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HOMOLOGY PRO STABILITY FOR TOR-UNITAL PRO RINGS

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Abstract. Let \( \{A_m\} \) be a pro system of associative commutative, not necessarily unital, rings. Assume that the pro systems \( \{\text{Tor}_i^{\mathbb{Z}, A_m}(\mathbb{Z}, \mathbb{Z})\}\) vanish for all \( i > 0 \). Then we prove that the sequence

\[
\{H_i(\text{GL}_n(A_m))\}_m \to \{H_i(\text{GL}_{n+1}(A_m))\}_m \to \{H_i(\text{GL}_{n+2}(A_m))\}_m \to \cdots
\]

stabilizes up to pro isomorphisms for \( n \) large enough than \( l \) and the stable range of \( A_m \)'s.

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0. Introduction

0.1. The homology stability for general linear groups is a simple but deep question on homological algebra. Let \( R \) be an associative unital ring. We consider the general linear groups \( \text{GL}_n(R) \) of \( R \) and their sequence

\[
\text{GL}_n(R) \hookrightarrow \text{GL}_{n+1}(R) \hookrightarrow \text{GL}_{n+2}(R) \hookrightarrow \cdots,
\]

where each embedding is given by sending \( \alpha \) to \(( \alpha 0 0 1 \)). The question is whether the induced sequence of the integral group homology

\[
H_i(\text{GL}_n(R)) \to H_i(\text{GL}_{n+1}(R)) \to H_i(\text{GL}_{n+2}(R)) \to \cdots
\]

stabilizes for \( n \) large enough than \( l \). There have been many works on this problem, and the most striking result was obtained by Suslin.

Theorem 0.1 (Suslin [Su82]). Let \( R \) be an associative unital ring and \( l \geq 0 \). Then the canonical map

\[
H_i(\text{GL}_n(R)) \to H_i(\text{GL}_{n+1}(R))
\]

is surjective for \( n \geq \max(2l, l + \text{sr}(R) - 1) \) and bijective for \( n \geq \max(2l + 1, l + \text{sr}(R)) \), where \( \text{sr}(R) \) is the stable range of \( R \).

Things become much harder and interesting if we consider non-unital rings. Then the homology stability is strongly related with the \( K \)-theory excision and the Tor-unitality.

Let \( R \) be an associative unital ring and \( A \) a two-sided ideal of \( R \). We define the \( n \)-th relative \( K \)-group by

\[
K_n(R, A) := \pi_n \text{hofib}(B \text{GL}(R)^+ \to B \text{GL}(R/A)^+).
\]
We say that \( A \) satisfies the \( K \)-theory excision if, for every unital ring \( R \) which contains \( A \) as a two-sided ideal and for every \( n \geq 1 \), the canonical map

\[
K_n(\mathbb{Z} \times A, A) \xrightarrow{\sim} K_n(R, A)
\]

is an isomorphism. It is well-known that the \( K \)-theory excision fails in general. However, if the homology \( H_i(\text{GL}_n(A)) \) stabilizes for \( n \) large enough, then \( A \) satisfies the \( K \)-theory excision. Such being the case, the homology stability for non-unital rings fails in general, even if the stable range of \( A \) is finite.

On the other hand, in [Su95], Suslin has completely determined the obstruction to the \( K \)-theory excision: An associative ring \( A \) satisfies the \( K \)-theory excision if and only if \( A \) is Tor-unital, i.e. \( \text{Tor}_i^{\mathbb{Z}[A]}(\mathbb{Z}, \mathbb{Z}) = 0 \) for all \( i > 0 \). Hence, we may hope that Tor-unital rings satisfy the homology stability. Again, Suslin has given a partial solution.

**Theorem 0.2** (Suslin [Su96]). Let \( A \) be a Tor-unital \( \mathbb{Q} \)-algebra, \( r = \max(\text{sr}(A), 2) \) and \( l \geq 0 \). Then the canonical map

\[
H_i(\text{GL}_n(A)) \rightarrow H_i(\text{GL}_{n+1}(A))
\]

is surjective for \( n \geq 2l + r - 2 \) and bijective for \( n \geq 2l + r - 1 \).

Unfortunately, commutative rings really happen to be Tor-unital. Instead, a recent trend has been to think Tor-unital pro rings. We say that a pro system \( \{A_m\} \) of associative rings is Tor-unital if the pro system \( \{\text{Tor}_i^{\mathbb{Z}[A_m]}(\mathbb{Z}, \mathbb{Z})\}_m \) vanish for all \( i > 0 \). A notable discovery by Morrow [Mo15] is that, for any ideal \( A \) of a noetherian commutative ring, the pro ring \( \{A^m\}_{m \geq 1} \) is Tor-unital. Besides, Geisser-Hesselholt [GH06] has generalized Suslin’s excision theorem to the pro setting: If \( \{A_m\} \) is a Tor-unital pro ring then the canonical map

\[
\{K_n(\mathbb{Z} \times A_m, A_m)\}_m \xrightarrow{\sim} \{K_n(R_m, A_m)\}_m
\]

is a pro isomorphism for any pro system of unital rings \( \{R_m\} \) with a level map \( \{A_m\} \rightarrow \{R_m\} \) which exhibits each \( A_m \) as a two-sided ideal of \( R_m \).

Our main theorem is a pro version of Theorem 0.2.

**Theorem 0.3.** Let \( \{A_m\} \) be a commutative Tor-unital pro ring\(^1\), \( r = \max(\text{sr}(A_m), 2) \) and \( l \geq 0 \). Then the canonical map

\[
\{H_i(\text{GL}_n(A_m))\}_m \rightarrow \{H_i(\text{GL}_{n+1}(A_m))\}_m
\]

is a pro epimorphism for \( n \geq 2l + r - 2 \) and a pro isomorphism for \( n \geq 2l + r - 1 \).

It follows from Theorem 0.3 that if \( \{A_m\} \) is commutative Tor-unital then the action of \( \text{GL}_n(\mathbb{Z}) \) on \( \{H_i(\text{GL}_n(A_m))\}_m \) is pro trivial for \( n \geq 2l + r - 1 \), cf. Corollary 4.13. Together with the standard argument this reproves Geisser-Hesselholt’s pro excision theorem for commutative Tor-unital rings of finite stable range.

0.2. **Outline.** Suslin used the Malcev theory in the proof of Theorem 0.2, which works only for \( \mathbb{Q} \)-algebras. More precisely, he used the Malcev theory to get an acyclicity of the union of triangular spaces, cf. [Su96, Corollary 5.6]. We prove the pro version of the acyclicity by using a totally different method; it is closer to the methods developed in [Su82, Su95].

In §1, we prove the pro stability for \( H_i(\text{GL}_n) \), cf. Theorem 1.5. This essentially follows from Vaseršteı́n’s stability for relative \( K_1 \). In §2, we recall some properties of Tor-unital rings, which we need later. §3 is the technical heart of this paper. In this section, we study triangular spaces and prove a pro acyclicity of the union of triangular spaces, cf. Theorem 3.9. In §4, we complete the proof of Theorem 0.3, using the pro acyclicity of triangular spaces.

\(^1\)“commutative” means that each \( A_m \) is commutative. However, this condition may not be essential. We expect that the theorem is true without the commutativity assumption.
0.3. **Notation.**

1. A ring means an associative, not necessarily unital, ring.
2. \( sr(A) \) is the stable range of a ring \( A \), i.e. the minimum number \( r \geq 1 \) such that the stable range condition \([V_{a69}, (2.2)_n]\) holds for every \( n \geq r \).
3. Let \( A \) be a ring and \( n \geq 1 \).
   (a) The general linear group \( \text{GL}_n(A) \) is the kernel of the canonical map \( \text{GL}_n(\mathbb{Z} \ltimes A) \to \text{GL}_n(\mathbb{Z}) \).
   (b) The elementary subgroup \( E_n(A) \) is the subgroup of \( \text{GL}_n(A) \) generated by the elementary matrices \( e_{ij}(a) \) with \( a \in A \) and \( 1 \leq i \neq j \leq n \).
   We regard \( \text{GL}_n(A) \) as a subgroup of \( \text{GL}_{n+1}(A) \) by sending a matrix \( \alpha \) to \( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \). We write \( \text{GL}(A) = \bigcup_n \text{GL}_n(A) \) and \( E(A) = E_\infty(A) = \bigcup_n E_n(A) \).
4. A pro ring is a pro system of rings indexed by a filtered poset. Typically, we denote a pro ring by a bold letter \( \mathbf{A} = \{A_m\} \) and the structured maps \( A_m \to A_n \) by \( \iota_{m,n} \) or just by \( \iota \).
5. A unital (resp. commutative) pro ring is a pro ring which is levelwise unital (resp. commutative).
   Unless otherwise stated, we use standard operations of rings levelwise for pro rings: E.g. \( \text{GL}_n(A) = \{\text{GL}_n(A_m)\}_m \), \( \text{Tor}_{\mathbb{Z} \ltimes A}^2(Z, Z) = \{\text{Tor}_{\mathbb{Z} \ltimes A}^2(Z, Z)\}_m \), etc.
6. A left ideal of a pro ring \( \mathbf{A} = \{A_m\}_{m \in J} \) is a pro ring \( \mathbf{B} = \{B_m\}_{m \in J} \) with a level map \( \mathbf{B} \to \mathbf{A} \) which exhibits each \( B_m \) as a left ideal of \( A_m \).

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1. Pro stability for $K_1$

1.1. Vaseršteǐn’s stability. Let $R$ be a unital ring and $A$ a two-sided ideal of $R$. The normal elementary subgroup $E_n(R, A)$ is the smallest normal subgroup of $E_n(R)$ which contains $E_n(A)$. We write $E(R, A) = E_{\infty}(R, A) = \bigcup_n E_n(R, A)$. By Whitehead’s lemma, $E(R, A)$ is a normal subgroup of $GL(A)$. We define the relative $K_1$-group $K_1(R, A)$ to be the quotient group $GL(A)/E(R, A)$.

**Theorem 1.1** (Vaseršteǐn [Va69]). The canonical map

$$GL_n(A) \to K_1(R, A)$$

is surjective for $n \geq sr(A)$, and the kernel is $E_n(R, A)$ for $n \geq sr(A) + 1$.

1.2. Let $R$ be a unital ring and $A$ a two-sided ideal of $R$. The following lemma generalizes [Ti76, ??] for noncommutative rings.

**Lemma 1.2.** For $n \geq 3$, $E_n(R, A^2) \subset [E_n(A), E_n(A)]$.

**Proof.** Note the standard equality of elementary matrices;

$$[e_{ij}(a), e_{kl}(b)] = \begin{cases} 1 & \text{if } j \neq k, i \neq l \\ e_{il}(ab) & \text{if } j = k, i \neq l \\ e_{kj}(-ba) & \text{if } j \neq k, i = l, \end{cases}$$

which we use throughout the proof. One immediate consequence is that $E_n(A^2) \subset [E_n(A), E_n(A)]$ for $n \geq 3$.

For $r = (r_1, \ldots, r_n) \in R^n$ with $r_j = 1$, we write

$$X_j(r) := \prod_{k \neq j} e_{jk}(r_k) \quad \text{and} \quad X^j(r) := \prod_{k \neq j} e_{kj}(r_k).$$

Fix $1 \leq j \leq n$. It is easy to see that every $x \in E_n(R)$ has the form

$$x_{2m}(U) := X^1(u_{2m})X_j(u_{2m-1}) \cdots X^1(u_2)X_j(u_1)$$

for some $m > 0$ and $U = (u_1, u_2, \ldots, u_{2m}) \in (R^n)^{2m}$. We also set $x_0(\emptyset) := 1$ and

$$x_{2m-1}(V) := X^1(v_{2m-1})X_j(v_{2m-2}) \cdots X_j(v_2)X^1(v_1)$$

for $m > 0$ and $V = (v_1, v_2, \ldots, v_{2m-1}) \in (R^n)^{2m-1}$.

Consider the following assertion.

$(\triangledown)_N$ For every $U \in (R^n)^N$, $x_N(U)E_n(A^2)x_N(U)^{-1} \subset [E_n(A), E_n(A)]$.

We have seen $(\triangledown)_0$. Let $N > 0$ and suppose that $(\triangledown)_l$ holds for $l < N$. We shall prove $(\triangledown)_N$ in case $N$ even; the case $N$ odd is proved in the same way.

Let $U = (u_1, \ldots, u_N) \in (R^n)^N$ and $x := x_N(U)$. For $e_{ik}(a)$ with $a \in A^2$, $1 \leq i, k \leq n$ and $k \neq j$, we have $X_j(u_1)e_{ik}(a)X_j(-u_1) \in E_n(A^2)$ and thus by the induction hypothesis $xe_{ik}(a)x^{-1} \in [E_n(A), E_n(A)]$. For $e_{ij}(a)$ with $a \in A^2$ and $1 \leq i \neq j \leq n$, we have

$$X_j(u_1)e_{ij}(a)X_j(-u_1) = e_{ji}(u_{1,i})\left(\prod_{k \neq i,j} e_{ik}(-au_{1,k}) \cdot e_{ij}(a)\right)e_{ji}(-u_{1,i})$$

$$= \prod_{k \neq i,j} e_{ik}(-u_{1,i}au_{1,k})e_{ik}(-au_{1,k}) \cdot e_{ji}(u_{1,i})e_{ij}(a)e_{ji}(-u_{1,i}).$$

Hence, it follows from the induction hypothesis that $xe_{ij}(a)x^{-1}$ is generated by $y_ie_{ij}(a)y_i^{-1}$, $y_i = X^j(u_N)X_j(u_{N-1}) \cdots X^j(u_2)e_{ji}(u_{1,i})$, with $a \in A^2$ and $1 \leq i \neq j \leq n$ modulo $[E_n(A), E_n(A)]$.

