in\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ write

$$G_{\alpha,f}(\nu):=\|\nu\|^2_{\alpha}+2\langle f,\nu\rangle=\|\nu\|^{\cdot 2}_{\alpha}+2\langle f,\nu\rangle=\dot{G}_{\alpha,f}(\nu), \tag{5.14}$$

the last two equalities being obtained from (1.2) and (5.1).

As will be shown at the beginning of the proof of Theorem 5.11, for any $\mu\in\mathcal{E}^{\sigma}_{g,f}(A_1,1;D)$ (which exists by the permanent assumption (5.8)) we have $\mu-\mu'\in\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$. Hence

$$G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}):=\inf_{\nu\in\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})}G_{\alpha,f}(\nu)<\infty,$$

and the following minimum standard α-Riesz energy problem, which is actually [15, Problem 3.1], therefore makes sense.

Problem 5.10. *Does there exist $\lambda_{\mathbf{A},\alpha}\in\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ with*

$$G_{\alpha,f}(\lambda_{\mathbf{A},\alpha})=G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})?$$

By (5.13) and (5.14), this problem is in fact the $\mathcal{H}$-problem with $\mathcal{H}=\dot{\mathcal{E}}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$, and hence a solution to Problem 5.10 is *unique* provided that it exists (see Lemma 5.1).

Theorem 5.11. *If the separation condition* (1.3) *holds, then in Cases* I *and* II

$$G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})=G^{\sigma}_{g,f}(A_1,1;D), \tag{5.15}$$

$\sigma\in\mathfrak{C}(A_1)\cup\{\infty\}$ *being arbitrary. Furthermore, Problem* 5.10 *is* (*uniquely*) *solvable if and only if Problem* 5.5 *is so, and then their solutions are related to one another as follows:*

$$\lambda_{\mathbf{A},\alpha}=\lambda_{A_1,g}-\lambda'_{A_1,g}. \tag{5.16}$$

Proof. Any measure $\mu\in\mathcal{E}^{\sigma}_{g,f}(A_1,1;D)$ has finite standard α-Riesz energy, which is clear from the separation condition (1.3) by Lemma 3.11, and hence so does the α-Riesz swept measure μ' of μ onto $A_2=D^c$. Furthermore, $\mu'(A_2)=1$ according to Theorem 3.5, and so $\nu:=\mu-\mu'\in\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$. We therefore get from (5.7) and (5.14)

$$G_{g,f}(\mu)=\|\mu\|^2_g+2\langle f,\mu\rangle=\|\mu-\mu'\|^2_{\alpha}+2\langle f,\mu-\mu'\rangle=G_{\alpha,f}(\nu)\geqslant G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}), \tag{5.17}$$

the second equality being valid by Lemma 3.11, (5.5), and the c_{α}-absolute continuity of μ' [19, Corollary 3.19]. By letting here μ vary over $\mathcal{E}^{\sigma}_{g,f}(A_1,1;D)$, we obtain

$$G^{\sigma}_{g,f}(A_1,1;D)\geqslant G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}). \tag{5.18}$$

Further, fix $\nu\in\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$. Then $\nu^+\in\mathcal{E}^{\sigma}_{g,f}(A_1,1;D)$, for $E_g(\nu^+)\leqslant E_{\alpha}(\nu^+)$. Thus

$$\begin{aligned}G_{\alpha,f}(\nu)=\|\nu\|^2_{\alpha}+2\langle f,\nu^+\rangle&\geqslant\|\nu^+-(\nu^+)'\|^2_{\alpha}+2\langle f,\nu^+\rangle\\&=\|\nu^+\|^2_g+2\langle f,\nu^+\rangle\geqslant G^{\sigma}_{g,f}(A_1,1;D),\end{aligned} \tag{5.19}$$

where the former equality holds by (5.5) and the c_{α}-absolute continuity of ν^-, the former inequality is obtained from (3.4), and the latter equality is valid according to Lemma 3.10. Letting here ν range over $\mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ and then combining the inequality obtained with (5.18), we arrive at (5.15).

Suppose now that $\lambda_{\mathbf{A},\alpha}$ solves Problem 5.10. Substituting $\lambda_{\mathbf{A},\alpha}$ in place of ν into (5.19) and then combining this with (5.15), we see that in fact equality prevails in either of the two inequalities. Problem 5.5 has therefore a solution $\lambda_{A_1,g}$; and moreover $\lambda^+_{\mathbf{A},\alpha} = \lambda_{A_1,g}$ and $\lambda^-_{\mathbf{A},\alpha} = (\lambda^+_{\mathbf{A},\alpha})' = \lambda'_{A_1,g}$, which establishes (5.16).

To complete the proof, assume finally that $\lambda_{A_1,g} \in \mathcal{E}^\sigma_{g,f}(A_1, 1; D)$ solves Problem 5.5. Substituting $\lambda_{A_1,g}$ in place of μ into (5.17) and then combining this with (5.15), we see that the inequality in the relation thus obtained is in fact equality. Problem 5.10 has therefore a solution $\lambda_{\mathbf{A},\alpha}$, and moreover $\lambda_{\mathbf{A},\alpha} = \lambda_{A_1,g} - \lambda'_{A_1,g}$. $\square$

In all that follows, when speaking of Problem 5.10 *we shall always tacitly assume the separation condition* (1.3) *to hold.* Note that in the case where $f = 0$ and $\sigma = \infty$, Problem 5.10 reduces to problem (5.2), solved by [20, Theorem 6.2] quoted in Remark 5.4 above.

As seen from (5.16), Theorem 5.11 remains in force if the class $\mathcal{E}^\sigma_{\alpha,f}(\mathbf{A}, \mathbf{1})$ of admissible measures in Problem 5.10 is replaced by that of $\mu - \mu'$, where μ ranges over $\mathcal{E}^\sigma_{g,f}(A_1, 1; D)$. We are thus led to the study of the $\widetilde{\mathcal{H}}$-problem, $\widetilde{\mathcal{H}}$ being defined by (5.20) below.

5.4. **A minimum weak α-Riesz energy problem for a generalized condenser.** Omitting now the separation condition (1.3), we shall further show that the equivalence of [15, Problem 3.1] and [15, Problem 3.2], established in Theorem 5.11 above, remains valid if the concept of standard α-Riesz energy in [15, Problem 3.1] is replaced by that of weak α-Riesz energy, applied now to the (new) admissible measures $\nu \in \widetilde{\mathcal{H}}$ defined as follows:

$$\widetilde{\mathcal{H}} := \bigl\{\nu = \mu - \mu' : \ \mu \in \mathcal{E}^\sigma_{g,f}(A_1, 1; D)\bigr\}. \tag{5.20}$$

In view of the permanent assumption (5.8) the class $\mathcal{E}^\sigma_{g,f}(A_1, 1; D)$ is nonempty, and hence so is $\widetilde{\mathcal{H}}$, which obviously forms a convex cone. Furthermore,

$$\widetilde{\mathcal{H}} \subset \dot{\mathcal{E}}^\sigma_{\alpha,f}(\mathbf{A}, \mathbf{1}),$$

which is seen from Theorems 3.5, 4.2 and the equality $\langle f, \mu'\rangle = 0$, the last being obtained from (5.5) in view of the c_α-absolute continuity of μ' [19, Corollary 3.19].

For any $\nu = \mu - \mu' \in \widetilde{\mathcal{H}}$ we therefore get from (5.1), (4.2), and (5.7)

$$\dot{G}_{\alpha,f}(\nu) = \|\nu\|^{\cdot 2}_\alpha + 2\langle f, \nu\rangle = \|\mu\|^2_g + 2\langle f, \mu\rangle = G_{g,f}(\mu),$$

and hence

$$\dot{G}_{\alpha,f}(\widetilde{\mathcal{H}}) := \inf_{\nu \in \widetilde{\mathcal{H}}} \dot{G}_{\alpha,f}(\nu) = G^\sigma_{g,f}(A_1, 1; D).$$

Consider the $\widetilde{\mathcal{H}}$*-problem on the existence of* $\lambda_{\widetilde{\mathcal{H}}} \in \widetilde{\mathcal{H}}$ *with* $\dot{G}_{\alpha,f}(\lambda_{\widetilde{\mathcal{H}}}) = \dot{G}_{\alpha,f}(\widetilde{\mathcal{H}})$. According to Lemma 5.1 with $\mathcal{H} = \widetilde{\mathcal{H}}$, a solution $\lambda_{\widetilde{\mathcal{H}}}$ is unique (if it exists).

Summarizing what we have thus observed, we arrive at the following conclusion.

Theorem 5.12. *In both Cases* I *and* II *for any* $\sigma \in \mathfrak{C}(A_1) \cup \{\infty\}$ *we have*

$$\dot{G}_{\alpha,f}(\widetilde{\mathcal{H}}) = G^\sigma_{g,f}(A_1, 1; D).$$

The $\widetilde{\mathcal{H}}$*-problem has the* (*unique*) *solution* $\lambda_{\widetilde{\mathcal{H}}}$ *if and only if there is the* (*unique*) *solution* $\lambda_{A_1,g}$ *to Problem* 5.5, *and in the affirmative case the following formula holds:*

$$\lambda_{\widetilde{\mathcal{H}}} = \lambda_{A_1,g} - \lambda'_{A_1,g}. \tag{5.21}$$

5.5. **The case where $f = 0$ and $\sigma = \infty$.** We shall need the following particular case of Theorems 5.11 and 5.12, which can also be obtained from [20, Theorems 6.1, 6.2] quoted in Remark 5.4 above.

