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ON MODULES WITH SELF TOR VANISHING

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ABSTRACT. The long-standing Auslander and Reiten Conjecture states that a finitely generated module over a finite-dimensional algebra is projective if certain Ext-groups vanish. Several authors, including Avramov, Buchweitz, Iyengar, Jorgensen, Nasseh, Sather-Wagstaff, and Şega, have studied a possible counterpart of the conjecture, or question, for commutative rings in terms of vanishing of Tor. This has led to the notion of Tor-persistent rings. Our main result shows that the class of Tor-persistent local rings is closed under a number of standard procedures in ring theory.

1. INTRODUCTION

Inspired by work of Şega [22, para. preceding Thm. 2.6], Avramov, Iyengar, Nasseh, and Sather-Wagstaff raise in [6], the question of whether every commutative noetherian ring is Tor-persistent. A commutative ring $A$ is said to be Tor-persistent if every finitely generated $A$-module $M$ with $\text{Tor}^i_A(M, M) = 0$ for all $i \gg 0$, that is, $\text{Tor}^i_A(M, M)$ is bounded, has finite projective dimension. We refer to [6] and the precursor [5] (by the same authors) for a history/background of this question. The mentioned works also contain information about several interesting classes of rings which are known to be Tor-persistent. This includes Gorenstein rings with an exact zero divisor whose radical to the fourth power is zero [22, Thm. 2], complete intersection rings [15, Cor. (1.2)] (see also [3, Thm. IV] and [14, Thm. 1.9]) and Golod rings [16, Thm. 3.1].

In [6, Prop. 1.6] it is shown that a commutative noetherian ring $A$ is Tor-persistent if and only if the localization $A_m$ is so for every maximal ideal $m \subset A$; hence it suffices to study the question mentioned above for commutative noetherian local rings. Throughout this paper, $(R, m, k)$ denotes such a ring. Our main result is the following:

1.1 Theorem. The following conditions are equivalent:

(i) $R$ is Tor-persistent.

(ii) $\hat{R}$ is Tor-persistent.

(iii) $R[X_1, \ldots, X_n]$ is Tor-persistent.

(iv) $R[X_1, \ldots, X_n]_m[X_1, \ldots, X_n]$ is Tor-persistent.

While some papers in the literature approach the question raised in [6] by finding specific conditions that imply Tor-persistence, we show that Tor-persistence is a property preserved by standard procedures in local algebra. Our work is motivated by [10] where a result similar to Theorem 1.1 is proved for the so-called Auslander’s condition. However, our arguments are somewhat different since the techniques used in loc. cit. do not work in our setting; see Remark 2.3 and [10, Cor. (2.2)].

It should be noticed that there is some overlap between this paper and [5]. For example, the equivalence (i) $\Leftrightarrow$ (ii) in Theorem [1.1] is contained in [6, Prop. 1.5], and our Proposition 2.2 is akin to [6, Prop. 3.8]. However, the two papers have been written completely independently, indeed, [6] were only made available to us after we completed this work. Subsequently, we rewrote our introduction and adopted the terminology “Tor-persistent” coined in [6].

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Note that this work is announced under the different title Vanishing of endohomology over local rings in [5].
This short paper is organized as follows. In Section 2 we prove Theorem 1.1 and show how to construct new examples of Tor-persistent rings (Example 2.7). We also give a way to obtain certain kinds of regular sequences in power series rings (Lemma 2.6), which might be of independent interest. In Section 3 we consider another property for rings, called (TG); it is a slightly weaker property than Tor-persistence and it is related to the Gorenstein dimension. For this property we prove a result similar to Theorem 1.1 (see Theorem 3.2), and show that some results from Section 2 can be strengthened in this new setting.

2. MAIN RESULTS

2.1 Lemma. Let \((R, m, k) \rightarrow (S, n, ℓ)\) be a local homomorphism of commutative noetherian local rings. If \(S\) is Tor-persistent and has finite flat dimension over \(R\), then \(R\) is Tor-persistent.