For $U = (u_1, \ldots, u_N) \in (R^n)^N$ and $1 \leq p \leq N/2$, we set

$$y_i^{2p-1}(U) := X^j(u_N)X_j(u_{N-1}) \cdots X^j(u_{2p})e_{ji}(u_{2p-1,i}) \cdots e_{ij}(u_{2,i})e_{ji}(u_{1,i})$$

$$y_i^{2p}(U) := X^j(u_N)X_j(u_{N-1}) \cdots X_j(u_{2p+1})e_{ij}(u_{2p,i}) \cdots e_{ij}(u_{2,i})e_{ji}(u_{1,i}).$$

We claim that:
\[(\diamond)_{Q}\] For \(U \in (R^n)^N\), \(x_N(U)E_n(A^2)x_N(U)^{-1}\) is generated by \(y_i^Q(U)e_{ij}(a)y_i^Q(U)^{-1}\), \(a \in A^2\), \(1 \leq i \neq j \leq n\) modulo \([E_n(A), E_n(A)]\).

We have seen \((\diamond)_1\). Let \(Q > 1\) and suppose that \((\diamond)_1\) holds for \(l < Q\). We prove \((\diamond)_Q\) in case \(Q\) even; the case \(Q\) odd is proved in the same way.

Let \(U = (u_1, \ldots, u_N) \in (R^n)^N\). According to \((\diamond)_{Q-1}\), \(x_N(U)E_n(A^2)x_N(U)^{-1}\) is generated by \(y_i^{Q-1}(U)e_{ij}(a)y_i^{Q-1}(U)^{-1}\), \(a \in A^2\), \(1 \leq i \neq j \leq n\) modulo \([E_n(A), E_n(A)]\). We fix \(1 \leq i \neq j \leq n\) for a moment. Now, \(\Box\)

\[
X^j(u_{Q})e_{ji}(u_{Q-1,i}) = e_{ij}(u_{Q,i})e_{ji}(u_{Q-1,i}) \prod_{k \neq i,j} e_{kj}(u_{Q,k})e_{ki}(u_{Q,k}u_{Q-1,i}).
\]

Hence, by putting \(\bar{y} := \prod_{k \neq i,j} e_{ki}(u_{2p,k}u_{2p-1,i})\), we have

\[
y_i^{Q-1}(U) = X^j(u_{Q})X_j(u_{Q-1}) \cdots X_j(u_{Q+1})e_{ij}(u_{Q,i})e_{ji}(u_{Q-1,i})X^j(u_{Q-2}) \cdots X^j(u)X_j(u')\bar{y}
\]

for some \(u_1', \ldots, u_{Q-2}' \in R^n\) with \(u_{Q,i}' = u_{q,i}'\). For \(Q - 1 \leq q \leq N\), we set

\[
u_{q}' := \begin{cases} u_{q,i}e_i + e_j & \text{if } q = Q - 1, Q \\ u_q & \text{if } q > Q \end{cases}
\]

and \(U' := (u_1', \ldots, u_N')\), so that \(y_i^{Q-1}(U) = x_N(U')\bar{y}\) and \(y_i^Q(U') = y_i^Q(U)\) for \(q \geq Q\). By applying \((\diamond)_{Q-1}\) for \(U'\), we see that \(x_N(U')E_n(A^2)x_N(U')^{-1}\) is generated by \(y_i^Q(U')e_{ij}(a)y_i^Q(U')^{-1}\), \(a \in A^2\) modulo \([E_n(A), E_n(A)]\). Varying \(i\), this proves \((\diamond)_Q\) for the given \(U \in (R^n)^N\), and thus for all \(U \in (R^n)^N\).

According to \((\diamond)_N\), to prove \((\triangledown)_N\), it suffices to show that \(ye_{ij}(ab)y^{-1} \in [E_n(A), E_n(A)]\) for \(y = e_{ij}(r_N)e_{ji}(r_{N-1}) \cdots e_{ij}(r_2)e_{ji}(r_1)\) with \(a, b \in A\), \(r_1, \ldots, r_N \in R\) and \(1 \leq i \neq j \leq n\). Observe that we have

\[
e_{ij}(r_1)e_{ji}(ab)e_{ij}(r_1) = e_{ij}(r_1)e_{ji}(a)e_{ij}(b)e_{ij}(-r_1) = e_{ij}(r_1)e_{ji}(a)e_{ij}(-br_1)e_{ij}(b)
\]

for \(t \neq i, j\). Now, it is clear that \(y'[e_{ij}(r_1a)e_{ji}(a), e_{ij}(-br_1)e_{ij}(b)]y^{-1} \in [E_n(A), E_n(A)]\) for \(y' = e_{ij}(r_N)e_{ji}(r_{N-1}) \cdots e_{ij}(r_2)\), and thus we get \((\triangledown)_N\). \(\Box\)

**Corollary 1.3.** Let \(R = \{R_m\}\) be a unital pro ring and \(A = \{A_m\}\) a two-sided ideal of \(R\). Suppose that \(A/A^2 = \{A_m/A_m^2\} = 0\). Then, for \(3 \leq n \leq \infty\), the canonical maps

\[
E_n(A) \xrightarrow{\sim} E_n(R, A)
\]

\[
\uparrow \quad \uparrow
\]

\[
\uparrow \quad \uparrow
\]

\[
\sim \quad \sim
\]

\[
[E_n(A), E_n(A)] \xrightarrow{\sim} [E_n(R, A), E_n(R, A)]
\]

are pro isomorphisms.

**Proof.** Since all the indicated maps are injections, it suffices to show that the map \([E_n(A), E_n(A)] \to E_n(R, A)\) is a pro epimorphism. By the assumption \(A/A^2 = 0\), there exists \(s \geq m\) for each \(m\) such that \(\epsilon_{s,m}(A_m) \subset A_m^2\). Therefore,

\[
\epsilon_{s,m}E_n(R_s, A_s) \subset E_n(R_m, A_m^2) \subset [E_n(R_m, A_m), E_n(R_m, A_m)],
\]

where the last inclusion is by Lemma 1.2. This proves that \([E_n(A), E_n(A)] \to E_n(R, A)\) is a pro epimorphism. \(\Box\)
1.3. **Pro excision and pro stability.** Let \( R = \{ R_m \} \) be a unital pro ring and \( A = \{ A_m \} \) a two-sided ideal of \( R \). We define \( \text{sr}(A) := \max_m (\text{sr}(A_m)) \).

**Theorem 1.4** (Pro excision). Suppose that \( A/A^2 = 0 \). Then the canonical map
\[
H_1(\text{GL}(A)) \xrightarrow{\sim} K_1(R, A)
\]
is a pro isomorphism.

**Proof.** Since \( K_1(R, A) \) is levelwise abelian, we have a levelwise exact sequence
\[
0 \rightarrow H_1(E(R, A)) \rightarrow H_1(\text{GL}(A)) \rightarrow K_1(R, A) \rightarrow 0.
\]
According to Corollary 1.3, \( H_1(E(R, A)) = 0 \), and thus we get \( H_1(\text{GL}(A)) \simeq K_1(R, A) \).

**Theorem 1.5** (Pro stability). Suppose that \( A/A^2 = 0 \). Then the canonical map
\[
H_1(\text{GL}_n(A)) \rightarrow H_1(\text{GL}(A))
\]
is a pro epimorphism for \( n \geq \text{sr}(A) \) and a pro isomorphism for \( n \geq \max(3, \text{sr}(A) + 1) \).

**Proof.** The composite
\[
H_1(\text{GL}_n(A)) \rightarrow H_1(\text{GL}(A)) \xrightarrow{\sim} K_1(R, A)
\]
is a levelwise surjection for \( n \geq \text{sr}(A) \) by Theorem 1.1. Since the last map is a pro isomorphism by Theorem 1.4, the first map is a pro epimorphism for \( n \geq \text{sr}(A) \).

Consider the commutative diagram
\[
\begin{array}{ccccccccc}
H_1(E_n(R, A)) & \rightarrow & H_1(\text{GL}_n(A)) & \rightarrow & H_1(\text{GL}_n(R, A)/E_n(R, A)) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_1(E(R, A)) & \rightarrow & H_1(\text{GL}(A)) & \rightarrow & H_1(K_1(R, A)) & \rightarrow & 0
\end{array}
\]
with levelwise exact rows. The left terms are zero for \( n \geq 3 \) by Corollary 1.3. According to Theorem 1.1, the right vertical map is a levelwise bijection for \( n \geq \text{sr}(A) + 1 \). Hence, the middle term is a pro isomorphism for \( n \geq \max(3, \text{sr}(A) + 1) \).

**Theorem 1.6.** Set \( \bar{E}_n(A) := \text{GL}_n(A) \cap E(A) \). Suppose that \( A/A^2 = 0 \). Then the canonical map
\[
E_n(A) \rightarrow \bar{E}_n(A)
\]
is a pro isomorphism for \( n \geq \max(3, \text{sr}(A) + 1) \).

**Proof.** Let \( \bar{E}_n(R, A) := \text{GL}_n(A) \cap E(R, A) \). According to Theorem 1.1, the canonical map \( E_n(R, A) \rightarrow \bar{E}_n(R, A) \) is a levelwise bijection for \( n \geq \text{sr}(A) + 1 \). Hence, the theorem follows from Corollary 1.3.
2. Tor-unital pro rings

The treatment of this section closely follows Suslin [Su95] and Geisser-Hesselholt [GH06].

2.1. Definitions.

Definition 2.1. A pro ring $A = \{A_m\}$ is Tor-unital if

$$\text{Tor}_i^{Z \times A} (Z, Z) = \{\text{Tor}_i^{Z \times A_m} (Z, Z)\}_m = 0$$

as pro abelian groups for all $i > 0$.

Example 2.2.
(i) A unital pro ring, i.e. a pro system of unital rings, is Tor-unital.
(ii) (Morrow [Mo15]) Let $A$ be an ideal of a noetherian commutative ring, then $\{A_m\}_m \geq 1$ is Tor-unital.

Definition 2.3. Let $A = \{A_m\} \in J$ be a pro ring.
(i) A left $A$-module is a pro abelian group $M = \{M_m\}_m \in J$ with a level map $A \times M \to M$ which exhibits each $M_m$ as a left $A_m$-module. A morphism between left $A$-modules $M = \{M_m\}_m$ and $N = \{N_m\}_m$ is a level map $f : M \to N$ such that each $f_m : M_m \to N_m$ is a morphism of left $A_m$-modules.
(ii) A left $A$-module $P$ is pseudo-free if there is an isomorphism of left $A$-modules $A \otimes L \cong P$ for some pro system $L$ of free abelian groups. We call such an $L$ a free basis of $P$.
(iii) A morphism $f : P \to M$ of left $A$-modules is special if $P$ is pseudo-free with a free basis $L$ and $f$ is induced from a level morphism of pro abelian groups $L \to M$.

2.2. Pro resolution.

Proposition 2.4 (Suslin [Su95], Geisser-Hesselholt [GH06]). Let $A = \{A_m\}_m \in J$ be a Tor-unital pro ring. Suppose we are given an augmented complex

$$\ldots \to C_1 \to C_0 \xrightarrow{\epsilon} C_{-1}$$

of left $A$-modules such that: 

(i) Each $C_k$ with $k \geq -1$ is pseudo-free.
(ii) The homology $H_k(C_{\bullet}, m)$ is annihilated by $A_m$ for every $m$ and $k \geq -1$.

Then

$$H_k(C_{\bullet}) = \{H_k(C_{\bullet}, m)\}_m = 0$$

for all $k \geq -1$.

In fact, a finer result holds.

Proposition 2.5. Let $A = \{A_m\}_m \in J$ be a Tor-unital pro ring and $k \geq -1$. Then there exists $s(m) \geq m$ for each $m \in J$ such that the map

$$\iota_{s(m), m} : H_k(C_{\bullet, s(m)}) \to H_k(C_{\bullet, m})$$

is zero for all augmented complexes of left $A$-modules which satisfy the conditions (i) (ii).

Proof. Let $C$ be a pseudo-free left $A$-module with a free basis $L$. Then we have levelwise isomorphisms

$$\text{Tor}_q^{Z \times A} (Z, C) \cong \text{Tor}_q^{Z \times A} (Z, A \otimes L) \cong \text{Tor}_q^{Z \times A} (Z, A) \otimes L \cong \text{Tor}_q^{Z \times A} (Z, Z) \otimes L.$$

Since $A$ is Tor-unital, we see that

$$\text{Tor}_q^{Z \times A} (Z, C) = 0$$

for every $q \geq 0$. 

\footnote{We thank Takeshi Saito for pointing out an unnecessary condition, the augmentation $\epsilon$ is special, which was in the first draft and in [Su95, GH06] too.}
Let $Z_k$ and $B_{k-1}$ are the kernel and the image of $C_k \to C_{k-1}$ respectively. By the assumption (ii), we have a levelwise inclusion $AC_{-1} \subseteq B_{-1}$, and thus there is a levelwise surjection $C_{-1}/AC_{-1} \to H_{-1}(C_\ast)$. Since $C_{-1}$ is pseudo-free, $C_{-1}/AC_{-1} = \Tor_{0}^{Z \ltimes A}(Z, C_{-1}) = 0$. Therefore, $H_{-1}(C_\ast) = 0$.

Let $k \geq 0$ and suppose that $H_l(C_\ast) = 0$ for $l < k$. Consider the levelwise spectral sequence $E^1_{pq} = \begin{cases} \Tor_{q}^{Z \ltimes A}(Z, C_p) & \text{if } 0 \leq p \leq k \\ \Tor_{q}^{Z \ltimes A}(Z, Z_k) & \text{if } p = k + 1 \\ 0 & \text{otherwise} \end{cases}$ which arises from the brutal truncation of the complex

$$Z_k \to C_k \to C_{k-1} \to \ldots \to C_0.$$ 

By the induction hypothesis, the complex is pro quasi-isomorphic to $C_{-1}$ and thus $E^\infty_q \simeq \Tor_{q}^{Z \ltimes A}(Z, C_{-1}) = 0$ for $q \geq 0$. Since $C_p$ is pseudo-free, we also have $E^1_{pq} = 0$ for $0 \leq p \leq k$. Hence,

$$Z_k/AZ_k = \Tor_{0}^{Z \ltimes A}(Z, Z_k) = E^\infty_{k+1} = 0.$$ 

On the other hand, by the assumption (ii), we have $AZ_k \subseteq B_k$. Therefore, $H_k(C_\ast) = 0$.