Theorem 5.13. *Let $f = 0$ and $\sigma = \infty$ both hold, and let $\nu \in \widetilde{\mathcal{H}}$, resp. $\nu \in \mathcal{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$, be given. Then ν solves the $\widetilde{\mathcal{H}}$-problem, resp. Problem* 5.10, *if and only if assumption* (5.3) *holds, and in the affirmative case we have*

$$\nu = \mu_{A_1,g} - \mu'_{A_1,g} = (\gamma_{A_1,g} - \gamma'_{A_1,g})/c_g(A_1).$$

Proof. By Theorem 5.12, resp. Theorem 5.11, ν solves the $\widetilde{\mathcal{H}}$-problem, resp. Problem 5.10, if and only if ν^+ solves Problem 5.5, which in turn holds if and only if $c_g(A_1)$ is finite, and then $\nu^+ = \lambda_{A_1,g}$ is in fact the g-capacitary measure $\mu_{A_1,g}$ on A_1 (see Remark 5.9). Substituting (5.12) into (5.21), resp. (5.16), completes the proof. □

6. On the existence of minimizers

Let a generalized condenser $\mathbf{A} = (A_1, A_2)$, a constraint $\sigma \in \mathfrak{C}(A_1) \cup \{\infty\}$, and an external field f satisfy all the permanent assumptions indicated in Sections 4 and 5. Recall that when speaking of Problem 5.10, we always tacitly assume the separation condition (1.3) to hold.

Theorems 6.1, 6.6 and Corollary 6.7 below provide sufficient and/or necessary conditions for the solvability of the $\widetilde{\mathcal{H}}$-problem as well as Problem 5.10. See also Theorems 8.1, 8.4, 8.6, and 8.10 in Section 8 where such criteria are established in terms of variational inequalities for the α-Riesz potentials.

Theorem 6.1. *The $\widetilde{\mathcal{H}}$-problem as well as Problem* 5.10 *is* (*uniquely*) *solvable in both Cases* I *and* II *provided that $c_g(A_1) < \infty$ or $\sigma(A_1) < \infty$ holds.*

Proof. According to Theorems 5.11 and 5.12, it is enough to show that under the stated assumptions Problem 5.5 is solvable.[13]

A sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathcal{E}^{\sigma}_{g,f}(A_1, 1; D)$ is said to be *minimizing* in Problem 5.5 if

$$\lim_{k\to\infty} G_{g,f}(\mu_k) = G^{\sigma}_{g,f}(A_1, 1; D). \tag{6.1}$$

Let $\mathbb{M}^{\sigma}_{g,f}(A_1, 1; D)$ consist of all these $\{\mu_k\}_{k\in\mathbb{N}}$, which exist by the permanent assumption (5.8). We proceed the proof of Theorem 6.1 with the following auxiliary assertions.

Lemma 6.2. *For any $\{\mu_k\}_{k\in\mathbb{N}}$ and $\{\nu_k\}_{k\in\mathbb{N}}$ in $\mathbb{M}^{\sigma}_{g,f}(A_1, 1; D)$,*

$$\lim_{k\to\infty} \|\mu_k - \nu_k\|_g = 0. \tag{6.2}$$

Proof. Based on the convexity of the class $\mathcal{E}^{\sigma}_{g,f}(A_1, 1; D)$ and the pre-Hilbert structure on the space $\mathcal{E}_g(D)$, similarly as in the proof of Lemma 5.1 we obtain

$$0 \leqslant \|\mu_k - \nu_k\|^2_g \leqslant -4G^{\sigma}_{g,f}(A_1, 1; D) + 2G_{g,f}(\mu_k) + 2G_{g,f}(\nu_k).$$

Since $-\infty < G^{\sigma}_{g,f}(A_1, 1; D) < \infty$, we get (6.2) from (6.1) by letting here $k \to \infty$. □

Corollary 6.3. *Every $\{\mu_k\}_{k\in\mathbb{N}} \in \mathbb{M}^{\sigma}_{g,f}(A_1, 1; D)$ is a strong Cauchy sequence in $\mathcal{E}^+_g(D)$.*

[13] Sufficient conditions for the solvability of Problem 5.5, established in Theorem 6.1, are less demanding than those in [15] (cf. [15, Theorems 5.1, 5.2]), since our present proof is based on the perfectness of the α-Green kernel and the existence of the α-Green equilibrium measure [19, Theorem 4.11, 4.12].

Lemma 6.4. *The mapping $\mu \mapsto G_{g,f}(\mu)$ is vaguely l.s.c. on $\mathcal{E}^{\sigma}_{g,f}(A_1, 1; D)$ if Case* I *holds, and in Case* II *it is strongly continuous.*

Proof. Since the mapping $\mu \mapsto E_g(\mu)$ is vaguely l.s.c. on $\mathfrak{M}^+(D)$ according to [18, Lemma 2.2.1(e)] with $\kappa = g$, the former assertion follows from Lemma 2.1 with $\psi = f$. The latter assertion is obtained directly from the equality in (5.9). □

Let $\mathcal{E}^{\sigma}_{g,f}(A_1; D)$ consist of all $\nu \in \mathcal{E}^+_g(A_1; D)$ such that $\nu \leqslant \sigma$ and f is ν-integrable.

Lemma 6.5. *There is a unique measure $\theta \in \mathcal{E}^{\sigma}_{g,f}(A_1; D)$ such that every minimizing sequence $\{\mu_k\}_{k\in\mathbb{N}} \in \mathbb{M}^{\sigma}_{g,f}(A_1, 1; D)$ converges to θ both strongly and vaguely in $\mathcal{E}^+_g(D)$.*

Proof. Fix $\{\mu_k\}_{k\in\mathbb{N}} \in \mathbb{M}^{\sigma}_{g,f}(A_1, 1; D)$. Being bounded in the vague topology on $\mathfrak{M}^+(D)$, $\{\mu_k\}_{k\in\mathbb{N}}$ has a vague cluster point θ [7, Chapter III, Section 2, Proposition 9], and hence there is a subsequence $\{\mu_{k_j}\}_{j\in\mathbb{N}}$ of $\{\mu_k\}_{k\in\mathbb{N}}$ converging vaguely to θ. Thus $\mu_{k_j} \otimes \mu_{k_j} \to \theta \otimes \theta$ vaguely in $\mathfrak{M}^+(D \times D)$ [7, Chapter III, Section 5, Exercise 5]. Being strong Cauchy in $\mathcal{E}^+_g(D)$ according to Corollary 6.3, $\{\mu_k\}_{k\in\mathbb{N}}$ is strongly bounded. Applying Lemma 2.1 to $X = D \times D$ and $\psi = g$, we therefore get $\theta \in \mathcal{E}^+_g(D)$. Since A_1 is (relatively) closed in D, θ is carried by A_1. Also note that $\theta \leqslant \sigma$, for the vague limit of a sequence of positive measures likewise is positive.

By the perfectness of the α-Green kernel g, $\mu_k \to \theta$ strongly in $\mathcal{E}^+_g(D)$ when $k \to \infty$ (Definition 2.2). As g is strictly positive definite, θ must be a unique (strong and) vague cluster point of $\{\mu_k\}_{k\in\mathbb{N}}$. Since the vague topology is Hausdorff, a (unique) vague cluster point θ of $\{\mu_k\}_{k\in\mathbb{N}}$ must be its vague limit [6, Chapter I, Section 9, n° 1, Corollary].

If $\{\nu_k\}_{k\in\mathbb{N}}$ is another element of $\mathbb{M}^{\sigma}_{g,f}(A_1, 1; D)$, then $\nu_k \to \theta$ strongly in $\mathcal{E}^+_g(D)$, which is clear from the strong convergence of $\{\mu_k\}_{k\in\mathbb{N}}$ to θ and (6.2). Since the strong topology on $\mathcal{E}^+_g(D)$ is finer than the vague topology (Theorem 2.4), θ is also the vague limit of $\{\nu_k\}_{k\in\mathbb{N}}$.

To complete the proof, it remains to show that $\langle f, \theta\rangle < \infty$, or equivalently $G_{g,f}(\theta) < \infty$. As $\mu_k \to \theta$ strongly and vaguely in $\mathcal{E}^+_g(D)$, it follows from Lemma 6.4 and (6.1) that

$$G_{g,f}(\theta) \leqslant \lim_{k\to\infty} G_{g,f}(\mu_k) = G^{\sigma}_{g,f}(A_1, 1; D).$$

Since $G^{\sigma}_{g,f}(A_1, 1; D)$ is finite by the permanent condition (5.8), the lemma follows. □

We are now ready to complete the proof of Theorem 6.1. As seen from the last display, this will be done once we have proven that if $c_g(A_1) < \infty$ or $\sigma(A_1) < \infty$ holds, then actually $\theta \in \mathcal{E}^{\sigma}_{g,f}(A_1, 1; D)$, or equivalently

$$\theta(A_1) = 1, \tag{6.3}$$

θ being determined uniquely by Lemma 6.5.

