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FLOW EQUIVALENCE OF G-SFTS

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Abstract. In this paper, a $G$-shift of finite type ($G$-SFT) is a shift of finite type together with a free continuous shift-commuting action by a finite group $G$. We reduce the classification of $G$-SFTs up to equivariant flow equivalence to an algebraic classification of a class of poset-blocked matrices over the integral group ring of $G$. For a special case of two irreducible components with $G = \mathbb{Z}_2$, we compute explicit complete invariants. We relate our matrix structures to the Adler-Kitchens-Marcus group actions approach. We give examples of $G$-SFT applications, including a new connection to involutions of cellular automata.

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1. Introduction

The shifts of finite type (SFTs) are the fundamental building blocks of symbolic dynamics. Elaborations of these include the class of $G$-SFTs: SFTs equipped with a continuous action by a group $G$ which
commutes with the shift. Apart from a few remarks, in this paper \(G\)-SFT means \(G\)-SFT with \(G\) finite and acting freely \((gx = x\) only when \(g = e)\).

We will give an algebraic classification of these \(G\)-SFTs up to equivariant flow equivalence \((G\)-flow equivalence). This generalizes the \(G\)-flow equivalence classification for irreducible \(G\)-SFTs in [12] and the Huang flow equivalence classification for general SFTs without group action [4, 9].

Square matrices over \(\mathbb{Z}_+G\) (the positive cone in the integral group ring of \(G\)) present \(G\)-SFTs. When such matrices \(A\) and \(B\) present nontrivial mixing \(G\)-SFTs, these \(G\)-SFTs are \(G\)-flow equivalent if and only if there exist \(n\) in \(\mathbb{N}\) and identity matrices \(I_j, I_k\) such that there are matrices \(U, V\) in the elementary group \(\text{El}(n, \mathbb{Z}G)\) such that \(U((I - A) \oplus I_j)V = (I - B) \oplus I_k\). This reduces the dynamical classification of \(G\)-SFTs up to \(G\)-flow equivalence to the algebraic classification of square matrices over \(\mathbb{Z}G\) up to the stabilized elementary equivalence described above. The algebraic classification is in general highly nontrivial, but far more manageable than the dynamical problem.

For nonmixing \(G\)-SFTs, this stabilized elementary \(\mathbb{Z}G\) equivalence no longer implies \(G\)-flow equivalence. To get an analogous result (Theorem 5.1) for general \(G\)-SFTs, we consider \(G\)-SFTs presented by matrices in a special block triangular form, with entries in an \(ij\) block lying in \(\mathbb{Z}H_{ij}\) for some union \(H_{ij}\) of double cosets of \(G\), and their equivalence by elementary matrices from a restricted class subordinate to this blocked coset structure. Using this, we classify \(G\)-SFTs up to \(G\)-flow equivalence (Theorems 5.1 and 5.3).

The paper is organized as follows.

In Section 2, we discuss uses of \(G\)-SFTs, and application of the results of this paper. This includes a new use of the \(G\)-SFT structure, for cellular automata. Especially, the main theorem of the current paper is a key result for the classification up to flow equivalence of a large class of irreducible sofic shifts in [7]. In Section 3, we give a bare-bones review of the necessary background. In Section 4, we introduce the notion of coset structure, crucial for defining our restricted class of elementary matrices, and define various classes of related matrices. We prove Proposition 4.23 which tells us that, in order to classify \(G\)-SFTs up to \(G\)-flow equivalence, it is enough to work with square matrices over \(\mathbb{Z}_+G\) having a special block form.

In Section 5, we present our classification, with comments. The full classification statement is Theorem 5.3; the essence is the simpler statement of Theorem 5.1 for the case the \(G\)-flow equivalence respects the ordering of irreducible components. In each statement, for matrices in a suitable class chosen to present the \(G\)-SFTs, a matrix condition

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1By definition, an irreducible SFT is trivial iff it contains only one orbit. The only trivial mixing SFT is a single point; so, a trivial mixing \(G\)-SFT has \(G = \{e\}\).
is necessary and sufficient for the $G$-flow equivalence. In Section 6 we prove a strengthened version of necessity of the matrix condition, Theorem 6.2. In Section 7 we present the functorial Factorization Theorem 7.2, which we later use to prove sufficiency of the matrix condition in Theorem 5.3 and develop the setting for its proof. In Section 8 we give that proof. Section 9 contains the proof of Theorem 5.3 (a short argument appealing to Theorems 6.2 and 7.2), a result on range of invariants and a finiteness result. In Appendices A, B, and C we establish three types of positive Elp$\left(\mathcal{H}\right)$-equivalences which we use in the paper.

In Appendix D we relate the group actions viewpoint on $G$-SFTs developed in the Adler-Kitchens-Marcus paper [2] to our matrix-based setup. (This is analogous to the relation of matrices and linear transformations in elementary linear algebra.) Adler, Kitchens and Marcus were concerned only with nonwandering SFTs. We explain in Remark 4.25 how our coset structures give a kind of algebraic calculus to describe the transitions among irreducible components of a general $G$-SFT.

In Appendix E we work out algebraic invariants of $G$-flow equivalence for a special class of systems with exactly two irreducible components. For the subclass with $G = \mathbb{Z}_2$, we give a complete algebraic classification, with algorithms to answer all questions (at least, the natural ones we thought of). This supplies a tractable collection of examples and points to some of the issues involved in a general algebraic classification.

2. Applications of G-SFTs

G-SFTs arise in several contexts, including the following. (Below, we occasionally assume background reviewed in Section 3.)

(i) In [2], Adler, Kitchens and Marcus introduced invariants of nonwandering $G$-SFTs (and a more general class), and used these in [1] to classify factor maps between irreducible SFTs up to almost topological conjugacy. Here, a construction replaces a given factor map with a map $\varphi : S' \to S$, equivalent up to almost conjugacy, which is constant $n$-to-1. The map $\varphi$ gives rise to a continuous function $\tau : S \to S_n$ (with $S_n$ the group of permutations on $n$ symbols). This $\tau$ is used as a skewing function to generate a $G$-SFT (with $G = S_n$) extension $T$ of $S$, with maps $\pi : T \to S$ and $\alpha : T \to S'$ such that $\pi = \varphi \circ \alpha$. Group invariants of the $S_n$-action on $T$ are then related to $\varphi$ and used for the classification, which requires further constructions.

(ii) Field, Golubitsky and Nicol used $G$-SFTs in studies of “symmetry in chaos” and related equivariance [15, 16, 17].
(iii) The Livšic theorem, restricted to dimension zero, states that two real-valued Hölder functions on an irreducible SFT are cohomologous if and only if on each periodic orbit the sum of their output values is the same. G-SFTs arise as a case of the study of analogous rigidity possibilities for skew products in terms of group-valued skewing functions (see [24, 28, 10]).

(iv) In [31, 30, 32], Michael Sullivan introduced “twistwise flow equivalence”. Here $G = \mathbb{Z}_2$. For an SFT basic set of the return map to a cross-section under a flow on a 3-manifold, the return map to the cross-section induces a map on the local stable set which is orientation preserving or reversing. This additional data is given by a function $X \to \mathbb{Z}_2$, which can be encoded as a matrix $A$ over $\mathbb{Z}G$. Sullivan found G-FE invariants of $A$ to produce new invariants of the template (branched two-manifold fitted with an expansive semiflow) for the flow. The complete algebraic G-FE classification (for $G = \mathbb{Z}_2$) for the mixing case gave further invariants (see [12, Sec. 7] and [33] for more).

(v) The mapping class group of a nontrivial irreducible SFT (see [8]) is the countable group of homeomorphisms of the mapping torus of the SFT which respect orientation of the suspension flow, up to isotopy. This is a challenging group to understand. The classification of irreducible G-SFTs up to G-flow equivalence has provided at least a little information: for example (see [8 Theorem 8.6]), if the irreducible SFT is defined by a matrix $A$ such that $\det(I - A)$ is odd, then the free orientation preserving involutions of the mapping torus are contained in a finite set of conjugacy classes of this mapping class group. (For a speculative application related to mapping class groups, see Remark 7.3)

(vi) The main result of the current paper has already been applied to the classification of sofic shifts up to flow equivalence in [7]. We will give more detail on this below.

The automorphism group $\text{Aut}(S)$ of an SFT $S$ is the group of homeomorphisms commuting with $S$. The G-SFTs offer a different tool set to the study of finite subgroups of $\text{Aut}(S)$. For simplicity, we consider just the case of $G = \mathbb{Z}_2$. If $U$ is a free involution in $\text{Aut}(S)$, then the pair $(S, U)$ presents a G-SFT. When $S$ is $\sigma_n$, the full shift on $n$ symbols, this can also be described as a free involution of an invertible one-dimensional cellular automata. A longstanding question (recalled in [14, p.492]) asks for $n = 2$ whether two such involutions must be conjugate in $\text{Aut}(\sigma_n)$. This conjugacy in the group $\text{Aut}(\sigma_n)$ is equivalent to topological conjugacy of the corresponding G-SFTs. By Proposition 3.4.1, conjugacy in the group is equivalence to strong shift equivalence over $\mathbb{Z}_+G$ of presenting matrices.

A necessary condition for this conjugacy (a test) is that the G-SFTs be G-flow equivalent. (Equivalently, the induced involutions of the
mapping torus are conjugate in the mapping class group of $\sigma_n$.) We prove next that this necessary condition is satisfied.

For the proof, we assume the background and notation of Section 3. For a matrix $A$ over a commutative ring, the sequence $(\text{trace}(A^n))_{n \in \mathbb{N}}$ determines and is determined by the polynomial $\det(I - tA)$. We have $\text{trace}(A^n) = |\text{Fix}(\sigma_n^a)|$ when $\sigma_A$ is an SFT defined by a matrix $A$ over $\mathbb{Z}_+$. We assume in the proof some familiarity with the Bowen-Franks group invariant of flow equivalence. Finally, note that a full shift on an odd number of symbols has an odd number of fixed points, and therefore admits no free involution.

**Theorem 2.1.** Suppose $G = \mathbb{Z}_2$, $k$ is a positive integer and $\sigma_{2k}$ is the full shift on $2k$ symbols. Then there is a free involution commuting with $\sigma_{2k}$. For any two such involutions $U, U'$ the $G$-SFTs $(\sigma_{2k}, U), (\sigma_{2k}, U')$ are $G$-flow equivalent.

**Proof.** Let $G = \{e, g\}$. For the existence claim, choose a free order two permutation of the $2k$ symbols; this defines a one-block code which defines a free involution $U$ of $\sigma_{2k}$.

Now let $A$ be an $m \times m$ matrix over $\mathbb{Z}_+G$ presenting a skew product $G$-SFT isomorphic to one defined by $(U, \sigma_{2k})$. This $A$ presents a skewing function on an SFT $\overline{T}$ which is a factor of $\sigma_{2k}$ under the map $\pi$ which collapses $G$-orbits to points. Therefore (e.g. by [12] Theorem 1), $\det(I - tA) = 1 - 2kt$. So, for every $n$,

$$|\text{Fix}(T^n)| = |\text{Fix}(\overline{T}^n)| = (2k)^n.$$

Let $\text{trace}(A^n) = \alpha_n e + \beta_n g$. From the structure of the skew product construction, one can check that $\alpha_n$ is the number of fixed points of $\overline{T}^n$ whose preimages are contained in $\text{Fix}(T^n)$. As $\pi$ maps $\text{Fix}(T^n)$ 2-to-1 into $\text{Fix}(\overline{T}^n)$, we have $\alpha_n = (1/2)(2k)^n$, and therefore $\beta_n = (1/2)(2k)^n$, for each $n$. This forces $\det(I - tA) = \det(I - tB)$, for the $1 \times 1$ matrix $B = (k(e + g))$, because $B^n = ((1/2)(2k)^ne + (1/2)(2k)^ng)$. Therefore $\det(I - tA) = e - tk(e + g)$.

Because $G = \mathbb{Z}_2$ and $1 - 2k$ is odd, by [12] Theorem 8.1] the matrix $I - A$ is $\text{El}(m, \mathbb{Z}G)$-equivalent to a diagonal matrix, $D$, which must have determinant $e - k(e + g)$. It can happen, for some $k$, that $e - k(e + g)$ factors in $\mathbb{Z}G$. Nevertheless, $D$ must be $(e - k(e + g)) \oplus I_{m-1}$. This is because the Bowen-Franks group for $\overline{T}$ is isomorphic to $\mathbb{Z}_{2k-1}$ (because $\det(I - tA) = 1 - 2kt$) and is also isomorphic to $\text{cok}(I - A)$. So, the $\text{El}(\mathbb{Z}G)$ class of $I - A$ is the same for any choice of free involution $U$. It follows from [12] Theorem 6.4] (the case $\mathcal{P} = 1$ of Theorem 6.1) that all these $G$-SFTs fall in the same $G$-flow equivalence class. \qed

By definition, the involutions $U, U'$ are conjugate on periodic points if there is a bijection $\text{Per}(T) \rightarrow \text{Per}(T')$ which intertwines the actions of $(T, U)$ and $T, U'$. A consequence of Ulf Fiebig’s work [13] (or the
arguments above) is that free involutions of $\sigma_{2k}$ are conjugate on periodic points (for $k = 1$ this is explicitly contained in [14, Corollary 1.12]). Fiebig did much more, in particular for $G$ actions which need not be free. In contrast to the free case, we do not have a $\mathbb{Z}_+G$ matrix framework which handles the nonfree actions.

**Problem 2.2.** Develop a useful matrix framework of matrices over $\mathbb{Z}_+G$ for $G$-SFTs for which the $G$ action need not be free. The framework should in particular capture topological conjugacy and flow equivalence of the $G$-SFT.

The papers [14, 29] are relevant to Problem 2.2. A solution to Problem 2.2 would give a $\mathbb{Z}G$ matrix framework for the entire vast collection of finite subgroups of $\text{Aut}(\sigma_n)$. It would also naturally involve general, reducible $G$-SFTs.

The discussions above involved $G$-SFTs which are irreducible as SFTs. The advance of the current paper is in addressing $G$-flow equivalence of $G$-SFTs which are reducible. We expect the general reducible case to be meaningful to related applications (especially, given a solution to Problem 2.2), but for the most part we have not developed reducible applications related to the items above. However, our main result, Theorem 5.3, is an essential tool for our classification in [7] of a large collection of irreducible sofic shifts up to flow equivalence (those which are “point extension type”, or PET). We emphasize that our results on $G$-FE for reducible $G$-SFTs is used for the flow equivalence classification of sofic shifts which are irreducible (or, equivalently for flow equivalence, mixing).

For simplicity, we will describe only a subclass $C$ of the PET sofic shifts. Let $C$ be the class of nontrivial irreducible strictly sofic shifts such that for the right Fischer cover $\pi : X \to Y$, the set $M = \{x : |\pi^{-1}(\pi x)| > 1\}$ is a closed proper subset and satisfies the following condition for some finite group $G$: there is a shift-commuting embedding of $T$ to $M$ which takes $G$-orbits to the fibers of $\pi$. Given the easily computed flow equivalence class of the irreducible SFT $X$, we show in [7] that the $G$-flow equivalence class of $T$ is a complete classification invariant for the flow equivalence class of the sofic shift $Y$. For every $X$, every $G$-SFT $G$-flow equivalence class arises in this construction. This theorem appeals to the full strength of our classification result Theorem 5.3 (as noted in [7] Remark 7.14)).

We give next an example to indicate how this works.

**Example 2.3.** Let $A = \left(\begin{array}{cccc}
    a & b & c & d \\
    a & b & d & c \\
    0 & 0 & k & \ell \\
    0 & 0 & \ell & k
\end{array}\right)$ be a matrix in which the nonzero letters represent positive integers. There is a free involution $\gamma$ on the edge SFT $\sigma_A$ coming from a graph automorphism $\gamma$ corresponding to the involution of vertices $1 \leftrightarrow 2$, $3 \leftrightarrow 4$. (E.g., there are $d$ edges from vertex 1 to vertex 4, and these are mapped bijectively to the $d$ edges
from vertex 2 to vertex 3.) The edge $\sigma_A$, together with $\gamma$, is a $G$-SFT, with $G = \mathbb{Z}_2$.

Now take any $4 \times 4$ matrix $B$ which is entrywise greater than $A$. Define a one block code $\varphi$ from $X_B$ which maps two edges (the symbols of $X_B$) to the same symbol if and only if they are edges of $X_A$ paired by $\gamma$. The image shift is a mixing strictly sofic shift. Now suppose $A' = \begin{pmatrix} a & b & c' & d' \\ b & a & c' & d' \\ 0 & 0 & k & \ell \\ 0 & 0 & \ell & k \end{pmatrix}$, again with letters representing positive integers, and again with $B$ entrywise greater than $A'$. Construct $\varphi'$ from $X_B$ just as $\varphi$ was constructed. Are the two image sofic shifts flow equivalent? The result of [7] tell us they are if and only if the $G$-SFTs on $X_A$ and $X_A'$ are $G$-flow equivalent.

Here, for any choices of the letters, we can translate to the completely worked classification of Appendix E and look up the answer. For the translation, first $A$ becomes $\begin{pmatrix} ae + bg & ce + dg \\ 0 & ke + \ell g \end{pmatrix}$. There is an isomorphism from $\mathbb{Z}G$ to the subring $R$ of $\mathbb{Z}_2^2$ consisting of the $(\alpha, \beta)$ such that $\alpha \equiv \beta \mod 2$, given by $ae + bg \mapsto (a + b, a - b) := (\alpha, \beta)$. We apply this map entrywise to $A$ to arrive at a matrix $\begin{pmatrix} (\alpha_p, \beta_p) & (\alpha_q, \beta_q) \\ 0 & (\alpha_q, \beta_q) \end{pmatrix}$, where $(\alpha_p, \beta_p) = (a + b, a - b)$, $(\alpha_q, \beta_q) = (k + \ell, k - \ell)$ and $(\alpha, \beta) = (c + d, c - d)$. For $A'$ we do the same, arriving at the same matrix except that $(\alpha', \beta')$ replaces $(\alpha, \beta)$. From Proposition E.12 or Proposition E.13 we determine the ideal $J$ to which the classifying Theorem E.9 applies; and then we can determine from Theorem E.9 whether the $G$-flow equivalence holds. Example E.17 gives a complete discussion of the classification for the case $ae + bg = 54 - 42g$, $ke + \ell g = 16e - 8g$.

3. Background

This section provides a bare-bones review of the background material, assuming some familiarity with the subject. For basic background on shifts of finite type, see [19, 20]. For a detailed presentation with proofs of the basic theory of $G$-SFTs and $G$-flow equivalence for finite $G$, see [12]. The basic ideas of skew product constructions are of fundamental importance in various branches of dynamics; the exposition in [12] is tailored to our topic and also includes facts specific to it. See [10] for further developments, and a correction [10 Appendix A] to [12].

3.1. Shifts of finite type and matrices over $\mathbb{Z}_+$. Given an $n \times n$ square matrix $A$ over $\mathbb{Z}_+ = \{0, 1, \ldots\}$, let $\mathcal{G}_A$ be a graph (in this paper, graph means directed graph) with vertex set $\{1, \ldots, n\}$, edge set $\mathcal{E} = \mathcal{E}_A$ and adjacency matrix $A$. Define $X_A$ to be the subset of $\mathcal{E}^\mathbb{Z}$ realized by bi-infinite paths in $\mathcal{G}_A$. With the natural topology, $X_A$ is a zero-dimensional compact metrizable space. The homeomorphism $\sigma_A : X_A \to X_A$ given by the shift map $\sigma_A$, defined by $(\sigma_A(s))_i = s_{i+1}$,
is the edge SFT defined by $A$. Every SFT is topologically conjugate to some edge SFT.

3.2. Matrices over $\mathbb{Z}_+G$. Let $G$ be a finite group, let $\mathbb{Z}G$ be the integral group ring of $G$, and let $\mathbb{Z}_+G$ be the subset containing the elements $\sum_{g \in G} n_g g$ with $n_g \geq 0$ for all $g$. Suppose $A$ is a square matrix over $\mathbb{Z}_+G$. Let $\overline{A}$ denote the standard augmentation of $A$: the matrix over $\mathbb{Z}_+$ obtained by applying entrywise the standard augmentation map, $\sum_{g \in G} n_g g \mapsto \sum_g n_g$.

By an irreducible matrix $A$ over $\mathbb{Z}G$ we mean a square matrix over $\mathbb{Z}G$ whose augmentation $\overline{A}$ is an irreducible matrix. An irreducible component of $A$ is a maximal irreducible principal submatrix of $A$. A matrix $A$ is said to be essentially irreducible if it has a unique irreducible component. If $A$ is essentially irreducible, then its unique irreducible component is called the irreducible core of $A$.

An element $\sum_{g \in G} n_g g$ of $\mathbb{Z}G$ is $G$-positive when $n_g > 0$ for all $g \in G$.

3.3. $G$-SFTs. In this paper, by a $G$-SFT we mean an SFT together with a free continuous action on its domain by a finite group $G$ which commutes with the shift. (In general, a “$G$-SFT” is not restricted to free actions or finite groups.) Two $G$-SFTs are $G$-conjugate (isomorphic as $G$-SFTs) if there is a topological conjugacy between them which intertwines their $G$ actions. For a left $G$-SFT, the $G$ action is from the left: $gh : y \mapsto g(hy)$ ($h$ acts first). For a right $G$-SFT, the $G$ action is from the right: $gh : y \mapsto (yg)h$ ($g$ acts first).

Standing Convention 3.3.1. Unless mentioned otherwise, in this paper a $G$-SFT is a left $G$-SFT (although we might sometimes repeat the declaration for clarity). This is the choice which aligns with matrix invariants (see [10, Appendix A]). (The $G$-SFTs of [2] are implicitly left $G$-SFTs; the $G$-SFTs of [1, p.493] are right $G$-SFTs.)