A finer variant is also clear from this proof. \qed

Lemma 2.6. Let $A$ be a pro ring and $P$ a pseudo-free left $A$-module. Then there exists an augmented complex $P_\ast$ of left $A$-modules with $P_{-1} = P$ which satisfies the conditions (i), (ii) and (iii) The augmentation $\epsilon : P_0 \to P_{-1}$ is special.

Proof. Write $P = \{P_m\}$ and let $Z[P] = \{Z[P_m]\}$ be the pro system of the free abelian groups generated by the sets $P_m$. Then $P_0 := A \otimes Z[P]$ is a pseudo-free $A$-module and the canonical map $Z[P] \to P$ induces a special morphism $\epsilon : P_0 \to P$.

Let $R = \{R_m\}$ be the kernel of $\epsilon$, and $Z[R] = \{Z[R_m]\}$ the pro system of the free abelian group generated by $R_m$. Then $P_1 := A \otimes Z[R]$ is a pseudo-free $A$-module. Repeating this procedure, we obtain an augmented complex $P_\ast$ with $P_{-1} = P$ which satisfies the desired conditions. \qed
3. Pro acyclicity of triangular spaces

The goal of this section is to prove Theorem 3.9.

3.1. Preliminaries on homology.

3.1.1. For a simplicial set $X$, we denote by $C_*(X)$ the complex freely generated by $X_*$ with the differential being the alternating sum of the faces. We write $H_n(X) = H_n(C_*(X))$. Also, we write $\tilde{H}_n(X)$ for the reduced homology.

Let $G$ be a group. We write $EG$ for the simplicial set whose degree $n$-part is $G^{\times(n+1)}$ and whose $i$-th face (resp. the $i$-th degeneracy) omits the $i$-th entry (resp. repeats the $i$-th entry). We give a right $G$-action on $EG$ by $(g_0, \ldots, g_n) \cdot g := (g_0g, \ldots, g_ng)$. The classifying space $BG$ is defined to be $EG/G$.

3.1.2. By a pro object or pro system, we mean a pro object whose index category is a left filtered small category.

**Lemma 3.1.** Let $f: X \to Y$ be a morphism between pro systems of pointed simplicial sets. Suppose that $f$ induces pro isomorphisms

$$\pi_n(X) \sim \pi_n(Y)$$

for all $n \geq 0$. Then $f$ induces pro isomorphisms

$$H_n(X) \sim H_n(Y)$$

for all $n \geq 0$.

**Proof.** Since $\mathbb{Z}\pi_0(X) \simeq H_0(X)$, the assertion is clear for $n = 0$. Hence, by taking the connected components, we may assume that $X$ and $Y$ are connected.

Then, according to [Is01], for each $n \geq 1$, the induced map $P_n(X) \to P_n(Y)$ on the $n$-th Postnikov sections is a strict weak equivalence, i.e. isomorphic to a levelwise weak equivalence. Hence, the induced map $C_*(P_n(X)) \to C_*(P_n(Y))$ is isomorphic to a levelwise quasi-isomorphism.

On the other hand, by Hurewicz theorem and Serre spectral sequence, we have levelwise isomorphisms $H_k(X) \simeq H_k(P_n(X))$ for $k \leq n$. Now, in the commutative diagram

$$\begin{array}{ccc}
H_k(X) & \longrightarrow & H_k(Y) \\
\sim & & \sim \\
H_k(P_n(X)) & \longrightarrow & H_k(P_n(Y))
\end{array}$$

the vertical maps and the bottom map are pro isomorphisms for $k \leq n$, and so is the top map. Since $n$ is arbitrary, $H_k(X) \xrightarrow{\sim} H_k(Y)$ is a pro isomorphism for every $k \geq 0$. \qed

For a simplicial group $G$, we consider the bi-simplicial set $BG$ constructed degreewise. For a bi-simplicial set $X$, we denote by $C_*(X)$ the double-complex freely generated by $X_*$ with the differential being the alternating sum of the faces.

**Corollary 3.2.** Let $f: P \to Q$ be a morphism between pro systems of simplicial abelian groups. Suppose that $f$ induces pro isomorphisms

$$\pi_n(P) \sim \pi_n(Q)$$

for all $n \geq 0$. Then $f$ induces pro isomorphisms

$$H_n(\text{Tot } C_*(BP)) \sim H_n(\text{Tot } C_*(BQ))$$

for all $n \geq 0$.

**Proof.** Now, the morphism $B_kP \to B_kQ$ induces pro isomorphisms $\pi_n(B_kP) \to \pi_n(B_kQ)$ for all $n \geq 0$. Hence, by Lemma 3.1, the induced maps

$$H_n(C_*(B_kP)) \to H_n(C_*(B_kQ))$$

are pro isomorphisms for all $n \geq 0$. By the standard spectral sequence, we obtain the corollary. \qed
3.1.3. Let us quote a lemma from [Su95, §2].

**Lemma 3.3.** Let $G$ be a group and $H$ a group with a left $G$-action. Then there exists a natural quasi-isomorphism

$$C_\ast(B(G \rtimes H)) \simeq C_\ast(EG) \otimes_G C_\ast(BH).$$

Let $G = \{G_m\}$ be a pro group (= a pro system of groups). A left $G$-module $M$ is a pro abelian group $M = \{M_m\}$ with a level map $G \times M \to M$ which exhibits each $M_m$ as a left $G_m$-module. A morphism between left $G$-modules $M = \{M_m\}$ and $N = \{N_m\}$ is a level map $f : M \to N$ such that each $f_m : M_m \to N_m$ is a morphism of left $G_m$-modules. These form the category of left $G$-modules, and we consider simplicial objects in this category; simplicial left $G$-modules and morphisms between them.

**Corollary 3.4.** Let $G$ be a pro group. Let $P$ and $Q$ be simplicial left $G$-modules and $f : P \to Q$ a morphism between them. Suppose that $f$ induces pro isomorphisms

$$\pi_n(P) \xrightarrow{\sim} \pi_n(Q)$$

for all $n \geq 0$. Then $f$ induces pro isomorphisms

$$H_n(\operatorname{Tot} C_\ast(B(G \rtimes P))) \xrightarrow{\sim} H_n(\operatorname{Tot} C_\ast(B(G \rtimes Q)))$$

for all $n \geq 0$, where the semi-direct products are taken levelwise and degreewise.

**Proof.** This follows from Corollary 3.2 and Lemma 3.3. \qed

3.2. The key lemma.

3.2.1. Triangular spaces. Let $A$ be a ring and $P$ a left $A$-module. Let $\sigma$ be a finite poset. We define a group $T^\sigma(A, P)$ by

$$T^\sigma(A, P) := \prod_{i_0, j_0 \leq \max \sigma} A_{i_0, j_0} \times \prod_{i_0, j_0 \leq \max \sigma} P_{i_0, j_0},$$

where $A_{i_0, j_0}$ and $P_{i_0, j_0}$ are just the copies of $A$ and $P$ respectively. For $\alpha \in T^\sigma(A, P)$, we denote its $(i, j)$-th component by $\alpha_{i, j}$; thus $\alpha_{i, j} \in A$ if $j \not\in \max \sigma$, and $\alpha_{i, j} \in P$ if $j \in \max \sigma$. For $\alpha, \beta \in T^\sigma(A, P)$, we define the composition $\alpha \cdot \beta$ by

$$(\alpha \cdot \beta)_{i, j} = \alpha_{i, j} + \beta_{i, j} + \sum_{i_0 < k_0 < \max \sigma} \alpha_{i_0, k_0} \beta_{k_0, j}$$

for $i_0, j_0$. We shorten $T^\sigma(A) := T^\sigma(A, A)$.

Set $\sigma_0 := \sigma \setminus \max \sigma$. Then we have an identification

$$T^\sigma(A, P) = T^{\sigma_0}(A) \times M_{\sigma_0, \max \sigma}(P),$$

and canonical inclusion and projection

$$T^{\sigma_0}(A) \hookrightarrow T^\sigma(A, P) \twoheadrightarrow T^{\sigma_0}(A).$$

Let $\theta : \sigma \to \tau$ be a morphism of finite posets. Then it induces a morphism of groups

$$T^\theta : T^\sigma(A) \to T^\tau(A).$$

If $\theta^{-1}(\max \tau) = \max(\sigma)$, then it also induces a morphism $T^\sigma(A, P) \to T^\tau(A, P)$ for any left $A$-module $P$, which we also denote by $T^\theta$.

Let $f : P \to Q$ be a morphism of $A$-modules. Then it induces a morphism of groups

$$T^f : T^\sigma(A, P) \to T^\sigma(A, Q)$$

If $\theta : \sigma \to \tau$ satisfies $\theta^{-1}(\max \tau) = \max(\sigma)$, then we define

$$T^{f, \theta} : T^\sigma(A, P) \to T^\tau(A, Q)$$

to be the composite $T^f \circ T^\theta = T^\theta \circ T^f$. 

3.2.2. For a finite poset $\sigma$ and $p \geq 0$, let $[p]$ be the poset $0 < 1 < 2 < \cdots < p$ and endow $\sigma \times [p]$ with the lexicographical order. We define
\[
\sigma \star [p] := \sigma \times [p] \setminus \max \sigma \times \{1, \ldots, p\}.
\]
We denote by $\phi$ (resp. $\varphi$) the embedding $\sigma \rightarrow \sigma \times [p]$ (resp. $\sigma \rightarrow \sigma \star [p]$), $a \mapsto (a, 0)$. Note that $\varphi^{-1}(\max(\sigma \star [p])) = \max \sigma$ and that $(\sigma \star [p])_0 = \sigma \times [p]$.

The following lemma is a variant of Lemma 7.4 in [Su82].

**Lemma 3.5.** Let $\{A_m\}_{m \in \Xi}$ be a commutative Tor-unital pro ring and $l \geq 0$. Then there exist $p_l \geq 0$ and $s_l(m) \geq m$ for each $m \in \Xi$ such that:

(i) For all finite posets $\sigma$ and all pseudo-free $\{A_m\}$-modules $\{P_m\}$, the map
\[
t_{s_l(m), m} H_1(T^\sigma) : \tilde{H}_1(T^\sigma(A_{s_l(m)}, P_{s_l(m)})) \to \tilde{H}_1(T^\sigma \star [p_l](A_m, P_m))
\]
is equal to zero.

(ii) For all finite posets $\sigma$ and all special morphisms $f : \{P_m\} \rightarrow \{Q_m\}$ between pseudo-free $\{A_m\}$-modules, the map
\[
t_{s_l(m), m} H_1(T^{f \star \varphi}) : \tilde{H}_1(T^\sigma(A_{s_l(m)}, P_{s_l(m)})) \to \tilde{H}_1(T^\sigma \star [p_l](A_m, Q_m))
\]
is equal to zero.

**Proof.** We prove the lemma by induction on $l \geq 0$. The case $l = 0$ is clear, here we can take $p_0 = 0$ and $s_0(m) = m$. Let $L > 0$ and suppose that we have constructed $p_0 \leq p_1 \leq \cdots \leq p_{L-1}$ and $s_0(m) \leq s_1(m) \leq \cdots \leq s_{L-1}(m)$ which satisfy the conditions of the lemma.

Set $q := p_{L-1} + 1$ and $t(m) := s_{L-1}(m)$. First, we prove the following.

**Sublemma 3.6.** For all finite posets $\sigma$ and all special morphisms $f : \{P_m\} \rightarrow \{Q_m\}$ between pseudo-free $\{A_m\}$-modules, the diagram
\[
\begin{array}{ccc}
H_L(T^\sigma(A_{t(m)}, P_{t(m)})) & \xrightarrow{t_{s_l(m), m} H_1(T^{f \star \varphi})} & H_L(T^\sigma \star [q](A_m, Q_m)) \\
\downarrow H_L(T^{\sigma \circ_t}) & & \downarrow H_L(T^{\sigma \circ_t \star [q]}) \\
H_L(T^{\sigma \circ_t}(A_{t(m)})) & \xrightarrow{t_{s_l(m), m} H_1(T^\sigma)} & H_L(T^{\sigma \circ_t \star [q]}(A_m))
\end{array}
\]
commutes, where the vertical maps are the canonical projection and inclusion.

**Proof.** Let $f : \{P_m\} \rightarrow \{Q_m\}$ be a special morphism between pseudo-free $\{A_m\}$-modules and $\{L_m\}$ a free basis of $\{P_m\}$ such that $f$ is induced from a map $\{L_m\} \rightarrow \{Q_m\}$. Note that we have an equality $\lim_{i \uparrow} L_m^{(i)} = \{L_m\}$, where $\{L_m^{(i)}\}$ is a sub-system of $\{L_m\}$ such that each $L_m^{(i)}$ is finitely generated and the limit runs around all such systems. Hence, we have
\[
\lim_{i \uparrow} C_*(B M_{n,k}(A_m \otimes L_m^{(i)})) \simeq C_*(B M_{n,k}(P_m))
\]
for every $m$ and $n, k \geq 1$. It follows that
\[
C_*(B T^\sigma(A_m, P_m)) \simeq \lim_{i \uparrow} C_*(B T^\sigma(A_m, A_m \otimes L_m^{(i)}))
\]
and
\[
H_*(T^\sigma(A_m, P_m)) \simeq \lim_{i \uparrow} H_*(T^\sigma(A_m, A_m \otimes L_m^{(i)})).
\]

Consequently, to show the sublemma, we may assume that $\{P_m\} = \{A_m \otimes \mathbb{Z} L_m\}$ with $L_m$ a free abelian group of finite rank. We may also assume that $\{Q_m\} = \{A_m \otimes \mathbb{Z} M_m\}$ with $M_m$ a free abelian group of finite rank.