To this end consider an exhaustion of A_1 by an increasing sequence $\{K_j\}_{j\in\mathbb{N}}$ of compact sets, and fix $\{\mu_k\}_{k\in\mathbb{N}} \in \mathbb{M}^{\sigma}_{g,f}(A_1, 1; D)$. By Lemma 6.5, then $\mu_k \to \theta$ both strongly and vaguely in $\mathcal{E}^+_g(D)$. Since 1_{K_j} is upper semicontinuous (and bounded), while 1_D is (finitely) continuous on D, we obtain from Lemma 2.1 with $X = D$, applied subsequently to 1_D and -1_{K_j},

$$\begin{aligned} 1 = \lim_{k\to\infty} \mu_k(A_1) \geqslant \theta(A_1) = \lim_{j\to\infty} \theta(K_j) &\geqslant \lim_{j\to\infty} \limsup_{k\to\infty} \mu_k(K_j) \\ &= 1 - \lim_{j\to\infty} \liminf_{k\to\infty} \mu_k(A_1 \setminus K_j). \end{aligned}$$

Equality (6.3) will therefore follow if we prove the relation

$$\lim_{j\to\infty}\ \liminf_{k\to\infty}\ \mu_k(A_1\setminus K_j)=0. \tag{6.4}$$

Assume first that the constraint $\sigma=\xi\in\mathfrak{C}(A_1)$ is bounded. Since

$$\infty>\xi(A_1)=\lim_{j\to\infty}\ \xi(K_j),$$

it follows that

$$\lim_{j\to\infty}\ \xi(A_1\setminus K_j)=0.$$

When combined with $\mu_k(A_1\setminus K_j)\leqslant\xi(A_1\setminus K_j)$ for any $k,j\in\mathbb{N}$, this implies (6.4).

Assume finally that $c_g(A_1)<\infty$. For any $F\subset A_1$, relatively closed in D, then there exists the (unique) α-Green equilibrium measure γ_F on F, see Theorem 3.6.

Write $K_j^*:=\mathrm{Cl}_D(A_1\setminus K_j)$, $j\in\mathbb{N}$. By the monotonicity of K_j^* (as j ranges over $\mathbb{N}$) and (3.9) for $F=K_j^*$, the measure $\gamma_j:=\gamma_{K_j^*}$ belongs to $\Gamma_p:=\Gamma_{K_p^*}$ for all $p\geqslant j$, Γ_F being defined in Theorem 3.6, while γ_p solves problem (3.10) with $F=K_p^*$. Applying [18, Lemma 4.1.1] to the convex class Γ_p, we therefore obtain

$$\|\gamma_j-\gamma_p\|_g^2\leqslant\|\gamma_j\|_g^2-\|\gamma_p\|_g^2\quad\text{for all } p\geqslant j.$$

Furthermore, it is clear from (3.10) with $F=K_j^*$ that the sequence $\bigl\{\|\gamma_j\|_g^2\bigr\}_{j\in\mathbb{N}}$ is bounded and decreasing, and hence it is Cauchy in $\mathbb{R}$. The preceding inequality thus implies that $\{\gamma_j\}_{j\in\mathbb{N}}$ is strong Cauchy in $\mathcal{E}_g^+(D)$. Since it obviously converges vaguely to zero in $\mathfrak{M}^+(D)$, zero is also its strong limit because of the perfectness of the kernel g. Hence,

$$\lim_{j\to\infty}\ \|\gamma_j\|_g=0.$$

Integrating (3.9) with $F=K_j^*$ with respect to the measure μ_k (which is c_α-absolutely continuous, see the beginning of Section 5.2) and then applying the Cauchy–Schwarz (Bunyakovski) inequality in the pre-Hilbert space $\mathcal{E}_g(D)$, we get

$$\mu_k(A_1\setminus K_j)\leqslant\mu_k(K_j^*)=E_g(\gamma_j,\mu_k)\leqslant\|\gamma_j\|_g\|\mu_k\|_g\quad\text{for all } k,j\in\mathbb{N}.$$

As $\{\|\mu_k\|_g\}_{k\in\mathbb{N}}$ is bounded, combining the last two relations again results in (6.4), thus completing the proof of Theorem 6.1. □

As seen from the following Theorem 6.6, the sufficient conditions on the solvability, established in Theorem 6.1, are actually *sharp*.

Theorem 6.6. *Suppose Case* II *with $\zeta\geqslant0$ takes place. If moreover $c_g(A_1)=\infty$, then the $\widetilde{\mathcal{H}}$-problem as well as Problem* 5.10 *is unsolvable for every $\sigma\in\mathfrak{C}(A_1)\cup\{\infty\}$ such that $\sigma\geqslant\xi_0$, where $\xi_0\in\mathfrak{C}(A_1)$ with $\xi_0(A_1)=\infty$ is chosen properly.*

Proof. According to Theorems 5.11 and 5.12, it is enough to show that under the stated assumptions Problem 5.5 is unsolvable. Since Case II with $\zeta\geqslant0$ takes place,

$$G_{g,f}(\nu)=\|\nu\|_g^2+2E_g(\zeta,\nu)\geqslant\|\nu\|_g^2\geqslant0\quad\text{for all } \nu\in\mathcal{E}_g^+(D). \tag{6.5}$$

Consider an exhaustion of D by an increasing sequence $\{K_j\}_{j\in\mathbb{N}}$ of compact sets, and write $A_1^j:=A_1\cap K_j$. As $c_g(A_1^j)<\infty$ for every $j\in\mathbb{N}$, while $c_g(A_1)=\infty$, it follows from the subadditivity of $c_g(\cdot)$ on universally measurable sets [18, Lemma 2.3.5] that $c_g(A_1\setminus A_1^j)=\infty$. Hence, for every $j\in\mathbb{N}$ there is $\nu_j\in\mathcal{E}_g^+(A_1\setminus A_1^j,1;D)$ of compact support $S_D^{\nu_j}$ such that

$$\|\nu_j\|_g\leqslant1/j. \tag{6.6}$$

Clearly, the K_j can be chosen successively so that $A_1^j \cup S_D^{\nu_j} \subset A_1^{j+1}$. Any compact set $K \subset D$ is contained in a certain K_j with j large enough, and hence K has points in common with only finitely many $S_D^{\nu_j}$. Therefore ξ_0 defined by the relation

$$\xi_0(\varphi) := \sum_{j\in\mathbb{N}} \nu_j(\varphi) \quad \text{for any } \varphi \in C_0(D)$$

is a positive Radon measure on D carried by A_1. Furthermore, $\xi_0(A_1) = \infty$. For each $\sigma \in \mathfrak{C}(A_1) \cup \{\infty\}$ such that $\sigma \geqslant \xi_0$ we thus have

$$\nu_j \in \mathcal{E}^\sigma_{g,f}(A_1, 1; D) \quad \text{for all } j \in \mathbb{N}.$$

Therefore, by (6.6) and the Cauchy–Schwarz inequality,

$$\lim_{j\to\infty} G_{g,f}(\nu_j) = \lim_{j\to\infty} \bigl[\|\nu_j\|_g^2 + 2E_g(\zeta, \nu_j)\bigr] \leqslant \lim_{j\to\infty} \bigl[\|\nu_j\|_g^2 + 2\|\zeta\|_g\|\nu_j\|_g\bigr] = 0.$$

Combined with (6.5), this yields $G^\sigma_{g,f}(A_1, 1; D) = 0$. Relation (6.5) also implies that $G^\sigma_{g,f}(A_1, 1; D)$ can be attained only at zero measure, for the kernel g is strictly positive definite. Since $0 \notin \mathcal{E}^\sigma_{g,f}(A_1, 1; D)$, Problem 5.5 with the constraint σ specified above is unsolvable. $\square$

Combining Theorems 6.1 and 6.6 leads to the following assertion.

Corollary 6.7. *Let Case* II *with* $\zeta \geqslant 0$ *take place. Then the* $\widetilde{\mathcal{H}}$*-problem as well as Problem* 5.10 *is solvable for every* $\sigma \in \mathfrak{C}(A_1) \cup \{\infty\}$ *if and only if* $c_g(A_1)$ *is finite.*

Remark 6.8. Theorem 6.6 and Corollary 6.7 remain valid in Case I provided that $f(x) \to 0$ as $x \to \omega_D$, where ω_D is the Alexandroff point of the locally compact space D.

7. Description of the supports of the minimizers

Let $\nu \in \widetilde{\mathcal{H}}$, resp. $\nu \in \mathcal{E}^\sigma_{\alpha,f}(\mathbf{A}, \mathbf{1})$, where $\sigma \in \mathfrak{C}(A_1) \cup \{\infty\}$, solve the $\widetilde{\mathcal{H}}$-problem, resp. Problem 5.10. According to Theorem 5.12, resp. Theorem 5.11, there exists the (unique) solution $\lambda_{A_1,g}$ to Problem 5.5, and moreover

$$\nu^- = (\nu^+)' = \lambda'_{A_1,g}. \tag{7.1}$$

The following Theorem 7.1 describes $S^{\nu^-}_{\mathbb{R}^n}$, while a description of $S^{\nu^+}_D$ will be provided in Theorems 8.4 and 8.11 below.