Proof. Assume \(S\) is Tor-persistent and let \(M\) be a finitely generated \(R\)-module such that \(\text{Tor}_i^R(M, M) = 0\) for all \(i \gg 0\). We have \(\text{Tor}_i^R(M, S) = 0\) for each \(i > d\), where \(d\) is the flat dimension of \(S\) over \(R\). Replacing \(M\) by a sufficiently high syzygy we can (by dimension shifting) assume that \(\text{Tor}_i^R(M, M) = 0\) and \(\text{Tor}_i^R(M, S) = 0\) for every \(i > 0\). In this case there is an isomorphism \(M \otimes_R S \cong M \otimes_R S\) in the derived category over \(S\). This yields:

\[
(M \otimes_R S) \otimes_S S \cong (M \otimes_S S) \otimes_S S \cong (M \otimes_R S) \otimes_S S.
\]

As the complex \(M \otimes_R S\) is homologically bounded (its homology is even concentrated in degree zero) and since \(S\) has finite flat dimension over \(R\), the left-hand side is homologically bounded, and hence so is the right-hand side. That is, \(\text{Tor}_i^S(M \otimes_R S, M \otimes_R S) = 0\) for all \(i \gg 0\). As \(S\) is Tor-persistent, it follows that \(M \otimes_R S \cong M \otimes_R S\) has finite projective dimension over \(S\). It follows from \([4, (1.5.3)]\) that \(\text{pd}_R(M)\) is finite. □

2.2 Proposition. Let \((R, m, k)\) be a commutative noetherian local ring and let \(x = x_1, \ldots, x_n\) be an \(R\)-regular sequence. If \(R/(x)\) is Tor-persistent, then \(R\) is Tor-persistent. The converse is true if \(x_i \notin m^2 + (x_1, \ldots, x_{i-1})\) holds for every \(i = 1, \ldots, n\).

Proof. The first statement is a special case of Lemma 2.1. We now prove the (partial) converse. By assumption, \(x_i\) is a non zero-divisor on \(R/(x_1, \ldots, x_{i-1})\), which has the maximal ideal \(\bar{m} = m/(x_1, \ldots, x_{i-1})\). Since \(x_i \notin m^2 + (x_1, \ldots, x_{i-1})\) we have \(\bar{x}_i \notin \bar{m}^2\), so by induction it suffices to consider the case where \(n = 1\).

Let \(R\) be Tor-persistent and let \(x \in m \smallsetminus m^2\) be a non zero-divisor on \(R\). To see that \(R/(x)\) is Tor-persistent, let \(N\) be a finitely generated \(R/(x)\)-module with \(\text{Tor}_i^{R/(x)}(N, N) = 0\) for all \(i \gg 0\). By \([21, 11.65]\) (see also \([13, \text{Lem. 2.1.}]\)) there is a long exact sequence,

\[
\cdots \rightarrow \text{Tor}_i^{R/(x)}(N, N) \rightarrow \text{Tor}_i^{R}(N, N) \rightarrow \text{Tor}_i^{R/(x)}(N, N) \rightarrow \cdots.
\]

Therefore \(\text{Tor}_i^R(N, N) = 0\) for all \(i \gg 0\). Since \(R\) is Tor-persistent, we get that \(\text{pd}_R(N)\) is finite. As \(x \notin m^2\), it follows that \(\text{pd}_{R/(x)}(N)\) is finite; see e.g. \([2, \text{Prop. 3.3.5(1)}]\). □

2.3 Remark. It would be interesting to know if the last assertion in Proposition 2.2 holds without the assumption \(x_i \notin m^2 + (x_1, \ldots, x_{i-1})\), i.e. if Tor-persistence is preserved when passing to the quotient by an ideal generated by any regular sequence; cf. Proposition 3.3.1.