Fix $m \in \Xi$. Let $e_1, \ldots, e_l$ be a basis of $L_{t(m)}$ and $f_1, \ldots, f_j$ a basis of $M_m$. Since $f$ is special, the map $t_{s_l(m), m} f : P_{t(m)} \rightarrow Q_m$ is induced by a map $\alpha : L_{t(m)} \rightarrow Q_m$, which sends $e_i$ to $\sum_j \alpha_{i,j} f_j$ with $\alpha_{i,j} \in A_m$. We may also denote $t_{s_l(m), m} f$ by $\alpha$. 
If $\alpha = 0$, then the diagram

\[
\begin{array}{ccc}
H_L(T^\sigma(A_{t(m)}, P_{t(m)})^{\psi(A_{t(m)}, P_{t(m)})}) & \xrightarrow{\psi(A_{t(m)}, P_{t(m)})} & H_L(T^\sigma\star[q](A_m, Q_m)) \\
\downarrow & & \downarrow \\
H_L(T^\sigma\gamma(A_{t(m)})) & \xrightarrow{\psi(A_{t(m)}, P_{t(m)})} & H_L(T^\sigma\gamma\times[q](A_m))
\end{array}
\]

commutes, and thus the sublemma holds in this case. Let $(u, v) \in [1, I] \times [1, J]$ and suppose that the sublemma holds if $\alpha_{i,j} = 0$ for $(i, j) \geq (u, v)$ with respect to the lexicographical order. We prove the sublemma in case $\alpha_{i,j} = 0$ for $(i, j) > (u, v)$. We define a morphism $\beta: P_{t(m)} \to Q_m$ by sending $e_i$ to $\delta_{i,u} f_v$.

We define an embedding $\psi: \sigma \to \sigma \star [q]$ by

$$
\psi(x) = \begin{cases} 
(x, 0) & \text{if } x \in \max \sigma \\
(x, q) & \text{if } x \notin \max \sigma.
\end{cases}
$$

Then the image $\tau$ of $\psi$ intersects with $\sigma \star [q - 1] \subset \sigma \star [q]$ only at $\max \sigma \times \{0\}$, and thus the composite

$$
T^\sigma \xrightarrow{\psi} T^\sigma \times T^\sigma \star [q - 1] \times T^\sigma \prod T^\sigma \star [q]
$$

is a group homomorphism. Now, we have a group morphism

$$
T^\sigma(A_{t(m)}, P_{t(m)}) \xrightarrow{\psi A_{t(m)}, P_{t(m)}} T^\sigma \star [q - 1](A_m, Q_m) \times T^\sigma \star [q](A_m, Q_m),
$$

which we denote by $T^{\alpha, \psi} \cdot T^{\beta, \psi}$. Since $q - 1 = p_{L-1}$ and $t(m) = s_{L-1}(m)$, by the induction hypothesis and by the Künne formula, we obtain

$$
(3.1) \quad H_L(T^{\alpha, \psi} \cdot T^{\beta, \psi}) = H_L(T^{\alpha, \psi}) + H_L(T^{\beta, \psi}).
$$

Set

$$
\omega := \prod_{x \in \sigma_0} e_{\varphi(x), \psi(x)}(\alpha_{u,v}) \in T^\sigma \star [q](A_m).
$$

We define $(\alpha'_{i,j}) \in M_{L, I}(A^m)$ by $\alpha'_{i,j} = \alpha_{i,j}$ unless $(i, j) = (u, v)$ and $\alpha'_{u,v} = 0$, which induces a map $\alpha': Q_{t(m)} \to P_m$ by sending $e_i$ to $\sum_j \alpha'_{i,j} f_j$.

Claim 3.7. We have an equality $^3$

$$
\Ad(\omega) \circ (T^{\alpha', \psi} \cdot T^{\beta, \psi}) = T^{\alpha, \psi} \cdot T^{\beta, \psi}.
$$

We calculate the $(k, l)$-entry of (3.2) at $U \in T^\sigma(A_{t(m)}, P_{t(m)})$. It suffices to do this for:

1. $(k, l) = (\varphi(x), \varphi(y))$ with $x \in \sigma_0$ and $y \in \sigma$.
2. $(k, l) = (\varphi(x), \psi(y))$ with $x \in \sigma_0$ and $y \in \sigma_0$.
3. $(k, l) = (\psi(x), \psi(y))$ with $x \in \sigma_0$ and $y \in \sigma_0$.
4. $(k, l) = (\psi(x), \varphi(y))$ with $x \in \sigma_0$ and $y \in \sigma_0$.

---

$^3$Here is the only place we need the commutativity of pro rings
Case (1):
\[
(\text{Ad}(\omega) \circ (T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} \\
= ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} + \alpha_{u,v} \cdot ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} \\
= T^{\alpha',\varphi}(U)_{\varphi(x),\varphi(y)} + \alpha_{u,v} \cdot T^{\beta,\psi}(U)_{\varphi(x),\varphi(y)} \\
= \begin{cases} 
U_{x,y} & \text{if } y \in \sigma_0 \\
\alpha'(U_{x,y}) + \beta(U_{x,y})\alpha_{u,v} = \alpha(U_{x,y}) & \text{if } y \in \text{max } \sigma 
\end{cases} \\
= ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)}.
\]

Case (2):
\[
(\text{Ad}(\omega) \circ (T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} \\
= \alpha_{u,v} \cdot ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} - ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} \cdot \alpha_{u,v} \\
= \alpha_{u,v}U_{x,y} - U_{x,y}\alpha_{u,v} \\
= 0 \\
= ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)}.
\]

Case (3):
\[
(\text{Ad}(\omega) \circ (T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} \\
= ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} - ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} \cdot \alpha_{u,v} \\
= (T^{\alpha,\varphi} \cdot T^{\beta,\psi})(U)_{\varphi(x),\varphi(y)}.
\]

Case (4):
\[
(\text{Ad}(\omega) \circ (T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} = ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)} = ((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U))_{\varphi(x),\varphi(y)}.
\]

Consequently, we obtain the equality in the claim.

Again, by the induction hypothesis and by the Künneth formula, we obtain
\[
H_L(T^{\alpha,\varphi} \cdot T^{\beta,\psi}) = H_L(T^{\alpha',\varphi} \cdot T^{\beta,\psi}) \\
= H_L(T^{\alpha',\varphi}) + H_L(T^{\beta,\psi}).
\]

(3.2)

It follows from (3.1, 3.2) that
\[
H_L(T^{\alpha,\varphi}) = H_L(T^{\alpha',\varphi}).
\]

Therefore, by induction, we get the sublemma. \(\square\)

We return to the proof of Lemma 3.5. We prove (i) \(i=L\). Let \(\{P_m\}\) be a pseudo-free \(\{A_m\}\)-module.
Let \(\{P_m[-]\}\) be a pro resolution of \(\{P_m\}\), cf. Lemma 2.6. Then, by Proposition 2.4 and Corollary 3.4, \(\{P_m[-]_{\geq 0}\}\) \(\rightarrow\) \(\{P_m\}\) induces a pro isomorphism
\[
\Theta: \{H_L(T^\sigma (A_m, P_m[-]_{\geq 0}))\}m \xrightarrow{\sim} \{H_L(T^\sigma (A_m, P_m))\}m.
\]

In fact, for each \(m \in \Xi\) there exists \(r(m) \geq m\), which does not depend on \(\{P_m\}\), \(\{P_m[-]\}\) and \(\sigma\), such that the maps \(r_{(r(m))}: \ker \Theta_{(r(m))} \rightarrow \ker \Theta_{m}\) and \(r_{(r(m))}: \coker \Theta_{(r(m))} \rightarrow \coker \Theta_{m}\) are equal to zero, cf. Proposition 2.5.

We set
\[
p := p_L := \left( \prod_{l=1}^{L-1} (p_l + 1) \right) (q + 1) - 1, \\
s(m) := s_L(m) := r(s_1(\cdots (s_{L-1}(t(m)))\cdots)).
\]
We claim that \((i)_{i=L}\) holds with these definitions. We prove it by induction on \(n := \#\sigma \geq 1\). The case \(n = 1\) is clear, and so let \(n > 1\).

By Lemma 3.3, we have
\[
C_\ast(BT^\sigma(A_m, P_m[\geq 0])) = C_\ast(ET^\sigma(A_m)) \otimes_{T^\sigma(A_m)} C_\ast(BM_{\ast, \max}(P_m[\geq 0]))
\]
and thus we have a first quadrant spectral sequence
\[
(E^1_{s,t})^\sigma = H_i(T^\sigma(A_m, P_m[s])) \Rightarrow H_{s+t}(T^\sigma(A_m, P_m[\geq 0])).
\]
It is clear that \((E^2_{s,t})^\sigma = 0\) for \(s > 0\). Hence, the spectral sequence induces a filtration
\[
0 = F^\sigma_{-1,0} \subset F^\sigma_{0,0} \subset \cdots \subset F^\sigma_{L-1,0} = H_L(T^\sigma(A_m, P_m[\geq 0]))
\]
with \(F^\sigma_{i,m}/F^\sigma_{i-1,m}\) a subquotient of \(H_{L-i}(T^\sigma(A_m, P_m[i]))\).

Note that the map \(\varphi : \sigma \to \sigma \cdot p\) induces a morphism of spectral sequences
\[
(E^1_{s,t})^\sigma = H_i(T^\sigma(A_m, P_m[s])) \Rightarrow H_{s+t}(T^\sigma(A_m, P_m[\geq 0]));
\]
\[
(E^1_{s,t})^\sigma \cdot p = H_i(T^\sigma \cdot p(A_m, P_m[s])) \Rightarrow H_{s+t}(T^\sigma \cdot p(A_m, P_m[\geq 0])).
\]

By the induction hypothesis, the induced map
\[
F^\sigma_{i,sL-1,m}/F^\sigma_{i-1,sL-1,m} \to F^\sigma_{i,m}/F^\sigma_{i-1,m}
\]
is zero for \(1 \leq i \leq L - 1\). Also, observe that \((\sigma \cdot p)[0][p] = \sigma \cdot ((a + 1)(b + 1) - 1)\). It follows that, by putting \(s'(m) := s_1(\cdots(s_{L-1}(t(m)))\cdots)\) and \(p' := \prod_{l=1}^{L-1}(p_l + 1) - 1\), the canonical map
\[
\iota_{s'(m),t(m)}H_L(T^\sigma) : H_L(T^\sigma(A_{s'(m)}, P_{s'(m)}[\geq 0])) \to H_L(T^\sigma \cdot p')(A_{t(m)}, P_{t(m)}[\geq 0])
\]
factors through \(F^\sigma_{0,1,t(m)}\).

Now, we have lifts in the commutative diagram

Consider the following diagram

\[
\begin{array}{ccc}
H_L(T^\sigma(A_{s(m)}, P_{s(m)}[0])) & \xrightarrow{\Theta} & H_L(T^\sigma \cdot p')(A_{t(m)}, P_{t(m)}[\geq 0]) \\
H_L(T^\sigma(A_{s(m)}, P_{s(m)})) & \xrightarrow{\Theta} & H_L(T^\sigma \cdot p')(A_{t(m)}, P_{t(m)})) \\
\end{array}
\]
where we omit the structured maps $\iota_{s,*}$. The right rectangle commutes by Sublemma 3.6, though the lower right square may not commute. It follows from a simple diagram chase that the bottom rectangle commutes. Therefore, by the induction hypothesis for $n = \#\sigma$, the middle composite $\iota_{s,(m),m}H_L(T^\sigma)$ equals zero.

Finally, $(ii)_{i,L}$ follows immediately from $(i)_{i,L}$ and Sublemma 3.6.

3.2.3. Let $\sigma$ be a partial ordering on $\{1, \ldots, n\}$. Then we can naturally regard $T^\sigma(A)$ as a subgroup of $GL_n(A)$. For $k \geq 0$, we define $k\bar{\sigma}$ to be the partial ordering on $\{1, \ldots, n + k\}$ obtained from $\sigma$ by adding the relations $i < n + j$ for $i \in \{1, \ldots, n\}$ and $1 \leq j \leq k$.

Set
\[ k\bar{T}^\sigma(A, P) := \begin{cases} T^\sigma(A) & \text{if } k = 0, \\ T^{k\bar{\sigma}}(A, P) & \text{if } k \geq 1. \end{cases} = T^\sigma(A) \times M_{n,k}(P). \]

We write $\Pi_n$ for the set of all partial orderings on $\{1, \ldots, n\}$.

**Corollary 3.8.** Let $A$ be a commutative Tor-nilpotent pro ring and $l, n \geq 0$. Let $\sigma_1, \ldots, \sigma_t \in \Pi_n$. Then there exists $p \geq 0$ such that the canonical map
\[ \tilde{H}_t\left( \bigcup_{i=1}^t B^{k\bar{T}^\sigma_i}(A, P) \right) \to \tilde{H}_t\left( \bigcup_{i=1}^t B^{k\bar{T}^\sigma_i \times [p]}(A, P) \right) \]
is equal to zero as a pro morphism for all $k \geq 0$ and all pseudo-free $A$-modules $P$, where the unions are taken in $B(GL_n(A) \times M_{n,k}(P))$ and $B(GL_n^{(p+1)}(A) \times M_{n,(p+1),k}(P))$ respectively.

In particular, there exists $N_n \geq n$ such that the canonical map
\[ \tilde{H}_t\left( \bigcup_{\sigma \in \Pi_n} B^{k\bar{T}^\sigma}(A, P) \right) \to \tilde{H}_t\left( \bigcup_{\sigma \in \Pi_{N_n}} B^{k\bar{T}^\sigma}(A, P) \right) \]
is equal to zero for all $k \geq 0$ and all pseudo-free $A$-modules $P$.