Let $\breve{A}_2$ denote the κ_α*-reduced kernel* of A_2 [23, p. 164], i.e. the set of all $x \in A_2$ such that $c_\alpha\bigl(B(x,r)\cap A_2\bigr) > 0$ for any $r > 0$, where $B(x,r) := \bigl\{y \in \mathbb{R}^n : |y-x| < r\bigr\}$.

For the sake of simplicity of formulation, in the following assertion we assume that in the case $\alpha = 2$ the domain D is simply connected.

Theorem 7.1. $S^{\nu^-}_{\mathbb{R}^n}$ *is described by*

$$S^{\nu^-}_{\mathbb{R}^n} = \begin{cases} \breve{A}_2 & \textit{if} \quad \alpha < 2, \\ \partial D & \textit{if} \quad \alpha = 2. \end{cases} \tag{7.2}$$

Proof. For any $x \in D$ let K_x denote the inverse of $\mathrm{Cl}_{\overline{\mathbb{R}^n}} A_2$ relative to $S(x, 1)$. Since K_x is compact, there is the (unique) κ_α-equilibrium measure $\gamma_x \in \mathcal{E}^+_\alpha(K_x; \mathbb{R}^n)$ on K_x with the properties $\|\gamma_x\|^2_\alpha = \gamma_x(K_x) = c_\alpha(K_x)$,

$$\kappa_\alpha\gamma_x = 1 \quad \text{n.e. on } K_x, \tag{7.3}$$

and $\kappa_\alpha\gamma_x \leqslant 1$ on $\mathbb{R}^n$. Note that $\gamma_x \neq 0$, for $c_\alpha(K_x) > 0$ in consequence of $c_\alpha(A_2) > 0$, see [23, Chapter IV, Section 5, n° 19]. We assert that under the stated requirements

$$S^{\gamma_x}_{\mathbb{R}^n} = \begin{cases} \breve{K}_x & \text{if} \quad \alpha < 2, \\ \partial K_x & \text{if} \quad \alpha = 2. \end{cases} \tag{7.4}$$

The latter identity in (7.4) follows from [23, Chapter II, Section 3, n° 13]. To establish the former identity,[14] we first note that $S^{\gamma_x}_{\mathbb{R}^n} \subset \breve{K}_x$ by the c_α-absolute continuity of γ_x. As for the converse inclusion, assume on the contrary that there is $x_0 \in \breve{K}_x$ such that $x_0 \notin S^{\gamma_x}_{\mathbb{R}^n}$. Choose $r > 0$ with the property $\overline{B}(x_0, r) \cap S^{\gamma_x}_{\mathbb{R}^n} = \varnothing$, where $\overline{B}(x_0, r) := \bigl\{y \in \mathbb{R}^n : |y - x_0| \leqslant r\bigr\}$. But $c_\alpha\bigl(B(x_0, r) \cap \breve{K}_x\bigr) > 0$, hence by (7.3) there is $y \in B(x_0, r)$ such that $\kappa_\alpha\gamma_x(y) = 1$. The function $\kappa_\alpha\gamma_x$ is α-harmonic on $B(x_0, r)$ [23, Chapter I, Section 5, n° 20], continuous on $\overline{B}(x_0, r)$, and takes at $y \in B(x_0, r)$ the maximum value 1. Applying [23, Theorem 1.28] we see that $\kappa_\alpha\gamma_x = 1$ holds m-a.e. on $\mathbb{R}^n$, hence everywhere on $(\breve{K}_x)^c$ by the continuity of $\kappa_\alpha\gamma_x$ on $\bigl(S^{\gamma_x}_{\mathbb{R}^n}\bigr)^c$ $\bigl[\supset (\breve{K}_x)^c\bigr]$, and altogether n.e. on $\mathbb{R}^n$ by (7.3). This means that γ_x serves as the α-Riesz equilibrium measure on the whole of $\mathbb{R}^n$, which is impossible.

Based on (7.1), (7.4) and the integral representation (3.3), we then arrive at (7.2) with the aid of the fact that for every $x \in D$, ε'_x is the Kelvin transform of the equilibrium measure γ_x, see [19, Section 3.3]. □

8. Description of the potentials of the minimizers

Let a generalized condenser $\mathbf{A} = (A_1, A_2)$, a constraint $\sigma \in \mathfrak{C}(A_1) \cup \{\infty\}$, and an external field f be as indicated in Sections 4 and 5. For any $\nu \in \dot{\mathcal{E}}^\sigma_{\alpha,f}(\mathbf{A}, \mathbf{1})$ define $W^\nu_{\alpha,f} := \kappa_\alpha\nu + f$; $W^\nu_{\alpha,f}$ is termed the *f-weighted α-Riesz potential* of ν. Since $\kappa_\alpha\nu$ is finite n.e. on $\mathbb{R}^n$ by the general convention (3.1), $W^\nu_{\alpha,f}$ is well defined n.e. on $\mathbb{R}^n$ and finite n.e. on A_1° (see (5.10)). Furthermore, $W^\nu_{\alpha,f}$ is $|\mu|$-measurable for every $\mu \in \mathfrak{M}(\mathbb{R}^n)$.

The purpose of this section is to establish sufficient and/or necessary conditions for the solvability of the $\widetilde{\mathcal{H}}$-problem as well as Problem 5.10 in terms of variational inequalities for the f-weighted α-Riesz potential.

8.1. Variational inequalities in the constrained minimum α-Riesz weak energy problems.

Consider first the $\widetilde{\mathcal{H}}$-problem as well as Problem 5.10 in the constrained setting ($\sigma \neq \infty$). Omitting now the requirement $G^\sigma_{g,f}(A_1, 1; D) < \infty$, for a given $\xi \in \mathfrak{C}(A_1)$ we assume instead that $E_g(\xi|_K) < \infty$ for every compact $K \subset A_1^\circ$. Also assume that

$$\xi(A_1 \setminus A_1^\circ) = 0, \tag{8.1}$$

which yields $\xi(A_1^\circ) > 1$. In view of Lemma 5.7, $\mathcal{E}^\xi_{g,f}(A_1, 1; D)$ is then nonempty, and hence the auxiliary Problem 5.5 makes sense. According to Theorem 5.12, resp. Theorem 5.11, so does the $\widetilde{\mathcal{H}}$-problem, resp. Problem 5.10.

Theorem 8.1. *Let f be lower bounded, and let $\nu \in \widetilde{\mathcal{H}}$, resp. $\nu \in \mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1})$, be given. Then ν solves the $\widetilde{\mathcal{H}}$-problem, resp. Problem* 5.10, *if and only it satisfies the*

[14] We have brought here this proof, since we did not find a reference for this possibly known assertion.

following three relations

$$\nu^- = (\nu^+)', \tag{8.2}$$

$$W^{\nu}_{\alpha,f} \geqslant w \quad (\xi - \nu^+)\text{-a.e.}, \tag{8.3}$$

$$W^{\nu}_{\alpha,f} \leqslant w \quad \nu^+\text{-a.e.} \tag{8.4}$$

with some $w \in \mathbb{R}$.[15]

Proof. We begin by establishing Theorem 8.3 below, related to the (auxiliary) Problem 5.5. When investigating Problem 5.5 we shall need the following assertion.

Lemma 8.2 (see [15, Lemma 4.3]). $\lambda \in \mathcal{E}^{\xi}_{g,f}(A_1, 1; D)$ *solves Problem* 5.5 *if and only if*

$$\langle W^{\lambda}_{g,f}, \mu - \lambda \rangle \geqslant 0 \quad \text{for all} \quad \mu \in \mathcal{E}^{\xi}_{g,f}(A_1, 1; D),$$

where it is denoted $W^{\lambda}_{g,f} := g\lambda + f|_D$.

Theorem 8.3. *Let* f *be lower bounded. A measure* $\lambda \in \mathcal{E}^{\xi}_{g,f}(A_1, 1; D)$ *is the* (*unique*) *solution to Problem* 5.5 *if and only if there is* $w \in \mathbb{R}$ *such that*

$$W^{\lambda}_{g,f} \geqslant w \quad (\xi - \lambda)\text{-a.e.}, \tag{8.5}$$

$$W^{\lambda}_{g,f} \leqslant w \quad \lambda\text{-a.e.} \tag{8.6}$$

Proof. We permanently use the fact that both ξ and λ are c_g-absolutely continuous, for they are of finite α-Green energy if restricted to any compact $K \subset A_1^{\circ}$. For any $c \in \mathbb{R}$ write

$$A_1^+(c) := \{x \in A_1 : \ W^{\lambda}_{g,f}(x) > c\} \quad \text{and} \quad A_1^-(c) := \{x \in A_1 : \ W^{\lambda}_{g,f}(x) < c\}.$$

Suppose first that λ solves Problem 5.5. Inequality (8.5) is valid with $w = L$, where

$$L := \sup\bigl\{q \in \mathbb{R} : \ W^{\lambda}_{g,f} \geqslant q \quad (\xi - \lambda)\text{-a.e.}\bigr\}.$$

In turn, (8.5) with $w = L$ implies that $L < \infty$ because $W^{\lambda}_{g,f} < \infty$ holds n.e. on A_1° and hence $(\xi-\lambda)$-a.e. on A_1°, while $(\xi-\lambda)(A_1^{\circ}) > 0$ by (8.1). Also note that $L > -\infty$, for $W^{\lambda}_{g,f}$ is lower bounded on A_1 by assumption.