Suppose $A$ is a square matrix over $\mathbb{Z}_+G$. Then $A$ can be interpreted as the adjacency matrix of a labeled graph $\mathcal{G}_A$, where the underlying graph is $\mathcal{G}_\tau$, and the label of an edge of $\mathcal{G}_\tau$ is the corresponding element of $G$ (so if the $(s, t)$ entry of $A$ is $\sum_{g \in G} n_g g$, then there is for each $g \in G$, $n_g$ is the number of edges from $s$ to $t$ with label $g$). The labeled graph defines a skewing function $\tau_A : X_\tau \rightarrow G$ which sends $x$ to the label of $x_0$. The skew product construction then gives a homeomorphism $T_A : X_\tau \times G \rightarrow X_\tau \times G$ defined by $(x, g) \mapsto (\sigma_\tau(x), g\tau_A(x))$, and $T_A$ is an SFT. (We consider every map topologically conjugate to an edge SFT to be SFT.) The continuous free left $G$ action $g : (x, g') \mapsto (x, gg')$ commutes with $T_A$. Together with this action, $T_A$ is a $G$-SFT. The map collapsing $G$-orbits to points is given by $(x, g) \mapsto x$; it defines a

\footnote{“$G$-positive” replaces the term “very positive” used in [12].}
factor map from the SFT $T_A$ to the edge SFT defined by $\overline{A}$. Every $G$-SFT is isomorphic to one presented as a group extension in this way by some $A$ over $\mathbb{Z}_+G$.

3.4. Cohomology. Continuous functions $\tau$ and $\rho$ from an SFT $(X,\sigma)$ into $G$ are cohomologous (written $\tau \sim \rho$) if there is another continuous function $\psi$ from $X$ into $G$ such that for all $x$ in $X$, $\tau(x) = [\psi(x)]^{-1}\rho(x)\psi(\sigma x)$. In this equation, the product on the right is a product in the group $G$. This is the form appropriate for our consideration of left $G$-SFTs (for which $\tau$ skews from the right). For right $G$-SFTs we would use instead the equation $\tau(x) = [\psi(\sigma x)]\rho(x)\psi(x)^{-1}$ for all $x$. For nonabelian $G$, these coboundary equations are not equivalent. The following result is fundamental for us.

Proposition 3.4.1. [12, Proposition 2.7.1] Suppose $G$ is a finite group and $A, B$ are square matrices over $\mathbb{Z}_+G$. Then the following are equivalent.

1. $A$ and $B$ are strong shift equivalent over $\mathbb{Z}_+G$.
2. There is a topological conjugacy $\varphi: X_A \to X_B$ such that $\tau_B \sim \tau_A \circ \varphi$.
3. The $G$-SFTs $T_A$ and $T_B$ are $G$-conjugate.

As seen in the Section 2, $G$-SFTs may arise in some setting directly, or in terms of a function from an SFT into $G$, corresponding to (2) above. Both possibilities are addressed by the matrix invariant (1).

3.5. Flow equivalence. Let $Y$ be a compact metrizable space. In this paper, a flow on $Y$ is a continuous $\mathbb{R}$-action on $Y$ with no fixed point. Two flows are topologically conjugate, or conjugate, if there is a homeomorphism intertwining their $\mathbb{R}$-actions. Two flows are equivalent if there is a homeomorphism between their domains taking $\mathbb{R}$-orbits to $\mathbb{R}$-orbits and preserving orientation (i.e., respecting the direction of the flow). A cross-section to a flow $\gamma: Y \times \mathbb{R} \to Y$ is a closed subset $C$ of $Y$ such that the restriction of $\gamma$ to $C \times \mathbb{R}$ is a surjective local homeomorphism onto $Y$. In that case, the return time function $\tau_C: C \to \mathbb{R}$ given by $\tau_C(x) = \min\{t > 0: \gamma(x, t) \in C\}$ is well defined and continuous. The map $r_C: C \to C$ given by $r_C(x) = \gamma(x, \tau_C(x))$ is call the return map of $C$. A section of a flow is the return map of a cross-section of the flow.

For $i = 1, 2$ suppose $S_i: X_i \to X_i$ is a homeomorphism of a compact metrizable space, and $Y_i$ is its mapping torus with the induced suspension flow. The homeomorphisms $S_1, S_2$ are flow equivalent if they are topologically conjugate to sections of a common flow; equivalently, after a continuous time change, the flows on $Y_1$ and $Y_2$ become topologically conjugate; equivalently, there is a homeomorphism $Y_1 \to Y_2$ which on each $Y_1$ flow orbit is an orientation preserving homeomorphism to a $Y_2$ flow orbit. A flow equivalence $S_1 \to S_2$ is such a homeomorphism.
By a $G$-flow we mean a flow together with a continuous free left $G$-action which commutes with the flow. A free $G$ action commuting with a section lifts to a free $G$ action commuting with the flow. Two $G$-flows are $G$-conjugate if the flows are topologically conjugate by a map which intertwines the $G$-actions. Two $G$-flows are $G$-equivalent if the flows are equivalent by a map which intertwines the $G$-actions (i.e., by a $G$-flow equivalence).

The standard theory carries over to the $G$ setting. We call two $G$-homeomorphisms $G$-flow equivalent if they are conjugate to $G$-sections of the same $G$-flow. $G$-sections of two $G$-flows are $G$-flow equivalent if and only if the flows are $G$-equivalent.

If $A$ and $B$ are square matrices over $\mathbb{Z}_+G$, then a $G$-flow equivalence $T_A \rightarrow T_B$ of their $G$-SFTs induces a flow equivalence of the SFTs defined by their standard augmentations $\overline{A}, \overline{B}$.

### 3.6. Positive equivalence

Suppose $B, B', U, V$ are $n \times n$ matrices over $\mathbb{Z}G$ with $U, V$ in $GL(n, \mathbb{Z}G)$. We say $(U, V) : B \rightarrow B'$ is an equivalence if $UBV = B'$. If $\{U, V\}$ is contained in a subset $\mathcal{M}$ of $GL(n, \mathbb{Z})$, then it is an $\mathcal{M}$-equivalence, and the matrices $B, B'$ are $\mathcal{M}$-equivalent.

A basic elementary matrix is a matrix $E_{st}(x)$, which denotes a square matrix equal to the identity except for perhaps the off-diagonal st entry (so, $s \neq t$), which is equal to an element $x$ of $\mathbb{Z}G$. Suppose $E = E_{st}(g)$ and $A$ is a square matrix over $\mathbb{Z}_+G$ such that $g$ is a summand of $A(i, j)$ (i.e., $g \in G$ and the coefficient of $g$ in $A(i, j)$ is positive). Then we say that each of the equivalences

$$(E, I) : (I - A) \rightarrow E(I - A), \quad (E^{-1}, I) : E(I - A) \rightarrow (I - A),$$

$$(I, E) : (I - A) \rightarrow (I - A)E, \quad (I, E^{-1}) : (I - A)E \rightarrow (I - A)$$

is a basic positive $\mathbb{Z}G$-equivalence. Here the equivalences $(E, I)$ and $(I, E)$ are forward and the other two are backward. An equivalence $(U, V) : (I - A) \rightarrow (I - B)$ is a positive $\mathbb{Z}G$-equivalence if it is a composition of basic positive equivalences.

A basic positive equivalence $(I - A) \rightarrow (I - B)$ induces a $G$-flow equivalence $T_A \rightarrow T_B$. Every $G$-flow equivalence $T_A \rightarrow T_B$ is induced (up to isotopy, see [4] Section 6) by a positive $\mathbb{Z}G$-equivalence. For a justification of this claim, we refer to [12]; for more on its place in the positive K-theory classifications for symbolic dynamics, see [3].

The elementary group El$(n, \mathbb{Z}G)$ is the group of $n \times n$ matrices which are products of basic elementary matrices. A positive equivalence $(I - A) \rightarrow (I - B)$ through $n \times n$ matrices is an El$(n, \mathbb{Z}G)$ equivalence, but in general, an El$(n, \mathbb{Z}G)$ equivalence need not be a positive $\mathbb{Z}G$ equivalence, even if $\overline{A}$ is primitive (see for instance [12] Example 4.3). Therefore, we do not in general have that an equivalence $(I - A) \rightarrow (I - B)$ induces a $G$-flow equivalence $T_A \rightarrow T_B$. Still, we will in Theorem 7.2...
show that if $A$ and $B$ satisfy specified conditions, and the equivalence $(U, V) : (I - A) \rightarrow (I - B)$ preserves specified structures (the poset structure, the cycle components (see later in this section) and the coset structure (see Section 4)), then it must be a positive $ZG$-equivalence and thus induce a $G$-flow equivalence $T_A \rightarrow T_B$ (see Theorem 5.3).

For the proofs in Appendices A and B we will use the graphical viewpoint described next (this description can also be found in [12]).

3.7. A row cut basic positive equivalence. Suppose $(E, I) : (I - A) \rightarrow (I - B)$ is a basic forward positive equivalence, $E = E_{st}(g)$. Then $A$ and $B$ agree except perhaps in row $s$, where

$$B(s, r) = A(s, r) + gA(t, r) \quad \text{if } r \neq t,$$
$$B(s, t) = A(s, t) + gA(t, t) - g.$$  

Consequently the labeled graph $G_B$ associated to $B$ is constructed from the labeled graph $G_A$ as follows. An edge $e$ from $s$ to $t$ with label $g$ is deleted from $G_A$. Then, for each $G_A$-edge $f$ beginning at $t$, an additional edge (called $[ef]$) from $s$ to $r$ with label $gh$ (where $h$ is the $G$-label of $f$ and $r$ is the terminal vertex of $f$) is added in to form $G_B$. We refer to this type of positive equivalence as a $(g, s, t)$ row cut (of the matrix $A$, or of an edge $e$ labeled $g$), or just a row cut. When $E(s, t) = p \leq A(s, t)$, we may likewise refer to the positive equivalence implemented by $(E, I)$ as a $(p, s, t)$ row cut.

See Figure 1 for an example of a $(g, s, t)$ row cut of an edge from $s$ to $t$ labeled $g$. For a matrix example corresponding to Figure 1 set to $r = 3$, we use matrices $E = E_{st}(g)$ and

$$A = \begin{pmatrix} p_{11} & g + p_{12} & p_{13} \\ 0 & h' & h'' \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} p_{11} & gh' + p_{12} & gh'' + p_{13} \\ 0 & h' & h'' \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

in which the $p_{ij}$ are arbitrary elements of $\mathbb{Z}_G$, suppressed from the figure, and row 2 has just two entries for simplicity. The change from $G_A$ to $G_B$ is the replacement of the dashed edge of the left graph with the dashed edges of the right graph. On the left, $g, h', h''$ are labels of edges $e, f', f''$; on the right $gh', gh''$ label edges named $[ef'], [ef'']$.

![Figure 1. A row cut of an edge from s to t.](image-url)
The correspondence of the graphs $G_A, G_B$ induces a bijection of $\sigma_A$-orbits and $\sigma_B$-orbits, e.g.

$$\cdots b e f'' c e f' f' d \cdots \leftrightarrow \cdots b[e f''] c[e f'] f' d \cdots .$$

This bijection of orbits does not arise from a bijection of points for the SFTs, but it does correspond to a $G$-equivariant homeomorphism of their mapping tori (after changing time by a factor of 2 over the clopen sets $\{x : x_0 = [e f]\}$, the new flow is conjugate to the old one), which lifts to a $G$-equivariant homeomorphism of the respective mapping tori.

### 3.8. A column cut basic positive equivalence

The other type of basic forward positive equivalence is $(I, E): (I - A) \rightarrow (I - B)$, with $E = E_{st}(g)$. Then $A$ and $B$ agree except perhaps in column $t$, where

$$B(r,t) = A(r,t) + A(r,s)g \quad \text{if } r \neq t , \quad \text{and}$$

$$B(s,t) = A(s,t) + A(t,t)g - g .$$

The labeled graph $G_B$ associated to $B$ is constructed from the labeled graph $G_A$ as follows. An edge $e$ from $s$ to $t$ with label $g$ is deleted from $G_A$. Then, for each $G_A$-edge $f$ ending at $s$, an additional edge (called $[f e]$) from $r$ to $t$ with label $h g$ (where $h$ is the $G$-label of $f$ and $r$ is the initial vertex of $f$) is added in to form $G_B$. We refer to this type of positive equivalence as a $(g, s, t)$ column cut (of the matrix $A$, or of an edge $e$ labeled $g$), or just a column cut. Figure 2 gives the column-cut analogue of Figure 1.

**Example 3.8.1.** When $A$ is a square matrix over $\mathbb{Z}_+G$ with some diagonal entry $A_{tt} = 0$, row cuts may be applied to zero out each entry of column $t$, and then column cuts to zero out each entry of row $t$, giving a permutation matrix $P$ and a smaller matrix $M$ such that $P^{-1}AP = M \oplus 0$. We call this type of operation a trim move. Note $A$ and $M$ define $G$-SFTs which are $G$-flow equivalent. For instance, with $G$ the symmetric group $S_3$, using $A_{22} = 0$ and the standard notation recalled in Definition 4.7, the matrix

$$A = \begin{pmatrix} 23 & 12 \\ 13 & 0 \end{pmatrix}$$
can be row cut, using \((E_{12}(12), I) : (I - A) \rightarrow (I - B)\), to the matrix

\[
B = \begin{pmatrix}
(23) + (132) & 0 \\
(13) & 0 \\
\end{pmatrix}
\]

which can be column cut, using \((I, E_{21}(13)) : (I - B) \rightarrow (I - C)\), to the matrix

\[
C = \begin{pmatrix}
(23) + (132) & 0 \\
0 & 0 \\
\end{pmatrix}
= ((23) + (132)) \oplus 0.
\]

3.9. **Poset-blocked matrices.** In order to handle general \(G\)-SFTs (having more than one irreducible component), as for the case \(G = \{e\}\) addressed in [4, 9] we need to consider matrices with block structures corresponding to irreducible components and transitions between them. Throughout this paper, \(P = \{1, \ldots, N\}\) is a poset (partially ordered set) with a partial order relation \(\preceq\) chosen such that \(i \preceq j\) \(\iff i \leq j\). We will write \(i < j\) if \(i \leq j\) and \(i \neq j\). For a vector of positive integers \(n = (n_1, \ldots, n_N)\), let \(n = \sum_{j=1}^{N} n_j\), and let \(I_i = \{((\sum_{j=1}^{i-1} n_j) + 1, (\sum_{j=1}^{i-1} n_j) + 2, \ldots, \sum_{j=1}^{i} n_j)\} \) for each \(i \in P\). If \(s \in \{1, 2, \ldots, n\}\), then we let \(i(s)\) be the unique integer such that \(s \in I_{i(s)}\). For an \(n \times n\) matrix \(A\) and \(i, j \in P\), we let \(A\{i, j\}\) denote the submatrix of \(A\) obtained by deleting the rows corresponding to indices not belonging to \(I_i\) and columns corresponding to indices not belonging to \(I_j\). The matrix \(A\) is called an \((n, P)\)-blocked matrix if \(A\{i, j\} \neq 0 \iff i \leq j\). For \(S\) a subset of \(ZG\), we let \(M_P(n, S)\) denote the set of \((n, P)\)-blocked matrices with entries in \(S\), and we let \(M_P(S)\) be the union over \(n\) of the sets \(M_P(n, S)\).

The set \(M_P^0(n, Z_+G)\) is the set of matrices \(A\) in \(M_P(n, Z_+G)\) satisfying the following conditions:

1. Each diagonal block \(A\{i, i\}\) is essentially irreducible.
2. If \(i < j\), then there are \(r > 0\), an index \(s\) corresponding to a row in the irreducible core of \(A\{i, i\}\), and an index \(t\) corresponding to a column in the irreducible core of \(A\{j, j\}\) such that \(A'(s, t) \neq 0\).

For \(A \in M_P^0(n, Z_+G)\), \(i\) in \(P\) corresponds explicitly to an irreducible component of the SFT defined by \(X_{\text{ir}}\), with \(i \prec j\) if and only there exists an orbit in \(X_{\text{ir}}\) backwardly asymptotic to component \(i\) and forwardly asymptotic to component \(j\). If \(A \in M_P^0(Z_+G)\) and \(A' \in M_P^0(Z_+G)\), then a flow equivalence \(T_A \rightarrow T_B\) induces a poset isomorphism \(P \rightarrow P'\). We say the flow equivalence respects the component order if this isomorphism is \(k \mapsto k, 1 \leq k \leq N\).

We let \(M_P^0(Z_+G)\) be the union over \(n\) of the sets \(M_P^0(n, Z_+G)\).

3.10. **Cycle components.** For a matrix \(A\) in \(M_P^0(n, Z_+G)\), a cycle component is a component \(i\) in \(P\) such that the irreducible core of
$A\{i,i\}$ is a cyclic permutation matrix. The cycle components contribute significantly to technical difficulties in the classification of $G$-SFTs up to $G$-flow equivalence. For $A$ in $M^0_P(n, Z_G)$, $C(A)$ denotes the set of its cycle components. For a subset $C$ of $P$,

$$M^0_P(C,n,Z_G) := \{A \in M^0_P(n, Z_G) : C(A) = C\}.$$

We let $M^0_P(C,Z_G)$ be the union over $n$ of the sets $M^0_P(C,n,Z_G)$.

3.11. Stabilizations. An unblocked matrix $A'$ is a stabilization (or $0$-stabilization) of an $m \times n$ matrix $A$ if $A$ equals an upper left corner of $A'$, and $A'$ is zero in all remaining entries. (I.e., $A'(s,t) = A(s,t)$ if $1 \leq s \leq m$ and $1 \leq t \leq n$, and otherwise $A'(s,t) = 0$.) A matrix $A'$ in $M_P(n,Z_G)$ is a stabilization (or $0$-stabilization) of a matrix $A$ in $M_P(n,Z_G)$ if $n' \geq n$ (i.e., $n'_i \geq n_i$ for $1 \leq i \leq N$) and for $1 \leq i,j \leq n$, the $i,j$ block submatrix of $A'$ is a stabilization of the $i,j$ block submatrix of $A$.

A matrix $M'$ in $M_P(n',Z_G)$ is a $1$-stabilization of a matrix $M$ in $M_P(n,Z_G)$ if $M' - I$ is a $0$-stabilization of $M - I$. When its entries come from $Z_G$, a matrix $A$ and its $0$-stabilizations define the same $G$-SFT, but it is $I - A$ and its $1$-stabilizations which will share the algebraic invariants for $G$-flow equivalence.

4. $(G, P)$ Coset Structures

The classification up to $G$-flow equivalence of $G$-SFTs $T_A$ defined by irreducible matrices $A$ over $Z_G$ required a reduction to the case that the “weights group” of $T_A$ is all of $G$. (This essentially amounts to reducing to the case that $T_A$ is mixing as an SFT, as recalled in Appendix [12]) For general $A$ over $Z_G$, we will need an analogous reduction on the irreducible components of $A$, and then we will capture invariants of transitions between components using double coset conditions. In this section we prepare the formal structure for this. We begin with the double coset conditions.

Below, $G$ is the given finite group and $P = \{1, \ldots, N\}$ is the given finite poset, with a partial order relation $\leq$ satisfying $i \leq j \implies i \leq j$. Let $H_i$ and $H_j$ be subgroups of $G$. An $(H_i, H_j)$ double coset is a nonempty set equal to $H_i gH_j$ for some $g$ in $G$.

Definition 4.1. A $(G, P)$ coset structure $\mathcal{H}$ is a function which assigns to each pair $(i,j)$ in $P \times P$ such that $i \leq j$ a nonempty subset $H_{ij}$ of $G$ such that

$$i \leq j \leq k \implies H_{ij}H_{jk} \subset H_{ik}.$$

Consequently, $H_i$ (also denoted $H_i$) is a subgroup of $G$ and for $i < j$ $H_{ij}$ is a nonempty union of $(H_i, H_j)$ double cosets.
If \(H_i, H_j\) are subgroups of an abelian group \(G\), then \(H_iH_j\) is a group, and a double coset \(H_igH_j\) is a coset \(gH_iH_j\). For general \(G\), \(\mathcal{H}\) is actually a double coset structure; we use “coset structure” for brevity.

**Definition 4.3.** (Notation) \(\mathcal{P}_n\) denotes the poset \(\{1, \ldots, n\}\) with the linear order: \(i < j\) iff \(i < j\).

In the next two examples, we consider \(\mathcal{H}\) a coset structure for \((G, \mathcal{P}_3)\), with \(G\) abelian, using additive notation. Here the possibilities for \(\mathcal{H}\) are as follows.

- \(H_1, H_2, H_3\) are arbitrary subgroups of \(G\),
- \(H_{12}\) is an arbitrary nonempty union of cosets of \(H_1 + H_2\),
- \(H_{23}\) is an arbitrary nonempty union of cosets of \(H_2 + H_3\),
- \(H_{13}\) is an arbitrary union of cosets of \(H_1 + H_3\) containing \(H_{12} + H_{23}\); because \(G\) is abelian, \(H_{12} + H_{23}\) is a union of cosets of \(H_1 + H_2 + H_3\).

**Example 4.4.** Let \(G = \mathbb{Z}/27\), \(H_1 = H_3 = 9G = \{0, 9, 18\}\), \(H_2 = H_{12} = H_{23} = 3G = \{0, 3, \ldots, 24\}\), \(H_{13} = \{1, 10, 19\} \cup 3G\). So, \(H_1 + H_2 + H_3 = 3G\). Now \(H_{13}\) contains a coset of \(H_1 + H_2 + H_3\), but \(H_{13}\) is not a union of cosets of \(H_1 + H_2 + H_3\).

**Example 4.5.** Let \(G = \mathbb{Z}/30\), \(H_1 = 15G\), \(H_2 = 6G\) and \(H_3 = 10G\). Then \(H_1 + H_2 = 3G\), \(H_2 + H_3 = 2G\), \(H_1 + H_3 = 5G\) and \(H_1 + H_2 + H_3 = G\). Because \(H_{13}\) contains a coset of \(H_1 + H_2 + H_3\), \(H_{13}\) must be \(G\).