**Proof.** Note that
\[ k\bar{T}^\sigma \times [p](A, P) = \begin{cases} T^\sigma[A,p] & \text{if } k = 0, \\ T^{k\bar{\sigma}}[A,p](A, P) & \text{if } k \geq 1. \end{cases} \]

Hence, the case $t = 1$ is true by Lemma 3.5. Let $t > 1$ and suppose that the corollary holds for $s < t$.

We abbreviate $k\bar{T}^\sigma(A, P)$ as $\bar{T}^\sigma$. Set $\sigma_{i,t} := \sigma_i \cap \sigma_t$. Then we have a commutative diagram
\[
\begin{array}{c}
\tilde{H}_t\left( \bigcup_{i=1}^{t-1} B^{\bar{T}^\sigma_i} \right) \oplus \tilde{H}_t\left( B^{\bar{T}^\sigma_t} \right) \to \tilde{H}_t\left( \bigcup_{i=1}^{t-1} B^{\bar{T}^\sigma_i} \right) \to \tilde{H}_{t-1}\left( \bigcup_{i=1}^{t-1} B^{\bar{T}^\sigma_i} \right)
\end{array}
\]
with exact rows. By the induction hypothesis, the right vertical map is zero for some $q \geq 0$. Thus, there exists a lift as indicated above. Again, by the induction hypothesis, there exists $q' \geq 0$ such that the map
\[
\tilde{H}_t\left( \bigcup_{i=1}^{t-1} B^{\bar{T}^\sigma_i} \right) \oplus \tilde{H}_t\left( B^{\bar{T}^\sigma_t} \right) \to \tilde{H}_t\left( \bigcup_{i=1}^{t-1} B^{\bar{T}^\sigma_i \times [q]} \right) \oplus \tilde{H}_t\left( B^{\bar{T}^\sigma_t \times [q]} \right)
\]
is zero. It follows from $(\sigma_i \times [q]) \times [q'] = \sigma_i \times [q'']$, $q'' := (q + 1)(q' + 1) - 1$, that the map
\[
\tilde{H}_t\left( \bigcup_{i=1}^{t-1} B^{\bar{T}^\sigma_i} \right) \to \tilde{H}_t\left( \bigcup_{i=1}^{t-1} B^{\bar{T}^\sigma_i \times [q'']} \right)
\]
is zero. This completes the proof. □
3.3. The pro acyclicity theorem. Recall that $k\tilde{T}^\sigma(A) = T^\sigma(A) \rtimes M_{n,k}(A)$ for $\sigma \in \Pi_n$.

**Theorem 3.9.** Let $A$ be a commutative Tor-unital pro ring and $l \geq 0$. Then:

(i) For $n \geq 2l + 1$ and for any $k \geq 0$,

$$\tilde{H}_l\left( \bigcup_{\sigma \in \Pi_n} B^k\tilde{T}^\sigma(A) \right) = 0,$$

where the union is taken in $B(GL_n(A) \rtimes M_{n,k}(A))$.

(ii) For $n \geq 2l$ and for any $k \geq 0$, the canonical map

$$H_l\left( \bigcup_{\sigma \in \Pi_n} BT^\sigma(A) \right) \to H_l\left( \bigcup_{\sigma \in \Pi_n} B^k\tilde{T}^\sigma(A) \right)$$

is a pro isomorphism.

**Proof.** We write $0^k\tilde{X}_n(A) = \bigcup_{\sigma \in \Pi_n} B^k\tilde{T}^\sigma(A)$ and $X_n(A) = 0^k\tilde{X}_n(A)$.

We prove the theorem by induction on $l$. The case $l = 0$ is trivial. Let $L > 0$ and suppose that the theorem holds for $l < L$.

**Sublemma 3.10.** Let $k \geq 0$. The canonical map

$$H_L(0^k\tilde{X}_n(A)) \to H_L(0^k\tilde{X}_{n+1}(A))$$

is a pro epimorphism for $n \geq 2L$ and a pro isomorphism for $n \geq 2L + 1$.

**Proof.** Let us introduce some notation. Let $A$ be a ring, $\sigma \in \Pi_n$ and $1 \leq i \leq n$. We define $T_\sigma^\alpha(A)$ be the subgroup of $T_n^\sigma(A)$ consisting of all $\alpha$ with $\alpha_{i,j} = \alpha_{j,i} = 0$ for $i \neq j$. For $k \geq 0$, we set

$$k\tilde{X}_n^i(A) := \bigcup_{\sigma \in \Pi_n} BT^k\tilde{T}^\sigma(A)$$

and write $k\tilde{X}^{i_1,\ldots,i_p}_n(A)$ for the intersection of $k\tilde{X}^i_n(A), \ldots, k\tilde{X}^i_n(A)$. Then it is easy to see that $k\tilde{X}^{i_1,\ldots,i_p}_n(A) \simeq k\tilde{X}_{n-p}(A)$.

Consider the spectral sequence

$$(3.3) \quad k\tilde{E}_{p,q}^1 = \bigcup_{i_0,\ldots,i_p} H_q\left( k\tilde{X}^{i_0,\ldots,i_p}_{n+1}(A) \right) \Rightarrow H_{p+q}\left( \bigcup_{1 \leq i \leq n+1} k\tilde{X}^i_{n+1}(A) \right).$$

Since $k\tilde{X}^{i_0,\ldots,i_p}_{n+1}(A) \simeq k\tilde{X}_{n-p}(A)$, it follows form the induction hypothesis that

$$k\tilde{E}_{0,L}^2 \simeq H_L(k\tilde{X}_n(A))$$

for $n \geq 2L$. Hence, the canonical map

$$H_L(k\tilde{X}_n(A)) \to H_L\left( \bigcup_{1 \leq i \leq n+1} k\tilde{X}^i_{n+1}(A) \right)$$

is a pro epimorphism for $n \geq 2L$ and a pro isomorphism for $n \geq 2L + 1$. According to [Su82, Corollary 6.6], see also the remark before Theorem 7.1\(^4\), the canonical map

$$H_L\left( \bigcup_{1 \leq i \leq n+1} k\tilde{X}^i_{n+1}(A) \right) \to H_L(k\tilde{X}_{n+1}(A))$$

is a levelwise surjection for $n \geq 2L$ and a levelwise bijection for $n \geq 2L + 1$. Bringing these together, we obtain the sublemma.

\(^4\)The proof works for non-unital rings as it is.
We show \((i)\) \(i = L\). Suppose that \(n \geq 2L + 1\). According to Corollary 3.8, the canonical map

\[
H_L(k \tilde{X}_n(A)) \to H_L(k \tilde{X}_N(A))
\]

is zero for some \(N \geq n\). On the other hand, by Sublemma 3.10, this map is a pro isomorphism, and thus \(H_L(k \tilde{X}_n(A)) = 0\).

To get \((ii)\) \(i = L\), it remains to show that the canonical map

\[
H_L(X_{2L}(A)) \to H_L(k \tilde{X}_{2L}(A))
\]

is a pro isomorphism. By the spectral sequences (3.3), we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & E^2_{2, L-1} & \to & H_L(X_{2L}(A)) & \to & H_L(X_{2L+1}(A)) & \to & 0 \\
0 & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \to & \kappa \tilde{E}^2_{2, L-1} \\
\kappa \tilde{E}^2_{2, L-1} & \to & H_L(k \tilde{X}_{2L}(A)) & \to & H_L(k \tilde{X}_{2L+1}(A)) & \to & 0 \\
\end{array}
\]

with exact rows. Hence, it is enough to show that \(E^2_{2, L-1} \to \kappa \tilde{E}^2_{2, L-1}\) is a pro isomorphism; equivalently it is a pro epimorphism. This follows from the diagram

\[
\begin{array}{ccccccccc}
E^1_{2, L-1} = \bigoplus H_{L-1}(X_{2L-2}(A)) & \to & E^2_{2, L-1} & \to & 0 \\
\downarrow & \simeq & \downarrow & & \downarrow & & \downarrow & & \downarrow & \to & \kappa \tilde{E}^2_{2, L-1} \\
\kappa \tilde{E}^1_{2, L-1} = \bigoplus H_{L-1}(k \tilde{X}_{2L-2}(A)) & \to & \kappa \tilde{E}^2_{2, L-1} & \to & 0 \\
\end{array}
\]

with exact rows. \(\square\)
4. Homology pro stability

In this section, we completes the proof of Theorem 0.3. We follow Suslin [Su96], generalizing his argument to the pro setting.

4.1. Volodin spaces. Let $G$ be a group and $\{G_i\}_{i \in I}$ a family of subgroups of $G$. We define the Volodin space $V(G, \{G_i\}_{i \in I})$ to be the simplicial subset of $EG$ formed by simplices $(g_0, \ldots, g_p)$ such that there exists $i \in I$ and $g_j g_k^{-1} \in G_i$ for all $0 \leq j, k \leq p$.

The simplicial subset $V(G, \{G_i\}_{i \in I}) \subset EG$ is stable under the right action of $G$, and $V(G, \{G_i\})/G = \bigcup_{i \in I} BG_i$. Hence, we have a spectral sequence

\[ E^2_{p,q} = H_p(G, H_q(V(G, \{G_i\}_{i \in I}))) \Rightarrow H_{p+q}(\bigcup_{i \in I} BG_i). \]

Let $A$ be a ring. We consider the Volodin space $V_n(A) := V(E_n(A), \{T^\sigma(A)\}_{\sigma \in \Pi_n})$.

The permutation group $\Sigma_n$ acts on $V_n(A)$ by conjugation, and $E_n(A)$ acts on $V_n(A)$ by the right multiplication.

Here are some properties of the Volodin spaces we need.

**Lemma 4.1** (Suslin-Wodzicki [SW92, Corollary 2.7]). Let $n \geq 1$ and $k \geq 0$. The canonical projection and the inclusion

$V_n(A) \rightrightarrows V \left( \left( \begin{array}{cc} E_n(A) & * \\ 0 & 1_k \end{array} \right), \left\{ \left( \begin{array}{cc} T^\sigma(A) & * \\ 0 & 1_k \end{array} \right) \right\}_{\sigma \in \Pi_n} \right)$

are mutually inverse homotopy equivalences.

**Lemma 4.2** (Suslin-Wodzick [SW92, Lemma 2.8]). For every $n, l \geq 0$, the action of $E_{n+1}(A^2)$ on the image of the canonical map

$H_l(V_n(A)) \to H_l(V_{n+1}(A))$

is trivial.

**Corollary 4.3.** Let $A$ be a pro ring such that $A / A^2 = 0$. Then, for every $n, l \geq 0$, the action of $E_{n+1}(A)$ on the image of the canonical map

$H_l(V_n(A)) \to H_l(V_{n+1}(A))$

is pro trivial.

**Proof.** Write $A = \{A_m\}$. By the assumption, there exists $s \geq m$ for each $m$ such that $\tau_{s,m} A_s \subset A^2_n$. Hence, given $x$ in the image of $H_l(V_n(A_s)) \to H_l(V_{n+1}(A_s))$ and $g \in E_n(A_s)$, we have $\tau_{s,m}(gx) = \tau_{s,m}(x)$. \qed

4.2. Van der Kallen’s acyclicity. Let $A$ be a ring and $n \geq 1$. Fix a unital ring $R$ which contains $A$ as a two sided ideal. Let $I$ be a finite subset of $\{1, \ldots, n\}$ and $R^n$ the free right $R$-module with basis $e_1, \ldots, e_n$. A map $f: I \to R^n$ is called an $A$-unimodular function if $\{f(i)\}_{i \in I}$ forms a basis of a free direct summand of $R^n$ and $f(i) \equiv e_i$ modulo $A$. We denote by $\text{Uni}_{A,n} = \text{Uni}_{A,n}(R)$ the set of all $A$-unimodular functions $f: I \to R^n$, which does not depend on $R$.

We define the associated semi-simplicial set as follows: A $p$-simplex is an $A$-unimodular function $f \in \text{Uni}_{A,n}$ with $|\text{dom} f| = p + 1$. The $i$-th face $d_i: (\text{Uni}_{A,n})_p \to (\text{Uni}_{A,n})_{p-1}, 0 \leq i \leq p$, is defined by

$(f, \text{dom} f = \{i_0, \ldots, i_p\}) \mapsto f|_{\{i_0, \ldots, i_k, \ldots, i_p\}}$.

As in the preceding section, for a semi-simplicial set $X$, we denote by $C_*(X)$ the complex freely generated by $X$, with the differential being the alternating sum of the faces.

The following result is proved by van der Kallen [vdK80] in case $A$ is unital, and the proof can be easily modified for non-unital rings. We can also find the complete proof in [Su96, §2].

**Theorem 4.4.** $\tilde{H}_l(C_*(\text{Uni}_{A,n})) = 0$ for $n \geq 1 + \text{sr}(A) + 1$. 
Let $\text{SU}_{n\downarrow n}(\mathbb{R})$ (resp. $\overline{\text{SU}}_{n\downarrow n}(\mathbb{R})$) be the set of all unimodular functions $f \in \text{Un}_n(R)$ for which there exists $\alpha \in E_n(A)$ (resp. $\alpha \in E_n(R, A)$) such that $f(i) = e_i\alpha$ for all $i \in \text{dom} f$. Then $\text{SU}_{n\downarrow n}$ and $\overline{\text{SU}}_{n\downarrow n}(\mathbb{R})$ are sub semi-simplicial sets of $\text{Un}_n(\mathbb{R})$.

**Corollary 4.5.**

(i) $\bar{H}_i(C_\ast(\text{SU}_{n\downarrow n}(\mathbb{R}))) = 0$ for $n \geq l + \text{sr}(A) + 1$.

(ii) Let $A$ be a pro ring such that $A/A^2 = 0$. Then

$$\bar{H}_i(C_\ast(\text{SU}_{n\downarrow n}(A))) = 0$$

as pro abelian groups for $n \geq l + \text{sr}(A) + 1$.