We next proceed by establishing (8.6) with $w = L$. Assume, on the contrary, that this fails, i.e. $\lambda(A_1^+(L)) > 0$. Since $W^{\lambda}_{g,f}$ is λ-measurable, one can choose $c_1 \in (L, \infty)$ so that $\lambda(A_1^+(c_1)) > 0$. At the same time, as $c_1 > L$, (8.5) with $w = L$ yields $(\xi - \lambda)(A_1^-(c_1)) > 0$. Therefore, there exist compact sets $K_1 \subset A_1^+(c_1)$ and $K_2 \subset A_1^-(c_1)$ such that

$$0 < \lambda(K_1) < (\xi - \lambda)(K_2).$$

Write $\tau := (\xi - \lambda)|_{K_2}$; then $E_g(\tau) < \infty$. Since $\langle W^{\lambda}_{g,f}, \tau\rangle \leqslant c_1\tau(K_2) < \infty$, we thus get $\langle f, \tau\rangle < \infty$. Define $\theta := \lambda - \lambda|_{K_1} + b\tau$, where $b := \lambda(K_1)/\tau(K_2) \in (0, 1)$ by the last display. Straightforward verification then shows that $\theta(A_1) = 1$ and $\theta \leqslant \xi$, and hence $\theta \in \mathcal{E}^{\xi}_{g,f}(A_1, 1; D)$. On the other hand,

$$\begin{aligned}\langle W^{\lambda}_{g,f}, \theta - \lambda\rangle &= \langle W^{\lambda}_{g,f} - c_1, \theta - \lambda\rangle \\ &= -\langle W^{\lambda}_{g,f} - c_1, \lambda|_{K_1}\rangle + b\langle W^{\lambda}_{g,f} - c_1, \tau\rangle < 0,\end{aligned}$$

which is impossible by Lemma 8.2 applied to λ and $\mu = \theta$. This establishes (8.6).

[15] The lower boundedness of f holds automatically whenever Case I takes place. Furthermore, in Case I (8.4) can be rewritten equivalently in the following apparently stronger form: $W^{\nu}_{\alpha,f} \leqslant w$ on $S^{\nu^+}_D$. Also note that when speaking of the $\widetilde{\mathcal{H}}$-problem, (8.2) holds automatically, see (5.20).

Conversely, let (8.5) and (8.6) both hold with some $w \in \mathbb{R}$. Then $\lambda(A_1^+(w)) = 0$ and $(\xi - \lambda)(A_1^-(w)) = 0$. For any $\nu \in \mathcal{E}^\xi_{g,f}(A_1, 1; D)$ we therefore obtain

$$\begin{aligned}\langle W^\lambda_{g,f}, \nu - \lambda\rangle &= \langle W^\lambda_{g,f} - w, \nu - \lambda\rangle \\ &= \langle W^\lambda_{g,f} - w, \nu|_{A_1^+(w)}\rangle + \langle W^\lambda_{g,f} - w, (\nu - \xi)|_{A_1^-(w)}\rangle \geqslant 0.\end{aligned}$$

Application of Lemma 8.2 shows that, indeed, λ is the solution to Problem 5.5. □

To complete the proof of Theorem 8.1, fix $\nu \in \widetilde{\mathcal{H}}$, resp. $\nu \in \mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1})$. Note that

(8.7) $$\nu^+ \in \mathcal{E}^\xi_{g,f}(A_1, 1; D),$$

which follows from (5.20) if $\nu \in \widetilde{\mathcal{H}}$, and otherwise it is obvious by $E_g(\nu^+) \leqslant E_\alpha(\nu^+)$.

Assume first that the given ν is the (unique) solution to the $\widetilde{\mathcal{H}}$-problem, resp. Problem 5.10. According to Theorem 5.12, resp. Theorem 5.11, then (8.2) holds with $\nu^+ = \lambda$, where λ is the (unique) solution to Problem 5.5. Therefore by Lemma 3.9

$$W^\nu_{\alpha,f} = g\lambda + f|_D = W^\lambda_{g,f} \quad \text{n.e. on } D.$$

Combined with (8.5) and (8.6), this leads to (8.3) and (8.4) (with the same w as in (8.5) and (8.6)). Here we have used the c_α-absolute continuity of λ and ξ, see footnote 9.

Conversely, let for the given ν all the relations (8.2), (8.3), and (8.4) hold true. By Lemma 3.9, then $W^\nu_{\alpha,f} = W^{\nu^+}_{g,f}$ n.e. on D, where ν^+ belongs to $\mathcal{E}^\xi_{g,f}(A_1, 1; D)$ according to relation (8.7). In view of the c_α-absolute continuity of ν^+ and ξ, we thus see from (8.3) and (8.4) that $W^{\nu^+}_{g,f}$ satisfies (8.5) and (8.6) (with the same w as in (8.3) and (8.4)), which by Theorem 8.3 implies that ν^+ is the solution λ to Problem 5.5. Substituting $\nu^+ = \lambda$ into (8.2) and then applying Theorem 5.12, resp. Theorem 5.11, we infer that ν is the solution to the $\widetilde{\mathcal{H}}$-problem, resp. Problem 5.10, thus completing the proof of Theorem 8.1. □

Theorem 8.4. *Let $f = 0$, and assume that $\nu \in \widetilde{\mathcal{H}}$, resp. $\nu \in \mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1})$, solves the $\widetilde{\mathcal{H}}$-problem, resp. Problem* 5.10. *Then, and only then, this ν satisfies* (8.2) *as well as the following two relations*

(8.8) $$\kappa_\alpha\nu = w \quad (\xi - \nu^+)\text{-a.e.},$$

(8.9) $$\kappa_\alpha\nu \leqslant w \quad \textit{on } \mathbb{R}^n$$

with some $w \in (0, \infty)$. If moreover $\alpha < 2$ and $m(D^c) > 0$, it is also necessary that

(8.10) $$S^{\nu^+}_D = S^\xi_D.$$

Proof. As seen from Theorem 8.1, the former part of the theorem will be established once we have shown that the number w from (8.3) and (8.4) is now > 0, while the quoted relations can be rewritten as (8.8) and (8.9). We obtain from (8.2) in view of Lemma 3.9

(8.11) $$W^\nu_{\alpha,f} = \kappa_\alpha\nu = g\nu^+ \quad \text{on } D.$$

Substituting this into (8.4) gives $w \in (0, \infty)$, while (8.4) itself now takes the form

$$\kappa_\alpha\nu^+ \leqslant w + \kappa_\alpha\nu^- \quad \nu^+\text{-a.e.}$$

Consider an exhaustion of D by an increasing sequence $\{K_j\}_{j\in\mathbb{N}}$ of compact sets, and denote $\nu^+_j := \nu^+|_{K_j}$. The last display then remains valid with ν^+ replaced by ν^+_j, i.e.

$$\kappa_\alpha\nu^+_j \leqslant w + \kappa_\alpha\nu^- \quad \nu^+_j\text{-a.e.}$$

Since $\nu_j^+ \in \mathcal{E}_\alpha^+(K_j;\mathbb{R}^n)$ and $w > 0$, the former being clear from $E_g(\nu_j^+) \leqslant E_g(\nu^+) < \infty$ and Lemma 3.11, application of [23, Theorems 1.27, 1.29] shows that the preceding display holds in fact everywhere on $\mathbb{R}^n$. As $\kappa_\alpha\nu_j^+ \uparrow \kappa_\alpha\nu^+$ pointwise on $\mathbb{R}^n$, letting here $j \to \infty$ results in (8.9). Combining (8.9) and (8.3) establishes (8.8).

Assuming now that the hypotheses of the latter part of the theorem be fulfilled, we proceed by establishing (8.10), which according to Theorem 5.12, resp. Theorem 5.11, can be rewritten in the form $S_D^\lambda = S_D^\xi$, where λ is the solution (which exists) to Problem 5.5. On the contrary, let there be $x_0 \in S_D^\xi$ such that $x_0 \notin S_D^\lambda$. Then one can choose $r > 0$ so that $\overline{B}(x_0,r) \subset D$ and $\overline{B}(x_0,r) \cap S_D^\lambda = \varnothing$. It follows that $(\xi-\lambda)(B(x_0,r)\cap S_D^\xi) > 0$. By (8.8), there is therefore $x_1 \in B(x_0,r)\cap S_D^\xi$ possessing the property

$$\kappa_\alpha\lambda(x_1) = w + \kappa_\alpha\lambda'(x_1). \tag{8.12}$$

As $\kappa_\alpha\lambda$ is α-harmonic on $B(x_0,r)$ and continuous on $\overline{B}(x_0,r)$, while $w+\kappa_\alpha\lambda'$ is α-superharmonic on $\mathbb{R}^n$, we see from (8.9) and (8.12) with the aid of [23, Theorem 1.28] that

$$\kappa_\alpha\lambda = w + \kappa_\alpha\lambda' \quad m\text{-a.e. on } \mathbb{R}^n.$$

This implies $w = 0$, for $\kappa_\alpha\lambda = \kappa_\alpha\lambda'$ holds n.e. on D^c, and hence m-a.e. on D^c. A contradiction. $\square$

Remark 8.5. Let $f = 0$, and let ν solve the $\widetilde{\mathcal{H}}$-problem, resp. Problem 5.10, with a *bounded* constraint ξ. Combining (8.8) and (8.11) shows that $g\nu^+ = w$ holds $(\xi-\nu^+)$-a.e. Integrating this equality with respect to the (bounded positive) measure $\xi-\nu^+$ implies that the number $w \in (0,\infty)$ from Theorem 8.4 can be written in the form

$$w = \frac{E_g(\nu^+,\xi-\nu^+)}{(\xi-\nu^+)(A_1)}.$$

Thus $E_g(\nu^+,\xi-\nu^+) < \infty$, though in general $\xi \notin \mathcal{E}_g^+(D)$.