**Remark 4.6.** For \(G\) not necessarily abelian, with subgroups \(H_i, H_j\), let us recall some elementary facts about the double cosets \(H_igH_j, g \in G\). As in Example 4.5, \(H_iH_j\) need not be a group; \(H_igH_j\) need not be a coset of a subgroup of \(G\); \(H_igH_j\) is a union of right cosets of \(H_i\) (or left cosets of \(H_j\)), but the union is not arbitrary. A double coset \(H_igH_j\) is the orbit of \(g\) under the (right) action on \(G\) by \(H_i \oplus H_j\) given by \((h, k) : g \mapsto h^{-1}gk\). Consequently, two \((H_i, H_j)\) double cosets are equal or disjoint. Thus, if there are exactly \(r\) \((H_i, H_j)\) double cosets in \(G\), there are exactly \(2^r - 1\) sets which are nonempty unions of \((H_i, H_j)\) double cosets. For \(g \in G\), define the subgroup

\[
M_{ij}(g) = \{h \in H_i : \exists k \in H_j, \ h^{-1}gk = g\} = \{h \in H_i : g^{-1}hg \in H_j\} = H_i \cap gH_jg^{-1}.
\]

Then the isotropy group of \(g\) for this action of \(H_i \oplus H_j\) on \(G\) is

\[
\{(h, k) \in H_i \oplus H_j : h \in M_{ij}(g), k = g^{-1}hg\},
\]

with cardinality \(|M_{ij}(g)|\), and therefore \(|H_igH_j| = |H_i||H_j|/|M_{ij}(g)|\).

In particular, as the cardinality \(|M_{ij}(g)|\) of the isotropy group might vary with \(g\), so might the cardinality of the double coset \(|H_igH_j|\).

**Definition 4.7.** (Notation) \(S_n\) is the group of permutations of \(\{1, \ldots, n\}\) (to avoid confusion, we reserve sans serif numbers to indicate the elements on which \(S_n\) acts). \(A_n\) is the alternating group in \(S_n\). We use
cycle notation to denote elements of $S_n$. For example, $(123)$ is the
cyclic permutation $1 \to 2 \to 3$; $(12)(13) = (132)$. We let $e$ denote
the identity. For a subset $K$ of a group, $\langle K \rangle$ denotes the subgroup
generated by $K$; for example, in $S_n$, $\langle (12) \rangle = \{ (12), e \}$.

**Example 4.8.** Suppose $(G, \mathcal{P}) = (S_3, \mathcal{P}_2)$ with coset structure $\mathcal{H}$. If $H_{11} = S_3$ or $H_{22} = S_3$, then $H_{12}$ must be $S_3$. At the opposite extreme, if $H_{11} = H_{22} = \{ e \}$, then $H_{12}$ can be any nonempty subset of $S_3$. Now suppose $H_1 = \langle (12) \rangle$ and $H_2 = \langle (13) \rangle$. For $g$ in $S_3$,

$$|H_1 g H_2| = |H_1||H_2|/|H_1 \cap g H_2 g^{-1}| = 4/|H_1 \cap g H_2 g^{-1}|.$$

So, $|H_1 g H_2| = 2$ if $g H_2 g^{-1} = H_1$, and otherwise $|H_1 g H_2| = 4$. There are exactly two double cosets $H_1 g H_2$: $D_1 = \{ (23), (132) \}$ and its com-
plement $D_2$ in $S_3$. Neither double coset is a subgroup of $S_3$. $D_1$ is the
only coset of $H_1$ which is a double coset. $D_2$ is neither a left coset nor
a right coset of a subgroup of $S_3$. $D_1$ is a right coset of $H_1$ and a left
coset of $H_2$, but it is neither a left coset of $H_1$ nor a right coset of $H_2$.

**Example 4.9.** Suppose $(G, \mathcal{P}) = (S_3, \mathcal{P}_3)$ with coset structure $\mathcal{H}$. Let $H_1 = H_{12} = \langle (12) \rangle$, $H_2 = \{ e \}$, $H_3 = \langle (13) \rangle$ with $H_{12} = H_1$ and $H_{23} = H_2 \cup (23) H_3$. As seen in Example 4.8 there are two $(H_1, H_3)$ double cosets: $\{ (23), (132) \}$, and its com-
plement in $S_3$. By Example 4.2, $H_{13}$ contains $H_{12} H_{23}$, which here contains $\{ e, (23) \}$. Therefore $H_{13}$ intersects both
$(H_1, H_3)$ double cosets, and must equal $S_3$.

**Definition 4.10.** Two $(G, \mathcal{P})$ coset structures $\mathcal{H}, \mathcal{H}'$ are $G$-cohomologous
if there exist elements $\gamma_1, \ldots, \gamma_N$ in $G$ such that

$$i \leq j \implies H_{ij} = \gamma_i^{-1} H_{ij}' \gamma_j.$$

The “$G$” in “$G$-cohomologous” matters (see Example 4.11). Still,
because $(G, \mathcal{P})$ is fixed, we sometimes write just “coset structure” in
place of “$(G, \mathcal{P})$ coset structure”.

**Example 4.11.** Let $\mathcal{P}$ be the trivial poset $\mathcal{P}_1$. Then $(G, \mathcal{P})$ coset
structures $\mathcal{H}, \mathcal{H}'$ are cohomologous iff the groups $H_1, H_1'$ are conjugate
subgroups of $G$. Define order two subgroups of $S_4$, $H_1 = \langle (12)(34) \rangle$ and
$H_1' = \langle (13)(14) \rangle$. Let $H$ (isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$) be the subgroup
generated by $H_1$ and $H_1'$. Then $H_1, H_1'$ are conjugate as subgroups of $S_4$, but not as subgroups of $H$.

**Example 4.12.** Suppose $G$ is abelian and $\mathcal{H}, \mathcal{H}'$ are $(G, \mathcal{P}_2)$ coset
structures such that $\mathcal{H}_1 = \mathcal{H}_1'$ and $\mathcal{H}_2 = \mathcal{H}_2'$. If $|\mathcal{H}_{12}| = 1 = |\mathcal{H}'_{12}|$, Then $\mathcal{H}$ and $\mathcal{H}'$ are $G$-cohomologous if and only if $H_{12} = H_{12}'$ for
some $g$ in $G$. This always holds if $|\mathcal{H}_{12}| = 1 = |\mathcal{H}'_{12}|$, but need not hold
if e.g. $|\mathcal{H}_{12}| = 2 = |\mathcal{H}'_{12}|$.

**Example 4.13.** Let $\mathcal{H}$ be a $(S_3, \mathcal{P}_2)$ coset structure, with $H_{11} = H_{22} = \{ e \}$. We noted in Example 4.8 that any nonempty subset of $S_3$ may
serve as $H_{12}$. When $H_{12}$ is another such subset, of course it is necessary
that $|H_{12}| = |H'_{12}|$ for the structures to be cohomologous. But this is not a sufficient condition; for instance $H_{12} = \{e, (12)\}$ gives a system which is not cohomologous to that from $H'_{12} = \{e, (123)\}$, as is seen by applying the sign function to the defining relation.

**Definition 4.14.** For a matrix $A$ over $\mathbb{Z}_+ G$, with $\tau_A$ the associated labeling of edges of $G$, the **weight** of a path of edges $p = p_1 p_2 \cdots p_k$ in $G$ is defined to be $\tau_A(p) = \tau_A(p_1) \tau_A(p_2) \cdots \tau_A(p_k)$.

**Definition 4.15.** Suppose $A \in \mathcal{M}_p^\mathbb{Z} (\mathbb{Z}_+ G)$. Then a $(G, \mathcal{P})$ coset structure $\mathcal{H}$ for $A$ is defined as follows.

1. For each $i \in \mathcal{P}$, choose a vertex $v(i)$ from the irreducible core of the block $A\{i,i\}$.

2. For $i \preceq j$, $H_{ij}$ is the set of weights of paths from $v(i)$ to $v(j)$. The group $H_{ii}$ (also denoted $H_i$) was called a **weights group** for $A_{ii}$ in $\mathcal{H}$.

**Example 4.16.** With $G = S_3$, consider the matrices

$$A_1 = \left( \begin{array}{cc} (12) & (132) \\ (123) & e \end{array} \right), A_2 = \left( \begin{array}{cc} (12) & (13) \\ 0 & (12) \end{array} \right), A_3 = \left( \begin{array}{ccc} 0 & (12) & 0 \\ (12) & e & (12) \\ 0 & 0 & e \end{array} \right).$$

For $A_1$, $\mathcal{P} = \mathcal{P}_1$. If $v(1) = 1$, then $H_1 = \langle (12) \rangle$; if $v(1) = 2$, then from $(132) = (123)^{-1}$ we have $H_1 = \langle (12) \rangle \langle (12) \rangle (123)^{-1} = \langle (13) \rangle$. For $A_2$ and $A_3$, $\mathcal{P} = \mathcal{P}_2$. For $A_2$, $H_1 = H_2 = \langle (12) \rangle$ and $H_{12} = S_3 \setminus \langle (12) \rangle$. For $A_3$, $H_1 = H_2 = \{e\}$; if $v(1) = 1$ then $H_{12} = \{e\}$, and if $v(1) = 2$ then $H_{12} = \{\langle (12) \rangle \}$.

Even though the example above shows that they may be different, all coset structures for $A$ will be $G$-cohomologous. To see this, suppose $v_1, v_2$ are vertices in the irreducible core of $A\{i, i\}$ and $\gamma_i$ is the weight of a path from $v_1$ to $v_2$. Replacing a choice $v(i) = v_2$ with the choice $v(i) = v_1$ has the effect of replacing $H_i$ with $H'_i := \gamma_i H_i \gamma_i^{-1}$, and replacing $H_{ij}$ with $H'_{ij} := \gamma_i H_{ij}$ when $i < j$. Therefore $\mathcal{H}$ and $\mathcal{H}'$ are $G$-cohomologous.

**Definition 4.17.** The $(G, \mathcal{P})$ coset structure class of $A$ is the $G$-cohomology class of a $(G, \mathcal{P})$ coset structure for $A$.

**Example 4.18.** Let $G = \mathbb{Z}/4\mathbb{Z}$ with generator $g$. By Example 4.12, \((\frac{1}{2}, \frac{1}{2})\) and \((\frac{2}{0}, \frac{g}{0})\) have the same coset structure class, but \((\frac{2}{0}, \frac{1+g}{2})\) and \((\frac{2}{0}, \frac{1+g^2}{2})\) do not have the same coset structure class.

**Example 4.19.** Suppose $G$ is a nontrivial group. Define $A = \left( \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{array} \right)$ and $A' = \left( \begin{array}{ccc} 1 & 1+g & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{array} \right)$. Coset structures $\mathcal{H}, \mathcal{H}'$ for $A, A'$ are given by $H_1 = H_2 = H_{12} = H'_1 = H'_2 = \{1\}$ and $H'_{12} = \{1, g\}$. Because $|H'_{12}| \neq |H_{12}|$, the coset structures $\mathcal{H}, \mathcal{H}'$ are not $G$-cohomologous (and
therefore, by our structure theorem, the matrices $A, A'$ define $G$-SFTs which are not $G$-flow equivalent. However, $(I - A)$ and $(I - A')$ are $\text{El}(\mathbf{n}, \mathcal{P}_2, \mathbb{Z}^+G)$-equivalent, with $\mathbf{n} = (1, 2)$, because

$$E_{13}(g)(I - A) = \begin{pmatrix} 1 & g \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1-g & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} = I - A'. $$

A smaller example is given by $C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, and $C' = \begin{pmatrix} 1 & 2+g \\ 0 & 2 \end{pmatrix}$, which respectively define $G$-SFTs which respectively are $G$-flow equivalent to those defined by $A$ and $A'$.

Example 4.19 shows that $\text{El}((\mathcal{P}, \mathbf{n}, \mathbb{Z}G)$ equivalence, even by matrices with $i$th diagonal block entries in $\mathbb{Z}^+H_i$ for all $i$, does not give an algebraic relation capturing $G$-flow equivalence. This is why we are led to introduce $\text{El}_P(\mathbf{n}, H)$ equivalence later in this section.

**Example 4.20.** Notice that it can happen that not every coset structure in the $(G, \mathcal{P})$ coset structure class of a matrix $A$ is a $(G, \mathcal{P})$ coset structure for $A$. If for example $(G, \mathcal{P}) = (S_3, \mathcal{P}_2)$ and

$$A = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix},$$

then $H_{11} = H_{12} = H_{22} = \{e\}$ is the only $(S_3, \mathcal{P}_2)$ coset structure for $A$; but $H'_{11} = H'_{12} = \{e\}, H'_{12} = \{(12)\}$ is also in the $(S_3, \mathcal{P}_2)$ coset structure class of $A$.

The classes $\mathcal{M}_P(\mathbf{n}, \mathbb{Z}G)$ and $\mathcal{M}_P^n(C, \mathbf{n}, \mathbb{Z}^+G)$ used below were defined in Subsections 3.9 and 3.10.

**Definition 4.21.** Let $\mathcal{H}$ be a $(G, \mathcal{P})$ coset structure.

- $\mathcal{M}_P(\mathbf{n}, \mathcal{H})$ is the set of matrices $A \in \mathcal{M}_P(\mathbf{n}, \mathbb{Z}G)$ such that for $i \leq j$, the entries of $A(i,j)$ belong to $\mathbb{Z}H_{ij}$.
- $\mathcal{M}_P^n(C, \mathbf{n}, \mathcal{H})$ is the set of $A$ in $\mathcal{M}_P(\mathbf{n}, \mathcal{H}) \cap \mathcal{M}_P^n(C, \mathbf{n}, \mathbb{Z}^+G)$ such that $\mathcal{H}$ is a $(G, \mathcal{P})$ coset structure for $A$.
- $\mathcal{M}_P^o(\mathcal{C}, \mathcal{H}) = \cup_{\mathbf{n}} \mathcal{M}_P^n(C, \mathbf{n}, \mathcal{H})$.
- $\mathcal{M}_P^o(\mathcal{H}) = \cup_{\mathcal{C}} \mathcal{M}_P^n(C, \mathcal{H})$.

**Definition 4.22.** A matrix in $\mathcal{M}_P^n(C, \mathbb{Z}^+G)$ satisfies Condition $C_1$ (or, is $C_1$) if it belongs to $\mathcal{M}_P^n(C, \mathbf{n}, \mathbb{Z}^+G)$ for $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ such that the following condition holds for all $i \in \mathcal{P}$: $n_i = 1$ if and only if $i \in \mathcal{C}$.

A square matrix $A$ over $\mathbb{Z}^+G$ is nondegenerate if it has no zero row and no zero column. Notice that this is equivalent to the graph $G_\mathcal{P}$ being nondegenerate (that is, every vertex of $G_\mathcal{P}$ belongs to a bi-infinite path). The next proposition applies in particular to a nondegenerate matrix $A$ (which cannot be nilpotent).

**Proposition 4.23.** Let $A$ be a nonnilpotent square matrix over $\mathbb{Z}^+G$. Then there is an $N$, a partial order $\preceq$ on $\mathcal{P} := \{1, \ldots, N\}$ satisfying $i \preceq j \implies i \leq j$, a subset $\mathcal{C}$ of $\mathcal{P}$, a $(G, \mathcal{P})$ coset structure $\mathcal{H}$, and a
matrix $B \in \mathcal{M}_P^o(C, \mathcal{H})$ with each diagonal block irreducible such that $T_A$ and $T_B$ are $G$-flow equivalent. $B$ can be produced algorithmically from $A$.

**Example 4.24.** The first two of the matrices

$$
\begin{pmatrix}
0 & (12) \\
(13) & 0 \\
\end{pmatrix},
\begin{pmatrix}
e & (12) \\
e + (12) & 0 \\
0 & e \\
0 & e \\
\end{pmatrix}
$$

are not $C_1$, and the third does not satisfy the diagonal block irreducibility condition of Proposition 4.23. The algorithm outlined in the proof below will give, respectively,

$$
\begin{pmatrix}
(123) \\
e & (12) \\
e & (12) \\
\end{pmatrix},
\begin{pmatrix}
e & (12) \\
(123) & 0 \\
0 & e \\
\end{pmatrix},
\begin{pmatrix}
e & (12) \\
(123) & 0 \\
0 & e \\
\end{pmatrix}
$$

which satisfy both conditions. See Example 3.8.1 for details of the first example.

For an example of the step $B = DAD^{-1}$ in the proof of Prop. 4.23 we use $\mathcal{P} = \mathcal{P}_2$, $G = S_3$, $H_1 = H_2 = H_{12} = A_3$, $\nu(1) = 1$ and $\nu(2) = 4$. Then

$$
B = DAD^{-1} = 
\begin{pmatrix}
e & e & e & (123) \\
(123) & (213) & (123) & e \\
e & e & e & (123) \\
0 & 0 & 0 & e \\
0 & 0 & 0 & e \\
\end{pmatrix}
= 
\begin{pmatrix}
e & (12) & e & (123) \\
e & (23) & (123) & (12) \\
e & (12) & e & e \\
0 & 0 & 0 & e \\
0 & 0 & 0 & e \\
0 & 0 & 0 & (23) \\
\end{pmatrix}
\begin{pmatrix}
e & 0 & 0 & 0 \\
0 & 12 & 0 & 0 \\
0 & 0 & 0 & e \\
0 & 0 & 0 & (12) \\
0 & 0 & 0 & (132) \\
\end{pmatrix}
\begin{pmatrix}
e & e & e & (123) \\
e & (123) & (123) & e \\
e & e & e & (123) \\
0 & 0 & 0 & e \\
0 & 0 & 0 & e \\
0 & 0 & 0 & e \\
\end{pmatrix}
\begin{pmatrix}
e & 0 & 0 & 0 \\
0 & 12 & 0 & 0 \\
0 & 0 & 0 & e \\
0 & 0 & 0 & (12) \\
0 & 0 & 0 & (132) \\
\end{pmatrix}
$$

Proof of Prop. 4.23. If a diagonal entry of $A$ is zero, then we may apply the trimming move of Example 3.8.1 to produce a smaller matrix over $\mathbb{Z}_s G$, which defines a flow equivalent $G$-SFT, and which is also not nilpotent. By iteration of this move, we may assume without loss of generality that every diagonal entry of $A$ is nonzero. Then there is a poset $\mathcal{P} = \{1, \ldots, N\}$, a vector $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ and a permutation matrix $P$ and such that $PAP^{-1} \in \mathcal{M}_P^o(\mathbf{n}, \mathbb{Z}_s G)$. We may assume $PAP^{-1} = A$.

Let $\mathcal{C}$ be the set of cycle components for $A$ in $\mathcal{P}$. For $1 \leq i \leq N$, set $m_i = \sum_{h<i} n_1$ and $\mathcal{I}_i = \{m_i + 1, \ldots, m_i + n_i\}$. Let $A\{i,j\}$ be the submatrix of $A$ on indices $\mathcal{I}_i \times \mathcal{I}_j$. For $i \in \mathcal{C}$, $\mathcal{I}_i$ must now be a singleton. If $i \notin \mathcal{C}$ and $\mathcal{I}_i$ is a singleton, then there is a state splitting $A \to A'+A''$ which increases $n_i$ from 1 to 2 such that every entry of $A'\{i,i\}$ is nonzero. On iteration of this process, passing to a new $A$ and a new $\mathbf{n}$, we have $A \in \mathcal{M}_P^o(\mathbf{n}, \mathbb{Z}_s G)$, with each $A\{i,i\}$ irreducible, such that $i \in \mathcal{C} \iff n_i = 1$.

We now turn to the coset structure. For each $i$ in $\mathcal{P}$, pick an index $\nu(i)$ in $\mathcal{I}_i$, and with these choices define a $(G, \mathcal{P})$ coset structure $\mathcal{H}$ for $A$ as in Definition 4.15. Next, for each $i \in \mathcal{P}$ and each index $s$ in $\mathcal{I}_i$,
because $\mathcal{I}_i = \mathcal{I}_i^*$ we may choose $d_s$ the weight of some path from $v(i)$ to $s$ and $b_s$ the weight of some path from $s$ to $v(i)$. Let $D$ be the diagonal matrix with $D(s,s) = d_s$. Define $B = DAD^{-1}$. There is a positive $ZG$-equivalence from $I - A$ to $I - B$ (Proposition A.1), so $T_B$ is $G$-flow equivalent to $T_A$.

Suppose $s \in \mathcal{I}_i$ and $t \in \mathcal{I}_j$. We claim that $B(s,t) \in ZH_{ij}$. To prove this claim, note that $b_t d_t$ is the weight of a path from $t$ to $t$. Pick $k > 0$ such that $(b_t d_t)^k = e$. Then $(b_t d_t)^{k-1}b_t = (d_t)^{-1}$, and therefore $(d_t)^{-1}$ is the weight of a path from $t$ to $v(j)$. Therefore $B(s,t) = d_s A(s,t)(d_t)^{-1}$ is the weight of a path from $v(i)$ to $v(j)$, and therefore is in $H_{ij}$.

Because $I - A$ and $I - B$ are positive $ZG$ equivalent, a coset structure for $B$ must be $G$-cohomologous to the coset structure $\mathcal{H}$ of $A$. By construction, a coset structure $\mathcal{H}'$ for $B$ defined from the vertex choices $v(i)$ has $\mathcal{H}'_{ij} \subset \mathcal{H}_{ij}$ if $i \leq j$. By the $G$-cohomology, this containment must be equality, so $\mathcal{H}$ is a coset structure for $B$, and $B \in \mathcal{M}_P(C, \mathcal{H})$.

Remark 4.25. Suppose $A \in \mathcal{M}_P^+(n, \mathcal{H})$ and $T_A$ is the $G$-SFT defined by $A$ in Section 3.3. Let $\pi: X_A \times G \to X_A^G$ be the map collapsing $G$-orbits. For $i \in \mathcal{P}$, let $\overline{T}_i : \overline{X}_i \to \overline{X}_i$ be the mixing $Z$-SFT defined by $A\{i,i\}$. Let $X_i = \overline{X}_i \times H_i$, and let $T_i$ be the restriction of $T_A$ to $X_i$. Then $H_i$ is the isotropy group of $X_i$, and $\pi^{-1}(\overline{X}_i)$ is the disjoint union of $|G/H_i|$ mixing $Z$-SFTs; the mixing SFTs $g_1 X_i, g_2 X_i$ are equal if and only if $g_1 H_i = g_2 H_i$. Let us write $g_1 X_i \to g_2 X_j$ if there exists a point backwardly asymptotic to $g_1 X_i$ and forwardly asymptotic to $g_2 X_j$. Then $g_1 X_i \to g_2 X_j$ if and only if $X_i \to (g_1)^{-1}g_2 X_j$. From the definition of $T_A$, we see $X_i \to g X_j$ iff $g \in H_{ij}$, and this holds for every element or no element of a double coset $H_i g H_j$.