2.4 Remark. The sequence \(X_1, \ldots, X_n\) is regular on \(R[X_1, \ldots, X_n]\) and \(X_1\) does not belong to \((m, X_1, \ldots, X_n)^2 + (X_1, \ldots, X_{i-1})\). It follows from Proposition 2.2 that \(R\) is Tor-persistent if and only if \(R[X_1, \ldots, X_n]\) is Tor-persistent.

Proposition 2.2 can be used to construct new examples of Tor-persistent rings from known examples; see Example 2.7. However, to do so it is useful to have a concrete way of constructing regular sequences with the property mentioned in 2.2. In Lemma 2.6 below we give one such construction.
If $A$ is a commutative ring and $a$ is an element in $A$, then it can happen, perhaps surprisingly, that $X - a$ is a zero-divisor on $A[[X]]$; see [12] p. 146 for an example. However, as is well-known, if $A$ is noetherian, then the situation is much nicer:

2.5. Let $A$ be a commutative noetherian ring and consider an element $f = f(X_1, \ldots, X_n)$ in $A[X_1, \ldots, X_n]$. It follows from [11] Thm. 5 that if $f$ has some coefficient which is a unit in $A$, then $f$ is a non zero-divisor on $A[X_1, \ldots, X_n]$.

2.6 Lemma. Let $(R, m, k)$ be a commutative noetherian local ring. Consider the power series ring $S = R[[X_1, \ldots, X_n]]$ and write $n = (m, X_1, \ldots, X_n)$ for its unique maximal ideal. Let $0 = m_0 < m_1 \cdots < m_{t-1} < m = n$ be integers and let $f_1, \ldots, f_t \in n$ be elements such that, for every $i = 1, \ldots, t$, the following conditions hold:

(a) $f_i \in R[X_1, \ldots, X_m] \subseteq S$.
(b) The element $\frac{\partial f_i}{\partial X_i}(0, \ldots, 0) \in R$ is a unit for some $m_i < j$.

Then $f_1, \ldots, f_t$ is a regular sequence on $R[[X_1, \ldots, X_n]]$ with $f_i \notin n^2 + (f_1, \ldots, f_{i-1})$ for all $i$.

Proof. First note that condition (b) implies:

The power series $f_i(0, \ldots, 0, X_{m_i+1}, \ldots, X_n)$ has a coefficient which is a unit in $R$. (2.1)

Indeed, if $m_i < j$, then $\frac{\partial f_i}{\partial X_i}(0, \ldots, 0)$ is a coefficient in $f_i(0, \ldots, 0, X_{m_i+1}, \ldots, X_n)$.

Next we show that $f_1, \ldots, f_t$ is a regular sequence. With $i = 1$ condition (2.1) says that $f_1(X_1, \ldots, X_n)$ has a coefficient which is a unit in $R$, and so $f_1$ is a non zero-divisor on $S$ by 2.5. Next we show that $f_{i+1}$ is a non zero-divisor on $S/(f_1, \ldots, f_i)$ where $i \geq 1$. Write

$$f_{i+1} = \sum_{j=m_{i+1}}^{m_n} h_{m_{i+1}, \ldots, m_n} X_{m_{i+1}}^{\nu_{m_{i+1}}} \cdots X_n^{\nu_n} \in S \cong R[X_1, \ldots, X_m][[X_{m_{i+1}}, \ldots, X_n]]$$

with $h_s \in R[X_1, \ldots, X_m]$. As $f_1, \ldots, f_i \in R[X_1, \ldots, X_m]$ by (a) there is an isomorphism:

$$S/(f_1, \ldots, f_i) \cong (R[X_1, \ldots, X_m]/(f_1, \ldots, f_i))[X_{m_{i+1}}, \ldots, X_n].$$

