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ON MODULES WITH SELF TOR VANISHING

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ABSTRACT. The long-standing Auslander and Reiten Conjecture states that a finitely generated module over a finite-dimensional algebra is projective if certain Ext-groups vanish. Several authors, including Avramov, Buchweitz, Iyengar, Jorgensen, Nasseh, Sather-Wagstaff, and Şega, have studied a possible counterpart of the conjecture, or question, for commutative rings in terms of vanishing of Tor. This has led to the notion of Tor-persistent rings. Our main result shows that the class of Tor-persistent local rings is closed under a number of standard procedures in ring theory.

1. INTRODUCTION

Inspired by work of Şega [22, para. preceding Thm. 2.6], Avramov, Iyengar, Nasseh, and Sather-Wagstaff raise in [6], the question of whether every commutative noetherian ring is Tor-persistent. A commutative ring \( A \) is said to be Tor-persistent if every finitely generated \( A \)-module \( M \) with \( \text{Tor}_i^A(M, M) = 0 \) for all \( i \gg 0 \), that is, \( \text{Tor}_i^A(M, M) \) is bounded, has finite projective dimension. We refer to [6] and the precursor [5] (by the same authors) for a history/background of this question. The mentioned works also contain information about several interesting classes of rings which are known to be Tor-persistent. This includes Gorenstein rings with an exact zero divisor whose radical to the fourth power is zero [22, Thm. 2], complete intersection rings [15, Cor. (1.2)] (see also [3, Thm. IV] and [14, Thm. 1.9]) and Golod rings [16, Thm. 3.1].

In [6, Prop. 1.6] it is shown that a commutative noetherian ring \( A \) is Tor-persistent if and only if the localization \( A_m \) is so for every maximal ideal \( m \subset A \); hence it suffices to study the question mentioned above for commutative noetherian \emph{local} rings. Throughout this paper, \((R, m, k)\) denotes such a ring. Our main result is the following:

1.1 Theorem. The following conditions are equivalent:

(i) \( R \) is Tor-persistent.
(ii) \( \hat{R} \) is Tor-persistent.
(iii) \( R[X_1,...,X_n] \) is Tor-persistent.
(iv) \( R[X_1,...,X_n]_{(m,X_1,...,X_n)} \) is Tor-persistent.

While some papers in the literature approach the question raised in [6] by finding specific conditions that imply Tor-persistence, we show that Tor-persistence is a property preserved by standard procedures in local algebra. Our work is motivated by [10] where a result similar to Theorem [1.1] is proved for the so-called Auslander’s condition. However, our arguments are somewhat different since the techniques used in \emph{loc. cit.} do not work in our setting; see Remark [2.3] and [10, Cor. (2.2)].

It should be noticed that there is some overlap between this paper and [5]. For example, the equivalence (i) \( \Leftrightarrow \) (ii) in Theorem [1.1] is contained in [6, Prop. 1.5], and our Proposition [2.2] is akin to [6, Prop. 3.8]. However, the two papers have been written completely independently, indeed, [6] were only made available to us after we completed this work. Subsequently, we rewrote our introduction and adopted the terminology “Tor-persistent” coined in [6].

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Note that this work is announced under the different title \emph{Vanishing of endohomology over local rings} in [5].
This short paper is organized as follows. In Section 2 we prove Theorem 1.1 and show how to construct new examples of Tor-persistent rings (Example 2.7). We also give a way to obtain certain kinds of regular sequences in power series rings (Lemma 2.6), which might be of independent interest. In Section 3 we consider another property for rings, called (TG); it is a slightly weaker property than Tor-persistence and it is related to the Gorenstein dimension. For this property we prove a result similar to Theorem 1.1 (see Theorem 3.2), and show that some results from Section 2 can be strengthened in this new setting.

2. MAIN RESULTS

2.1 Lemma. Let $(R,m,k) \to (S,n,l)$ be a local homomorphism of commutative noetherian local rings. If $S$ is Tor-persistent and has finite flat dimension over $R$, then $R$ is Tor-persistent.

Proof. Assume $S$ is Tor-persistent and let $M$ be a finitely generated $R$-module such that $\text{Tor}_i^R(M,M) = 0$ for all $i \gg 0$. We have $\text{Tor}_i^R(M,S) = 0$ for each $i > d$, where $d$ is the flat dimension of $S$ over $R$. Replacing $M$ by a sufficiently high syzygy we can (by dimension shifting) assume that $\text{Tor}_i^R(M,M) = 0$ and $\text{Tor}_i^R(M,S) = 0$ for every $i > 0$. In this case there is an isomorphism $M \otimes_R^L S \cong M \otimes_R S$ in the derived category over $S$. This yields:

$$(M \otimes_R M) \otimes_S^L S \cong (M \otimes_S^L S) \otimes_S^L (M \otimes_R S) \cong (M \otimes_R S) \otimes_S^L (M \otimes_R S).$$