**Proof.** (i) See [Su96, Corollary 2.8].

(ii) Let $R$ be a unital pro ring which contains $A$ as a two-sided ideal. By Corollary 1.3, the canonical map $\text{SU}_{n\downarrow n} \to \overline{\text{SU}}_{n\downarrow n}(\mathbb{R})$ is a pro isomorphism. Hence, (ii) follows from (i). $\square$

4.3. **Homology pro stability for $V_n$ and $E_n$.** We say that a levelwise action of a pro group $\{G_m\}$ on a pro object $\{M_m\}$ is **pro trivial** if there exists $s \geq m$ for each $m$ such that $s,m(gx) = ts,m(x)$ for all $g \in G_s$ and $x \in M_s$.

**Theorem 4.6.** Let $A$ be a commutative $\text{Tor}$-unital pro ring. Let $r = \max(\text{sr}(A), 2)$ and $l \geq 0$. Then:

(i) The canonical map

$$H_i(V_n(A)) \to H_i(V_{n+1}(A))$$

is a pro epimorphism for $n \geq 2l + r + 1$ and a pro isomorphism for $n \geq 2l + r + 2$.

(ii) The conjugate action of $\Sigma_n$ on $H_i(V_n(A))$ is pro trivial for $n \geq 2l + r + 2$.

(iii) The action of $E_n(A)$ on $H_i(V_n(A))$ is pro trivial for $n \geq 2l + r + 2$.

(iv) The canonical map

$$H_i(E_n(A)) \to H_i\left(\begin{pmatrix} E_n(A) & * \\ 0 & 1_k \end{pmatrix}\right)$$

is a pro isomorphism for $n \geq 2l + r - 2$ and for any $k \geq 0$.

(v) The conjugate action of $\Sigma_n$ on $H_i(E_n(A))$ is pro trivial for $n \geq 2l + r - 1$.

(vi) The canonical map

$$H_i(E_n(A)) \to H_i(E_{n+1}(A))$$

is a pro epimorphism for $n \geq 2l + r - 2$ and a pro isomorphism for $n \geq 2l + r - 1$.

We prove Theorem 4.6 by induction on $l$. The case $l = 0$ is clear. Also, (iv, v, vi)$_{l=1}$ holds for the obvious reasons: (vi, vi)$_{l=1}$ follows from that $H_1(E_n(A)) = 0$ for $n \geq 3$. For (vi)$_{l=1}$, note that we have a levelwise exact sequence

$$M_{n,k}(A)E_n(A) \to H_1(E_n(A) \ltimes M_{n,k}(A)) \to H_1(E_n(A)) \to 0,$$

and it is easy to see that $M_{n,k}(A)E_n(A) = 0$ for $n \geq 2$.

Let $L > 0$. The proof is divided into the four steps.

- **Step 1:** $(i, ii, iii)$_{l=L-1} $\Rightarrow$ $(iii)$_{l=L-1}$.
- **Step 2:** $(iii)$_{l=L-1}, (iv)$_{l=L+1} $\Rightarrow$ $(iv)$_{l=L+1}$.
- **Step 3:** $(iv)$_{l=L+1}, (v, vi)$_{l=L+1} $\Rightarrow$ $(v, vi)$_{l=L+1}$.
- **Step 4:** $(i, ii)$_{l=L-1}, (iii)$_{l=L-1}, (vi)$_{l=L+1} $\Rightarrow$ $(i, ii)$_{l=L-1}$.

**4.4. Step 1: Covering argument I.** Suppose that $(i, ii, iii)$_{l=L-1} hold. We show $(iii)$_{l=L-1}.$

\[5\text{In this step, we only need } \text{Tor}_{1}^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = A/A^2 = 0.\]
4.4.1. Covering spectral sequence. Let $A$ be a ring. For $I \subset \{1, \ldots, n\}$, let $\Pi^I$ be the set of all partial orderings $\sigma$ of $\{1, \ldots, n\}$ for which every $i \in I$ is maximal. Set $V_n(A)^I := \bigcup_{\sigma \in \Pi^I} \pi_1(V_n(A)^\sigma)$. Then $V_n(A) = \bigcup_{i=1}^n V_n(A)^i$, and there is a spectral sequence

$$E^1_{p,q}(A) = \bigcup_{|I|=p+1} H_q(V_n(A)^I) \Rightarrow H_{p+q}(V_n(A)).$$

Let $\text{SU}_A$ be the subset of $\text{SU}_A$ consisting of those functions $f$ with $\text{dom } f = I$. We define a map $\phi: V_n(A)^I \to \text{SU}_A$ by $\phi(\alpha_1, \ldots, \alpha_q)(i) = e_i \alpha_i, i \in I$. Then $\phi$ is a morphism of simplicial sets regarding $\text{SU}_A$ as a constant simplicial set, and the inverse image of the unimodular function $f_0: i \mapsto e_i = V(E_n(A)^I, \{T^\sigma(A)\}_{\sigma \in \Pi^I})$, where $E_n(A)^I$ is the subgroup of $E_n(A)$ generated by elementary matrices $\alpha$ such that $e_i \alpha = e_i$ for all $i \in I$. For each $f \in \text{SU}_A$, choose $\Lambda(f) \in E_n(A)$ with $f(i) = e_i \Lambda(f), i \in I$. Since the map $\phi$ is $E_n(A)$-equivariant, $\Lambda(f)$ gives an isomorphism $\phi^{-1}(f_0) \simeq \phi^{-1}(f)$ and

$$\text{SU}_A \times V(E_n(A)^I, \{T^\sigma(A)\}_{\sigma \in \Pi^I}) \xrightarrow{\sim} V_n(A)^I, \quad (f, u) \mapsto u \Lambda(f).$$

Also, the conjugation by the shuffle permutation $\sigma_1, \sigma_1 \{n-p, \ldots, n\} = I$, gives an isomorphism

$$V(E_n(A)^{n-p, \ldots, n}, \{T^\sigma(A)\}_{\sigma \in \Pi^I}) \xrightarrow{\sim} V(E_n(A)^I, \{T^\sigma(A)\}_{\sigma \in \Pi^I}).$$

Hence, we get an isomorphism

$$\Phi_A: C_p(\text{SU}_A) \otimes H_q(V(E_n(A)^{n-p, \ldots, n}, \{T^\sigma(A)\}_{\sigma \in \Pi^I}) \xrightarrow{\sim} E^1_{p,q}(A).$$

For another choice of $\Lambda'$, there exists $\{\gamma(f) \in E_n(A)^{n-p, \ldots, n}\}_{f \in \text{SU}_A}$ such that $\Phi_{A'}(f, u) = \Phi_A(f, u \gamma(f))$.

Under the isomorphism $\Phi_A$, the differential $d^1: E_{p,q} \to E_{p-1,q}$ is given by, for $f \in \text{SU}_A$ and $u \in H_q(V(E_n(A)^{n-p, \ldots, n}, \{T^\sigma(A)\}_{\sigma \in \Pi^I})$,

$$(4.2) \quad d^1(f \otimes u) = \sum_{k=0}^p (-1)^k d_k f \otimes \tau_{I,k}(0) \alpha_k.$$  

Here, $\alpha_k$ is a certain element in $E_n(A)^{n-p+1, \ldots, n}, \tau_{I,k} := \sigma_{1}^{-1} \sigma_I, \text{ and } \delta$ is the map induced from the canonical embedding $E_n(A)^{n-p, \ldots, n} \to E_n(A)^{n-p+1, \ldots, n}$.

4.4.2. Pro arguments. We write $A = \{A_m\}_{m \in \mathbb{Z}}$.

Set $E_n(A) := \text{GL}_n(A) \cap E(A)$, then the canonical maps

$$H_q(V(\bar{E}_{n-p-1}(A), \{T^\sigma(A)\}_{\sigma \in \Pi^{n-p-1}}))$$

$$H_q(V(\bar{E}_n(A)^{n-p, \ldots, n}, \{T^\sigma(A)\}_{\sigma \in \Pi^{n-p, \ldots, n}}))$$

$$H_q \left( V \left( \left( \begin{array}{cc} \bar{E}_{n-p-1}(A) & \ast \\ 0 & 1_{p+1} \end{array} \right), \left( \begin{array}{cc} T^\sigma(A) \ast \\ 0 & 1_{p+1} \end{array} \right) \right)_{\sigma \in \Pi^{n-p-1}} \right)$$

are levelwise isomorphisms. Indeed, the second map is an isomorphism by definition and the composite is an isomorphism by Lemma 4.1. Hence, by Theorem 1.6, the canonical map

$$\lambda: H_q(V_{n-p-1}(A)) \to H_q(V(E_n(A)^{n-p, \ldots, n}, \{T^\sigma(A)\}_{\sigma \in \Pi^{n-p, \ldots, n}}))$$

is a pro isomorphism for $n-p-1 \geq r+1$.

Suppose that $q < L-1$ and $n-p-1 \geq 2q+r+2$. Then, by (iii) $L-1$, the action of $E_n(A)$ on $H_q(V_{n-p-1}(A))$ is pro trivial. Hence, there exists $s(m) \geq m$ for each $m \in \mathbb{E}$ such that the composite
Hence, in this case, the vanishing of $E_{\phi}$, hence $\psi$ is a pro epimorphism. These imply that
commutes, cf. the formula (4.2). The horizontal maps are the maps $s$ and it is a pro isomorphism.

Since $\lambda$ does not depend on the choice of $\Lambda$, we obtain a morphism of pro complexes

\[ \Psi_m : C_p(\text{Uni}_{A_{s(m)}, n}) \otimes H_q(V_{n-p-1}(A_{s(m)})) \rightarrow E^1_{p,q}(A_m) \]

commutes, cf. the formula (4.2). The horizontal maps are the maps $\Psi_m$ unless $n - p - 1 = 2q + r + 2$; in the last case only the bottom horizontal map can be identified with $\Psi_m$. Consequently, for $q < L - 1$, we obtain a morphism of pro complexes

(4.3) \[ \Psi : \sigma_{\leq n-2q-r-3}(C_*(\text{Uni}_{A,n}) \otimes H_q(V_{n-1-\bullet}(A))) \rightarrow \sigma_{\leq n-2q-r-3}E^1_{*,q}(A) \]

and it is a pro isomorphism.

**Claim 4.7.** For $q < L - 1$ and $0 < p < n - 2q - r - 3$, \[ E^2_{p,q}(A) = 0. \]

**Proof.** Suppose that $q < L - 1$ and $0 < p < n - 2q - r - 3$. Put $F_{p,q}(A) := C_p(\text{Uni}_{A,n}) \otimes H_q(V_{n-p-1}(A))$, which we regard as a complex in $p$ with the differential $\partial := \sum (-1)^k d_k \otimes \delta$. First, we show that $H_p(F_{*,q}(A)) = 0$.

By (i)_{<L-1}, the canonical map $H_q(V_{n-p-1}(A)) \rightarrow H_q(V_{n-p}(A))$ is a pro isomorphism, and thus

\[ \ker(F_{p,q}(A) \rightarrow F_{p-1,q}(A)) \cong Z_p(\text{Uni}_{A,n}) \otimes H_q(V_{n-p-1}(A)), \]

where $Z_p(\text{Uni}_{A,n}) := \ker(C_p(\text{Uni}_{A,n}) \rightarrow C_{p-1}(\text{Uni}_{A,n}))$. According to Corollary 4.5, the differential

\[ C_{p+1}(\text{Uni}_{A,n}) \rightarrow Z_p(\text{Uni}_{A,n}) \]

is a pro epimorphism. Also, by (i)_{<L-1}, the canonical map

\[ H_q(V_{n-p-2}(A)) \rightarrow H_q(V_{n-p-1}(A)) \]

is a pro epimorphism. These imply that $\partial : F_{p+1}(A) \rightarrow \ker(F_{p,q}(A) \rightarrow F_{p-1,q}(A))$ is a pro epimorphism, hence $H_p(F_{*,q}(A)) = 0$.

If $p < n - 2L - r - 3$, then $\Psi$ (4.3) induces a pro isomorphism

\[ H_pF_{*,q}(A) \cong E^2_{p,q}(A). \]

Hence, in this case, the vanishing of $E^2_{p,q}(A)$ follows from the one of $H_p(F_{*,q}(A))$. 
Finally, let \( p = n - 2q - r - 3 \). Then we have a commutative diagram

\[
\begin{array}{ccc}
F_{p+1,q}(A_{s(m)}) & \xrightarrow{\iota_{s(m),m}} & E_{p+1,q}(A_m) \\
\downarrow \theta & & \downarrow d^1 \\
F_{p,q}(A_{s(m)}) & \xrightarrow{\Psi_m} & E_{p,q}(A_m).
\end{array}
\]

Let \( m' \geq m \) and \( x \in \ker(E_{p,q}(A_{m'})) \to E_{p-1,q}(A_{m'}) \). Since \( \Psi \) is a pro isomorphism, if we have taken \( n' \) large enough, \( \iota_{m',m}x \) lifts to \( y \in \ker(F_{p,q}(A_{s(m)}) \to F_{p-1,q}(A_{s(m)})) \) along \( \Psi_m \). Further, since \( H_{p,q}(F_{\bullet,q}(A)) = 0 \), we may assume that \( y = \partial z \) for some \( z \in F_{p+1,q}(A_{s(m)}) \). Hence, \( \iota_{m',m}x \) is in the image of the differential \( d^1 \). This proves \( E_{p,q}^2(A) = 0 \).

4.4.3. Conclusion. Suppose that \( n \geq 2L + r \). If \( p + q = L - 1 \) and \( p > 0 \), then \( q < L - 1 \) and \( 0 < p \leq n - 2q - r - 3 \). Hence, by Claim 4.7, the \( E_{p,q}^2 \) terms with \( p + q = L - 1 \) are zero unless \( E_{0,L-1}^1 \), and the edge map

\( E_{0,L-1}^1(A) \to H_{L-1}(V_n(A)) \)

is a pro epimorphism.