We now give terminology for the equivalences fundamental to our results. For a positive vector $n = (n_1, \ldots, n_N)$, let $\text{El}_P(n, \mathcal{H})$ be the group of matrices generated by the basic elementary matrices in $\mathcal{M}_P(n, \mathcal{H})$. We define an $\text{El}_P(n, \mathcal{H})$-equivalence to be an equivalence $(U,V) : (I - A) \to (I - B)$ with $U,V$ in $\text{El}_P(n, \mathcal{H})$ and $A,B$ in $\mathcal{M}_P(n, \mathcal{H})$.

A basic positive $\text{El}_P(n, \mathcal{H})$-equivalence is a basic positive $ZG$-equivalence which is also an $\text{El}_P(n, \mathcal{H})$-equivalence. A positive $\text{El}_P(n, \mathcal{H})$-equivalence is defined to be a composition of basic positive $\text{El}_P(n, \mathcal{H})$-equivalences.

An (positive) $\text{El}_P(\mathcal{H})$-equivalence from $I - A$ to $I - B$ is defined to be any (positive) $\text{El}_P(n, \mathcal{H})$-equivalence $(U,V) : (I - A') \to (I - B')$ such that $A', B'$ are stabilizations of $A,B$. It is easy to check that if there is a (positive) $\text{El}_P(\mathcal{H})$-equivalence from $I - A$ to $I - B$, and a (positive) $\text{El}_P(\mathcal{H})$-equivalence from $I - B$ to $I - C$, then there is a (positive) $\text{El}_P(\mathcal{H})$-equivalence from $I - A$ to $I - C$. 

5. The main results

We can now state the primary result of the paper. The $C_1$ condition in the statement was given in Definition 4.22. Given $\gamma = (\gamma_1, \ldots, \gamma_N) \in G^N$, we define $D^n_\gamma \in M_P(n, \mathbb{Z} + G)$ to be the diagonal matrix $D$ such that $D(s,s) = \gamma_{i(s)}$ (i.e., for $1 \leq i \leq N$, the diagonal block $D\{i, i\}$ is $\gamma_iI$).

**Theorem 5.1.** Suppose $G$ is a finite group; $\mathcal{P} = \{1, \ldots, N\}$ is a poset; $\mathcal{H}$ and $\mathcal{H}'$ are $(G, \mathcal{P})$ coset structures; $A$ and $B$ are nondegenerate $C_1$ matrices in $M_P^n(C, n, \mathcal{H})$ and $M_P^n(C, n', \mathcal{H}')$, respectively. Then the following are equivalent.

1. There is a $G$-flow equivalence of the $G$-SFTs $T_A$ and $T_B$ which respects the component ordering.
2. For some $\gamma \in G$, for $C = (D^n_\gamma)^{-1}BD^n_\gamma$ there exist $m$ and $C_1$ stabilizations $A^{<0>}, C^{<0>}$ of $A, C$ in $M_P^n(C, m, \mathcal{H})$ such that the matrices $I - A^{<0>}$ and $I - C^{<0>}$ are $El_P(m, \mathcal{H})$-equivalent.

By condition (2) of Theorem 5.1 there is an immediate corollary.

**Corollary 5.2.** The $(G, \mathcal{P})$ coset structure class is an invariant of component-order-respecting $G$-flow equivalence.

We will give a more complicated statement next for a flow equivalence which need not respect component order. By Proposition 4.22, $G$-SFTs can be presented by matrices in the form addressed by Theorem 5.3. Therefore, Theorem 5.1 (as elaborated in Theorem 5.3) gives a classification of $G$-SFTs up to $G$-flow equivalence.

If $\mathcal{P} = \{1, \ldots, N\}$ and $\mathcal{P}' = \{1, \ldots, N\}$ are finite posets given by partial order relations $\preceq$ satisfying $i \preceq j \implies i \leq j$, $\alpha : \mathcal{P} \to \mathcal{P}'$ is a poset isomorphism, and $n = (n_1, \ldots, n_N)$ is a positive vector, then we denote by $\alpha^*(n)$ the vector $(m_1, \ldots, m_N)$ with $m_i = n_{\alpha^{-1}(i)}$, and we let $Q_\alpha^n$ be the $n \times n$ matrix (where $n = \sum_{j=1}^N n_j = \sum_{i=1}^N m_i$) with entries $q_{rs} = 1$ if $r = l + \sum_{j=1}^{\alpha^{-1}(i)} m_j$ and $s = l + \sum_{k=1}^{\alpha^{-1}(j)} n_k$ for some $i \in \mathcal{P}$ and some $l \in \{1, \ldots, n_i\}$, and 0 otherwise. Then $(Q_\alpha^n)^{-1}AQ_\alpha^n \in M_P(n, \mathbb{Z} + G)$ whenever $A \in M_P(\alpha^*(n), \mathbb{Z} + G)$.

**Theorem 5.3.** Let $G$ be a finite group, let $\mathcal{P} = \{1, \ldots, N\}$ and $\mathcal{P}' = \{1, \ldots, N'\}$ be finite posets given by partial order relations $\preceq$ satisfying $i \preceq j \implies i \leq j$, let $C$ and $C'$ be subsets of $\mathcal{P}$ and $\mathcal{P}'$ respectively, and let $\mathcal{H} = \{H_{ij}\}_{i,j \in \mathcal{P}}$ and $\mathcal{H}' = \{H'_{ij}\}_{i,j \in \mathcal{P}'}$ be $(G, \mathcal{P})$ and $(G, \mathcal{P}')$ coset structures.

Suppose $A \in M_P^n(C, n, \mathcal{H})$ and $B \in M_{P'}^n(C', n', \mathcal{H}')$ are $C_1$ stabilizations of nondegenerate matrices. Then the following are equivalent.

1. The $G$-SFTs $T_A$ and $T_B$ are $G$-flow equivalent.
2. There exists a poset isomorphism $\alpha : \mathcal{P} \to \mathcal{P}'$ such that $\alpha(C) = C'$, and there exists $\gamma \in G^N$ such that $H_{ij} = \gamma_i^{-1}H'_{\alpha(i)\alpha(j)}\gamma_j$ for
i, j ∈ \mathcal{P} \text{ with } i \preceq j, \text{ and such that for the matrix}
\begin{align*}
C &= (D_{\gamma}^{n})^{-1}(Q_{\alpha}^{n})^{-1} B Q_{\alpha}^{n} D_{\gamma}^{n}
\end{align*}
the following holds: there exist \textbf{m} and \mathcal{C}_1 \text{ stabilizations } A^{<0>}, C^{<0>} \in \mathcal{M}_P(C, \textbf{m}, \mathcal{H}) \text{ of } A, C \text{ such that the following holds: the matrices } I - A^{<0>} \text{ and } I - C^{<0>} \text{ are } \text{ElP}(\textbf{m}, \mathcal{H})\text{-equivalent.}

In Theorem 5.3, the implication (1) \implies (2) is a part of the more general result Theorem 6.2. The implication (2) \implies (1) is a consequence of a much stronger constructive statement, the Factorization Theorem 7.2. We therefore postpone the proof of Theorem 5.3 to Section 9.

After finite reductions, it is now clear that there is a procedure for determining \textit{G}-flow equivalence of \(T_A\) and \(T_B\) (hence of \textit{G}-SFTs) if there is a procedure for answering the following.

**Question 5.4.** Let \(G\) be a finite group. Given a \((G, \mathcal{P})\) coset structure \(\mathcal{H}, \mathcal{C} \subset \mathcal{P}\) and \(\mathcal{C}_1\) matrices \(A, C\) in \(\mathcal{M}_P(C, \textbf{m}, \mathcal{H})\), does there exist a procedure to decide whether the following holds: there exist \(m_i \in C\) such that \(m_i = 1\) for \(i \in C\), and 1-stabilizations \(L^{<1>}, M^{<1>}\) of \(L, M\) in \(\mathcal{M}_P(\textbf{m}, \mathcal{H})\), such that the matrices \(L^{<0>}\) and \(M^{<0>}\) are \(\text{ElP}(\textbf{m}, \mathcal{H})\)-equivalent.

We will give a more algebraically phrased question next for future reference. An answer yes to Question 5.5 gives an answer yes to Question 5.4 (using \(L = I - A, M = I - B\)).

**Question 5.5.** Let \(G\) be a finite group. Given a \((G, \mathcal{P})\) coset structure \(\mathcal{H}, \mathcal{C} \subset \mathcal{P}\), and \(\textbf{n}\) such that \(n_i = 1\) for \(i \in \mathcal{C}\), and matrices \(L, M\) in \(\mathcal{M}_P(\textbf{n}, \mathcal{H})\), is there a procedure to decide whether the following holds: there exist \(m_i \in C\) such that \(m_i = 1\) for \(i \in C\), and 1-stabilizations \(L^{<1>}, M^{<1>}\) of \(L, M\) in \(\mathcal{M}_P(\textbf{m}, \mathcal{H})\), such that the matrices \(L^{<0>}\) and \(M^{<0>}\) are \(\text{ElP}(\textbf{m}, \mathcal{H})\)-equivalent.

There is an affirmative answer to Question 5.5 for \(G = \{e\}\) (see [11]), and perhaps for all \(G\). In some cases the invariants for \(G\)-SFTs (which we can regard as \(G\)-SFTs with \(G = \{e\}\)) have allowed practical computation of examples and subclasses (see e.g. [18, 9]). We note that if \(G\) is abelian, then there are only finitely many \(G\)-flow equivalence classes of \(G\)-SFTs defined by matrices \(A\) for which the diagonal block determinants of \(I - A\) are prescribed regular elements (i.e., are not zero-divisors) in \(\mathbb{Z}G\) (Theorem 9.3).

For the case \(G = \{e\}\), there is a complicated diagram of homomorphisms of finitely generated abelian groups (the reduced \(K\)-web of [9], useful for applications to Cuntz-Krieger algebras as explained in [3]) which (with regard to an appropriate notion of diagram isomorphism) is a complete invariant for the \(\text{ElP}(\mathcal{H})\)-equivalence. For general \(G\), one can define a \(K\)-web invariant in the same way, using \(\mathbb{Z}G\)-modules.
and module homomorphisms in place of abelian groups and homomorphisms of abelian groups. However, this invariant is no longer complete, because new obstructions arise to passing from diagram isomorphism to the elementary matrix equivalence. (We thank Takeshi Katsura for showing us examples of this.) Developing a complete invariant from the $\mathbb{Z}G$ $K$-web by characterizing the allowed diagram isomorphisms is a nontrivial but perhaps accessible problem.

We use the $C_1$ condition in Theorem 5.3 to get a precise characterization in terms of matrix equivalence. To see that Theorem 5.3 would be false if the $C_1$ condition were dropped, consider the following example:

$$(I - A)U = I - B$$

$$\begin{pmatrix}
1 - g & -e & -e & 0 \\
0 & 0 & 0 & -e \\
0 & 0 & 0 & -e \\
0 & 0 & 0 & 1 - h
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 - g & -3e & -2e & 0 \\
0 & 0 & 0 & -e \\
0 & 0 & 0 & -e \\
0 & 0 & 0 & 1 - h
\end{pmatrix}.$$

Here $e$ is the identity in $G$ ($e = 1$ in $\mathbb{Z}G$) and $U \in \text{El}_P(H)$, for a coset structure $H$ for $A$. But $A$ and $B$ present skewing functions on SFTs with 2 and 5 infinite orbits, respectively, so $T_A$ and $T_B$ are certainly not $G$-flow equivalent.

In the case $G = \{e\}$ we understand by [4, Theorem 3.3] exactly when an $\text{El}_P(H)$-equivalence is a positive $\text{El}_P(H)$-equivalence (i.e., arises from a $G$-flow equivalence). We do not know how to do the same for nontrivial $G$, if a block $A\{i,i\}$ defining a cycle component is allowed to be larger than $1 \times 1$. We need to require the $C_1$ condition to be able to prove in the Factorization Theorem 7.2 below, that every $\text{El}_P(H)$-equivalence is a positive $\text{El}_P(H)$-equivalence.

Partly because of complications arising from cycle components, we avoid the infinite matrices used in [4] and [12] to describe stabilizations.

6. From $G$-flow equivalence to positive $\text{El}_P(H)$ equivalence

We will now present and prove Theorem 6.2 from which we directly get the implication $(1) \implies (2)$ in Theorem 5.3.

First we introduce Condition $C_{1+}$, which we shall use in the proof of Theorem 6.2.

**Definition 6.1.** Let $G$ be a finite group, let $\mathcal{P} = \{1, \ldots, N\}$ be a finite poset given by a partial order relation $\leq$ satisfying $i \leq j \implies i \leq j$, and let $n = (n_1, \ldots, n_N)$ be a vector of positive integers.

Given $C \subset \mathcal{P}$, a matrix $M$ in $\mathcal{M}_{\mathcal{P}}(n, \mathbb{Z}G)$ satisfies Condition $C_{1+}$ if for every $i$ in $C$ there exists $s_i$ in $\mathcal{I}_i$ such that the following hold.

1. If $\{s, t\} \subset \mathcal{I}_i$ and $M(s, t) \neq 0$, then $(s, t) = (s_i, s_i)$.
2. If $s \in \mathcal{I}_i$, with $M(s, t) \neq 0$ or $M(t, s) \neq 0$, then $s = s_i$. 
Notice that if $M$ is a stabilization of a $C_1$ matrix, then $M$ satisfies condition $C_{1+}$.

**Theorem 6.2.** Let $G$ be a finite group, let $\mathcal{P} = \{1, \ldots, N\}$ and $\mathcal{P}' = \{1, \ldots, N'\}$ be finite posets given by partial order relations $\preceq$ satisfying $i \preceq j \implies i \leq j$, let $\mathcal{C}$ and $\mathcal{C}'$ be subsets of $\mathcal{P}$ and $\mathcal{P}'$ respectively, and let $\mathcal{H} = \{H_{ij}\}_{i,j \in \mathcal{P}}$ and $\mathcal{H}' = \{H'_{ij}\}_{i,j \in \mathcal{P}'}$ be $(G, \mathcal{P})$ and $(G, \mathcal{P}')$ coset structures.

Suppose $A \in \mathcal{M}_p^\rho(\mathcal{C}, n, \mathcal{H})$ and $B \in \mathcal{M}_p^\rho(\mathcal{C}', n', \mathcal{H}')$ are stabilizations of nondegenerate matrices, and $T_A$ and $T_B$ are $G$-flow equivalent. Then there exist a poset isomorphism $\alpha : \mathcal{P} \to \mathcal{P}'$ such that $\alpha(\mathcal{C}) = \mathcal{C}'$ and $\gamma = (\gamma_1, \ldots, \gamma_N) \in G^N$ (where $N$ is the number of elements of $\mathcal{P}$) such that $H_{ij} = \gamma_i^{-1} H'_{\alpha(i)\alpha(j)} \gamma_j$ for $i, j$ in $\mathcal{P}$ with $i \leq j$, and such that for the matrix

$$C = (D^\mathcal{C})^{-1}(Q^\mathcal{C})^{-1}BQ^\mathcal{C}D^\mathcal{C}$$

the following holds: there exist $m$ and stabilizations $A^{<0>}, C^{<0>}$ in $\mathcal{M}_p^\rho(\mathcal{C}, m, \mathcal{H})$ of $A, C$ with a positive $El_\mathcal{P}(\mathcal{m}, \mathcal{H})$-equivalence $(U, V) : (I - A^{<0>}) \to (I - C^{<0>})$.

Moreover, if $A$ and $B$ satisfy Condition $C_1$, then the matrices $A^{<0>}$, $C^{<0>}$ can be chosen to satisfy Condition $C_1$.

Before we give the proof, which is a nontrivial elaboration of the proof for the case that $A, B$ are essentially irreducible [12 Proposition 4.7], we outline the argument.

A discrete cross-section for a homeomorphism $T : X \to X$ of a compact zero-dimensional metrizable space $X$ is a clopen subset $K \subset X$ such that every point of $X$ is mapped into $K$ by some positive power of $T$. In this case, for $x \in K$ there is a smallest positive integer $\rho_K(x)$ such that $T^{\rho_K(x)}(x)$ is in $K$ and the return map $R_K : K \to K$ is then the map $x \to T^{\rho_K(x)}(x)$ (see for example [14] for details). If $T$ is an SFT, then $R_K$ is again SFT. If $K$ is a $G$-invariant discrete cross-section for $T_A$, then there is a (unique) discrete cross-section $C$ for $\sigma_{\mathcal{X}}$ such that $K = C \times G$ and

$$R_K((x, g)) = (R_C(x), g\tau_A(x)\tau_A(\sigma_{\mathcal{X}}(x)) \ldots \tau_A(\sigma_{\mathcal{X}}^{\rho_C(x)-1}(x)))$$

for $x \in C$ and $g \in G$.

The Parry-Sullivan argument [26] shows that any flow equivalence of mapping tori of SFTs is isotopic to one which is induced by a conjugacy of return maps to discrete cross-sections (again, see for example [3] for details). It follows that since $T_A$ and $T_B$ are $G$-flow equivalent, there exist $G$-invariant discrete cross-sections $K_A$ and $K_B$ for $T_A$ and $T_B$ such that the return maps $R_{K_A}$ and $R_{K_B}$ are $G$-conjugate. Let $C_A$ and $C_B$ be discrete cross-sections for $\sigma_{\mathcal{X}}$ and $\sigma_{\mathcal{Y}}$ such that $K_A = C_A \times G$ and $K_B = C_B \times G$.

Our strategy is to first construct matrices $A^{<1>}, A^{<2>} \in \mathcal{M}_p^\rho(\mathcal{C}, \mathcal{H})$ such that $A^{<2>}$ presents the $G$-SFT $R_{K_A}$, and such that there are
positive El\(_P(\mathcal{H})\)-equivalences from \(I - A\) to \(I - A^{<1>}\) and from \(I - A^{<1>}\) to \(I - A^{<2>}\) (this is done in Step 1 and Step 2). Similarly, we get matrices \(B^{<1>}, B^{<2>} \in M^p(\mathcal{C}', \mathcal{H}')\) such that there is a positive El\(_P(\mathcal{H}')\)-equivalence \((I - B) \rightarrow (I - B^{<1>})\) and a positive El\(_P(\mathcal{H}')\)-equivalence \((I - B^{<1>}) \rightarrow (I - B^{<2>})\), and such that \(T_{B^{<2>}}\) is \(G\)-conjugate to \(R_{K_B}\). Since \(R_{K_A}\) and \(R_{K_B}\) are \(G\)-conjugate, it follows that \(T_{A^{<2>}}\) and \(T_{B^{<2>}}\) are \(G\)-conjugate. We use this in Step 3 to construct matrices \(A^{<3>} \in M_p(\mathcal{C}, \mathcal{r}, \mathcal{H}), B^{<3>} \in M^p(\mathcal{C}', \mathcal{r}', \mathcal{H}')\) and a poset isomorphism \(\alpha : \mathcal{P} \rightarrow \mathcal{P}'\) such that \(\alpha'(\mathcal{r}) = \mathcal{r}'\) and \(\alpha(\mathcal{C}) = \mathcal{C}'\), and such that there is a positive El\(_P(\mathcal{H})\)-equivalence from \(I - A^{<2>}\) to \(I - A^{<3>}\), a positive El\(_P(\mathcal{H}')\)-equivalence from \(I - B^{<2>}\) to \(I - B^{<3>}\), and such that \(A^{<3>} = (Q^r_\alpha)^{-1}B^{<3>}Q^s_\alpha\) and on this common domain the matrices \(A^{<3>}\) and \((Q^r_\alpha)^{-1}B^{<3>}Q^s_\alpha\) define skewing functions which are cohomologous. In Step 4 we then find a vector \(\gamma \in G^n\) such that \(H_{ij} = \gamma_i^{-1}H'_{(i)a(j)}\gamma_j\) for \(i, j \in \mathcal{P}\) with \(i \leq j\), and such that there are positive El\(_P(\mathcal{H})\)-equivalences

\[
(I - A^{<3>}) \rightarrow (I - (D^r_\gamma)^{-1}(Q^r_\alpha)^{-1}B^{<3>}Q^s_\alpha D^s_\gamma) \quad \text{and} \quad (I - (D^r_\gamma)^{-1}(Q^r_\alpha)^{-1}B^{<3>}Q^s_\alpha D^s_\gamma) \rightarrow (I - C),
\]

where \(C = (D^m_\gamma)^{-1}(Q^m_\alpha)^{-1}BQ^m_\alpha D^m_\gamma\). This completes the proof of the first half of the theorem.

To show that the matrices \(A^{<0>}, C^{<0>}\) can be chosen to satisfy Condition \(C_1\) if \(A\) and \(B\) satisfy Condition \(C_1\), we refine the construction of Steps 1–4 in order to obtain stabilizations \(\tilde{A}^{<0>}, \tilde{C}^{<0>} \in M_p(\mathcal{C}, \mathcal{H})\) of \(A\) and \(C\) and a positive El\(_P(\mathcal{H})\)-equivalence

\[
(\tilde{U}, \tilde{V}) : (I - \tilde{A}^{<0>}) \rightarrow (I - \tilde{C}^{<0>})
\]

such that for every \(i \in \mathcal{C}\) the matrices \(\tilde{U}(i, i), \tilde{V}(i, i)\) are the identity matrix. We then get \(C_1\) stabilizations \(A^{<0>}, C^{<0>}\) in \(M^p(\mathcal{C}, \mathcal{m}, \mathcal{H})\) of \(A, C\) and with a positive El\(_P(\mathcal{m}, \mathcal{H})\)-equivalence

\[
(U, V) : (I - A^{<0>}) \rightarrow (I - C^{<0>})
\]

as wanted by letting \(A^{<0>}, C^{<0>}, U, W\) be principal submatrices of \(\tilde{A}^{<0>}, \tilde{C}^{<0>}, \tilde{U}, \tilde{V}\). This is done in Steps 5–8.