As the complex $M \otimes_R M$ is homologically bounded (its homology is even concentrated in degree zero) and since $S$ has finite flat dimension over $R$, the left-hand side is homologically bounded, and hence so is the right-hand side. That is, $\text{Tor}_i^S(M \otimes_R S, M \otimes_R S) = 0$ for all $i \gg 0$. As $S$ is Tor-persistent, it follows that $M \otimes_R S \cong M \otimes_R S$ has finite projective dimension over $S$. It follows from [4, (1.5.3)] that $\text{pd}_R(M)$ is finite. \(\square\)

2.2 Proposition. Let $(R,m,k)$ be a commutative noetherian local ring and let $x = x_1, \ldots, x_n$ be an $R$-regular sequence. If $R/(x)$ is Tor-persistent, then $R$ is Tor-persistent. The converse is true if $x_i \not\in m^2 + (x_1, \ldots, x_{i-1})$ holds for every $i = 1, \ldots, n$.

Proof. The first statement is a special case of Lemma 2.1. We now prove the (partial) converse. By assumption, $x_i$ is a non zero-divisor on $R/(x_1, \ldots, x_{i-1})$, which has the maximal ideal $m = m/(x_1, \ldots, x_{i-1})$. Since $x_i \not\in m^2 + (x_1, \ldots, x_{i-1})$ we have $x_i \not\in m^2$, so by induction it suffices to consider the case where $n = 1$.

Let $R$ be Tor-persistent and let $x \in m \setminus m^2$ be a non zero-divisor on $R$. To see that $R/(x)$ is Tor-persistent, let $N$ be a finitely generated $R/(x)$-module with $\text{Tor}_i^{R/(x)}(N,N) = 0$ for all $i \gg 0$. By [21, 11.65] (see also [13, Lem. 2.1]) there is a long exact sequence,

$$\cdots \to \text{Tor}_i^{R/(x)}(N,N) \to \text{Tor}_i^{R}(N,N) \to \text{Tor}_i^{R/(x)}(N,N) \to \cdots.$$ 

Therefore $\text{Tor}_i^{R}(N,N) = 0$ for all $i \gg 0$. Since $R$ is Tor-persistent, we get that $\text{pd}_R(N)$ is finite. As $x \not\in m^2$, it follows that $\text{pd}_{R/(x)}(N)$ is finite; see e.g. [2] Prop. 3.3.5(1). \(\square\)

2.3 Remark. It would be interesting to know if the last assertion in Proposition 2.2 holds without the assumption $x_i \not\in m^2 + (x_1, \ldots, x_{i-1})$, i.e. if Tor-persistence is preserved when passing to the quotient by an ideal generated by any regular sequence; cf. Proposition 3.2.

2.4 Remark. The sequence $X_1, \ldots, X_n$ is regular on $R[X_1, \ldots, X_n]$ and $X_i$ does not belong to $(m,X_1, \ldots, X_{i-1})^2 + (X_1, \ldots, X_{i-1})$. It follows from Proposition 2.2 that $R$ is Tor-persistent if and only if $R[X_1, \ldots, X_n]$ is Tor-persistent.

Proposition 2.2 can be used to construct new examples of Tor-persistent rings from known examples; see Example 2.7. However, to do so it is useful to have a concrete way of constructing regular sequences with the property mentioned in 2.2. In Lemma 2.6 below we give one such construction.
If $A$ is a commutative ring and $a$ is an element in $A$, then it can happen, perhaps surprisingly, that $X - a$ is a zero-divisor on $A[X]$; see [12] p. 146 for an example. However, as is well-known, if $A$ is a noetherian ring, then the situation is much nicer.

2.5. Let $A$ be a commutative noetherian ring and consider an element $f = f(X_1, \ldots, X_n)$ in $A[X_1, \ldots, X_n]$. It follows from [11] Thm. 5 that if $f$ has some coefficient which is a unit in $A$, then $f$ is a non zero-divisor on $A[X_1, \ldots, X_n]$.

2.6 Lemma. Let $(R, m, k)$ be a commutative noetherian local ring. Consider the power series ring $S = R[[x_1, \ldots, x_n]]$ and write $n = (a_1, \ldots, a_n)$ for its unique maximal ideal.

Let $0 = m_0 < m_1 < \cdots < m_r = n$ be integers and let $f_i, \ldots, f_i \in n$ be elements such that, for every $i = 1, \ldots, r$, the following conditions hold:

(a) $f_i \in R[x_1, \ldots, x_m] \subseteq S$.
(b) The element $\frac{\partial f_i}{\partial x_0}(0, \ldots, 0) \in R$ is a unit for some $m_i - j < j$.