In particular, the image $f^*_{i+1}$ of $f_{i+1}$ in $S/(f_1, \ldots, f_i)$ can be identified with the element

$$f^*_{i+1} = \sum_{j=m_{i+1}}^{m_n} \bar{h}_{m_{i+1}, \ldots, m_n} X_{m_{i+1}}^{\nu_{m_{i+1}}} \cdots X_n^{\nu_n}$$

in the right-hand side of (2.3), where $\bar{h}_s$ is the image of $h_s$ in $R[X_{m_{i+1}}, \ldots, X_n]/(f_1, \ldots, f_i)$. Hence, to show that $\bar{f}_{i+1}$ is a non zero-divisor, it suffices by 2.5 to argue that one of the coefficients $\bar{h}_s$ is a unit. By (2.1) we know that $f_{i+1}(0, \ldots, 0, X_{m_{i+1}}, \ldots, X_n)$ has a coefficient which is a unit in $R$, and by (2.2) this means that one of the elements $h_{m_{i+1}, \ldots, m_n}(0, \ldots, 0) \in R$ is a unit. Consequently $\bar{h}_{m_{i+1}, \ldots, m_n} = h_{m_{i+1}, \ldots, m_n}(X_{m_{i+1}}, \ldots, X_m)$ will be a unit in $R[X_{m_{i+1}}, \ldots, X_n]$, so its image $\bar{h}_{m_{i+1}, \ldots, m_n}$ is also a unit, as desired.

Next we show that $f_i \notin n^2 + (f_1, \ldots, f_{i-1})$ holds for all $i$. Suppose for contradiction that:

$$f_i = \sum p_i q_i + \sum_{n=1}^{i-1} g_n f_n,$$

where $p_i, q_i \in n$ and $g_n \in S$.

By assumption (b) we have that $\frac{\partial f_i}{\partial X_i}(0, \ldots, 0) \in R$ is a unit for some $m_i < j$. It follows from the identity above that:

$$\frac{\partial f_i}{\partial X_i} = \sum p_i \frac{\partial f_i}{\partial X_i}(0, \ldots, 0) + \sum_{n=1}^{i-1} \left( \frac{\partial g_n}{\partial X_i}(0, \ldots, 0) f_n(0, \ldots, 0) + \frac{\partial g_n}{\partial X_i}(0, \ldots, 0) \right).$$

As already mentioned, the left-hand side is a unit, and this contradicts the right-hand side belongs to $m$. Indeed, we have $p_i(0, \ldots, 0), f_n(0, \ldots, 0) \in m$ as $p_i, q_i, f_n \in n$. Furthermore, $f_1, \ldots, f_{i-1}$ only depend on the variables $X_1, \ldots, X_{m_{i-1}}$ by (a), so every $\frac{\partial g_n}{\partial X_i}$ is zero.

2.7 Example. In $R[U, V, W]$ the following (more or less arbitrarily chosen) sequence, corresponding to $t = 2$ and $m_1 = 2$, satisfies the assumptions of Lemma 2.6:

$$f_1 = a + U^3 + UV + V \quad \text{and} \quad f_2 = b + UV^2 + W + W^2 \quad (a, b \in m).$$

Indeed, (a) is clear and (b) holds since $\frac{\partial f_2}{\partial X_i}(0, 0, 0) = 1 = \frac{\partial f_1}{\partial X_i}(0, 0, 0)$. So Proposition 2.2 implies that if $R$ is Tor-persistent, then so is $A = R[U, V, W]/(f_1, f_2)$. 

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Note that the fiber product ring
\[ R = k[[X]]/(X^4) \times_k k[[Y]]/(Y^2) \cong k[[X,Y]]/(X^4, Y^2, XY) \]
is artinian, not Gorenstein, and by [13, Thm. 1.1] it is Tor-persistent. Hence the following ring (where we have chosen \( a = Y^2 \) and \( b = X^2 \)) is Tor-persistent as well:
\[ A = k[X, Y, U, V, W]/(X^4, Y^2, XY, Y^2 + U\beta + V, X^2 + UV + W, W + W) . \]

