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ON MODULES WITH SELF TOR VANISHING

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ABSTRACT. The long-standing Auslander and Reiten Conjecture states that a finitely generated module over a finite-dimensional algebra is projective if certain Ext-groups vanish. Several authors, including Avramov, Buchweitz, Iyengar, Jorgensen, Nasseh, Sather-Wagstaff, and Şega, have studied a possible counterpart of the conjecture, or question, for commutative rings in terms of vanishing of Tor. This has led to the notion of Tor-persistent rings. Our main result shows that the class of Tor-persistent local rings is closed under a number of standard procedures in ring theory.

1. INTRODUCTION

Inspired by work of Şega [22, para. preceding Thm. 2.6], Avramov, Iyengar, Nasseh, and Sather-Wagstaff raise in [6], the question of whether every commutative noetherian ring is Tor-persistent. A commutative ring \( A \) is said to be Tor-persistent if every finitely generated \( A \)-module \( M \) with \( \text{Tor}^i_A(M,M) = 0 \) for all \( i \gg 0 \), that is, \( \text{Tor}^i(M,M) \) is bounded, has finite projective dimension. We refer to [6] and the precursor [5] (by the same authors) for a history/background of this question. The mentioned works also contain information about several interesting classes of rings which are known to be Tor-persistent. This includes Gorenstein rings with an exact zero divisor whose radical to the fourth power is zero [22, Thm. 2], complete intersection rings [15, Cor. (1.2)] (see also [3, Thm. IV] and [14, Thm. 1.9]) and Golod rings [16, Thm. 3.1].

In [6, Prop. 1.6] it is shown that a commutative noetherian ring \( A \) is Tor-persistent if and only if the localization \( A_m \) is so for every maximal ideal \( m \subset A \); hence it suffices to study the question mentioned above for commutative noetherian local rings. Throughout this paper, \( (R, m, k) \) denotes such a ring. Our main result is the following:

1.1 Theorem. The following conditions are equivalent:

(i) \( R \) is Tor-persistent.

(ii) \( \hat{R} \) is Tor-persistent.

(iii) \( R[\![X_1,\ldots,X_n]\!] \) is Tor-persistent.

(iv) \( R[\![X_1,\ldots,X_n]\!]/(m,X_1,\ldots,X_n) \) is Tor-persistent.

While some papers in the literature approach the question raised in [6] by finding specific conditions that imply Tor-persistence, we show that Tor-persistence is a property preserved by standard procedures in local algebra. Our work is motivated by [10] where a result similar to Theorem 1.1 is proved for the so-called Auslander’s condition. However, our arguments are somewhat different since the techniques used in loc. cit. do not work in our setting; see Remark 2.3 and [10, Cor. (2.2)].

It should be noticed that there is some overlap between this paper and [5]. For example, the equivalence \( (i) \iff (ii) \) in Theorem 1.1 is contained in [6, Prop. 1.5], and our Proposition 2.2 is akin to [6, Prop. 3.8]. However, the two papers have been written completely independently, indeed, [6] were only made available to us after we completed this work. Subsequently, we rewrote our introduction and adopted the terminology “Tor-persistent” coined in [6].
This short paper is organized as follows. In Section 2 we prove Theorem 1.1 and show how to construct new examples of Tor-persistent rings (Example 2.7). We also give a way to obtain certain kinds of regular sequences in power series rings (Lemma 2.6), which might be of independent interest. In Section 3 we consider another property for rings, called (TG); it is a slightly weaker property than Tor-persistence and it is related to the Gorenstein dimension. For this property we prove a result similar to Theorem 1.1 (see Theorem 3.2), and show that some results from Section 2 can be strengthened in this new setting.

2. MAIN RESULTS

2.1 Lemma. Let \((R,m,k) \rightarrow (S,n,\ell)\) be a local homomorphism of commutative noetherian local rings. If \(S\) is Tor-persistent and has finite flat dimension over \(R\), then \(R\) is Tor-persistent.