*Proof. Step 1: Higher block presentation.* We begin by constructing a higher block presentation \(A^{<1>}\) of \(T_A\) such that the discrete cross-section \(C_A\) corresponds to a union of vertices in the graph \(\mathcal{G}_A\), and such that there is a positive El\(_P(\mathcal{H})\)-equivalence from \(I - A\) to \(I - A^{<1>}\).

There is a \(k\) and a subset \(S\) of the \(2k + 1\)-blocks of \(X_\mathcal{A}\) such that \(C_A = \{x \in X_\mathcal{A} : x[-k, k] \in S\}\). Let \(P_{2k+1}\) be the set of paths in \(\mathcal{G}_A\) of length \(2k + 1\). For \(p \in P_{2k+1}\), let \(s(p)\) be the initial vertex of the middle edge of \(p\). Index the elements of \(P_{2k+1} = \{p_1, p_2, \ldots, p_n\}\) such that if \(s(p_i) < s(p_j)\), then \(i < j\). We will now construct an \(n \times n\) matrix \(A^{<1>}\) over \(\mathbb{Z}_4G\) (actually it will be a matrix over \(G\)). Let \(1 \leq s, t \leq n\). If
there exist edges $e, f$ in $G_A$ such that $p_e = f p_t$, then the $(s, t)$ entry of $A^{<1>}$ is the label of the middle edge of $p_s$. If there are no edges $e, f$ in $G_A$ such that $p_e = f p_t$, then the $(s, t)$ entry of $A^{<1>}$ is 0. It follows from Proposition 11 that $A^{<1>} \in \mathcal{M}_p^2(C, H)$ and that there is a positive $\text{El}_p(H)$-equivalence from $I - A$ to $I - A^{<1>}$, Since $A$ is a stabilization of a nondegenerate matrix, it follows that $A^{<1>}$ is nondegenerate.

**Step 2: Discrete cross-section.**

In this step we produce a nondegenerate matrix $A^{<2>} \in \mathcal{M}_p^2(C, H)$ which presents the $G$-SFT $R_{K_A}$, and explain that there is a positive $\text{El}_p(H)$-equivalence from $I - A^{<1>}$ to $I - A^{<2>}$. The matrix $A^{<2>}$ is the adjacency matrix of the labelled graph which has vertex set $\{s \in \{1, 2, \ldots, n\} : p_s \in S\}$ and there for each path $p$ in $G_{A^{<1>}}$ which starts and ends in vertices $s$ and $t$ for which $p_s, p_t \in S$, but which otherwise go through vertices $v$ for which $p_v \notin S$, is an edge from $s$ to $t$ with label equal to the weight $\tau_{A^{<1>}}(p)$ of $p$ (so in particular, if $e$ is an edge in $G_{A^{<1>}}$ which starts and ends in vertices $s$ and $t$ for which $p_s, p_t \in S$, then there is an edge in $G_{A^{<2>}}$ from $s$ to $t$ with the same label as $e$). We then have that $T_{A^{<2>}}$ is $G$-conjugate to $R_{K_A}$.

We will now construct a positive $\text{El}_p(H)$-equivalence from $I - A^{<1>}$ to $I - A^{<2>}$. This is accomplished by iterating a matrix move. Given a matrix $M$ in $\mathcal{M}_p^2(C, m, H)$ and a vertex $s$ such that $M(s, s) = 0$, the move produces a positive $\text{El}_p(m, H)$-equivalence $(I - M) \rightarrow (I - M_s)$ for a related matrix $M_s$ in $\mathcal{M}_p^2(C, m, H)$, where $M_s$ has row $s$ and column $s$ zero.

For a description of this move, let $M(r, s) = p \neq 0$, let $E_r$ be the basic elementary matrix $E_{r,s}(p)$. Let $U$ be the product of these $E_r$ (so, $U(r, s) = M(r, s)$ if $r \neq s$, and in other entries $U = I$) and let $M'$ be the matrix such that $U(I - M) = I - M'$. Then $(U, I) : (I - M) \rightarrow (I - M')$, as a composition of the equivalences $(E_r, I)$, is a positive $\text{El}_p(m, H)$ equivalence. Column $s$ of $M'$ is zero. Row $s$ of $M'$ equals row $s$ of $M$.

Next, we zero out row $s$ of $M'$. Define $V(s, t) = M(s, t)$ if $s \neq t$ and $V = I$ otherwise. Let $M(s)$ be the matrix such that $(I - M')V = I - M(s)$. Then $(I, V) : (I - M') \rightarrow (I - M(s))$ is a positive $\text{El}_p(m, H)$-equivalence. We have

$$M(s)(r, t) = \begin{cases} M(r, t) + M(r, s)M(s, t) & \text{if } r \neq s \neq t \\ 0 & \text{if } r = s \text{ or } s = t. \end{cases}$$

The matrix $M(s)$ presents a skewing function into $G$ induced by the return map to the clopen set of points $x$ for which the initial and terminal vertices of $x_0$ do not equal $s$.

Altogether, for $\{1, \ldots, n\} \setminus S = \{s(1), \ldots, s(k)\}$, set $M_0 = A^{<1>}$, and set $M_i = (M_{i-1})_{(s(i))}$ for $1 \leq i \leq k$. Then we have a positive $\text{El}_p(H)$-equivalence from $I - A^{<1>} = 1 - M_0$ to $I - M_k = I - A^{<2>}$. 


Step 3: The resolving tower and matrix cohomology.

Similar to how we constructed $A^{<1>}$ and $A^{<2>}$, we can construct nondegenerate matrices $B^{<1>}, B^{<2>} \in \mathcal{M}_p^0(\mathcal{C}, \mathcal{H})$ such that there is a positive Elp($\mathcal{H}$)-equivalence $(I - B) \rightarrow (I - B^{<1>})$ and a positive Elp($\mathcal{H}$)-equivalence $(I - B^{<1>}) \rightarrow (I - B^{<2>})$, and such that $T_B^{<2>}$ is $G$-conjugate to $R_{K_B}$. Since $R_{K_A}$ and $R_{K_B}$ are $G$-conjugate, it follows that $T_{A^{<2>}}$ and $T_{B^{<2>}}$ are $G$-conjugate. It therefore follows from Proposition 3.4.1 that there is a topological conjugacy

$$\varphi : X_{A^{<2>}} \rightarrow X_{B^{<2>}}$$

which takes the skewing function $\tau_{A^{<2>}}$ to a function cohomologous to $\tau_{B^{<2>}}$. In this step we will construct matrices $A^{<3>} \in \mathcal{M}_p^0(\mathcal{C}, \mathcal{r}, \mathcal{H})$, $B^{<3>} \in \mathcal{M}_p^0(\mathcal{C}, \mathcal{r}', \mathcal{H}')$ and a poset isomorphism $\alpha : \mathcal{P} \rightarrow \mathcal{P}'$ such that $\alpha^*(r) = r'$ and $\alpha(C) = C'$, and such that there is a positive Elp($\mathcal{H}'$)-equivalence from $I - A^{<2>}$ to $I - A^{<3>}$, a positive Elp($\mathcal{H}'$)-equivalence from $I - B^{<3>}$ to $I - B^{<3>}$, and such that $A^{<3>} = (Q^r_\alpha)^{-1}B^{<3>}Q^r_\alpha$ and $B^{<3>} = (Q^r_\alpha)^{-1}A^{<3>}Q^r_\alpha$ define skewing functions which are cohomologous.

There is a standard decomposition for the topological conjugacy $\varphi$ (see also [20, Theorem 7.1.2]). It follows from this that there is a matrix $M$ over $\mathbb{Z}_+$, with one-block conjugacies (given by graph homomorphisms) $\varphi_1 : \sigma_M \rightarrow \sigma_{A^{<3>}}$ and $\varphi_2 : \sigma_M \rightarrow \sigma_{B^{<3>}}$ such that

1. $\varphi$ is $\varphi_1^{-1}$ followed by $\varphi_2$,
2. $\varphi_1$ is left resolving, with $\varphi_1^{-1}$ a composition of conjugacies given by row splittings,
3. $\varphi_2$ is right resolving, with $\varphi_2^{-1}$ a composition of conjugacies given by column splittings.

Since $\varphi_1^{-1}$ a composition of conjugacies given by row splittings, it follows that there is a ZG-matrix $A^{<3>}$ such that $\varphi_1^{-1}$ can be lifted to a $G$-conjugacy $\psi_A : T_{A^{<2>}} \rightarrow T_{A^{<3>}}$, also given by row splittings, and a permutation matrix $P_A$ such that $P_A^{-1}A^{<3>}P_A = M$ and $\psi_A((x, g)) = (\eta_{P_A}(\varphi_1^{-1}(x)), g)$ for $(x, g) \in X_{A^{<3>}} \times G$, where $\eta_{P_A} : \sigma_M \rightarrow \sigma_{A^{<3>}}$ is the conjugacy given by $P_A$. It follows from Proposition 3.4.1 that $A^{<3>} \in \mathcal{M}_p^0(\mathcal{C}, \mathcal{H})$ and that there is a positive Elp($\mathcal{H}$)-equivalence from $I - A^{<2>}$ to $I - A^{<3>}$. Since $A^{<2>}$ is nondegenerate, so is $A^{<3>}$. Choose $r$ such that $A^{<3>} \in \mathcal{M}_p^0(\mathcal{C}, \mathcal{r}, \mathcal{H})$.

Similarly, there is a vector $r'$, a nondegenerate ZG-matrix $B^{<2.5>} \in \mathcal{M}_p^0(\mathcal{C}', \mathcal{r}', \mathcal{H}')$, a $G$-conjugacy $\psi_B : T_{B^{<2>}} \rightarrow T_{B^{<2.5>}}$, a permutation matrix $P_B$ such that $P_B^{-1}B^{<2.5>}P_B = M$ and $\psi_B((x, g)) = (\eta_{P_B}(\varphi_2^{-1}(x)), g)$ for $(x, g) \in X_{B^{<2.5>}} \times G$, where $\eta_{P_B} : \sigma_M \rightarrow \sigma_{B^{<2.5>}}$ is the conjugacy given by $P_B$, and a positive Elp($\mathcal{H}'$)-equivalence from $I - B^{<2>}$ to $I - B^{<2.5>}$. Let $P = P_BP_A^{-1}$. Then $A^{<3>} = P^{-1}B^{<2.5>}P$. It follows that there is a poset isomorphism $\alpha : \mathcal{P} \rightarrow \mathcal{P}'$ such that $\alpha^*(r) = r'$ and $\alpha(C) = C'$, and a permutation matrix $P' \in \mathcal{M}_p(\mathcal{r}', \mathbb{Z}_+)$ such that $P = \cdots$
Let $B^{<3>}_r = (P'_r)^{-1} B^{<2,5>}_r P' \in \mathcal{M}^0_{P'}(C', r', \mathcal{H}')$. It then follows from Proposition 3.1 that there is a positive $\text{El}_{P'}(r', \mathcal{H}')$-equivalence $(I - B^{<2,5>}_r) \rightarrow (I - B^{<3>}_r)$. We furthermore have that $(Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha} \in \mathcal{M}^0_{P'}(C, r, \mathcal{H})$, $(Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha} = A^{<3>}_r$, and $\tau_{(Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha}}$ are cohomologous.

**Step 4:** $\text{El}_{P}(\mathcal{H})$-equivalence.

In this step we complete the proof apart from the (nontrivial) “moreover” statement. We continue with the notation of the last step.

Let $\psi$ be the continuous function from $X_{A^{<3>}_r}$ into $G$ such that

$$
\tau_{A^{<3>}_r} (x) = (\psi(x))^{-1} \tau_{(Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha}} (x) \psi(\sigma_{A^{<3>}_r}(x))
$$

for all $x \in X_{A^{<3>}_r}$. Then proof of Parry for [24] Lemma 9.1 (translated from his vertex SFTs to our edge SFTs) shows that if $x \in X_{A^{<3>}_r}$, then $\psi(x)$ is determined by the initial vertex of the edge $x_0$. Because $A^{<3>}_r$ and $(Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha}$ are nondegenerate, this implies that there is a diagonal matrix $D$, with each diagonal element an element of $G$, such that

$$
D^{-1} A^{<3>}_r D = (Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha}.
$$

We let $M_i$ denote a diagonal block $M\{i, i\}$ of a $P$-blocked matrix. The matrices $A^{<3>}_r$ and $((Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha})_i$ are essentially irreducible and the group $H_i$ is a weights group for $A^{<3>}_r$ and $((Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha})_i$. It then follows from the proof of Theorem 4.7 of [12] that there exists $\gamma \in G^N$ such that for each $i$ the diagonal matrix $(DD^r_i)_i$ has every entry in $H_i$. Then

$$
(D^r_i)^{-1} (Q^m_{\alpha})^{-1} B^{<3>}_r Q^m_{\alpha} D^r_i = (D^r_i)^{-1} (D^{-1} A^{<3>}_r D) D^r_i = (DD^r_i)^{-1} A^{<3>}_r (DD^r_i).
$$

Applying Proposition 3.1, we have $(D^r_i)^{-1} (Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha} D^r_i \in \mathcal{M}^0_{P}(\mathcal{H})$, with a positive $\text{El}_{P}(\mathcal{H})$-equivalence

$$
(I - A^{<3>}_r) \rightarrow (I - (D^r_i)^{-1} (Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha} D^r_i).
$$

So, $H_{ij} = \gamma_i^{-1} H_{(\alpha(i)\alpha(j))} \gamma_j$ for $i, j \in P$ with $i \leq j$, and we have positive $\text{El}_{P}(\mathcal{H})$-equivalences

$$
(I - A) \rightarrow (I - A^{<3>}_r) \rightarrow (I - (D^r_i)^{-1} (Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha} D^r_i).
$$

Let $C = (D^n_{\alpha})^{-1} (Q^n_{\alpha})^{-1} B Q^n_{\alpha} D^n_{\alpha}$. Then $C \in \mathcal{M}^0_{P}(\mathcal{H})$ because $B \in \mathcal{M}^0_{P'}(C', \mathcal{H}')$, $\alpha$ is a poset isomorphism from $P$ to $P'$ such that $\alpha(C) = C'$ and $H_{ij} = \gamma_i^{-1} H_{(\alpha(i)\alpha(j))} \gamma_j$ for $i, j \in P$ with $i \leq j$.

It remains to show that there is a positive $\text{El}_{P}(\mathcal{H})$-equivalence from $I - C$ to $I - (D^r_i)^{-1} (Q^r_{\alpha})^{-1} B^{<3>}_r Q^r_{\alpha} D^r_i$. We have proved that there is a positive $\text{El}_{P}(\mathcal{H})$-equivalence from $I - B$ to $I - B^{<3>}_r$ given by a path

$$
(I - B') \xrightarrow{(E_1,F_1)} \cdots \xrightarrow{(E_r,F_r)} (I - (B^{<3>}_r)')
$$
in which $B' \in \mathcal{M}^p_{D}(\mathbf{m}', \mathcal{H}')$ is a stabilization of $B$, $(B^{<3>})' \in \mathcal{M}^o_{D}(\mathbf{m}', \mathcal{H}')$ is a stabilization of $B^{<3>}$, and the $E_t$ and $F_t$ are basic elementary matrices in $\text{El}_p(\mathbf{m}', \mathcal{H}')$. But then $E'_t := (D^p_t)^{-1}(Q^p_t)^{-1}E_tQ^p_oD^p_o$ and $F'_t := (D^p_t)^{-1}(Q^p_t)^{-1}F_tQ^p_oD^p_o$ are basic elementary matrices in $\text{El}_p(\mathcal{H})$, and we have a positive $\text{El}_p(\mathcal{H})$-equivalence

$$(I - (D^p_t)^{-1}(Q^p_t)^{-1}B'Q^p_oD^p_o) \xrightarrow{(E'_t,F'_t)} \cdots \xrightarrow{(E'_t,F'_t)} (I - (D^p_t)^{-1}(B^{<3>})'(Q^p_oD^p_o)).$$

Since $(D^p_t)^{-1}(Q^p_t)^{-1}B'Q^p_oD^p_o$ is a stabilization of $C$ and

$$(D^p_t)^{-1}(Q^p_t)^{-1}(B^{<3>})'(Q^p_oD^p_o)$$

is a stabilization of $(D^p_t)^{-1}(Q^p_t)^{-1}B^{<3>}Q^p_oD^p_o$, this shows that there is a positive $\text{El}_p(\mathcal{H})$-equivalence from $I - C$ to $I - (D^p_t)^{-1}(Q^p_t)^{-1}B^{<3>}Q^p_oD^p_t$ and thus that there is a positive $\text{El}_p(\mathcal{H})$-equivalence from $I - A$ to $I - C$.

**Moreover**. For the rest of the proof, we assume that $A$ and $B$ satisfy Condition $\mathcal{C}_1$. It remains to show that we can find stabilizations $A^{<0>}, C^{<0>} \in \mathcal{M}^p_{D}(\mathcal{C}, \mathbf{m}, \mathcal{H})$ of $A$, $C$ with a positive $\text{El}_p(\mathbf{m}, \mathcal{H})$-equivalence $(U, V) : (I - A^{<0>}) \to (I - C^{<0>})$.

For this we refine the construction of Steps 1-4 in order to obtain stabilizations $\tilde{A}^{<0>}, \tilde{C}^{<0>} \in \mathcal{M}^p_{D}(\mathcal{C}, \mathbf{t}, \mathcal{H})$ of $A$ and $C$ and a positive $\text{El}_p(\mathbf{t}, \mathcal{H})$-equivalence $(\tilde{U}, \tilde{V}) : (I - \tilde{A}^{<0>}) \to (I - \tilde{C}^{<0>})$ such that for every $i \in \mathcal{C}$ the matrices $\tilde{U}\{i, i\}, \tilde{V}\{i, i\}$ are the identity matrix. We then get $\mathcal{C}_1$ stabilizations $A^{<0>}, C^{<0>}$ in $\mathcal{M}^p_{D}(\mathcal{C}, \mathbf{m}, \mathcal{H})$ of $A$, $C$ and with a positive $\text{El}_p(\mathbf{m}, \mathcal{H})$-equivalence $(U, V) : (I - A^{<0>}) \to (I - C^{<0>})$ as wanted by letting $A^{<0>}, C^{<0>}, U, W$ be principal submatrices of $\tilde{A}^{<0>}, \tilde{C}^{<0>}, \tilde{U}, \tilde{V}$.

**Step 5: Getting a $\mathcal{C}_a$-equivalence** $(I - A^{<0>}) \to (I - A^{<3>})$.

In this step we will show that the positive equivalence $I - A \to I - A^{<3>}$ of Steps 1-3 can be chosen such that there are stabilizations $A''$, $A^{<3>}'' \in \mathcal{M}^p_{D}(\mathcal{C}, \mathbf{s}, \mathcal{H})$ of $A$ and $A^{<3>}$ and a positive $\text{El}_p(\mathbf{s}, \mathcal{H})$-equivalence $(U_A, V_A) : (I - A'') \to (I - A''^{<3>})$ such that for every $i \in \mathcal{C}$ the matrices $U_A\{i, i\}, V_A\{i, i\}$ are unipotent upper triangular.

Recall that the positive $\text{El}_p(\mathcal{H})$-equivalence from $I - A$ to $I - A^{<1>}$ is the composition of positive $\text{El}_p(\mathcal{H})$-equivalences $(I - A_t) \to (I - A_{t+1})$ obtained by applying Proposition [C.1]. At each stage the $\mathcal{P}$ blocking of $A_{t+1}$ is the lift of the $\mathcal{P}$ blocking of $A_t$. At step $t$, there is an index $s_t$ such that either row $s_t$ of $A_t$ is split into two rows, or column $s_t$ is split into two columns. We will choose $s_t, 1 + s_t$ to be the indices associated to the splitting (so, if an index $j$ of $A_t$ is greater than $s$, then it corresponds to index $j + 1$ of $A_{t+1}$). In the case that $s_t$ is an index in a cycle component we place additional conditions as follows. If $A_t(s_t, s_t) = g \neq 0$, then we require that $A_{t+1}(s_t + 1, s_t + 1) = g$ and $A_{t+1}(s_t, s_t) = 0$.
in the case \(A_t \to A_{t+1}\) is a column splitting, and we require
\[
A_{t+1}(s_t + 1, s_t + 1) = 0 \quad \text{and} \quad A_{t+1}(s_t, s_t) = g
\]
in the case \(A_t \to A_{t+1}\) is a row splitting.

When \(A_t\) is upper triangular in its cycle component diagonal blocks, it follows from Proposition C.1 that \(A_{t+1}\) is as well, and that there are unipotent upper triangular matrices \(U_t, V_t\) such that
\[
(U_t, V_t) : (I - A_t) \to (I - A_{t+1})
\]
is a positive \(\mathcal{E}_P(\mathcal{H})\) equivalence. By induction, each cycle component block of \(A^{<1>}\) is upper triangular, and there is a positive \(\mathcal{E}_P(\mathcal{H})\) equivalence
\[
(U_1, V_1) : (I - A') \to (I - A^{<1>'})
\]
where \(A'\) is a stabilization of \(A\), \(A^{<1>}'\) is a stabilization of \(A^{<1>}\), and for every \(i \in \mathcal{C}\) the matrices \(U_1\{i, i\}, V_1\{i, i\}\) are unipotent upper triangular.