**Proof of Theorem 1.1** The equivalence \((i) \iff (iii)\) is noted in Remark 2.2. Let \( a_1, \ldots, a_n \) be a set of elements that generate \( m \). We have \( \hat{R} \cong R[X_1, \ldots, X_n]/(X_1 - a_1, \ldots, X_n - a_n) \) by [17, Thm. 8.12]. The sequence \( f_i = X_i - a_i \) clearly satisfies the assumptions in Lemma 2.6 so the equivalence \((i) \iff (ii)\) follows. Note that \( R[X_1, \ldots, X_n]/(m, x_1, \ldots, x_n) \) and \( R[X_1, \ldots, X_n]/\mathfrak{a} \) have isomorphic completions (both are isomorphic to \( \hat{R}[X_1, \ldots, X_n] \)), so the equivalence \((iii) \iff (iv)\) follows from the already established equivalence between \((i)\) and \((ii)\).

\[ \square \]

3. Connections with the Gorenstein dimension

In this section, we give a few remarks and observations pertaining Aulander’s G-dimension [11] and self Tor vanishing. For a commutative noetherian local ring \((R, m, k)\), we consider the following property (which \( R \) may, or may not, have):

\(\text{\text{(TG)}}\) Every finitely generated \( R \)-module \( M \) satisfying \( \text{Tor}^R_i(M, M) = 0 \) for all \( i \gg 0 \) has finite \( G \)-dimension, that is, \( \text{G-dim}_R(M) < \infty \).

Every Tor-persistent ring has the property \(\text{(TG)}\), see [9, Prop. (1.2.10)], and the converse holds if the maximal ideal \( m \) is decomposable; see [20, Thm. 5.5].

Testing finiteness of the \( G \)-dimension via the vanishing of Tor, in some form, is an idea pursued in a number of papers. For example, in [7, Thm. 3.11] it was proved that a finitely generated module \( M \) over a commutative noetherian ring \( R \) has finite \( G \)-dimension if and only if the stable homology \( \text{Tor}^R_i(M, R) \) vanishes for every \( i \in \mathbb{Z} \). Furthermore, finitely generated modules testing finiteness of the \( G \)-dimension via the vanishing of absolute homology, i.e. Tor, were also examined in [8].

For the property \(\text{(TG)}\) we have the following stronger version of Proposition 2.2.

**3.1 Proposition.** Let \( (R, m, k) \) be a commutative noetherian local ring and let \( \underline{a} = x_1, \ldots, x_n \) be an \( R \)-regular sequence. Then \( \hat{R} \) has the property \(\text{(TG)}\) if and only if \( R/(\underline{a}) \) has it.

**Proof.** For the “if” part we proceed as in the proof of Lemma 2.1 with \( S = R/(\underline{a}) \). Note that having replaced \( M \) with a sufficiently high syzygy, the sequence \( \underline{a} \) becomes regular on \( M \) (this is standard but see also [19, Lem. 5.1]). From the finiteness of \( \text{G-dim}_{R/(\underline{a})}(M/(\underline{a})M) \) we infer the finiteness of \( \text{G-dim}_R(M) \) from [9, Cor. (1.4.6)]. For the “only if” part proceed as in the proof of Proposition 2.2. From the finiteness of \( \text{G-dim}_R(M) \) one always gets finiteness of \( \text{G-dim}_{R/(\underline{a})}(N) \) (the assumption \( x \notin m^2 \) is not needed) by [9, Thm. p. 39]. \[ \square \]

Now the arguments in the proof of Theorem 1.1 applies and give the following.

**3.2 Theorem.** Let \( (R, m, k) \) be a commutative noetherian local ring. The following conditions are equivalent:

\( i \) \( R \) has the property \(\text{(TG)}\).
\( ii \) \( \hat{R} \) has the property \(\text{(TG)}\).
\( iii \) \( R[X_1, \ldots, X_n] \) has the property \(\text{(TG)}\).
\( iv \) \( R[X_1, \ldots, X_n]/(m, x_1, \ldots, x_n) \) has the property \(\text{(TG)}\).

\[ \square \]

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