Proof. Assume \(S\) is Tor-persistent and let \(M\) be a finitely generated \(R\)-module such that \(\text{Tor}_i^R(M,M) = 0\) for all \(i \gg 0\). We have \(\text{Tor}_i^R(M,S) = 0\) for each \(i > d\), where \(d\) is the flat dimension of \(S\) over \(R\). Replacing \(M\) by a sufficiently high syzygy we can (by dimension shifting) assume that \(\text{Tor}_i^R(M,M) = 0\) and \(\text{Tor}_i^R(M,S) = 0\) for every \(i > 0\). In this case there is an isomorphism \(M \otimes_R S \cong M \otimes_R S\) in the derived category over \(S\). This yields:

\[
(M \otimes_R M) \otimes_S S \cong (M \otimes_R S) \otimes_S S \cong (M \otimes_R S) \otimes_S (M \otimes_R S).
\]

As the complex \(M \otimes_R M\) is homologically bounded (its homology is even concentrated in degree zero) and since \(S\) has finite flat dimension over \(R\), the left-hand side is homologically bounded, and hence so is the right-hand side. That is, \(\text{Tor}_i^S(M \otimes_R S, M \otimes_R S) = 0\) for all \(i \gg 0\). As \(S\) is Tor-persistent, it follows that \(M \otimes_R S \cong M \otimes_R S\) has finite projective dimension over \(S\). It follows from \([4, (1.5.3)]\) that \(\text{pd}_R(M)\) is finite. \(\Box\)

2.2 Proposition. Let \((R,m,k)\) be a commutative noetherian local ring and let \(x=x_1,\ldots,x_n\) be an \(R\)-regular sequence. If \(R/(x)\) is Tor-persistent, then \(R\) is Tor-persistent. The converse is true if \(x_i \notin m^2 + (x_1,\ldots,x_{i-1})\) holds for every \(i=1,\ldots,n\).

Proof. The first statement is a special case of Lemma 2.1. We now prove the (partial) converse. By assumption, \(x_i\) is a non zero-divisor on \(R/(x_1,\ldots,x_{i-1})\), which has the maximal ideal \(m=m/(x_1,\ldots,x_{i-1})\). Since \(x_i \notin m^2 + (x_1,\ldots,x_{i-1})\) we have \(x_i \notin m^3\), so by induction it suffices to consider the case where \(n=1\).

Let \(R\) be Tor-persistent and let \(x \in m \setminus m^2\) be a non zero-divisor on \(R\). To see that \(R/(x)\) is Tor-persistent, let \(N\) be a finitely generated \(R/(x)\)-module with \(\text{Tor}_i^{R/(x)}(N,N) = 0\) for all \(i \gg 0\). By \([21, 11.65]\) (see also \([13, \text{Lem. 2.1}]\)) there is a long exact sequence,

\[
\cdots \rightarrow \text{Tor}_{i-1}^{R/(x)}(N,N) \rightarrow \text{Tor}_i^R(N,N) \rightarrow \text{Tor}_i^{R/(x)}(N,N) \rightarrow \cdots.
\]

Therefore \(\text{Tor}_i^R(N,N) = 0\) for all \(i \gg 0\). Since \(R\) is Tor-persistent, we get that \(\text{pd}_R(N)\) is finite. As \(x \notin m^2\), it follows that \(\text{pd}_{R/(x)}(N)\) is finite; see e.g. \([4, \text{Prop. 3.3.5(1)}]\). \(\Box\)

2.3 Remark. It would be interesting to know if the last assertion in Proposition 2.2 holds without the assumption \(x_i \notin m^2 + (x_1,\ldots,x_{i-1})\), i.e. if Tor-persistence is preserved when passing to the quotient by an ideal generated by any regular sequence; cf. Proposition 3.3.1.