8.2. **Variational inequalities in the unconstrained minimum α-Riesz weak energy problems.** In this section we shall consider the unconstrained case ($\sigma = \infty$). The results obtained then take a simpler form if compared with those in the constrained case, while they provide us with much more detailed information about the potentials and the supports of the minimizers. When $\sigma = \infty$ serves as a superscript, we shall omit it in the notations.

Theorem 8.6. *Let $\nu \in \widetilde{\mathcal{H}}$, resp. $\nu \in \mathcal{E}_{\alpha,f}(\mathbf{A},\mathbf{1})$, be given. Then ν solves the $\widetilde{\mathcal{H}}$-problem, resp. Problem* 5.10, *if and only if it satisfies* (8.2) *as well as the following two relations*

$$W_{\alpha,f}^\nu \geqslant w' \quad \text{n.e. on } A_1, \tag{8.13}$$

$$W_{\alpha,f}^\nu = w' \quad \nu^+\text{-a.e.,} \tag{8.14}$$

where $w' \in \mathbb{R}$.

Proof. We first establish the following theorem, related to Problem 5.5 (with $\sigma = \infty$).

Theorem 8.7. *A measure $\lambda \in \mathcal{E}^+_{g,f}(A_1, 1; D)$ is the* (*unique*) *solution to Problem* 5.5 *if and only if there is $w' \in \mathbb{R}$ such that*[16]

$$W^\lambda_{g,f} \geqslant w' \quad \text{n.e. on } A_1, \tag{8.15}$$

$$W^\lambda_{g,f} = w' \quad \lambda\text{-a.e.} \tag{8.16}$$

Proof. This theorem is in fact a very particular case of [33, Theorems 7.1, 7.3] (see also Theorem 1 and Proposition 1 in the earlier paper [30]). □

Theorem 8.6 can now be obtained from Theorem 8.7 with the aid of Theorems 5.11 and 5.12 in the same manner as Theorem 8.1 has been derived from Theorem 8.3. □

Remark 8.8. The number w' from Theorems 8.6 and 8.7 is unique (whenever it exists) and can be written in the form

$$w' = \langle W^\nu_{\alpha,f}, \nu^+\rangle = \langle W^\lambda_{g,f}, \lambda\rangle,$$

ν and λ being as indicated in Theorems 8.6 and 8.7, respectively.

Definition 8.9. For a generalized condenser $\mathbf{A}$ in $\mathbb{R}^n$, $\theta = \theta_{\mathbf{A},\alpha} \in \mathfrak{M}(\mathbf{A})$ is said to be a *condenser measure* if $\kappa_\alpha\theta$ takes the value 1 and 0 n.e. on A_1 and A_2, respectively, and

$$0 \leqslant \kappa_\alpha\theta \leqslant 1 \quad \text{n.e. on } \mathbb{R}^n.$$

As seen from the above definition and [23, p. 178, Remark], a condenser measure is *unique* provided that it is c_α-absolutely continuous.

If A_1 and A_2 are compact disjoint sets, then the existence of a condenser measure was established by Kishi [22], actually even in the general setting of a function kernel on a locally compact Hausdorff space. See also [13], [23], [4], [3] where the existence of condenser potentials was analyzed in the framework of Dirichlet spaces. An intimate relation between a condenser measure $\theta_{\mathbf{A},\alpha}$, $\mathbf{A}$ being a generalized condenser in $\mathbb{R}^n$, and the solution to the $\ddot{\mathcal{E}}_\alpha(\mathbf{A}, \mathbf{1})$-problem (see Remark 5.4) has been established in our preceding study [20].

Theorem 8.10. *Let $f = 0$, and let $\nu \in \widetilde{\mathcal{H}}$, resp. $\nu \in \mathcal{E}_{\alpha,f}(\mathbf{A}, \mathbf{1})$, be given. Then the following four assertions are equivalent:*

(i) *ν solves the $\widetilde{\mathcal{H}}$-problem, resp. Problem* 5.10.

(ii) *ν satisfies* (8.2) *as well as the following two relations*

$$\kappa_\alpha\nu = w' \quad \text{on} \quad A_1 \setminus I_{A_1,\alpha}, \tag{8.17}$$

$$\kappa_\alpha\nu \leqslant w' \quad \text{on} \quad \mathbb{R}^n, \tag{8.18}$$

where $w' \in (0, \infty)$.

(iii) *There is a* (*unique*) *bounded c_α-absolutely continuous condenser measure $\theta_{\mathbf{A},\alpha}$.*

(iv) *$c_g(A_1) < \infty$.*

If any of these (i)-(iv) *holds, then the number w' appearing in* (ii) *is unique and given by*[17]

$$w' = \dot{E}_\alpha(\nu) = E_g(\mu_{A_1,g}) = 1/c_g(A_1). \tag{8.19}$$

[16] Suppose Case I holds. Then (8.16) leads to $W^\lambda_{g,f} \leqslant w'$ on S^λ_D, which together with (8.15) gives $W^\lambda_{g,f} = w'$ n.e. on S^λ_D. Similarly, (8.14) in Theorem 8.6 takes the following apparently stronger form: $W^\nu_{\alpha,f} = w'$ n.e. on $S^{\nu^+}_D$. Also note that the number w' from Theorems 8.6 and 8.7 is then > 0.

[17] If the separation condition (1.3) holds, then $\dot{E}_\alpha(\nu)$ in (8.19) can be replaced by $E_\alpha(\nu)$ (see Lemma 4.1).

Furthermore, ν *and* $\theta_{\mathbf{A},\alpha}$ *are related to one another by the formula*

$$\theta_{\mathbf{A},\alpha} = c_g(A_1)\nu = \gamma_{A_1,g} - \gamma'_{A_1,g}.$$

Here $\mu_{A_1,g}$*, resp.* $\gamma_{A_1,g}$*, is the* g*-capacitary, resp.* g*-equilibrium, measure on* A_1*.*

Proof. Let the assumptions of the theorem be fulfilled, and let (i) hold. According to Theorem 5.13, (i) is equivalent to (iv), and moreover

$$\nu = \mu_{A_1,g} - \mu'_{A_1,g} = (\gamma_{A_1,g} - \gamma'_{A_1,g})/c_g(A_1). \tag{8.20}$$

Since $f = 0$, we thus have by Lemma 3.9

$$W^{\nu}_{\alpha,f} = \kappa_\alpha\nu = g\mu_{A_1,g} = g\gamma_{A_1,g}/c_g(A_1) \ \text{ on } D. \tag{8.21}$$

Combined with (3.11) for $F = A_1$, this shows that (8.17) holds with $w' = E_g(\mu_{A_1,g}) = 1/c_g(A_1) \in (0,\infty)$. But according to Theorem 4.2, $E_g(\mu_{A_1,g}) = \dot{E}_\alpha(\mu_{A_1,g} - \mu'_{A_1,g}) = \dot{E}_\alpha(\nu)$, which together with the preceding relation establishes (8.19).

Since ν is c_α-absolutely continuous, we see from (8.17) that $\kappa_\alpha\nu^+ = w' + \kappa_\alpha\nu^-$ holds ν^+-a.e. Applying now to ν^+ the same arguments as in the first paragraph of the proof of Theorem 8.4, we therefore arrive at (8.18), thus completing the proof of the implication (i)$\Rightarrow$(ii). The converse implication follows directly from Theorem 8.6.

Assuming now again that (i) holds, we next prove that $\theta := c_g(A_1)\nu$ is a condenser measure. Combining $\kappa_\alpha\theta = c_g(A_1)\kappa_\alpha\nu$ with (8.17)–(8.19) shows that $\kappa_\alpha\theta$ equals 1 n.e. on A_1 and $\leqslant 1$ on $\mathbb{R}^n$. But this θ can be written as $\gamma_{A_1,g} - \gamma'_{A_1,g}$, see (8.20), and hence $\kappa_\alpha\theta$ equals 0 n.e. on A_2 by (3.2). Noting that $\kappa_\alpha\theta = g\gamma_{A_1,g} > 0$ on D, see (8.21), we conclude that this θ is indeed a condenser measure, which in addition is bounded and c_α-absolutely continuous. It has thus been proven that (i) implies (iii) with $\theta_{\mathbf{A},\alpha} := \theta$.