Next consider the restriction of the Step 2 move \(M \to M(s)\) to a diagonal block \(M\{i, i\}\) with \(i \in \mathcal{C}\). Clearly, if \(M\{i, i\}\) is upper triangular, then the trimming matrices implementing the positive equivalence \((I - M) \to (I - M_s)\) are unipotent upper triangular, and \(M(s)\{i, i\}\) is upper triangular. Because \(A^{<1>}\{i, i\}\) is upper triangular, it follows by induction that \(A^{<2>}\{i, i\}\) is upper triangular and that there is a positive \(\mathcal{E}_P(\mathcal{H})\) equivalence
\[
(U_2, V_2) : (I - A^{<1>'}) \to (I - A^{<2>''})
\]
where \(A^{<1>'}\) is a stabilization of \(A^{<1>}\), \(A^{<2>''}\) is a stabilization of \(A^{<2>}\), and for every \(i \in \mathcal{C}\) the matrices \(U_2\{i, i\}, V_2\{i, i\}\) are unipotent upper triangular. By the argument for Step 1, we will likewise have that \(A^{<3>}\) upper triangular in each cycle component block and that there is a positive \(\mathcal{E}_P(\mathcal{H})\) equivalence
\[
(U_3, V_3) : (I - A^{<2>''}) \to (I - A^{<3>''})
\]
where \(A^{<2>''}\) is a stabilization of \(A^{<2>}\), \(A^{<3>''}\) is a stabilization of \(A^{<3>}\), and for every \(i \in \mathcal{C}\) the matrices \(U_3\{i, i\}, V_3\{i, i\}\) are unipotent upper triangular. Putting this together we get there is a stabilization \(A''' \in \mathcal{M}_P^0(\mathcal{C}, s, \mathcal{H})\) of \(A\) and a stabilization \(A^{<3>'''} \in \mathcal{M}_P^0(\mathcal{C}, s, \mathcal{H})\) of \(A^{<3>} \) such that there is a positive \(\mathcal{E}_P(s, \mathcal{H})\) equivalence
\[
(U_A, V_A) : (I - A''') \to (I - A^{<3>'''})
\]
such that for every \(i \in \mathcal{C}\) the matrices \(U_A\{i, i\}, V_A\{i, i\}\) are unipotent upper triangular.

By the same argument, there are stabilizations \(B'''\), \(B^{<2.5>'''} \in \mathcal{M}_P^0(\mathcal{C}', s', \mathcal{H}')\) of \(B, B^{<2.5>}\) and a positive \(\mathcal{E}_P(s', \mathcal{H}')\) equivalence
\[
(U_B, V_B) : (I - B''') \to (I - B^{<2.5>'''})
\]
such that for every $i \in C$ the matrices $U_B\{i,i\}$, $V_B\{i,i\}$ are unipotent upper triangular.

**Step 6: Clearing out diagonal $C$ blocks in $(U,V)$.**

From Step 5 we have a stabilizations $A'^m$, $A'^{<3>}m \in \mathcal{M}_p(C,s,H)$ of $A,A'^{<3>}$ and the positive $E_p(s,H)$-equivalence $[6,4]$ such that for every $i \in C$ the matrices $U_A\{i,i\}$, $V_A\{i,i\}$ are unipotent upper triangular. In this step we will show there is a positive $E_p(s,H)$-equivalence $(I - A'^m) \rightarrow (I - A'^{<3>}m)$ such that precomposing $(U,V)$ with this equivalence produces an equivalence

$$(\tilde{U}_A, \tilde{V}_A) : (I - A'^m) \rightarrow (I - A'^{<3>}m)$$

such that for all $i$ in $C$, the diagonal blocks $\tilde{U}_A\{i,i\}$ and $\tilde{V}_A\{i,i\}$ are the identity matrix.

So, consider $i \in C$. Restricted to the block $(I - A'^m)\{i,i\} : = (I - M)$, our equivalence $U_A(I - A'^m)V_A = (I - A'^{<3>}m)$ has the following block triangular form, with central block $1 \times 1$:

$$
\begin{pmatrix}
U_{11} & U_{12} & U_{13} \\
0 & U_{22} & U_{23} \\
0 & 0 & U_{33}
\end{pmatrix} \begin{pmatrix}
I & 0 & 0 \\
0 & 1 - g & 0 \\
0 & 0 & I
\end{pmatrix} \begin{pmatrix}
V_{11} & V_{12} & V_{13} \\
0 & V_{22} & V_{23} \\
0 & 0 & V_{33}
\end{pmatrix} = \begin{pmatrix}
I & 0 & 0 \\
0 & 1 - g & 0 \\
0 & 0 & I
\end{pmatrix}.

The form is determined by placing the unique entry $1 - g$ as the central block. Suppose $\{s,t\} \subset I_i$, $s < t$, $s \neq s_i \neq t$ and $E$ is a basic elementary matrix of size matching $A'^m$ with $E(s,t) = \pm h$ for some $h$ in $G$. (For $E$ in $E_p(H)$, $h$ must be $g^k$ for some $k$.) Then, because $A$ satisfies condition $C_1$, $(E,E^{-1}) : (I - A'^m) \rightarrow (I - A'^m)$ is a positive $E_p(n,H)$-equivalence. After precomposing $(U,V)$ with a suitable composition of these, we may assume $U_{11} = I$, $U_{33} = I$ and $U_{13} = 0$. Our matrix equivalence now has the following form

(6.5)

$$
\begin{pmatrix}
I & x & 0 \\
0 & 1 & U_{23} \\
0 & 0 & I
\end{pmatrix} \begin{pmatrix}
I & 0 & 0 \\
0 & 1 - g & 0 \\
0 & 0 & I
\end{pmatrix} \begin{pmatrix}
V_{11} & V_{12} & V_{13} \\
0 & V_{22} & V_{23} \\
0 & 0 & V_{33}
\end{pmatrix} = \begin{pmatrix}
I & 0 & 0 \\
0 & 1 - g & 0 \\
0 & 0 & I
\end{pmatrix},

$$

which multiplies out to give

$$
\begin{pmatrix}
V_{11} + x(1-g) & V_{12} + x(1-g)y \\
0 & 1 - g \\
0 & 0
\end{pmatrix} \begin{pmatrix}V_{13} + x(1-g)y \\
0 & (1-g)y + U_{23}V_{33} \\
V_{33}\end{pmatrix} = \begin{pmatrix}
I & 0 & 0 \\
0 & 1 - g & 0 \\
0 & 0 & I
\end{pmatrix}.

Consequently we can rewrite the left side of (6.5) as

$$
\begin{pmatrix}
I & x & 0 \\
0 & 1 & (g - 1)y \\
0 & 0 & I
\end{pmatrix} \begin{pmatrix}
I & 0 & 0 \\
0 & 1 - g & 0 \\
0 & 0 & I
\end{pmatrix} \begin{pmatrix}
I & x(g - 1) & x(g - 1)y \\
0 & 1 & y \\
0 & 0 & I
\end{pmatrix}.
$$
This equivalence \((U,V):(I-M)\to(I-M)\) is a composition of two equivalences, \((U_1,V_1)\) followed by \((U_2,V_2)\), where

\[
U_1 = \begin{pmatrix} I & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad V_1 = \begin{pmatrix} I & x(g-1) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix}, \\
U_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & (g-1)y \\ 0 & 0 & I \end{pmatrix}, \quad V_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & I \end{pmatrix}.
\]

We will see how these equivalences are related to positive equivalences \((I-M)\to(I-M)\).

Consider a term \(-h\ (h \in G)\) which is part of an entry of \(y\) in \(V_2\), say the \((s,t)\) entry of \(V\). Recall \(E_{s,t}(\delta)\) denotes a basic elementary matrix with off-diagonal entry \(\delta\) in position \((s,t)\). We define now \(n \times n\) matrices \(E_1,\ldots,E_4\), \(E_1(r,t) = -M(r,s)h\) if \(r \notin \{s,t\}\); in other entries, \(E_1 = I\). \(E_2 = E_{s,t}(-gh)\); \(E_3 = E_{s,t}(-h)\); \(E_4 = E_{s,t}(h)\). Then \(E_4E_2E_1(I-M)E_3 = (I-M)\), and applying the multiplications in the order indexed gives a positive \(\text{El}_P(H)\)-equivalence. It may be easiest to see the argument by restricting to a \(4 \times 4\) principal submatrix. For this, suppose \(2 = s, 3 = t\) and \(M(1,2) \neq 0 \neq M(2,4)\). Then these principal submatrices (with names unchanged for simplicity) have the forms

\[
I - M = \begin{pmatrix} 1-x & -w & 0 & -u \\ 0 & 1-g & 0 & -z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 & 0 & -wh & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -gh & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -h & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (E_4)^{-1}.
\]

For the \(n \times n\) matrices, we have \((E_4E_2E_1, E_3) = (E_4E_{s,t}((g-1)h), E_{s,t}(-h))\).

For the case the term is \(h\), there is similarly a positive equivalence \(F_4F_3F_1(I-M)F_2 = (I-M)\), in which \(F_1 = E_3, F_2 = (E_3)^{-1}, F_3 = E_2^{-1}\) and \(F_4 = E_1^{-1}\). In the \(4 \times 4\) sample, this has the form

\[
F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -h & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
F_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & gh & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_4 = \begin{pmatrix} 1 & 0 & wh & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]
Here $(F_1 F_2 F_1, F_2) = (F_1 E_{s,t}((g - 1)h), E_{s,t}(h))$.

A suitable composition of the above equivalences is an equivalence which in the $\{i, i\}$ block matches $(U_1, V_1)$. Precomposing $(U_A, V_A)$ with the inverse of this composition gives the required matrix $(\tilde{U}_A, \tilde{V}_A)$.

Similarly, there is a positive $\text{El}_{P'}(s', \mathcal{H}')$-equivalence
\[(\tilde{U}_B, \tilde{V}_B) : (I - B^m) \rightarrow (I - B^{<2.5>^{m'}})\]
such that for every $i \in C'$ the matrices $\tilde{U}_B\{i, i\}$, $\tilde{V}_B\{i, i\}$ are the identity matrix.

**Step 7: Cohomology.**

In this step we will show that there are stabilizations $B_{1}^{<2.5>, B_{<3>}} \in \mathcal{M}_{M}(C', s'_i, \mathcal{H}')$ of $B^{<2.5>}$, $B^{<3>}$ and a positive $\text{El}_{P'}(s'_i, \mathcal{H}')$-equivalence
\[(\tilde{U}'_i, \tilde{V}'_i) : (I - B_{1}^{<2.5>}) \rightarrow (I - B_{1}^{<3>})\]
such that for every $i \in C'$ the matrices $\tilde{U}'_i\{i, i\}$, $\tilde{V}'_i\{i, i\}$ are the identity matrix, and we will show that there are stabilizations $A_{1}^{<3>}$, $M_1 \in \mathcal{M}_{M}(C, s_1, \mathcal{H})$ of $A^{<3>}$, $(D'_\gamma)^{-1}(Q'^{m'}_\alpha)^{-1}B^{<3>^{m'}}Q'^{m'}_\alpha D'_\gamma$ and a positive $\text{El}_{P}(s_1, \mathcal{H})$-equivalence
\[(\tilde{U}_1, \tilde{V}_1) : (I - A_{1}^{<3>}) \rightarrow (I - M_1)\]
such that for every $i \in C$ the matrices $\tilde{U}_1\{i, i\}$, $\tilde{V}_1\{i, i\}$ are the identity matrix.

Recall that $B^{<3>} = (P')^{-1}B^{<2.5>^{P}}$ where $P' \in \mathcal{M}_{P'}(r', \mathbb{Z}_+)$ is a permutation matrix. Since $B$ satisfies condition $C_1$,
\[(\tilde{U}_B, \tilde{V}_B) : (I - B^m) \rightarrow (I - B^{<2.5>^{m'}})\]
is a positive $\text{El}_{P'}(s', \mathcal{H}')$-equivalence such that for every $i \in C'$ the matrices $\tilde{U}_B\{i, i\}$, $\tilde{V}_B\{i, i\}$ are the identity matrix, and $B^{<2.5>}$ is nondegenerate, it follows that $B^{<2.5>}$ satisfies condition $C_1$, and thus that $P'\{i, i\} = 1$ for every $i \in C'$. It therefore follows from Proposition [B.1] that there are stabilizations $B_{1}^{<2.5>, B_{<3>}}$ of $B^{<2.5>}$, $B^{<3>}$ and a positive $\text{El}_{P'}(s'_i, \mathcal{H}')$-equivalence
\[(\tilde{U}'_i, \tilde{V}'_i) : (I - B_{1}^{<2.5>}) \rightarrow (I - B_{1}^{<3>})\]
such that for every $i \in C'$ the matrices $\tilde{U}'_i\{i, i\}$, $\tilde{V}'_i\{i, i\}$ are the identity matrix.

Recall that $(DD'_\gamma)^{-1}A^{<3>} (DD'_\gamma) = (D'_\gamma)^{-1}(Q'^{m'}_\alpha)^{-1}B^{<3>^{m'}}Q'^{m'}_\alpha D'_\gamma$. Since $A$ satisfies condition $C_1$,
\[(\tilde{U}_A, \tilde{V}_A) : (I - A^m) \rightarrow (I - A^{<3>^{m'}})\]
is a positive $\text{El}_{P}(s, \mathcal{H})$-equivalence such that for every $i \in C$ the matrices $\tilde{U}_A\{i, i\}$, $\tilde{V}_A\{i, i\}$ are the identity matrix, and $A^{<3>}$ is nondegenerate, it follows that $A^{<3>}$ satisfies condition $C_1$. It then follows from the proof of Theorem 4.7 of [12] that $\gamma \in G^N$ can be chosen such that for each $i \in C$ the diagonal matrix $(DD'_\gamma)_i$ is 1. Applying Proposition [A.1],
we get stabilizations $A_1^{<3>}$, $M_1$ of $A^{<3>}$, $(D_\gamma^e)^{-1}(Q_\alpha^e)^{-1}B^{<3>}Q_\alpha^eD_\gamma^e$ and a positive Elp($s_1, \mathcal{H}$)-equivalence

$$(\tilde{U}_1, \tilde{V}_1) : (I - A_1^{<3>}) \rightarrow (I - M_1)$$

such that for every $i \in \mathcal{C}$ the matrices $\tilde{U}_1\{i, i\}$, $\tilde{V}_1\{i, i\}$ are the identity matrix.

**Step 8: Conclusion.**

By composing the equivalences $(\tilde{U}_B, \tilde{V}_B) : (I - B^{m'}) \rightarrow (I - B^{<2,5>})$ and $(\tilde{U}_1', \tilde{V}_1') : (I - B_1^{<2,5>}) \rightarrow (I - B_1^{<3>})$, we get stabilizations $B_2, B_2^{<3>} \in \mathcal{M}_{2n}(C', s_2', \mathcal{H}')$ of $B, B^{<3>}$ and a positive Elp($s_2', \mathcal{H}'$)-equivalence

$$(\tilde{U}_2', \tilde{V}_2') : (I - B_2) \rightarrow (I - B_2^{<3>})$$

such that for every $i \in \mathcal{C}'$ the matrices $\tilde{U}_2'\{i, i\}$, $\tilde{V}_2'\{i, i\}$ are the identity matrix. By multiplying this equivalence with $(D_\gamma^{e'})^{-1}(Q_\alpha^{e'})^{-1}$ on the left and $Q_\alpha^{e'}D_\gamma^{e'}$ on the right we get stabilizations $C_2, M_2 \in \mathcal{M}_{2n}(C, s_2, \mathcal{H})$ of $C, (D_\gamma^{e'})^{-1}(Q_\alpha^{e'})^{-1}B^{<3>}Q_\alpha^{e'}D_\gamma^{e'}$ (where $\alpha^*(s_2) = s_2'$) and a positive Elp($s_2, \mathcal{H}$)-equivalence

$$(\tilde{U}_2, \tilde{V}_2) : (I - C_2) \rightarrow (I - M_2)$$

such that for every $i \in \mathcal{C}$ the matrices $\tilde{U}_2\{i, i\}$, $\tilde{V}_2\{i, i\}$ are the identity matrix. By composing the inverse of this equivalence with the equivalences $(\tilde{U}_A, \tilde{V}_A) : (I - A^{m'}) \rightarrow (I - A^{<3>})$ and $(\tilde{U}_1, \tilde{V}_1) : (I - A_1^{<3>}) \rightarrow (I - M_1)$, we get stabilizations $\tilde{A}^{<0>}, \tilde{C}^{<0>} \in \mathcal{M}_{2n}(\mathcal{C}, \mathcal{H})$ of $A, C$ and a positive Elp($s_3, \mathcal{H}$)-equivalence

$$(\tilde{U}, \tilde{V}) : (I - \tilde{A}^{<0>}) \rightarrow (I - \tilde{C}^{<0>})$$

such that for every $i \in \mathcal{C}$ the matrices $\tilde{U}\{i, i\}$, $\tilde{V}\{i, i\}$ are the identity matrix.

It remains to obtain the equivalence in $\mathcal{C}_1$ form. Let $\mathcal{I}$ be the index set of $\tilde{A}^{<0>}$ (and $\tilde{C}^{<0>}$), and let $\mathcal{I}_{sec}$ be the set of elements $s \in \mathcal{I}$ such that $i(s) \in \mathcal{C}$ and $\tilde{A}^{<0>}(s, t) = \tilde{A}^{<0>}(t, s) = 0$ for all $t \in \mathcal{I}$. Since $\tilde{U}(I - \tilde{A}^{<0>})\tilde{V} = I - \tilde{C}^{<0>}$, $\tilde{U}\{i, i\}$ and $\tilde{V}\{i, i\}$ are the identity matrix for every $i \in \mathcal{C}$, and $\tilde{A}^{<0>}$ and $\tilde{C}^{<0>}$ satisfy condition $C_{1+}$ (because $A$ and $C$ satisfy condition $C_1$), it follows that $\mathcal{I}_{sec}$ is equal to the set of $s \in \mathcal{I}$ such that $i(s) \in \mathcal{C}$ and $\tilde{C}^{<0>}(s, t) = \tilde{C}^{<0>}(t, s) = 0$ for all $t \in \mathcal{I}$. Let $\mathcal{I}_{prim}$ be the complement in $\mathcal{I}$ of the $\mathcal{I}_{sec}$. Let $W = \tilde{V}^{-1}$ and write the equivalence in the form

$$(6.6) \quad \tilde{U}(I - \tilde{A}^{<0>}) = (I - \tilde{C}^{<0>})\tilde{W}.$$
then we have the required $C_1$-equivalence. For a verification, suppose $t, u \in I_{\text{prim}}$. Then

$$
(U(I - A^{<0>})(t, u) = U(t, u) - \sum_{s \in I_{\text{prim}}} U(t, s) A^{<0>}(s, u)
$$

$$
= \tilde{U}(t, u) - \sum_{s \in I_{\text{prim}}} \tilde{U}(t, s) \tilde{A}^{<0>}(s, u)
$$

$$
= \tilde{U}(t, u) - \sum_{s \in I} \tilde{U}(t, s) \tilde{A}^{<0>}(s, u)
$$

$$
= (\tilde{U}(I - \tilde{A}^{<0>}))(t, u).
$$

Likewise,

$$
(I - C^{<0>})W(t, u) = (I - \tilde{C}^{<0>})\tilde{W}(t, u).
$$

The required equality now follows from (6.6). \hfill \Box

### 7. The Factorization Theorem: Setting

In this section we present the Factorization Theorem 7.2 which we shall use to prove the implication (2) \implies (1) in Theorem 5.3 and establish the setting for the proof of Theorem 7.2.

As before, $G$ is a finite group, $P = \{1, \ldots, N\}$ is a finite poset given by a partial order relation $\preceq$ satisfying $i \preceq j \implies i \leq j$, $C$ is a subset of $P$, and $H = \{H_{ij}\}_{i,j \in P}$ is a $(G, P)$ coset structure. For the class $\mathcal{M}_P(C, n, H)$, recall Definition 4.21.

**Definition 7.1.** A matrix $A$ in $\mathcal{M}_P(C, n, H)$ satisfies condition $C_2$ if the following holds: if $i \in P$ and $i$ is not a cycle component of $A$, then there are matrices $U_i, V_i$ in $\text{El}(n_i, H_i)$ such that $U_i(I - A_i)V_i$ is a block diagonal matrix with one summand a $2 \times 2$ identity matrix.

**Theorem 7.2** (Factorization Theorem). Suppose $A$ and $A'$ are matrices in $\mathcal{M}_P(C, n, H)$ which satisfy conditions $C_1$ and $C_2$. Then the following are equivalent.

1. $(U, V) : (I - A) \to (I - A')$ is an El$_P(n, H)$-equivalence.
2. $(U, V) : (I - A) \to (I - A')$ is a positive El$_P(n, H)$-equivalence.

We do not have a sharp statement as to which general El$_P(H)$-equivalences are positive El$_P(H)$ equivalences. However, the restriction above to matrices satisfying $C_1$ and $C_2$ is rather mild. Condition $C_2$ is a harmless technical condition (achievable by replacing $A$ with a larger stabilization) which is needed below to apply the Factorization Theorem proved in [12] for the case that the presenting matrix $A$ over $\mathbb{Z}_+G$ is essentially irreducible. Any $G$-SFT can be presented by a matrix in
some $\mathcal{M}_p^o(C, n, \mathcal{H})$ (by Proposition 4.23) which satisfies $C_1$ and $C_2$ (by Proposition 7.11). Before turning to the setting of Theorem 7.2 we remark on a possible future application.

**Remark 7.3.** Let $S$ be a nontrivial mixing SFT defined by a matrix $A$ over $\mathbb{Z}_+$. There is a “Bowen-Franks” representation, a homomorphism $\beta$ from the mapping class group of $S$ to the group of automorphisms of $\text{cok}(I - A)$. This map $\beta$ is fundamental to understanding the mapping class group; for example, quite possibly its kernel is a simple group. That $\beta$ is surjective is a corollary of the Factorization Theorem of [4] (as proved in [4]), because every automorphism of $\text{cok}(I - A)$ is induced by an elementary equivalence over $\mathbb{Z}$.

The mapping class group of a $G$-SFT, an SFT $T$ with $G$ acting by automorphisms, is naturally defined as the centralizer in the mapping class group of $T$ of the subgroup of elements induced by the $G$-action. For a mixing $G$-SFT we likewise have a map $\beta$ from this centralizer into the automorphism group of $\text{cok}(I - A)$. The Factorization Theorem tells us that the automorphisms in the range of $\beta$ are those induced by elementary $\mathbb{Z}G$ equivalences. The problem of characterizing these has not to our knowledge been addressed. Whatever analogous algebraic structures are developed for the case of reducible $G$-SFTs, Theorem 7.2 will be a tool for characterizing the range of invariants.

It will be convenient to have a setting in which we work with equivalences $(A - I) \rightarrow U(A - I)V$ (rather than $(I - A) \rightarrow U(I - A)V$), within a class of matrices $A - I$ which are “as positive as possible”. We develop the apparatus for this next.