2.4 Remark. The sequence \(X_1,\ldots,X_n\) is regular on \(R[X_1,\ldots,X_n]\) and \(X_0\) does not belong to \((m,X_1,\ldots,X_n)^2 + (X_1,\ldots,X_n,0)\). It follows from Proposition 2.2 that \(R\) is Tor-persistent if and only if \(R[X_1,\ldots,X_n]\) is Tor-persistent.

Proposition 2.2 can be used to construct new examples of Tor-persistent rings from known examples; see Example 2.7. However, to do so it is useful to have a concrete way of constructing regular sequences with the property mentioned in 2.2. In Lemma 2.6 below we give one such construction.
If $A$ is a commutative ring and $a$ is an element in $A$, then it can happen, perhaps surprisingly, that $X - a$ is a zero-divisor on $A[[X]]$; see [12] p. 146 for an example. However, as is well-known, if $A$ is noetherian, then the situation is much nicer:

2.5. Let $A$ be a commutative noetherian ring and consider an element $f = f(X_1, \ldots, X_n)$ in $A[X_1, \ldots, X_n]$. It follows from [11] Thm. 5 that if $f$ has some coefficient which is a unit in $A$, then $f$ is a non zero-divisor on $A[X_1, \ldots, X_n]$.

2.6 Lemma. Let $(R, m, k)$ be a commutative noetherian local ring. Consider the power series ring $S = R[[X_1, \ldots, X_n]]$ and write $n = (m, X_1, \ldots, X_n)$ for its unique maximal ideal. Let $0 \leq m_0 < m_1 < \cdots < m_i < n$ be integers and let $f_1, \ldots, f_i \in n$ be elements such that, for every $i = 1, \ldots, n$, the following conditions hold:

(a) $f_i \in R[X_1, \ldots, X_m] \subseteq S$.
(b) The element $\frac{\partial f_i}{\partial x_j}(0, \ldots, 0) \in R$ is a unit for some $m_i < j$.

Then $f_1, \ldots, f_i$ is a regular sequence on $R[[X_1, \ldots, X_n]]$ with $f_i \notin n^2 + (f_1, \ldots, f_i-1)$ for all $i$. 

Proof. First note that condition (b) implies:

The power series $f_i(0, \ldots, 0, X_{m_i+1}, \ldots, X_n)$ has a coefficient which is a unit in $R$. (2.1)

Indeed, if $m_i < j$, then $\frac{\partial f_i}{\partial x_j}(0, \ldots, 0)$ is a coefficient in $f_i(0, \ldots, 0, X_{m_i+1}, \ldots, X_n)$.

Next we show that $f_1, \ldots, f_i$ is a regular sequence. With $i = 1$ condition (2.1) says that $f_1(X_1, \ldots, X_n)$ has a coefficient which is a unit in $R$, and so $f_1$ is a non zero-divisor on $S$ by 2.5. Next we show that $f_{i+1}$ is a non zero-divisor on $S/(f_1, \ldots, f_i)$ where $i \geq 1$. Write

$f_{i+1} = \sum v_{m_i+1}^{m_i+1} \ldots v_{m_i}^{m_i+1} X_{m_i+1}^{v_{m_i+1}} \cdots X_n^{v_n} \in S \cong R[[X_1, \ldots, X_m]][X_{m_i+1}, \ldots, X_n]$ (2.2)

with $v_s \in R[X_1, \ldots, X_m]$. As $f_1, \ldots, f_i \in R[X_1, \ldots, X_m]$ by (a) there is an isomorphism:

$S/(f_1, \ldots, f_i) \cong (R[[X_1, \ldots, X_m]]/(f_1, \ldots, f_i))[X_{m_i+1}, \ldots, X_n]$ (2.3)

In particular, the image $\bar{f}_{i+1}$ of $f_{i+1}$ in $S/(f_1, \ldots, f_i)$ can be identified with the element