To complete the proof, it is enough to show that (iii) implies (iv). By Definition 8.9, $\kappa_\alpha\theta_{\mathbf{A},\alpha} = 0$ n.e. on A_2, which in view of the stated c_α-absolute continuity of $\theta_{\mathbf{A},\alpha}$ implies that $\theta^-_{\mathbf{A},\alpha}$ is the α-Riesz swept measure of $\theta^+_{\mathbf{A},\alpha}$ onto $A_2 = D^c$ (see Section 3). Applying Lemma 3.9 to the (bounded, hence extendible) measure $\theta^+_{\mathbf{A},\alpha}$, we thus get

$$g\theta^+_{\mathbf{A},\alpha} = \kappa_\alpha\theta_{\mathbf{A},\alpha} = 1 \ \text{ n.e. on } A_1.$$

Integrating this equality with respect to the (c_α-absolutely continuous, bounded) measure $\theta^+_{\mathbf{A},\alpha}$ implies that $\theta^+_{\mathbf{A},\alpha} \in \mathcal{E}^+_g(A_1; D)$. Applying [18, Lemma 3.2.2] with $\kappa = g$, we therefore see from the last display that $c_g(A_1) < \infty$, which is (iv). □

In the following assertion we require that in the case $\alpha < 2$, $m(D^c) > 0$. For the sake of simplicity of formulation, we also assume that in the case $\alpha = 2$, $D \setminus \breve{A}_1$ is simply connected, where $\breve{A}_1$ denotes the κ_α-reduced kernel of A_1 (see Section 7).

Theorem 8.11. *Let* $f = 0$*, and let* $\nu \in \widetilde{\mathcal{H}}$*, resp.* $\nu \in \mathcal{E}_{\alpha,f}(\mathbf{A},\mathbf{1})$*, solve the* $\widetilde{\mathcal{H}}$*-problem, resp. Problem* 5.10*. In addition to* (8.17) *and* (8.18)*, then*

$$\kappa_\alpha\nu < w' \ \text{ on } D \setminus \breve{A}_1, \tag{8.22}$$

w' *being defined by* (8.19)*, and also*

$$S^{\nu^+}_D = \begin{cases} \breve{A}_1 & \text{if } \alpha < 2, \\ \partial_D\breve{A}_1 & \text{if } \alpha = 2. \end{cases} \tag{8.23}$$

Proof. Under the stated hypotheses, $\nu^+ = \mu$, where $\mu := \mu_{A_1,g}$ is the g-capacitary measure on A_1, and $g\mu = \kappa_\alpha \nu$ on D. Assuming first $\alpha < 2$, we begin by showing that

$$g\mu < w' \quad \text{on } D \setminus S^\mu_D. \tag{8.24}$$

Suppose on the contrary that this fails for some $x_0 \in D \setminus S^\mu_D$. By (8.18), then $g\mu(x_0) = w'$, or equivalently

$$\kappa_\alpha \mu(x_0) = w' + \kappa_\alpha \mu'(x_0). \tag{8.25}$$

Choose $\varepsilon > 0$ so that $\overline{B}(x_0, \varepsilon) \subset D \setminus S^\mu_D$. Since $\kappa_\alpha \mu$ is α-harmonic on $B(x_0, \varepsilon)$ and continuous on $\overline{B}(x_0, \varepsilon)$, while $w' + \kappa_\alpha \mu'$ is α-superharmonic on $\mathbb{R}^n$, we conclude from (8.18) and (8.25) with the aid of [23, Theorem 1.28] that $\kappa_\alpha \mu = w' + \kappa_\alpha \mu'$ m-a.e. on $\mathbb{R}^n$. As $\kappa_\alpha \mu = \kappa_\alpha \mu'$ holds n.e. on D^c, hence m-a.e. on D^c, we thus get $w' = 0$. A contradiction.

We next proceed by proving the former identity in (8.23). Let, on the contrary, there exist $x_1 \in \breve{A}_1$ such that $x_1 \notin S^\mu_D$, and let $V \subset D \setminus S^\mu_D$ be an open neighborhood of x_1. By (8.24), then $g\mu < w'$ on V. On the other hand, since $V \cap A_1$ has nonzero capacity, $g\mu(x_2) = w'$ for some $x_2 \in V$ by (8.17). The contradiction obtained shows that, indeed, $S^\mu_D = \breve{A}_1$. Substituting this into (8.24) establishes (8.22) for $\alpha < 2$.

In the rest of the proof, let $\alpha = 2$. To verify (8.22), assume on the contrary that it fails for some x_3 in the domain $D_0 := D \setminus \breve{A}_1$. By (8.18), then $g\mu(x_3) = w'$, which by the maximum principle applied to the harmonic function $g\mu$ on D_0 yields $g\mu = w'$ on D_0. Together with (8.17) this shows that $g\mu = w'$ n.e. on D, so that μ/w' serves as the g-equilibrium measure on the whole of D, which is impossible.

By [23, Theorem 1.13], we see from (8.17) that the restriction of μ to $\breve{A}_1 \setminus \partial_D \breve{A}_1$ equals 0, and so $S^\mu_D \subset \partial_D \breve{A}_1$. Thus, if we prove the converse inclusion, then the latter identity in (8.23) follows. On the contrary, assume there is a point $y \in \partial_D \breve{A}_1$ such that $y \notin S^\mu_D$; then one can choose a neighborhood $V_1 \subset D$ of y with the property $V_1 \cap S^\mu_D = \varnothing$. As $c_\alpha(V_1 \cap A_1) > 0$, (8.17) implies that $g\mu(y_1) = w'$ for some $y_1 \in V_1$. Taking (8.18) into account and applying the maximum principle to the harmonic function $g\mu$ on V_1, we thus get $g\mu = w'$ on V_1. Since $V_1 \cap D_0 \neq \varnothing$, this contradicts (8.22). □

9. Comments

Remark 9.1. Based on Theorem 8.11 and its proof, we are led to the following assertion, providing a description of the α-Green equilibrium measure γ on F. As usual, we denote by $\breve{F}$ the κ_α-reduced kernel of F (see Section 7).

Theorem 9.2. *Let F be a relatively closed subset of D with $c_g(F) < \infty$. If $\alpha < 2$, assume additionally that $m(D^c) > 0$, while in the case $\alpha = 2$ let $D \setminus \breve{F}$ be simply connected. Then the support of the α-Green equilibrium measure γ on F is given by*

$$S^\gamma_D = \begin{cases} \breve{F} & \text{if } \alpha < 2, \\ \partial_D \breve{F} & \text{if } \alpha = 2. \end{cases}$$

Remark 9.3. As seen from the results obtained, the generalized condensers such that the unconstrained minimum weak α-Riesz energy problems are solvable differ drastically from those for which the solvability occurs in the constrained setting. Indeed, if the constraint $\xi \in \mathfrak{C}(A_1)$ is *bounded*, then the $\widetilde{\mathcal{H}}$-problem as well as Problem 5.10 is solvable in either Case I or Case II even if $c_\alpha(A_2 \cap \mathrm{Cl}_{\mathbb{R}^n} A_1) > 0$ (actually, even if $A_1 = D$; see Theorem 6.1). However, if $f = 0$ and $\sigma = \infty$ (*no* external field and *no* active constraint), then the $\widetilde{\mathcal{H}}$-problem as well as Problem 5.10 reduces to problem (2.1) with $Q = A_1$ and $\kappa = g$, or equivalently to the problem on the existence of the α-Green equilibrium measure γ_{A_1}, while the solvability of the latter necessarily implies that $c_\alpha(A_2 \cap \mathrm{Cl}_{\mathbb{R}^n} A_1) = 0$ (see Corollary 3.8).

10. An example of a Green equilibrium measure with infinite Newtonian energy

Let $n = 3$, $\alpha = 2$, and let $D = \bigl\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 > 0\bigr\}$. We construct a relatively closed 2-regular subset F of D with $0 < c_g(F) < \infty$ such that its (classical) Green equilibrium measure $\gamma = \gamma_F$ (which exists, see Theorem 3.6) has infinite Newtonian energy. The present example is a strengthening of the example in [16, Appendix], quoted in the Introduction, because the measure in question now is an equilibrium measure.

This example shows that the results of the present paper, related to minimum weak α-Riesz problems over subclasses of $\mathfrak{M}(\mathbf{A})$, in general fail if we replace the weak energy by the standard α-Riesz energy. This is also the case for [20, Theorem 6.1], quoted in Remark 5.4 above. This justifies the need for the concept of weak α-Riesz energy when dealing with condenser problems.

Example 10.1. Let D be a domain in $\mathbb{R}^3$, specified above. The boundary ∂D is then the plane $\{x_1 = 0\}$. For $r > 0$ write $K_r := \bigl\{(0, x_2, x_3) \in \mathbb{R}^3 : \ x_2^2 + x_3^2 \leqslant r^2\bigr\}$; then K_r is the closed disc in the plane ∂D of radius r centered at $(0, 0, 0)$. Write briefly $K := K_1$. For $\varepsilon \in \mathbb{R}$ let K_r^ε denote the translation of K_r by the vector $(\varepsilon, 0, 0)$. Thus $K_r^\varepsilon \subset D$ when $\varepsilon > 0$. For $\varepsilon, s > 0$ denote by $K_{r,s}^\varepsilon \subset D$ the translation of K_r by $(\varepsilon, s, 0)$.