**Definition 7.4.** Given $\mathcal{H}$ and $i < j$, $D_{ij} = D_{ij}(\mathcal{H})$ is the set of $(H_i, H_j)$ double cosets contained in $H_{ij}$, and $\mathcal{R}_{ij} = \mathcal{R}_{ij}(\mathcal{H})$ is the set of $D \in D_{ij}$ such that

$$i < k < j \implies H_{ik}H_{kj} \cap D = \emptyset.$$  

(7.5)

We also define

$$\mathcal{R}^c = \{(i, j, D) : D \in \mathcal{R}_{ij}, i \in \mathcal{C} \text{ and } j \in \mathcal{C}\}.$$  

(7.6)

When $\mathcal{H}$ is the coset structure of a matrix $A$ in $\mathcal{M}_p^o(n, \mathcal{H})$, and $g \in D \in \mathcal{R}_{ij}$, then $g$ cannot be the weight of a path which goes from $\mathcal{I}_i$ to $\mathcal{I}_j$ without passing some other $\mathcal{I}_k$.

**Example 7.7.** With $G = S_3$, for the G-SFT given by

$$
\begin{pmatrix}
e & e & e + (12) \\
e & 0 & e \\
0 & 0 & e
\end{pmatrix}
$$

we have $\mathcal{C} = \{1, 2, 3\}$, $H_1 = H_2 = \{e\}$, $H_3 = A_3$, $H_{12} = \{e\}$, $H_{13} = S_3$, $H_{23} = A_3$. $H_{13}$ contains both $(H_1, H_3)$ double cosets, namely $D_1 = A_3$ and $D_2 = S_3 \setminus A_3$. We have $(1, 3, D_1) \notin \mathcal{R}^c$ and $(1, 3, D_2) \in \mathcal{R}^c.$
If $D$ is a nonempty subset of $G$, then $\pi_D$ is the projection

$$\pi_D : \sum_{g \in G} n_g g \mapsto \sum_{g \in D} n_g g.$$ 

An element $\sum_{g \in G} n_g g$ is $D$-positive if $n_g \geq 0$ for any $g$ and $n_g > 0$ precisely when $g \in D$. The terms are used for matrices when the conditions hold entrywise.

**Definition 7.8.** $\mathcal{M}_P^+(C, n, \mathcal{H})$ is the set of matrices $M$ in $\mathcal{M}_P(n, \mathcal{H})$ whose blocks $M\{i, j\}$ satisfy the following conditions:

- (1) $M + I \in \mathcal{M}_P^+(C, n, \mathcal{H})$.
- (2) $M\{i, i\}$ is $H_i$-positive if $i \notin C$.
- (3) If $i < j$ and $D \in D_{ij}$, then

  (i) $(i, j, D) \notin \mathcal{R}^C \implies \pi_D M\{i, j\}$ is $D$-positive

  (ii) $(i, j, D) \in \mathcal{R}^C \implies \pi_D M\{i, j\} > 0$.

By definition, the condition $\pi_D M\{i, j\} > 0$ means that every entry of $\pi_D M\{i, j\}$ is nonnegative and nonzero. Note, Condition 1 above implies that $M\{i, i\}$ has the form $(g_i - 1)$ for some $g_i \in G$ if $i \in C$.

**Definitions 7.9.** An elementary positive equivalence in $\mathcal{M}_P^+(C, n, \mathcal{H})$ is an El$_P(n, \mathcal{H})$ equivalence $(U, V) : B \rightarrow B' = UBV$ such that the following hold:

- (1) $B, B' \in \mathcal{M}_P^+(C, n, \mathcal{H})$;
- (2) one of $U, V$ equals $I$;
- (3) one of $U, V$ is a basic elementary matrix.

A positive equivalence in $\mathcal{M}_P^+(C, n, \mathcal{H})$ is a composition of elementary positive equivalences in $\mathcal{M}_P^+(C, n, \mathcal{H})$. For such an equivalence, we use notations such as

$$(U, V) : B \xrightarrow{+} B'$$

or

$$B \xrightarrow{(U, V)} B'$$

or

$$B \xrightarrow{+} B'.$$

**Observation 7.10.** Suppose $A, A'$ are in $\mathcal{M}_P^+(C, n, \mathcal{H})$; $B = A - I$, $B' = A' - I$; and

$$(U, V) : B \xrightarrow{+} B'.$$

Then $(U, V) : I - A \rightarrow I - A'$ is a positive El$_P(n, \mathcal{H})$-equivalence.

**Proposition 7.11.** Suppose $A \in \mathcal{M}_P^+(C, n, \mathcal{H})$.

- (1) Suppose $A^{<0>}$ is the stabilization of $A$ in $\mathcal{M}_P^+(C, k, \mathcal{H})$, where $k_i = n_i + 2$ if $i \notin C$ and $k_i = n_i$ if $i \in C$. Define $m = (k_1, \ldots, k_n)$ by $m_i = k_i$ if $i \notin C$, $m_i = 1$ if $i \in C$. Then there is a matrix $A'$ in $\mathcal{M}_P^+(C, m, \mathcal{H})$, satisfying $C_1$ and $C_2$, such that the 1-stabilization in $\mathcal{M}_P^+(C, k, \mathcal{H})$ of $I - A'$ is positive El$_P(k, \mathcal{H})$-equivalent to $I - A^{<0>}$. 

(2) Suppose $A$ satisfies $C_1$ and $C_2$. Then there is a matrix $A'$, satisfying $C_1$ and $C_2$, such that $A' - I \in \mathcal{M}_{p}^{++}(\mathcal{C}, \mathcal{n}, \mathcal{H})$ and there is a positive $\text{El}_{p}(\mathcal{n}, \mathcal{H})$-equivalence from $I - A$ to $I - A'$.

We postpone the proof of Proposition 7.11 to later in this section.

Proposition 7.11 along with Observation 7.10 tells us that (after accepting that we might need to pass to slightly larger matrices to satisfy $C_1$) we have reduced the problem of showing every $\text{El}_{p}(\mathcal{n}, \mathcal{H})$-equivalence of matrices $I - A$ (with $A$ from $\mathcal{M}_{p}^{++}(\mathcal{C}, \mathcal{n}, \mathcal{H})$) is positive to the problem of showing every $\text{El}_{p}(\mathcal{n}, \mathcal{H})$-equivalence of matrices $M$ (with $M$ from $\mathcal{M}_{p}^{++}(\mathcal{C}, \mathcal{n}, \mathcal{H})$) is a positive equivalence in $\mathcal{M}_{p}^{++}(\mathcal{C}, \mathcal{n}, \mathcal{H})$.

Unfortunately, as we see in Example 7.12 below, Proposition 7.11 would not be true if in the definition of $\mathcal{M}_{p}^{++}(\mathcal{C}, \mathcal{n}, \mathcal{H})$ the condition “$\pi_3 M \{i,j\} > 0$” in 3(ii) were strengthened to $D$-positivity. This complication accounts for a good deal of the difficulty of arguments to come.

**Example 7.12.** With $G = S_3$, define matrices over $\mathbb{Z}_+ G$,

$$A_1 = \begin{pmatrix} 12 & e \\ 0 & 12 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} e & e & e + 12 \\ 0 & e & e \\ 0 & 0 & 123 \end{pmatrix}.$$

For $A_1$, $(G, P) = (S_3, P_2)$; $\mathcal{H}$ is given by $\langle (12) \rangle = H_1 = H_2 = H_{12}$; $\mathcal{C} = \{1, 2\}$; and $D = H_{12}$ is an $(H_1, H_2)$ double coset, with $(1, 2, D) \in \mathcal{R}^c$. A basic positive $\text{El}_{p}(\mathcal{n}, \mathcal{H})$-equivalence $(I - A) \to (I - B)$ must be given by right or left multiplication of $(I - A)$ by one of four matrices, $\begin{pmatrix} 1 & \pm g \\ 0 & 1 \end{pmatrix}$ with $g \in S_3$. The orbit of $I - A_1$ under such equivalences consists of just $I - A_1$ and $I - A_3$, where $A_3 = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$. Neither $A_1 - I$ nor $A_3 - I$ is $D$-positive. (Alternately, we can see that a matrix $B = \begin{pmatrix} 12 \\ x \end{pmatrix}$ defining a $G$-SFT which is $G$-flow equivalent to $T_{A_1}$, with $x = n_1 e + n_2 (12) \in \mathbb{Z}_+ G$, must satisfy $n_1 + n_2 = 1$, because the augmentations $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & n_1 + n_2 \\ 0 \end{pmatrix}$ must define flow equivalent SFTs.)

For $A_2$, continuing from Example 7.11 we have $(1, 3, D_2) \in \mathcal{R}^c$. A nonidentity basic positive $\text{El}_{p}(\mathcal{n}, \mathcal{H})$-equivalence can only be given by multiplication by a matrix $E_{s tj}(\pm g)$, with $g \in S_3$ and $s < t$. It is straightforward to check that if $I - B$ is in the orbit of $I - A_1$ under such positive equivalences, then $B(s, t) \in \mathbb{Z}_+ A_3$ if $(s, t) \neq (1, 3)$, and for $B(1, 3) = \sum_{g \in G} n_g g$ we have $\sum_{g \in A_3} n_g = 1$. Thus, $B - I$ cannot be $D_2$-positive.

---

3From its proof, one sees that Prop. 7.11 can be strengthened such that, given $k \in \mathbb{N}$, for $D \in \mathcal{D}_j \setminus \mathcal{R}^c$, entrywise $A\{i,j\} \geq k \sum_{g \in D} g$. Such flexibility is not available for $D$ in $\mathcal{R}^c$. 
Before giving the proof of Proposition 7.11, we introduce further notation.

**Definition 7.13.** We define
\[
\delta_{ij} = \sum_{g \in H_{ij}} g \quad \text{if } i < j
\]
\[
\delta_i = \sum_{g \in H_i} g
\]

If \( i \in C \), then \( \delta_i = \sum_{g=0}^{\kappa(g)-1} (g_i)^m \), where \( \kappa(g) \) is the order of \( g \) in \( G \).

**Definition 7.14.** Let \( S = \{(i, j) \in P \times P : i < j\} \). Define \( S_0 = \emptyset \). Inductively, given \( S_m \), define \( S_{m+1} \) to be the set of \((i, j)\) in \( S \setminus \bigcup_{t \leq m} S_t \) such that
\[
i < k < j \implies \{(i, k), (k, j)\} \subset \bigcup_{k=0}^{m} S_k.
\]

Define \( \rho(i, j) = m \) if \((i, j) \in S_m \).

To make the sets \( S_m \) more concrete, we prove the next proposition. For this, we define a path of length \( \ell \) in \( P \) from \( i_0 \) to \( i_\ell \) to be a string \( (i_0, i_1, \ldots, i_{\ell}) \) such that \( i_{t-1} < i_t \) for \( 1 \leq t \leq \ell \). Such a path is **maximal** if \((i_{t-1}, i_t) \in S_1 \) for \( 1 \leq t \leq \ell \). For \( i < j \), define \( \rho(i, j) \) to be the maximum length of a path from \( i \) to \( j \).

**Proposition 7.15.** \((i, j) \in S_m \) if and only if \( \rho(i, j) = m \).

**Proof.** The proof is by induction on \( m \). The basis step \( m = 1 \) is straightforward. Assume the claim is true for \( m \). If \( \rho(i, j) = m+1 \), then clearly \((i, j) \in \bigcup_{t \leq m+1} S_t \); by the induction hypothesis, \((i, j) \notin \bigcup_{t \leq m} S_t \), and therefore \((i, j) \in S_{m+1} \). Now suppose \((i, j) \in S_{m+1} \). Then for every maximal path \((i, i_1, \ldots, i_k, j)\), we have by the induction hypothesis that \( \rho(i, i_k) \leq m \), and therefore \( k \leq m \) and the length of \((i, i_1, \ldots, i_k, j)\) is at most \( m+1 \). Therefore \( \rho(i, j) \leq m+1 \). By the induction hypothesis, \( \rho(i, j) > m \), and therefore \( \rho(i, j) = m+1 \).

Note, \( i < k < j \) implies \( \max\{\rho(i, k), \rho(k, j)\} < \rho(i, j) \). This is a key point for the proof by induction below.

**Example 7.16.** With \( P = P_4 \), we have \( S \) partitioned into \( S_1 = \{(1, 2), (2, 3), (3, 4)\} \), \( S_2 = \{(1, 3), (2, 4)\} \) and \( S_3 = \{(1, 4)\} \).

**Proof of Prop. 7.11.** (1) Clearly \( A^{<0>} \) satisfies \( C_2 \), as does any matrix \( \text{El}_P(k, H) \)-equivalent to \( A^{<0>} \). Now apply the initial part of the proof of Proposition 4.23 to \( A^{<0>} \), iterating the trimming move of Example 3.8.1 to produce a matrix \( A_1 \) in which a diagonal entry is zero iff the row and column through that entry are zero. By induction, these trimming moves are positive \( \text{El}_P(k, H) \)-equivalences in \( M^p_P(C, n, H) \), because \( A_1 \in M^p_P(C, n, H) \). If \( i \in C \), then the nonzero diagonal entry
of \( A_1 \{i, i\} \) must be unique. It follows that \( A_1 \) is the stabilization of a matrix \( A' \) as claimed.

For the proof of (2), suppose \( A \in \mathcal{M}^p_{\mathcal{H}}(\mathcal{C}, \mathcal{H}) \), satisfying \( C_1 \) and \( C_2 \). Without loss of generality, suppose also that a diagonal entry in \( A \) is zero iff the row and column through that entry are zero. Given \( i \notin \mathcal{C} \), let \( C_i = A \{i, i\} \). Steps 0,3,4,5,6 of \( 12 \) Lemma 6.6] give an explicit string of basic positive \( \mathcal{Z}\mathcal{H}_r \)-equivalences which send \( I - C_i \) to a matrix \( I - B_i \) such that \( B_i - I \) is \( H_r \)-positive. These basic positive equivalences are given by row and column cuts (recall subsections 3.7,3.8) and therefore the elementary row and column operations defining them give a string of basic positive \( \mathcal{B} \) equivalences for every \( M \). Loss of generality that \( \mathcal{A} \) induction, addressing the sets \( \mathcal{D} \) may assume Condition 3 holds for perhaps condition \( \mathcal{E} \).

To arrange for condition (3), we consider double cosets \( D \in \mathcal{D}_{ij} \) by induction, addressing the sets \( \mathcal{D}_{ij} \) in an order with \( \rho(i,j) \) nondecreasing. The positive equivalences below are compositions of \( \text{El}_p(\mathcal{N}, \mathcal{H}) \)-equivalences, of the form \((I - B) \rightarrow (I - C)\) with \( B \leq C \). So, all the matrices \( C \) continue to satisfy Conditions 1 and 2. Also, if Condition 3 is satisfied by \( B \) for \( D \in \mathcal{D}_{i'j'} \), then this remains true for \( C \). In particular, when considering \( D \in \mathcal{D}_{ij} \), if e.g. \( \rho(i,k) < \rho(i,j) \) then we may assume without loss of generality that \( A \) satisfies all conditions of Definition 7.8 except perhaps condition (3).

We begin with a claim.

**Claim 1.** Suppose \( i \notin \mathcal{C} \) or \( j \notin \mathcal{C} \); \((s,t) \in \mathcal{I}_i \times \mathcal{I}_j ; g \in \mathcal{H}_{ij} \); and \( A(s,t) \geq g \in \mathcal{H}_{ij} \). Let \( D = H_i g H_j \). Then there is a positive equivalence \((I - A) \rightarrow (I - A')\) where \( A' \geq A \) and \( A' \{i, j\} \) is \( \mathcal{D} \)-positive.

**Proof of Claim 1.** We assume \( j \notin \mathcal{C} \) (the proof for the case \( i \notin \mathcal{C} \) is similar). Let \( E = E_{st}(g) \). There is a positive equivalence \((I - A) \rightarrow (I - B)\) (a \( g, s, t \) row cut equivalence, as in Sec 3.7) implemented by \( I - B = E(I - B) \). Here row \( s \) of \( B \{i, j\} \) is \( gH_j \)-positive, because

\[
B(s,t) = A(s,t) - g + gA(t,t) \geq g \delta_j ,
\]

\[
B(s,t') = A(s,t') + gA(t,t') \geq g \delta_j \quad \text{if } t' \in \mathcal{I}_j \text{ and } t' \neq t ,
\]

and the inequalities hold because \((A - I) \{j, j\} \) is \( H_j \)-positive. For use in Subcase 2, after performing a second \((g, s, t) \) row cut equivalence (if necessary), we obtain \( B' \) with

\[
B'(s,t') = B(s,t') + gB(t,t') \geq A(s,t') + 2g \delta_j \quad \text{if } t' \in \mathcal{I}_j .
\]

There are two cases for the rest of the proof.

Subcase 1: \( i \notin \mathcal{C} \). Suppose \( t' \in \mathcal{I}_j \); \( B(s,t') \geq h \); and \( E = E_{st} \). Choose \( gh \in gH_j \). Then there is a positive equivalence to \((I - B) \rightarrow (I - C)\) (a \( gh, s, t' \) column cut equivalence, as in Sec 3.8) implemented by \((I - C) = (I - B)E \). Here column \( t' \) of \( C \{i, j\} \) is \( H_j g H_j \) positive,
because
\[ C(s, t') = B(s, t') - gh + B(s, s)A(s, t') \geq \delta_i g \delta_j , \]
\[ C(s', t') = B(s', t') + B(s', s)A(s, t') \geq \delta_i g \delta_j \]
where the inequalities hold because \((B - I)\{i, i\}\) is \(H_i\)-positive. So, after implementing \((h, s, t')\) column cuts for each \(t'\) in \(I_i\), we pass to \(I - A\) where \(A\{i, j\}\) is \(D\)-positive.

Subcase 2: \(i \in C\). Now \(I_i = \{s\}\). Let \(B(s, s) = g_i\). Suppose \(t' \in I_j\). Let \((g_i)_{\ell} = 1\). Starting from \(B'\), apply \((g_i)^{k}, s, t'\) column cuts, for \(k = 0, 1, \ldots, \ell - 1\), to produce \(C_0, \ldots, C_{\ell - 1}\). Because \(B'(s, t') \geq 2g\delta_j\), we have

\[ C_0(s, t') = B'(s, t') - g + g_iB'(s, t') > g\delta_j + g_i g 2\delta_j \]
\[ C_1(s, t') = C_0(s, t') - g_i g + g_i C_0(s, t') > g\delta_j + g_i g \delta_j + (g_i)^2 2g \delta_j \]
\[ \ldots \]
\[ C_{\ell - 1}(s, t') > g\delta_j + g_i g \delta_j + \cdots + (g_{i-1})^{\ell - 1}(\delta_j - 1) = \delta_i g \delta_j \]

Now we consider \(D \in D_{ij}\). If \(D \in R_{ij}\), then there must be some \(g\) in \(D\) and some \((s, t)\) in \(I_s \times I_t\) such that \(A(s, t) \geq g\). If \(\{i, j\} \subset C\), then \(A\{i, j\}\) has only one entry, and therefore Condition 3(ii) holds for \(D\). If \(\{i, j\}\) is not contained in \(C\), then by Claim 1 there is a positive equivalence to some \(A'\) with \(A'\{i, j\}\) \(D\)-positive.

From here, suppose \((i, j, D) \in R\). Because \(H\) is a coset structure for \(A\), we deduce that there exists \(k\) such that \(i < k < j\) and \(D \subset H_{ik} H_{kj}\). So, given \(i < k < j\), we will finish by producing a positive equivalence replacing \(A\) with a matrix \(A'\), \(A' \geq A\), such that \(A'\{i, j\}\) is \(H_{ik} H_{kj}\)-positive. This argument goes by cases. For an element \(i\) of \(C\), we let \(g_i\) be the unique entry of \(A\{i, i\}\).

**Case 1:** \(k \notin C\), or \(\{i, j\} \cap C = \emptyset\). Suppose \((s, r) \in I_s \times I_t\). Perform an \((A(s, r), s, r)\) row cut equivalence on \(A\) to produce a matrix \(B\) such that

\[ B(s, r) = A(s, r)B(r, r) \geq \delta_{ik} \]
\[ B(s, t) = A(s, t) + A(s, r)B(r, t) \geq \delta_{ik} \delta_{kj} \quad \text{if} \quad t \in I_j \]
where the inequalities hold because, by the induction hypothesis, \((A - I)\{i, k\}\) is \(H_{jk}\) positive and \((A - I)\{k, j\}\) is \(H_{kj}\) positive. Thus row \(s\) of \(B\{i, j\}\) is now \(H_{ik} H_{kj}\) positive. Repeat as needed for all \(s\) in \(I_i\) to obtain \(A'\) such that \(A'\{i, j\}\) is \(H_{ik} H_{kj}\) positive.

**Case 2:** \(\{i, k, j\} \subset C\). Let \(I_i \times I_k \times I_j = \{(s, r, t)\}\).

Looking only at the principal submatrices on indices \(\{s, r, t\}\), let \(A = \begin{pmatrix} g_i & b & c \\ 0 & g_k & d \\ 0 & 0 & g_j \end{pmatrix}\). By the induction hypothesis, for \(D \in H_{ik}\) we have \(\pi_D(b) > 0\), and for \(D \in H_{kj}\) we have \(\pi_D(d) > 0\). Therefore \(H_{ik} H_{kj} \subset H_{ik} b H_k d H_j\).
So, it suffices to give a string of positive equivalences producing \( A' \) with \( A'(s,t) \geq \delta_b d_k d_j \).