Now, the composite

\[
\begin{array}{ccc}
C_0(S\text{Uni}_{A_m,n}) \otimes H_{L-1}(V_{n-1}(A_m)) & \xrightarrow{id \otimes \lambda} & C_0(S\text{Uni}_{A_m,n}) \otimes H_{L-1}(V(E_n(A_m)^{1,n}), \{T^\sigma(A_m)\}_{\sigma \in \Pi_n}) \\
& \xrightarrow{\Phi_\lambda} & E_{0,L-1}^1(A_m) \\
& \xrightarrow{\text{edge}} & H_{L-1}(V_n(A_m))
\end{array}
\]

is given by \( f \otimes u \mapsto \sigma(i)(\delta u)\sigma^{-1}(i)A(f) \), where \( f \) is an \( A_m \)-unimodular function with \( \text{dom} f = \{i\} \) and \( u \in H_{L-1}(V_{n-1}(A_m)) \). Since the action of \( E_n(A) \) on the image of \( \delta: H_{L-1}(V_{n-1}(A)) \to H_{L-1}(V_n(A)) \) is pro trivial by Corollary 4.3, the above composite yields a pro morphism

(4.4) \( C_0(S\text{Uni}_{A_m,n}) \otimes H_{L-1}(V_{n-1}(A_m)) \to H_{L-1}(V_n(A)), \ f \otimes u \mapsto \sigma(i)(\delta u)\sigma^{-1}(i). \)

Furthermore, since the edge map is a pro epimorphism, \( \Phi_\lambda \) is an isomorphism and \( (id \otimes \lambda) \) is a pro isomorphism, we see that (4.4) is a pro epimorphism.

By Corollary 4.3 again, we conclude that the action of \( E_n(A) \) on \( H_{L-1}(V_n(A)) \) is pro trivial. This proves (iii)\(_{L=L+1}\).

4.5. Step 2: \( V \) to \( E \). Suppose that (iii)\(_{L \leq L+1}\) and (vi)\(_{L \leq L+1}\) hold. We show (vi)\(_{L=L+1}\).

Suppose that \( n \geq 2L + r \) and fix \( k \geq 0 \). We set

\[
\tilde{E}_n(A) := \begin{pmatrix} E_n(A) & * \\ 0 & 1_k \end{pmatrix}, \quad \tilde{T}^\sigma(A) := \begin{pmatrix} T^\sigma(A) & * \\ 0 & 1_k \end{pmatrix}
\]

and \( \tilde{V}_n(A) := V(\tilde{E}_n(A), \{\tilde{T}^\sigma(A)\}_{\sigma \in \Pi_n}) \).

By Lemma 4.1, the canonical inclusion and projection \( V_n(A) \to \tilde{V}_n(A) \) are mutually inverse homotopy equivalences. It follows that the action of \( \begin{pmatrix} 1_k & 0 \\ 0 & 1_k \end{pmatrix} \) on \( H_{L}(\tilde{V}_n(A)) \) is trivial. By (iii)\(_{L \leq L-1}\), the action of \( E_n(A) \) on \( H_q(V_n(A)) \simeq H_q(\tilde{V}_n(A)) \) is pro trivial for \( q \leq L - 1 \). Hence, the action of \( \tilde{E}_n(A) \) on \( H_q(\tilde{V}_n(A)) \) is pro trivial for \( q \leq L - 1 \).
Consider the spectral sequences (4.1) and the canonical map between them:
\[ E^2_{p,q}(A) = H_p(E_n(A), H_q(V_n(A))) \rightleftharpoons \bar{H}_{p+q}(\bigcup_{\sigma \in \Pi_n} BT^\sigma(A)) \]
\[ \tilde{E}^2_{p,q}(A) = H_p(\tilde{E}_n(A), H_q(\tilde{V}_n(A))) \rightleftharpoons \bar{H}_{p+q}(\bigcup_{\sigma \in \Pi_n} B\tilde{T}^\sigma(A)). \]

For \( q \leq L - 1 \), the \( E^2 \)-terms fit into the extensions
\[ 0 \rightarrow H_p(E_n(A)) \otimes H_q(V_n(A)) \rightarrow E^2_{p,q}(A) \rightarrow \text{Tor}(H_{p-1}(E_n(A)), H_q(V_n(A))) \rightarrow 0 \]
\[ 0 \rightarrow H_p(\tilde{E}_n(A)) \otimes H_q(\tilde{V}_n(A)) \rightarrow \tilde{E}^2_{p,q}(A) \rightarrow \text{Tor}(H_{p-1}(\tilde{E}_n(A)), H_q(\tilde{V}_n(A))) \rightarrow 0. \]

By (iv) \( _{L} \), the canonical map \( H_p(E_n(A)) \to \tilde{H}_p(E_n(A)) \) is a pro isomorphism for \( p \leq L \). Hence, the canonical map
\[ E^2_{p,q}(A) \rightarrow \tilde{E}^2_{p,q}(A) \]
is a pro isomorphism for \( p \leq L \) and \( q \leq L - 1 \). Also, \( E^2_{p,q}(A) \simeq \tilde{E}^2_{p,q}(A) \) for all \( q \geq 0 \), since \( H_*(V_n(A)) \simeq H_*(\tilde{V}_n(A)) \). Finally, by Theorem 3.9, the canonical map \( E^\infty_l(A) \to \tilde{E}^\infty_l(A) \) is a pro isomorphism for \( n \geq 2l \).

Bringing these together, we have:
1. \( E^2_{p,q}(A) \simeq \tilde{E}^2_{p,q}(A) \) for \( p + q = L - 1 \).
2. \( E^2_{p,q}(A) \simeq \tilde{E}^2_{p,q}(A) \) for \( p + q = L \).
3. \( E^2_{p,q}(A) \simeq \tilde{E}^2_{p,q}(A) \) for \( p + q = L + 1 \) and \( p \geq 2 \) and \( q \geq 1 \).
4. \( E^\infty_l(A) \simeq \tilde{E}^\infty_l(A) \) and \( E^\infty_{L+1}(A) \simeq \tilde{E}^\infty_{L+1}(A) \).

Then, by Lemma 4.8 below, we conclude that
\[ E^2_{L+1,0}(A) \rightarrow \tilde{E}^2_{L+1,0}(A) \]
is a pro epimorphism, and thus a pro isomorphism. This proves (vi) \( _{L+1} \).

**Lemma 4.8 ([Su96, Remark A.5]).** Let \( A \) be an abelian category. Let \( f : E \to \tilde{E} \) be a morphism of first quadrant homological spectral sequence in \( A \), and let \( L \geq 0 \). Assume that \( f \) induces:
1. A monomorphism \( E^2_{p,q} \hookrightarrow \tilde{E}^2_{p,q} \) for \( p + q = L - 1 \).
2. An isomorphism \( E^2_{p,q} \cong \tilde{E}^2_{p,q} \) for \( p + q = L \).
3. An epimorphism \( E^2_{p,q} \twoheadrightarrow \tilde{E}^2_{p,q} \) for \( p + q = L + 1 \), \( q \geq 1 \) and \( p \geq 2 \).
4. An isomorphism \( E^\infty_{L} \cong \tilde{E}^\infty_{L} \) and an epimorphism \( E^\infty_{L+1} \twoheadrightarrow \tilde{E}^\infty_{L+1} \).

Then \( f \) induces an epimorphism
\[ E^2_{L+1,0} \rightarrow \tilde{E}^2_{L+1,0}. \]

**4.6. Step 3: Covering argument II.** Suppose that (iv) \( _{L+1} \) and (v) \( _{L+1} \) hold. We show (vi) \( _{L+1} \).

**Sublemma 4.9.** For \( l \leq L + 1 \) and \( n \geq 2l + r - 2 \), the conjugate action of \( \text{GL}_{n+1}(\mathbb{Z}) \) on the image of
\[ H_l(E_n(A)) \rightarrow H_l(E_{n+1}(A)) \]
is pro trivial.

**Proof.** The case \( l = 0, 1 \) is clear. Suppose that \( 2 \leq l \leq L + 1 \) and \( n \geq 2l + r - 2 \).

Since \( \text{GL}_{n+1}(\mathbb{Z}) = \mathbb{Z} \times \text{SL}_{n+1}(\mathbb{Z}) \) and \( \text{SL}_{n+1}(\mathbb{Z}) = E_{n+1}(\mathbb{Z}) \), \( \text{GL}_{n+1}(\mathbb{Z}) \) is generated by \( e_{i,n+1}(1), e_{n+1,i}(1), 1 \leq i \leq n \), and \( \text{diag}(1, \ldots, 1, -1) \). It is obvious that \( \text{diag}(1, \ldots, 1, -1) \) acts trivially on the image of \( H_l(E_n(A)) \rightarrow H_l(E_{n+1}(A)) \).

We show the triviality of the conjugate action of \( e_{i,n+1}(1) \); the one of \( e_{n+1,i}(1) \) is similar. By Corollary 1.3, it suffices to show that the action on the image of
\[ H_l(E_n(R, A)) \rightarrow H_l(E_{n+1}(R, A)) \]
is pro trivial for some unital pro ring $R$ which contains $A$ as a two-sided ideal. The inclusion $E_n(R, A) \hookrightarrow E_{n+1}(R, A)$ factors through
\[
\tilde{E}_n(R, A) := \begin{pmatrix} E_n(R, A) \\ 0 \end{pmatrix} \subset E_{n+1}(R, A)
\]
and it is normalized by $e_{i,n+1}(1)$. Hence, it suffices to show that $e_{i,n+1}(1)$ acts pro trivially on the image of $H_i(E_n(R, A)) \to H_i(\tilde{E}_n(R, A))$. Now, we have a commutative diagram
\[
\begin{array}{c}
H_i(E_n(R, A)) \\ \downarrow \\
H_i(E_n(R, A)) \\
\end{array} \xrightarrow{e_{i,n+1}(1)} \begin{array}{c}
H_i(\tilde{E}_n(R, A)) \\ \downarrow \\
H_i(E_n(R, A)) \\
\end{array}
\]
and the vertical maps, the canonical inclusion and projection, are pro isomorphisms by $(iv)_{\leq L+1}$. This implies that $e_{i,n+1}(1)$ acts pro trivially on the image of $H_i(E_n(R, A)) \to H_i(\tilde{E}_n(R, A))$. \hfill $\square$

We consider the hyperhomology spectral sequence
\[
E^1_{p,q}(A) = H_q(E_{n+1}(A), C_p(S\text{Uni}_{A,n+1})) \Rightarrow H_{p+q}(E_{n+1}(A), C_{\ast}(S\text{Uni}_{A,n+1})).
\]
The $C_p(S\text{Uni}_{A,n+1})$ decomposes into a direct sum of $E_{n+1}(A)$-submodules $C_p(S\text{Uni}_{A,n+1}^I)$ with $|I| = p + 1$, and we have a levelwise isomorphism $\mathbb{Z}E_{n+1}(A) \otimes \mathbb{Z}E_{n+1}(A)^I \mathbb{Z} \xrightarrow{\sim} C_p(S\text{Uni}_{A,n+1}^I)$, which sends $\alpha \in E_{n+1}(A)$ to the unimodular function $i \mapsto e_i\alpha$, $i \in I$. Hence,
\[
\bigsqcup_{|I|=p+1} H_q(E_{n+1}(A)^I) \simeq E^1_{p,q}(A).
\]
Let $\Delta^n$ be the nerve of the partially ordered set $\{1 < 2 < \cdots < n + 1\}$. We define level maps $E_{n-p}(A) \to E_{n+1}(A)^I$ by sending $\alpha$ to $\sigma_I(n-p+1,\ldots,n+1) = I$. These maps yield
\[
\Psi: \Delta^n \otimes H_q(E_{n-p}(A)) \simeq \bigsqcup_{|I|=p+1} H_q(E_{n-p}(A)) \xrightarrow{\sim} \bigsqcup_{|I|=p+1} H_q(E_{n+1}(A)^I) \simeq E^1_{p,q}(A).
\]
It follows from Theorem 1.6 and $(iv)_{\leq L+1}$ that $\Psi$ is a pro isomorphism for $q \leq L+1$ and $n-p \geq \max(2q+r-2,r+1)$. Furthermore, by Sublemma 4.9, we see that the diagram
\[
\begin{array}{cccc}
\Delta^n_{p+1} \otimes H_q(E_{n-p-1}(A)) \\ \downarrow \sum k=0 (-1)^k d_k \otimes \delta \downarrow \\
\Delta^n_{p} \otimes H_q(E_{n-p}(A)) \\
\end{array} \xrightarrow{\Psi} \begin{array}{c}
E^1_{p+1,q}(A) \\
E^1_{p,q}(A) \\
\end{array}
\]
commutes for $q \leq L+1$ and $n-p \geq 2q+r-1$, where $d_k$ are the face maps of $\Delta^n$ and $\delta$ is the canonical map $H_q(E_{n-p-1}(A)) \to H_q(E_{n-p}(A))$.

Claim 4.10. For $q \leq L$ and $0 < p \leq n - 2q - r + 1$,
\[
E^2_{p,q}(A) = 0.
\]

Proof. Suppose that $q \leq L$ and $0 < p \leq n - 2q - r + 1$. Put $F_{p,q}(A) := \Delta^n_p \otimes H_q(E_{n-p}(A))$, which we regard as a complex in $p$ with differential $\sum_{k=0}^{p+1} (-1)^k d_k \otimes \delta$. Then, by $(vii)_{\leq L+1}$, we have
\[
\ker(F_{p,q}(A) \to F^{-1,q}(A)) \simeq \ker(\mathbb{Z}\Delta^n_p \to \mathbb{Z}\Delta^n_{p-1}) \otimes H_q(E_{n-p}(A)).
\]
Again by $(vii)_{\leq L+1}$, the canonical map
\[
H_q(E_{n-p-1}(A)) \to H_q(E_{n-p}(A))
\]
is a pro epimorphism. Since $\Delta^n$ is contractible, we conclude that $H_p(F_{*q}(A)) = 0$. 
Now, we have a pro isomorphism
\[ E^2_{p,q}(A) \simeq H_p(F_{\ast,q}(A)) \]
for \( n - p - 1 \geq r + 1 \). Our assumption says \( n - p - 1 \geq 2q + r - 2 \); hence, in case \( 2q + r - 2 \geq r + 1 \), the vanishing of \( E^2_{p,q}(A) \) follows form the one of \( H_p(F_{\ast,q}(A)) \).