Then $f_1, \ldots, f_r$ is a regular sequence on $R[[x_1, \ldots, x_n]]$ with $f_i \not\in n^2 + (f_1, \ldots, f_{i-1})$ for all $i$.

Proof. First note that condition (b) implies:

The power series $f_i(0, \ldots, 0, X_{m_i+1}, \ldots, X_n)$ has a coefficient which is a unit in $R$. (2.1)

Indeed, if $m_i - 1 < j$, then $\frac{\partial f_i}{\partial x_j}(0, \ldots, 0)$ is a coefficient in $f_i(0, \ldots, 0, X_{m_i+1}, \ldots, X_n)$.

Next we show that $f_1, \ldots, f_r$ is a regular sequence. With $i = 1$ condition (2.1) says that $f_1(X_1, \ldots, X_n)$ has a coefficient which is a unit in $R$, and so $f_1$ is a non zero-divisor on $S$ by (2.5).

Next we show that $f_{i+1}$ is a non zero-divisor on $S/(f_1, \ldots, f_i)$ where $i \geq 1$. Write

\[ f_{i+1} = \sum_{v_{m_i+1} - \cdots - v_n} h_{v_{m_i+1} - \cdots - v_n} x_{m_i+1}^{v_{m_i+1}} \cdots x_n^{v_n} \in S \cong R[[x_1, \ldots, x_m]][[x_{m+1}, \ldots, x_n]] \]

(2.2)

with $h_s \in R[x_1, \ldots, x_m]$. As $f_1, \ldots, f_i \in R[x_1, \ldots, x_m]$ by (a) there is an isomorphism:

\[ S/(f_1, \ldots, f_i) \cong (R[[x_1, \ldots, x_m]]/(f_1, \ldots, f_i))[x_{m+1}, \ldots, x_n] \].

(2.3)

In particular, the image $f_{i+1}$ of $f_{i+1}$ in $S/(f_1, \ldots, f_i)$ can be identified with the element

\[ f_{i+1} = \sum_{v_{m_i+1} - \cdots - v_n} h_{v_{m_i+1} - \cdots - v_n} x_{m_i+1}^{v_{m_i+1}} \cdots x_n^{v_n} \]

in the right-hand side of (2.3), where $h_s$ is the image of $h_s$ in $R[[x_1, \ldots, x_m]]/(f_1, \ldots, f_i)$. Hence, to show that $f_{i+1}$ is a non zero-divisor, it suffices by (2.5) to argue that one of the coefficients $h_s$ is a unit. By (2.1) we know that $f_{i+1}(0, \ldots, 0, X_{m+1}, \ldots, X_n)$ has a coefficient which is a unit in $R$, and by (2.2) this means that one of the elements $h_{v_{m_i+1} - \cdots - v_n}(0, \ldots, 0) \in R$ is a unit. Consequently $h_{v_{m_i+1} - \cdots - v_n}(X_1, \ldots, x_m)$ will be a unit in $R[x_1, \ldots, x_m]$, so its image $h_{v_{m_i+1} - \cdots - v_n}$ also a unit, as desired.

Next we show that $f_i \not\in n^2 + (f_1, \ldots, f_{i-1})$ holds for all $i$. Suppose for contradiction that:

\[ f_i = \sum_{r} p_r q_r + \sum_{v} g_v f_v, \text{ where } p_r, q_r \in n \text{ and } g_v \in S. \]

By assumption (b) we have that $\frac{\partial f_i}{\partial x_0}(0, \ldots, 0) \in R$ is a unit for some $m_i - 1 < j$. It follows from the identity above that:

\[ \frac{\partial f_i}{\partial x_0}(0) = \sum_{r} \frac{\partial f_i}{\partial x_r}(0) q_r(0) + \sum_{v} \frac{\partial g_v}{\partial x_0}(0) f_v(0) + \sum_{v} g_v(0) \frac{\partial f_v}{\partial x_0}(0). \]

As already mentioned, the left-hand side is a unit, and this contradicts that the right-hand side belongs to $m$. Indeed, we have $p_r(0) q_r(0), f_v(0) \in m$ as $p_r, q_r, f_v \in n$. Furthermore, $f_1, \ldots, f_{i-1}$ only depend on the variables $X_1, \ldots, X_{m-1}$ by (a), so every $\frac{\partial f_v}{\partial x_0}$ is zero. \hfill $\square$

2.7 Example. In $R[U, V, W]$ the following (more or less arbitrarily chosen) sequence, corresponding to $t = 2$ and $m_1 = 2$, satisfies the assumptions of Lemma 2.6:

\[ f_1 = a + U^3 + UV + V \quad \text{and} \quad f_2 = b + UV^2 + W + W^2 \quad (a, b \in m). \]