$\bar{f}_{i+1} = \sum v_{m_i+1}^{m_i+1} \ldots v_{m_i}^{m_i+1} X_{m_i+1}^{v_{m_i+1}} \cdots X_n^{v_n}$

in the right-hand side of (2.3), where $h_s$ is the image of $h_s$ in $R[[X_1, \ldots, X_m]]/(f_1, \ldots, f_i)$. Hence, to show that $\bar{f}_{i+1}$ is a non zero-divisor, it suffices by 2.5 to argue that one of the coefficients $h_s$ is a unit. By (2.1) we know that $f_{i+1}(0, \ldots, 0, X_{m_i+1}, \ldots, X_n)$ has a coefficient which is a unit in $R$, and by (2.2) this means that one of the elements $h_{m_{i+1}}, \ldots, h_n(0, \ldots, 0) \in R$ is a unit. Consequently $h_{m_{i+1}}, \ldots, h_n(X_1, \ldots, X_m)$ will be a unit in $R[X_1, \ldots, X_m]$, so its image $\bar{h}_{m_{i+1}}, \ldots, \bar{h}_n$ is also a unit, as desired.

Next we show that $f_i \notin n^2 + (f_1, \ldots, f_i-1)$ holds for all $i$. Suppose for contradiction that:

$f_i = \sum r_s q_s + \sum_{n=1}^{i-1} g_n f_s$, where $p_s, q_s \in n$ and $g_n \in S$.

By assumption (b) we have that $\frac{\partial f_i}{\partial x_j}(0, \ldots, 0) \in R$ is a unit for some $m_i < j$. It follows from the identity above that:

$\frac{\partial f_i}{\partial x_j}(0, \ldots, 0) = \sum r_s (\frac{\partial r_s}{\partial x_j}(0, \ldots, 0) q_s(0, \ldots, 0) + p_s(0, \ldots, 0) \frac{\partial q_s}{\partial x_j}(0, \ldots, 0)) + \sum_{n=1}^{i-1} \left( \frac{\partial g_n}{\partial x_j}(0, \ldots, 0) f_s(0, \ldots, 0) + g_n(0, \ldots, 0) \frac{\partial f_s}{\partial x_j}(0, \ldots, 0) \right)$.

As already mentioned, the left-hand side is a unit, and this contradicts that the right-hand side belongs to $m$. Indeed, we have $p_s(0, \ldots, 0), q_s(0, \ldots, 0), f_s(0, \ldots, 0) \in m$ as $p_s, q_s, f_s \in n$. Furthermore, $f_1, \ldots, f_i-1$ only depend on the variables $X_1, \ldots, X_{m_i-1}$ by (a), so every $\frac{\partial f_s}{\partial x_j}$ is zero. \qed

2.7 Example. In $R[U, V, W]$ the following (more or less arbitrarily chosen) sequence, corresponding to $t = 2$ and $m_1 = 2$, satisfies the assumptions of Lemma 2.6

$f_1 = a + U^3 + UV + V$ and $f_2 = b + UV^2 + W + W^2$ (a, b \in m).

Indeed, (a) is clear and (b) holds since $\frac{\partial f}{\partial x}(0, 0, 0) = 1 = \frac{\partial f}{\partial x}(0, 0, 0)$. So Proposition 2.2 implies that if $R$ is Tor-persistent, then so is $A = R[U, V, W]/(f_1, f_2)$. 

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Note that the fiber product ring
\[ R = k[[X]]/(X^4) \times_k k[[Y]]/(Y^3) \cong k[[X,Y]]/(X^4, Y^3, XY) \]
is artinian, not Gorenstein, and by [18, Thm. 1.1] it is Tor-persistent. Hence the following ring (where we have chosen \( a = Y^2 \) and \( b = X^2 \)) is Tor-persistent as well:
\[ A = k[[X,Y,U,V,W]]/(X^4, Y^3, XY, Y^2 + U^3 + UV + V, X^2 + UV^2 + W + W^2). \]