Since $0 < c_2(K) < \infty$ (in fact $c_2(K) = 2/\pi^2$, see e.g. [23, Chapter II, Section 3, n° 14]), there exists the (unique) κ_2-capacitary measure μ on K (see Remarks 2.3 and 2.6). Its Newtonian potential $\kappa_2\mu$ on $\mathbb{R}^3$ is then constant everywhere on the disc K, e.g. by the Wiener criterion, and equals there the Newtonian energy $E_2(\mu) = 1/c_2(K)$. By the continuity principle [23, Theorem 1.7], $\kappa_2\mu$ is (finitely) continuous on $\mathbb{R}^3$, and even uniformly since $\kappa_2\mu(x) \to 0$ uniformly as $|x| \to \infty$, the support $S_{\mathbb{R}^3}^\mu$ being compact (actually, $S_{\mathbb{R}^3}^\mu = K$).

For any positive Radon measure ν on $\mathbb{R}^3$ we denote by $\hat{\nu}$ the image of ν under the reflection $(x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3)$ with respect to ∂D. Similarly, for any $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ write $\hat{y} = (-y_1, y_2, y_3)$. The 2-*Green kernel* $g_D^2 = g$ on the half-space D is then given by

$$g(x, y) = \kappa_2(x, y) - \kappa_2(x, \hat{y}) = \kappa_2\varepsilon_y(x) - \kappa_2\widehat{\varepsilon_y}(x),$$

see e.g. [1, Theorem 4.1.6]. This yields the homogeneity relation

$$g(rx, ry) = r^{-1}g(x, y) \ \text{ for } x, y \in D, \ r > 0. \tag{10.1}$$

Writing for brevity $\psi(\varepsilon) := c_g(K_1^\varepsilon)$, we have for $\varepsilon, r \in (0, \infty)$

$$c_g(K_r^\varepsilon) = r c_g(K_1^{\varepsilon/r}) = r\psi\Bigl(\frac{\varepsilon}{r}\Bigr). \tag{10.2}$$

In fact, if γ_ε denotes the g-equilibrium measure on $K_1^{\varepsilon/r}$, then the image $r \circ \gamma_\varepsilon$ of γ_ε under the homothety $x \mapsto rx$, $x \in D$ (which preserves the total mass) is carried by K_r^ε and has constant g-potential $1/r$ in view of (10.1). Therefore, $r(r \circ \gamma_\varepsilon)$ is the g-equilibrium measure on K_r^ε, and hence $c_g(K_r^\varepsilon)$ equals the total mass $r\gamma_\varepsilon(K_1^{\varepsilon/r}) =$

$rc_g(K_1^{\varepsilon/r})$. In particular,

$$\psi(\delta) = c_g(K_1^\delta) = \delta c_g(K_{1/\delta}^1) \text{ for any } \delta \in (0,\infty). \tag{10.3}$$

Lemma 10.2. *The function ψ is (finitely) continuous. Moreover,*

$$\lim_{\delta\to 0} \psi(\delta) = \infty.$$

Proof. The value $\psi(\delta)$ is finite because K_1^δ is compact in D. In view of (10.2) and (10.3), for continuity of ψ it suffices to establish continuity of $r \mapsto c_g(K_r^1)$ for $r \in (0,\infty)$. This mapping is continuous from the right at any r_0 because $K_{r_0}^1 = \bigcap_{r>r_0} K_r^1$ with K_r^1 compact and decreasing for $r \downarrow r_0$ (see [18, Lemma 4.2.1]). For any $r \in (0,\infty)$ denote by ∂_r the boundary of K_r^1 relative to the plane $\{x_1 = 1\}$, i.e. $\partial_r := \partial_{\{x_1=1\}} K_r^1$. For continuity of $c_g(K_r^1)$ from the left, note that $K_{r_0}^1 \setminus \partial_{r_0} = \bigcup_{r<r_0} K_r^1$ (for example, through an increasing sequence of r), hence $c_g(K_{r_0}^1) = \sup_{r<r_0} c_g(K_r^1)$ by [18, Theorem 4.2] because the Newtonian capacity of ∂_{r_0}, imbedded into $\mathbb{R}^3$, equals 0, and hence so does $c_g(\partial_{r_0})$ (footnote 9).

Denoting by μ^δ the image of μ under the translation $(\delta, 0, 0)$, μ being the κ_2-capacitary measure on K, we get for any $\delta > 0$

$$\begin{aligned} E_g(\mu^\delta) &= \int g\mu^\delta \, d\mu^\delta = \int \Big(\kappa_2\mu^\delta - \kappa_2\widehat{\mu^\delta} \Big) \, d\mu^\delta \\ &= \int \kappa_2\mu \, d\mu - \int \kappa_2\mu(-2\delta, x_2, x_3) \, d\mu(x_1, x_2, x_3). \end{aligned} \tag{10.4}$$

Note that $\kappa_2\mu(-2\delta, x_2, x_3) \to \kappa_2\mu(0, x_2, x_3)$ uniformly with respect to $(0, x_2, x_3) \in K$ as $\delta \to 0$, which is seen from the uniform continuity of $\kappa_2\mu$ on $\mathbb{R}^3$ established above. In view of (10.4), we therefore get

$$\psi(\delta) = c_g(K_1^\delta) \geqslant 1/E_g(\mu^\delta) \to \infty \text{ as } \delta \to 0,$$

μ^δ being the κ_2-capacitary measure on K_1^δ. □

For the construction of the desired relatively compact subset F of D with finite $c_g(F)$ we consider sequences of numbers $\varepsilon_j > 0$, $r_j = j^{-3}$, and $s_j := aj \uparrow \infty$ for some constant $a \geqslant 4$. The set F will be of the form

$$F := \bigcup_j F_j, \quad F_j := K_{r_j,s_j}^{\varepsilon_j}. \tag{10.5}$$

Such F is indeed relatively closed in D, for if a sequence $\{x_k\}_{k\in\mathbb{N}} \subset F$ converges to $x \in D$, then all the x_k lie in a suitable (compact) finite union of sets F_j because $s_j \to \infty$.

Define $b := \inf_\delta \psi(\delta) < \infty$. In view of Lemma 10.2 we choose for every integer $j > b$ a number δ_j such that $\psi(\delta_j) = j$. Next choose $\varepsilon_j = r_j\delta_j = j^{-3}\delta_j$ for $j > b$ and note that with summation over $j > b$ we have by (10.2), (10.3), and (10.5)

$$\sum_j c_g(F_j) = \sum_j c_g(K_{r_j}^{\varepsilon_j}) = \sum_j r_j c_g(K_1^{\varepsilon_j/r_j}) = \sum_j j^{-3}\psi(\delta_j) = \sum_j j^{-2} < \infty.$$

In view of the countable subadditivity of $c_g(\cdot)$ on universally measurable sets, see [18, Lemma 2.3.5], it follows that $c_g(F) < \infty$, and so there exists the (unique) g-equilibrium measure $\gamma = \gamma_F$ on F. This positive measure γ has constant g-potential 1 everywhere on F because every point of F (that is, of some F_j) is 2-regular, as noted earlier.

Denoting $\gamma_j := \gamma|_{F_j}$ (where $F_j = K^{\varepsilon_j}_{r_j,s_j}$), we next show that

$$\frac{1}{2} \leqslant g\gamma_j \leqslant 1 \quad \text{on } F_j. \tag{10.6}$$

Fix $a := \max\{\gamma(F), 4\}$ for our choice of the sequence $s_j = aj$. For any $x \in F_j$,

$$\sum_{k\neq j} g\gamma_k(x) \leqslant \sum_{k\neq j} \kappa_2\gamma_k(x) = \sum_{k\neq j} \int_{F_k} \frac{1}{|x-y|}\, d\gamma_k(y) \leqslant \sum_{k\neq j} \frac{\gamma(F_k)}{a-2} \leqslant \frac{\gamma(F)}{a/2} \leqslant \frac{1}{2} \tag{10.7}$$

after using that $r_j, r_k \leqslant 1$ and $|x-y| \geqslant |s_j - s_k| - (r_j + r_k) \geqslant a-2 \geqslant a/2 \geqslant 2$ for $x \in F_j$, $y \in F_k$. It follows by subtraction that $g\gamma_j \geqslant 1/2$ on F_j and of course $g\gamma_j \leqslant g\gamma \leqslant 1$ on D, thus establishing (10.6).

Since $g(2\gamma_j) \geqslant 1$ on F_j by (10.6) and

$$c_g(F_j) = \inf\bigl\{\nu(F_j) :\ \nu \in \mathfrak{M}^+(F_j),\ g\nu \geqslant 1 \text{ n.e. on } F_j\bigr\}$$

(see e.g. [1, Theorem 5.5.5(ii)] or [14, p. 243]), we thus get

$$c_g(F_j) \leqslant 2\gamma(F_j). \tag{10.8}$$

As γ_j is carried by $F_j = K^{\varepsilon_j}_{r_j,s_j}$ with diameter $2r_j$, we have for the Newtonian energy $E_2(\gamma)$ after summation over $j > b$:

$$E_2(\gamma) \geqslant \sum_j E_2(\gamma_j) \geqslant \sum_j \frac{\gamma(F_j)^2}{2r_j} \geqslant \sum_j \frac{c_g(F_j)^2}{8r_j} = \sum_j \frac{(r_j\psi(\delta_j))^2}{8r_j} = \sum_j \frac{1}{8j} = \infty,$$

where the third inequality holds by (10.8), the first equality by (10.2) and (10.3), and the second equality by our choices $r_j = j^{-3}$ and $\psi(\delta_j) = j$.

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