It remains to show the case \( q = 1 \). However, in this case,
\[ E^1_{p,1}(A) \xrightarrow{\sim} \Delta^n \otimes H_1(E_{n-p}(A)) = 0. \]
This completes the proof. \( \Box \)

Suppose that \( n \geq 2L + r \). Then the \( E^2 \)-terms with \( p + q = L + 1 \) are zero unless \( E^2_{0,L+1}(A) \). Hence, the edge map
\[ E^1_{0,L+1}(A) \to E^\infty_{L+1}(A) \]
is a pro epimorphism. The left hand side is pro isomorphic to \( \Delta^n \otimes H_{L+1}(E_n(A)) \) by \( \Psi \). According to Corollary 4.5, \( H_i(C_n(SUn_{A,n+1})) = 0 \) for \( n \geq i + r \). Hence, we have a pro isomorphism
\[ E^\infty_{L+1}(A) = H_{L+1}(E_{n+1}(A), C_n(SUn_{A,n+1})) \simeq H_{L+1}(E_{n+1}(A)). \]
By using Sublemma 4.9, we see that the edge map
\[ \Delta^n \otimes H_{L+1}(E_n(A)) \to H_{L+1}(E_{n+1}(A)) \]
coincides as a pro morphism with the sum of the canonical map \( \delta : H_{L+1}(E_n(A)) \to H_{L+1}(E_{n+1}(A)) \).

Hence, the \( \delta \) is a pro epimorphism. This proves the first half of \( (vi)_{l=L+1} \).

Next, suppose that \( n \geq 2L + r + 1 \). Then by Claim 4.10, \( E^s_{n,L-s+2}(A) = 0 \) for \( s \geq 2 \). Hence, we have an exact sequence
\[ \Delta^n \otimes H_{L+1}(E_{n-1}(A)) \to \Delta^n \otimes H_{L+1}(E_n(A)) \to H_{L+1}(E_{n+1}(A)) \to 0. \]
Since \( H_{L+1}(E_{n-1}(A)) \to H_{L+1}(E_n(A)) \) is a pro epimorphism, we conclude that the canonical map
\[ H_{L+1}(E_n(A)) \xrightarrow{\sim} H_{L+1}(E_{n+1}(A)) \]
is a pro isomorphism. This proves the second half of \( (vi)_{l=L+1} \).

Finally, since \( H_{L+1}(E_{n-1}(A)) \to H_{L+1}(E_n(A)) \) is a pro epimorphism, Sublemma 4.9 implies that the action of \( \Sigma_n \) on \( H_{L+1}(E_n(A)) \) is pro trivial. This proves \( (v)_{l=L+1} \).

4.7. Step 4: \( E \to V \). Suppose that \( (i, ii)_{l \leq L-1}, (iii)_{l \leq L-1} \), and \( (vi)_{l \leq L+1} \) hold. We show \( (i, ii)_{l \leq L-1} \).

Suppose that \( n \geq 2L + r \). Consider the spectral sequences \( (4.1) \) and the canonical morphism between them:
\[ nE^2_{p,q}(A) = H_p(E_n(A), H_q(V(A))) \to H_{p+q}(\bigcup_{\sigma \in \Pi_n} BT^\sigma(A)) \]
\[ n+1 E^2_{p,q}(A) = H_p(E_{n+1}(A), H_q(V_{n+1}(A))) \to H_{p+q}(\bigcup_{\sigma \in \Pi_{n+1}} BT^\sigma(A)). \]
By \( (iii)_{l \leq L-1} \), for \( q \leq L - 1 \), the \( E^2 \)-terms fit into the extensions
\[ 0 \to H_p(E_n(A)) \otimes H_q(V_n(A^m)) \to nE^2_{p,q}(A) \to \text{Tor}(H_{p-1}(E_n(A)), H_q(V(A))) \to 0. \]
\[ 0 \to H_p(E_{n+1}(A)) \otimes H_q(V_{n+1}(A)) \to n+1 E^2_{p,q}(A) \to \text{Tor}(H_{p-1}(E_{n+1}(A)), H_q(V_{n+1}(A))) \to 0. \]
Hence, it follows from \( (i)_{l \leq L-1} \) and \( (vi)_{l \leq L+1} \) that the map
\[ nE^2_{p,q}(A) \to n+1 E^2_{p,q}(A) \]
is a pro epimorphism for \( q < L - 1 \) and \( p \leq L + 1 \), and it is a pro isomorphism if further \( n \geq 2p + r - 1 \).
Finally, by Theorem 3.9, \( nE^\infty(A) \simeq n+1 E^\infty(A) \) for \( n \geq 2i + 1 \).

Bringing these together, we have:
(1) \( nE^2_{p,q}(A) \simeq n+1 E^2_{p,q}(A) \) for \( p + q = L - 1 \) and \( p \geq 1 \).
(2) \( nE^2_{p,q}(A) \simeq n+1E^2_{p,q}(A) \) for \( p+q = L \) and \( p \geq 2 \).

(3) \( nE^2_{p,q}(A) \rightarrow n+1E^2_{p,q}(A) \) for \( p+q = L+1 \) and \( p \geq 3 \).

(4) \( nE^\infty_{L-1}(A) \simeq n+1E^\infty_{L-1}(A) \) and \( nE^\infty_L(A) \simeq n+1E^\infty_L(A) \).

Then, by Lemma 4.11 below, we conclude that the canonical map

\[
E^2_{0,L-1}(A) \rightarrow n+1E^2_{0,L-1}(A)
\]

is a pro isomorphism. By (iii) \( \leq L-1 \), the left hand side (resp. right hand side) is pro isomorphic to \( H_{L-1}(V_n(A)) \) (resp. \( H_{L-1}(V_{n+1}(A)) \)). Hence, we get the second part of (i) \( l=L-1 \).

Next, we show (ii) \( l=L-1 \). Now, the canonical map

\[
H_{L-1}(V_n(A)) \rightarrow H_{L-1}(V_{n+2}(A))
\]

is a \( \Sigma \)-equivariant pro isomorphism. Hence, it suffices to show that \( \Sigma \) acts pro trivially on \( H_{L-1}(V_{n+2}(A)) \). Now, the permutation \( \tau_{L+1,n+2} \) acts pro trivially on \( H_{L-1}(V_{n+2}(A)) \), since it acts trivially on the image of the above map. Since \( \Sigma \) acts pro trivially on \( H_{L-1}(V_{n+2}(A)) \).

In Step 1, we have seen that the map (4.4)

\[
C_0(SU_{A,n}) \otimes H_{L-1}(V_{n-1}(A)) \rightarrow H_{L-1}(V_{n}(A))
\]

sending \( f \otimes u \rightarrow \sigma_{(i)}(\delta u)^{-1}(\text{dom } f = (i)) \) is a pro epimorphism for \( n \geq 2L+r \). Now, we know that \( \sigma_{(i)}(\delta u)^{-1}(\text{dom } f = (i)) = \delta u \). Therefore, the canonical map \( \delta : H_{L-1}(V_{n-1}(A)) \rightarrow H_{L-1}(V_{n}(A)) \) is a pro epimorphism. This completes the proof of (i) \( l=L-1 \).

**Lemma 4.11 (Su96, Theorem A.6).** Let \( A \) be an abelian category. Let \( f : E \rightarrow \tilde{E} \) be a morphism of first quadrant homological spectral sequence in \( A \), and let \( L > 0 \). Assume that \( f \) induces:

1. A monomorphism \( E^2_{p,q} \rightarrow \tilde{E}^2_{p,q} \) for \( p+q = L-1, p \geq 1 \).
2. An isomorphism \( E^2_{p,q} \rightarrow \tilde{E}^2_{p,q} \) for \( p+q = L, p \geq 2 \).
3. An epimorphism \( E^2_{p,q} \rightarrow \tilde{E}^2_{p,q} \) for \( p+q = L+1, p \geq 3 \).
4. Isomorphisms \( E^\infty_{L-1} \rightarrow \tilde{E}^\infty_{L-1} \) and \( E^\infty_L \rightarrow \tilde{E}^\infty_L \).

Then \( f \) induces an isomorphism

\[
E^2_{0,L-1} \simeq \tilde{E}^2_{0,L-1}.
\]

**4.8. Homology pro stability for \( GL_n \).** Now, we can prove Theorem 0.3. We restate it here.

**Theorem 4.12.** Let \( A \) be a commutative Tor-unital pro ring. Let \( r = \max(\text{sr}(A), 2) \) and \( l \geq 0 \). Then the canonical map

\[
H_l(GL_n(A)) \rightarrow H_l(GL_{n+1}(A))
\]

is a pro epimorphism for \( n \geq 2l+r-2 \) and a pro isomorphism for \( n \geq 2l+r-1 \).

**Proof.** The case \( l = 0 \) is clear. The case \( l = 1 \) is proved in Theorem 1.5. Let \( l \geq 2 \) and \( n \geq 2l+r-2 \). Then, by Theorem 1.5 and Corollary 1.3, the sequence

\[
\begin{array}{c}
0 \rightarrow E_0(A) \rightarrow \rightarrow GL_n(A) \rightarrow H_1(GL(A)) \rightarrow 0.
\end{array}
\]

is exact up to pro isomorphisms. Now, we have a morphism of spectral sequences:

\[
\begin{array}{c}
nE^2_{p,q} = H_p(H_1(GL(A)), H_q(E_n(A))) \rightarrow H_{p+q}(GL_n(A)) \\
n+1E^2_{p,q} = H_p(H_1(GL(A)), H_q(E_{n+1}(A))) \rightarrow H_{p+q}(GL_{n+1}(A)).
\end{array}
\]

Using these spectral sequences, we can easily deduce the theorem from Theorem 4.6 (vi). \( \square \)

**Corollary 4.13.** Let \( B \) be a pro ring with a two-sided ideal \( A \) and \( r = \max(\text{sr}(A), 2) \). Suppose that \( A \) is commutative and Tor-unital. Then the conjugate action of \( GL_n(B) \) on \( H_1(GL_n(A)) \) is pro trivial for \( n \geq 2l+r-1 \).
Proof. Let $\alpha$ (resp. $\beta$) be the map $GL_n \to GL_{2n}$ given by
\[
g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1_n \end{pmatrix} \quad \text{resp.} \quad g \mapsto \begin{pmatrix} 1_n & 0 \\ 0 & g \end{pmatrix}.
\]
According to the Theorem 4.12, the induced maps
\[
\alpha, \beta : H_l(GL_n(A)) \simto H_l(GL_{2n}(A))
\]
are pro isomorphisms for $n \geq 2l + r - 1$.

Write $B = \{B_m\}_{m \in J}$ and $A = \{A_m\}_{m \in J}$. For each $m \in J$, choose $s(m) \geq m$ such that if $\alpha(a) = 0$ with $a \in H_l(GL_n(A_{s(m)}))$ then $\iota_{s(m),m}(a) = 0$. Next, choose $t(m) \geq s(m)$ such that for every $x \in H_l(GL_n(A_{t(m)}))$ there exists $y \in H_l(GL_n(A_{s(m)}))$ with $\iota_{t(m),s(m)}(\alpha(x)) = \beta(y)$. Then, for $g \in GL_n(B_{t(m)})$ and $x \in H_l(GL_n(A_{t(m)}))$,
\[
\alpha(\iota_{t(m),s(m)}(gx)) = \alpha(\iota_{t(m),s(m)}(g))\beta(y) = \beta(y) = \alpha(\iota_{t(m),s(m)}(x)).
\]
Hence, $\iota_{t(m),m}(gx) = \iota_{t(m),m}(x)$. This completes the proof. \(\square\)

Suslin has shown that if a ring $A$ is Tor-unital then for every ring $B$ which contains $A$ as a two-sided ideal the conjugate action of $GL(B)$ on $H_l(GL(A))$ is trivial, cf. [Su95, Corollary 4.5], see also [SW92, Corollary 1.6]. Geisser-Hesselholt has generalized Suslin’s result to a pro setting, cf. [GH06, Proposition 1.3]. Here is a straightforward generalization of their result.

Theorem 4.14 (Suslin, Geisser-Hesselholt). Let $B$ be a pro ring with a two-sided ideal $A$. Suppose that $A$ is Tor-unital. Then the conjugate action of $GL(B)$ on $H_l(GL(A))$ is pro trivial for all $l \geq 0$.

By using Theorem 4.14, we can strengthen our main theorem, Theorem 4.12.

Theorem 4.15. Let $A$ be a commutative Tor-unital pro ring, $r = \max(sr(A), 2)$ and $l \geq 0$. Suppose that there exists a unital pro ring $R$ with $sr(R) < \infty$ which contains $A$ as a two-sided ideal. Then the canonical map
\[
H_l(GL_n(A)) \to H_l(GL(A))
\]
is a pro epimorphism for $n \geq 2l + r - 2$ and a pro isomorphism for $n \geq 2l + r - 1$.

Proof. Let $R$ be a unital pro ring as in the statement. Consider the commutative diagram
\[
\begin{array}{ccc}
GL_n(A) & \longrightarrow & GL_n(R) \\
\downarrow & & \downarrow \\
GL(A) & \longrightarrow & GL(R)
\end{array}
\]
with exact rows. Now, the second and the third maps induce isomorphisms on homology for $n$ large enough. Also, the action of $GL_n(R)$ on $H_l(GL_n(A))$ is pro trivial for $n$ large enough (Theorem 4.13) and for $n = \infty$ (Theorem 4.14). Consequently, the canonical map
\[
H_l(GL_n(A)) \simto H_l(GL(A))
\]
is a pro isomorphism for $n$ large enough. Combining it with Theorem 4.12, we get the result. \(\square\)
References


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