**Proof of Theorem**[7,7] The equivalence \((i) \Leftrightarrow (iii)\) is noted in Remark 2.4. Let \( a_1, \ldots, a_n \) be a set of elements that generate \( m \). We have \( \tilde{R} \cong R[[X_1, \ldots, X_n]]/(X_1 - a_1, \ldots, X_n - a_n) \) by [17] Thm. 8.12. The sequence \( f_i = X_i - a_i \) clearly satisfies the assumptions in Lemma 2.6 so the equivalence \((i) \Leftrightarrow (ii)\) follows. Note that \( R[X_1, \ldots, X_n]_{(m,X_1,\ldots,X_n)} \) and \( R[X_1, \ldots, X_n] \) have isomorphic completions (both are isomorphic to \( \tilde{R}[X_1, \ldots, X_n] \)), so the equivalence \((iii) \Leftrightarrow (iv)\) follows from the already established equivalence between \((i)\) and \((ii)\). □

### 3. Connections with the Gorenstein Dimension

In this section, we give a few remarks and observations pertaining Aulander’s G-dimension [1] and self Tor vanishing. For a commutative noetherian local ring \((R, m, k)\), we consider the following property (which \( R \) may, or may not, have):

(TG) Every finitely generated \( R \)-module \( M \) satisfying Tor\(_{i}^{R}(M, M) = 0 \) for all \( i \gg 0 \) has finite G-dimension, that is, \( \text{G-dim}_{R}(M) < \infty \).

Every Tor-persistent ring has the property (TG), see [9, Prop. (1.2.10)], and the converse holds if the maximal ideal \( m \) is decomposable; see [20] Thm. 5.5.

Testing finiteness of the G-dimension via the vanishing of Tor, in some form, is an idea pursued in a number of papers. For example, in [7, Thm. 3.11] it was proved that a finitely generated module \( M \) over a commutative noetherian ring \( R \) has finite G-dimension if and only if the stable homology Tor\(_{i}^{R}(M, R) \) vanishes for every \( i \in \mathbb{Z} \). Furthermore, finitely generated modules testing finiteness of the G-dimension via the vanishing of absolute homology, i.e. Tor, were also examined in [8].

For the property (TG) we have the following stronger version of Proposition 2.2.

**3.1 Proposition.** Let \((R, m, k)\) be a commutative noetherian local ring and let \( \underline{x} = x_1, \ldots, x_n \) be an \( R \)-regular sequence. Then \( R \) has the property (TG) if and only if \( R/\langle \underline{x} \rangle \) has it.

**Proof.** For the “if” part we proceed as in the proof of Lemma 2.1, with \( S = R/\langle \underline{x} \rangle \). Note that having replaced \( M \) with a sufficiently high syzygy, the sequence \( \underline{x} \) becomes regular on \( M \) (this is standard but see also [19, Lem. 5.1]). From the finiteness of G-dim\(_{R/\langle \underline{x} \rangle}(M/\langle \underline{x} \rangle M)\) we infer the finiteness of G-dim\(_{R}(M)\) from [9, Cor. (1.4.6)]. For the “only if” part proceed as in the proof of Proposition 2.2. From the finiteness of G-dim\(_{R}(N)\) one always gets finiteness of G-dim\(_{R/\langle \underline{x} \rangle}(N)\) (the assumption \( x \notin \mathfrak{m}^3 \) is not needed) by [9, Thm. p. 39]. □

Now the arguments in the proof of Theorem 1.1 applies and give the following.

**3.2 Theorem.** Let \((R, m, k)\) be a commutative noetherian local ring. The following conditions are equivalent:

(i) \( R \) has the property (TG).
(ii) \( \tilde{R} \) has the property (TG).
(iii) \( R[X_1, \ldots, X_n] \) has the property (TG).
(iv) \( R[X_1, \ldots, X_n]_{(m,X_1,\ldots,X_n)} \) has the property (TG). □

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