Indeed, (a) is clear and (b) holds since $\frac{\partial f}{\partial x_0}(0, 0, 0) = 1 = \frac{\partial f}{\partial x_0}(0, 0, 0)$. So Proposition 2.2 implies that if $R$ is Tor-persistent, then so is $A = R[U, V, W]/(f_1, f_2)$.
Note that the fiber product ring
\[ R = k[[X]]/(X^4) \times_k k[[Y]]/(Y^2) \cong k[[X,Y]]/(X^4, Y^3, XY) \]
is artinian, not Gorenstein, and by [13, Thm. 1.1] it is Tor-persistent. Hence the following ring (where we have chosen \( a = Y^2 \) and \( b = X^2 \)) is Tor-persistent as well:
\[ A = k[X, Y, U, V, W]/(X^4, Y^3, XY, Y^2 + U^3 + UV + V, X^2 + UV^2 + W + W^2). \]

**Proof of Theorem** [17]. The equivalence (i) \( \Leftrightarrow \) (iii) is noted in Remark [2.2]. Let \( a_1, \ldots, a_n \) be a set of elements that generate \( m \). We have \( \tilde{R} \cong R[[X_1, \ldots, X_n]]/(X_1 - a_1, \ldots, X_n - a_n) \) by [17, Thm. 8.12]. The sequence \( f_i = X_i - a_i \) clearly satisfies the assumptions in Lemma [2.6] so the equivalence (i) \( \Leftrightarrow \) (ii) follows. Note that \( R[X_1, \ldots, X_n][m, x_1, \ldots, x_n] \) and \( R[X_1, \ldots, X_n] \) have isomorphic completions (both are isomorphic to \( \hat{R}[[X_1, \ldots, X_n]] \)), so the equivalence (iii) \( \Leftrightarrow \) (iv) follows from the already established equivalence between (i) and (ii). \( \square \)

## 3. Connections with the Gorenstein Dimension

In this section, we give a few remarks and observations pertaining Aulander’s G-dimension [1] and self Tor vanishing. For a commutative noetherian local ring \((R, m, k)\), we consider the following property (which \( R \) may, or may not, have):

(TG) Every finitely generated \( R \)-module \( M \) satisfying \( \text{Tor}_i^R(M, M) = 0 \) for all \( i > 0 \) has finite G-dimension, that is, \( \text{G-dim}_R(M) < \infty \).

Every Tor-persistent ring has the property (TG), see [9, Prop. (1.2.10)], and the converse holds if the maximal ideal \( m \) is decomposable; see [20, Thm. 5.5].

Testing finiteness of the G-dimension via the vanishing of Tor, in some form, is an idea pursued in a number of papers. For example, in [7, Thm. 3.11] it was proved that a finitely generated module \( M \) over a commutative noetherian ring \( R \) has finite G-dimension if and only if the stable homology \( \text{Tor}^R_i(M, R) \) vanishes for every \( i \in \mathbb{Z} \). Furthermore, finitely generated modules testing finiteness of the G-dimension via the vanishing of absolute homology, i.e. Tor, were also examined in [8].

For the property (TG) we have the following stronger version of Proposition [2.2].

### 3.1 Proposition.** Let \((R, m, k)\) be a commutative noetherian local ring and let \( \underline{x} = x_1, \ldots, x_n \) be an \( R \)-regular sequence. Then \( \tilde{R} \) has the property (TG) if and only if \( R/\underline{x} \) has it.

**Proof.** For the “if” part we proceed as in the proof of Lemma [2.1] with \( S = R/\underline{x}. \) Note that having replaced \( M \) with a sufficiently high syzygy, the sequence \( \underline{x} \) becomes regular on \( M \) (this is standard but see also [19, Lem. 5.1]). From the finiteness of \( \text{G-dim}_{R/\underline{x}}(M/\underline{x}M) \) we infer the finiteness of \( \text{G-dim}_R(M) \) from [9, Cor. (1.4.6)]. For the “only if” part proceed as in the proof of Proposition [2.2]. From the finiteness of \( \text{G-dim}_{R/\underline{x}}(N) \) one always gets finiteness of \( \text{G-dim}_{R/\underline{x}}(N) \) (the assumption \( x \notin m^2 \) is not needed) by [9, Thm. 39]. \( \square \)

Now the arguments in the proof of Theorem [1.1] applies and give the following.

### 3.2 Theorem.** Let \((R, m, k)\) be a commutative noetherian local ring. The following conditions are equivalent:

(i) \( R \) has the property (TG).
(ii) \( \tilde{R} \) has the property (TG).
(iii) \( R[X_1, \ldots, X_n] \) has the property (TG).
(iv) \( R[X_1, \ldots, X_n][m, x_1, \ldots, x_n] \) has the property (TG). \( \square \)

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