Let \( \ell = \kappa(g_i) \). We apply in order \( (b(g_k)^t, s, r) \) row cuts, for \( t = 0, 1, \ldots, \ell - 1 \), starting with \( A \). The first equivalence \( (t = 0) \) is implemented by

\[
\begin{pmatrix}
1 & b & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 - g_i & -b & -c \\
0 & 1 - g_k & -d \\
0 & 0 & 1 - g_j
\end{pmatrix}
= \begin{pmatrix}
1 - g_i & -b g_k & -c - b d \\
0 & 1 - g_k & -d \\
0 & 0 & 1 - g_j
\end{pmatrix}
\]

After the last \( (t = \ell - 1) \) equivalence we reach

\[
I - B = \begin{pmatrix}
1 - g_i & -b(g_k)^t & -c' \\
0 & 1 - g_k & -d \\
0 & 0 & 1 - g_j
\end{pmatrix}
\]

where \( B \geq A \) (because \( b(g_k)^t = b \) and \( c' = -A(s,t) - b(1 + g_k + \cdots + (g_k)^{t-1}) d = A(s,t) + b b_k d \). We follow this with a column cut equivalence,

\[
\begin{pmatrix}
1 - g_i & -b & -c' \\
0 & 1 - g_k & -d \\
0 & 0 & 1 - g_j
\end{pmatrix}
\begin{pmatrix}
1 & 0 & b d_k \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 - g_i & -b & -c'' \\
0 & 1 - g_k & -d \\
0 & 0 & 1 - g_j
\end{pmatrix}
\]

and then repeat the first move to reach a matrix \( I - C, C \geq A \), with \( C(s,t) = A(s,t) + (1 + g_i) b d_k d \). Iterating this process leads to a matrix \( I - D \) such that \( D(s,t) = A(s,t) + \delta b d_k d := d' \). Setting \( \delta'' = \delta b d_k d \), we then apply the equivalence

\[
\begin{pmatrix}
1 & 0 & \delta'' \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 - g_i & -b & -d' \\
0 & 1 - g_k & -d \\
0 & 0 & 1 - g_j
\end{pmatrix}
= \begin{pmatrix}
1 - g_i & -b & -d'' \\
0 & 1 - g_k & -d \\
0 & 0 & 1 - g_j
\end{pmatrix}
\]

to reach a matrix \( I - E \) for which \( E(s,t) = A(s,t) + \delta b d_k d g_j \). Iterating the move that produced \( E \) from \( A \), we arrive at \( A' \) with \( A'(s,t) = A(s,t) + \delta b d_k d g_j \).

**Case 3:** \( \{i, k\} \subset C, j \not\subset C \). Suppose \( t \in \mathcal{I}_j \). Use the Case 2 moves which led the matrix \( D \) with \( D(s,t) = A(s,t) + \delta b d_k d \). By the induction hypothesis, \( d \) is \( H_{kj} \) positive, and therefore \( E(s,t) \) is \( H_{ik} H_{kj} \) positive. Iterating this move over \( t \in \mathcal{I}_j \), we reach \( A' \) such that \( A' \{i, j\} \) is \( H_{ik} H_{kj} \) positive.

**Case 4:** \( \{j, k\} \subset C, i \not\subset C \). The proof here is similar to the proof for Case 3.

\[ \square \]

8. The Factorization Theorem: proof

This section is devoted to the proof of the Factorization Theorem \[7.2\]. We use and generalize proof techniques from \[12\] and \[4\]. We will prove three lemmas, and then use them in a short argument to finish the proof of Theorem \[7.2\].
Let \( \mathcal{U}_P(n, \mathcal{H}) \) be the set of matrices \( M \) in \( \text{El}_P(n, \mathcal{H}) \) such that every diagonal block \( M\{i, i\} \) is the identity matrix. We will address equivalences \((U, V)\) for matrices \( U, V \) in \( \mathcal{U}_P(n, \mathcal{H}) \).

In Lemma 8.1, we will consider \( 2 \times 2 \) upper triangular matrices. Recalling \( P_2 = \{1, 2\} \) with \( 1 < 2 \).

**Definition 8.1.** Suppose \( m = (m_1, m_2) \); \( \mathcal{H} \) is a \((G, P_2)\) coset structure; and \( B, B' \) are in \( \mathcal{M}_{P_2}^+(m, \mathcal{H}) \). A string of basic elementary positive \( \mathcal{U}_{P_2}(\mathcal{H})\)-equivalences

\[
B \xrightarrow{(E_1, F_1)} B_1 \xrightarrow{(E_2, F_2)} \cdots \xrightarrow{(E_t, F_t)} B'
\]

is extendable if the matrix products \( E_1 \cdots E_t \) and \( F_1 \cdots F_t \) are nonnegative, \( 1 \leq i \leq t \). In this case, with \((U, V) = (E_t \cdots E_2 E_1, F_1 F_2 \cdots F_t)\), we also say that \((U, V) : B \rightarrow C \) is an extendable positive equivalence.

Our interest in extendable equivalences is the following. Suppose \( M, M' \) are in \( \mathcal{M}_{P_2}^+(C, n, \mathcal{H}) \) with \( 2 \times 2 \) principal submatrices \( B, B' \) on the same coordinate indices \( s, t \) contained in a block \( \{i, j\} \) with \( i < j \). Then an extendable positive equivalence of \( B, B' \) (with respect to the restriction of \( \mathcal{H} \)) will give (by the same elementary operations) a positive equivalence from \( M \) to \( M' \) in \( \mathcal{M}_{P_2}^+(C, n, \mathcal{H}) \).

**Lemma 8.2.** Given matrices \( B, B' \) in \( \mathcal{M}_{P_2}^+(C, n, \mathcal{H}) \), suppose \((U, V) : B \rightarrow B' \) is a \( \mathcal{U}_P(\mathcal{H})\)-equivalence which only differs from the identity at blocks \( U\{i, j\} \) and \( V\{i, j\} \), and that every entry of \( U\{i, j\} \) and \( V\{i, j\} \) lies in \( \mathbb{Z}_+ G \). Then \((U, V) : B \rightarrow B' \) is an extendable positive \( \text{El}_P(\mathcal{H}) \) equivalence, and consequently

\[
(U, V) : B \xrightarrow{} B'
\]

**Proof.**

**Case 1** \( i \notin C, j \notin C \):
Since \( B, B' \) have positive entries in all relevant diagonal blocks we can simply decompose \( U \) and \( V \) one entry at a time, thus obtaining an extendable positive \( H_{ij} \)-equivalence at every step.

**Case 2** \( i \in C, j \notin C \):
As in Case 1, \((U, I) : B \xrightarrow{U} B \). So, without loss of generality, we can assume \((U, V) = (I, V) \). Considering compositions, it is enough to address the case that \( V \) has a single nonzero offdiagonal entry, say, \( V(1, 2) = s \neq 0 \). Then, in the principal submatrices on indices \( \{1, 2\} \), the equivalence \((I, V) : B \rightarrow BV \) has the form \( B = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \rightarrow \begin{pmatrix} g^{-1} r + (g-1)s \\ 0 \\ 0 \end{pmatrix} \).

Choose \( p \) in \( \mathbb{Z}_+ H_{ij} \) such that \( p > s \); then \( ph > s \), because \( h \) is \( H_{ij} \)-positive. Applying the equivalences \((E_{12}(p), I) \), \((I, V) \), \((E_{12}(-p)) \) produces

\[
B \rightarrow \begin{pmatrix} g^{-1} r + ph \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} g^{-1} r + ph + (g-1)s \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} g^{-1} r + (g-1)s \\ 0 \\ 0 \end{pmatrix}.
\]
Let \( E_{12}(p) = E_k \ldots E_1, E_{12}(s) = F_1 \ldots F_k \) be factorizations into non-negative matrices with offdiagonal entries in \( \mathcal{H}_i \). Then \((I, V)\) is the composition \((E_1, I), \ldots, (E_k, I), (I, F_1), \ldots, (I, F_k), (E_k^{-1}, I), \ldots, (E_1^{-1}, I)\) and therefore \((I, V)\) is extendable.

**Case 3** \( i \notin C, j \in C \):
The proof here is essentially as for Case 2.

**Case 4** \( i \in C, j \in C \):
Note first that in this case, the nontrivial matrices \( U\{i, j\} \) and \( V\{i, j\} \) are \( 1 \times 1 \). Let \( p \) be the entry of \( U\{i, j\} \) and \( s \) the entry of \( V\{i, j\} \); we have assumed that \( p, s \in \mathbb{Z} \).

The proof is by induction on \( K = \frac{p}{s} + \frac{1}{s} \), and the lemma is true for \( K = 0 \). Suppose \( \frac{p}{s} + \frac{1}{s} = K > 0 \) and the lemma holds if \( \frac{p}{s} + \frac{1}{s} < K \).

Here the submatrix of the equivalence \( UBV = B' \) containing any change has the form

\[
(8.3) \quad \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g - 1 & r \\ 0 & h - 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g - 1 & r' \\ 0 & h - 1 \end{pmatrix}.
\]

where

\[
(8.4) \quad r' = r + p(h - 1) + (g - 1)s.
\]

We use \( E(x) \) to denote a matrix \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \); e.g., \( U = E(p) \). For any \( x, y, z \),

\[
(8.5) \quad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g - 1 & y \\ 0 & h - 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g - 1 & y + x(h - 1) + (g - 1)z \\ 0 & h - 1 \end{pmatrix}
\]

so here the pair \((E(x), E(z))\) acts by adding \( x(h - 1) + (g - 1)z \) to the \((1, 2)\) entry. The equivalence given by \((U, V)\) is a composition of basic elementary equivalences, given by \((I, E(w))\) or \((E(w), I)\), with \( w \in G \) a summand of \( p \) or \( s \). Such an equivalence acts by adding a term \( w' - w \) to the \((1, 2)\) position, where \( w' \) is \( wh \) or \( gw \).

**Case 4(i):**
Assume \( r \neq r' \). Let \( r = \sum_{w} n_{w} w \) and \( r' = \sum_{w} n_{w}' w \). The images \( \mathcal{r} \) and \( \mathcal{r}' \) under the augmentation must be equal. So, there must be some \( w \in G \) such that \( n_{w} > n_{w}' \). Therefore \( w \) must be a summand of \( p \) or \( s \), and \((I, E(w))\) or \((E(w), I)\) applied to \( B \) is a positive equivalence in \( \mathcal{M}_{\mathcal{P}}^{\mathcal{P}}(\mathcal{C}, \mathcal{n}, \mathcal{H}) \). Now the equivalence given by \((U, V)\) is this positive equivalence followed by one satisfying the induction hypothesis. A composition of extendable equivalences is extendable. This completes the inductive step if \( r \neq r' \).

**Case 4(ii):**
Assume \( r = r' \) and note that in this case

\[
(8.6) \quad p + s = ph + gs
\]

according to \((8.4)\).

Suppose \( w_{0} \) is a summand of \( p + s \). Then \( w_{0} \in H_{12} \), since \((U, V) : B \rightarrow B' \) is a \( \mathcal{U}_{\mathcal{P}}(\mathcal{H})\)-equivalence. Because \( B \in \mathcal{M}_{\mathcal{P}}^{\mathcal{P}}(\mathcal{C}, \mathcal{n}, \mathcal{H}) \), there
must be a summand \( x \) of \( r \) and \( i, j \) such that \( w_0 = g^jxh^i \). We then have a positive equivalence \((E, F) : B \to B_0\) defined by
\[
\begin{pmatrix} g - 1 & r \\ 0 & h - 1 \end{pmatrix} \frac{(E, F)}{} + \frac{(I, E(x))}{(I, E(gx))} \cdots \frac{(I, E(g^{r-1}x))}{(E(g^r)x, F)} \frac{(E(g^r)x, I)}{(E(g^r)xh), I} \cdots \frac{(E(g^{r-1}xh), I)}{(E(g^r)xh^{r-1}, I)} \frac{(g - 1 & r + g^jxh^i - x \\ 0 & h - 1 \end{pmatrix}
\]

Let \((E_t, F_t)\) be the \( t \)th of these basic positive equivalences, so, \((E, F) = (E_{t+j} \cdots E_1, F_1 \cdots F_{i+j})\). Define
\[
(E'_1, F'_1) = \begin{cases} (E(w_0), I) & \text{if } w_0 \text{ is a summand of } p \\ (I, E(w_0)) & \text{otherwise} \end{cases}
\]

Now \((E'_1, F'_1) : B_0 \to E'_1B_0F'_1 =: B_1\), with \(B_1(1, 2) = B_0(1, 2) + w_1 - w_0\), where \(w_1 = w_0h\) if \(w'_1 = E(w_0)\) and \(w_1 = gw_0\) if \(F'_1 = E(w_0)\). In either case, according to (8.6) and the definition of \((E'_1, F'_1)\), \(w_1\) is a summand of \(p + s\). Since \(B_1(1, 2) = B_0(1, 2) + w_1 - w_0\), if \(w_1 \neq w_0\) then \(w_1\) must be a summand of \(p + s - w_0\), and we may construct a positive equivalence \((E'_2, F'_2) : B_1 \to B_2\) as before, with \(w_1\) in place of \(w_0\) and \(B_2(1, 2) = B_1(1, 2) + w_2 - w_1 = B_0(1, 2) + w_2 - w_0\). Since \(p + s\) is finite and \(r = r'\), this process must reach some \(w_m = w_0\), and we set \((E', F') = (E_m \cdots E'_1, F'_1 \cdots F'_m)\). Because \(B_m = B_0\), we have by composition a positive equivalence
\[
(E', F') : B \xrightarrow{(E, F)} B_0 \xrightarrow{(E', F')} B_0 \xrightarrow{(E^{-1}, F^{-1})} B.
\]

The equivalence \(B \xrightarrow{(E, F)} B_0 \xrightarrow{(E', F')} B_0\) is extendable because the matrices \(E_t, F_t, E'_t, F'_t\) are nonnegative. Extendability through the remaining basic equivalences holds because
\[
\begin{align*}
(E_t^{-1} \cdots E_1^{-1} E' E, & \quad FF'_1 \cdots F^{-1}_t) \\
& \quad (E' E_t^{-1} \cdots E_1^{-1} E, \quad FF'_1 \cdots F^{-1}_t F') \\
& \quad (E' E_{t+1} \cdots E_{i+j}, \quad F_{t+1} \cdots F_{i+j} F')
\end{align*}
\]
and all the matrices in the last line are nonnegative. The equivalence \((U(E')^{-1}, (F')^{-1}V) : B \to B\) has the form \((E(p'), E(s'))\) with \(p' + s' = p + s - m < p + s\), and therefore is extendable by the induction hypothesis. This finishes the inductive step for Case 2(ii).

\[\text{Lemma 8.7. Suppose } U \text{ and } V \text{ are matrices in } U_P(n, H), \text{ and } B \text{ and } B' \text{ are in } M^+_H(C, n, H), \text{ and } UBV = B'. \text{ Then } (U, V) : B \to B'.\]

\[\text{Proof. Recalling Definition } [7,14] \text{ let } (i_1, j_1), \ldots, (i_r, j_r) \text{ be an enumeration of elements of } S \text{ such that } t \leq s \implies \rho(i_t, j_t) \leq \rho(i_s, j_s).\]
We will define various matrices by induction, beginning with $B_0 = B$, $B'_0 = B'$, $U_0 = U$, $V_0 = V$. For $1 \leq s \leq r$, given $B_{s-1}, B'_{s-1}, U_{s-1}, V_{s-1}$ with $B_{s-1}, B'_{s-1} \in \mathcal{M}^{\mathcal{P}+}_n(\mathcal{C}, \mathcal{n}, \mathcal{H})$ and $U_{s-1}, V_{s-1} \in \mathcal{U}_p(\mathcal{n}, \mathcal{H})$, we choose matrices $P_s, Q_s$ in $\mathcal{U}_p(\mathcal{n}, \mathcal{H})$, equal to $I$ outside block $\{i_s, j_s\}$, such that the following Positivity Conditions hold:

1. For some nonnegative integer $M_s$, every entry of $P_s\{i_s, j_s\}$ and every entry of $Q_s\{i_s, j_s\}$ equals $M_s \delta_{i,j} \delta_{j,i}$.
2. The blocks $(P_sU_{s-1})\{i_s, j_s\}$ and $(V_{s-1}Q_s)\{i_s, j_s\}$ have all entries in $\mathbb{Z}_+ H_{ij}$.

We note that by taking $M_s$ large in (1), we can achieve (2). We then define matrices $W_s, X_s$ in $\mathcal{U}_p(\mathcal{n}, \mathcal{H})$, equal to $I$ outside block $\{i_s, j_s\}$, by setting

\[
W_s\{i_s, j_s\} = (P_sU_{s-1})\{i_s, j_s\} \\
X_s\{i_s, j_s\} = (V_{s-1}Q_s)\{i_s, j_s\}.
\]

Finally we define

\[
U_s = P_sU_{s-1}W_s^{-1} \\
B_s = W_sB_{s-1}X_s \\
V_s = X_s^{-1}V_{s-1}Q_s \\
B'_s = P_sB'_{s-1}Q_s
\]

Then $U_s, V_s \in \mathcal{U}_p(\mathcal{n}, \mathcal{H})$ and $B_s, B'_s \in \mathcal{M}^{\mathcal{P}+}_n(\mathcal{C}, \mathcal{n}, \mathcal{H})$, $0 \leq s \leq r$.

For $1 \leq s \leq r$, we will verify the following claims by induction.

\begin{enumerate}
    \item\hspace{0.5cm} $(a)$ $U_sB_sV_s = B'_s$
    \item\hspace{0.5cm} $(b)$ $B'_{s-1} \xrightarrow{(P_s,Q_s)} B'_s$
    \item\hspace{0.5cm} $(c)$ $B_{s-1} \xrightarrow{(W_s,X_s)} B_s$
    \item\hspace{0.5cm} $(d)$ $U_s = P_s \cdots P_1 U W^{-1}_1 \cdots W^{-1}_s$ and $V_s = X^{-1}_s \cdots X^{-1}_1 V Q_1 \cdots Q_s$
    \item\hspace{0.5cm} $(e)$ $U_s\{i_t, j_t\} = 0 = V_s\{i_t, j_t\}$ if $1 \leq t \leq s$.
\end{enumerate}

Before proving (a)-(e), suppose all these claims hold. Define $P = P_1 P_{r-1} \cdots P_1$ and $Q = Q_1 Q_2 \cdots Q_r$. From (b), we have

\[
B' = B'_0 \xrightarrow{(P,Q)} B'_1 \xrightarrow{(P,Q)} B'_2 \cdots \xrightarrow{(P,Q)} B'_r = PB'Q
\]

and therefore $B' \xrightarrow{(P,Q)} PB'Q$. Similarly, let $W = W_r \cdots W_1$ and $X = X_1 \cdots X_r$. From (c), we have

\[
B = B_0 \xrightarrow{(W,X)} B_1 \xrightarrow{(W,X)} B_2 \cdots \xrightarrow{(W,X)} B_r = WBX
\]
and therefore $B \xrightarrow{(W,X)} WBX$. Because $\{U_r, V_r\} \subset U_P(n, \mathcal{H})$, from (e) with $s = r$ we get $U_r = I = V_r$. Using (d) at $s = r$, we then get

$$I = U_r = P_r \cdots P_1 UW_1^{-1} \cdots W_r^{-1} = PUW^{-1}$$

and similarly $I = V_r = X^{-1}VQ$. Therefore $(PU, VQ) = (W, X)$ and

$$B \xrightarrow{(PU,VQ)} PUBVQ = B'Q \xrightarrow{(P^{-1}U,V^{-1})} B'.$$

This shows $(U, V) : B \to B'$ is a positive equivalence.

To finish the proof it remains to verify (a)-(e) for $1 \leq s \leq r$.

**Proof of (a).** We have $U_0B_0V_0 = B'_0$. Suppose $0 < s \leq r$ and (a) holds at $s - 1$. Then

$$U_sB_sV_s = \left(P_sU_{s-1}W_s^{-1}\right)\left(W_sB_{s-1}X_s\right)\left(X_s^{-1}V_{s-1}Q_s\right)$$

$$= P_s \left(U_{s-1}B_{s-1}V_{s-1}\right)Q_s$$

$$= P_sB'_{s-1}Q_s = B'_s.$$

**Proof of (b).** We have $B'_0 = B' \in \mathcal{M}^+_P(C, n, \mathcal{H})$. Suppose $1 \leq s \leq r$ and $B'_{s-1} \in \mathcal{M}^+_P(C, n, \mathcal{H})$. An entry in $B'_{s-1}$ could decrease in $P_sB_{s-1}Q_s$ only as a result of addition of a term $(M_s\delta_i\delta_j\delta_l)(g_j - 1)$ or $(g_i - 1)(M_s\delta_i\delta_j\delta_l)$. But, these terms vanish, as $(g_i - 1)\delta_i = 0 = \delta_j(g_j - 1)$. Therefore $P_sB_{s-1}Q_s \geq B_{s-1}$. Because $B_{s-1} \in \mathcal{M}^+_P(C, n, \mathcal{H})$, this implies $P_sB_{s-1}Q_s \in \mathcal{M}^+_P(C, n, \mathcal{H})$.

Now enumerate the coordinates of the nonzero off-diagonal entries of $Q_s$ as $(a_1, b_1), \ldots, (a_T, b_T)$. For $1 \leq t \leq T$, let $E_t$ be the basic elementary matrix such that $E_t(a_i, b_i) = Q_s(a_i, b_i)$. Because these entries lie in blocks $\{i_s, j\}$ with $i_s < j$, we have $Q_s = \prod_{t=1}^T E_t$. This $(I, Q) : B'_{s-1} \to B'_{s-1}Q$ is a composition of equivalences

$$B'_{s-1} := B'_{s-1,0} \xrightarrow{(I, E_1)} B'_{s-1,1} \xrightarrow{(I, E_2)} \cdots \xrightarrow{(I, E_T)} B'_{s-1,T} = B_{s-1}Q_s$$

By induction, for $1 \leq t \leq T$, $E_t$ is nonnegative and $B'_{s-1, t} \in \mathcal{M}^+_P(C, n, \mathcal{H})$. It then follows from Lemma[8,2] that each $(I, E_t)$ gives a positive equivalence. Thus $B'_{s-1} \xrightarrow{(I,Q_s)} B_{s-1}Q_s$, and similarly $B'_{s-1}Q_s \xrightarrow{(P_s,I)} PB'_{s-1}Q_s$.

By composition, $B'_{s-1}Q \xrightarrow{(P_s,Q_s)} P_sB'_{s-1}Q_s$.

**Proof of (c).** We have $B_0 \in \mathcal{M}^+_P(C, n, \mathcal{H})$. Now suppose $1 \leq s \leq r$ and $B_{s-1} \in \mathcal{M}^+_P(C, n, \mathcal{H})$. The matrices $W_s$ and $X_s$ are in $U_P(n, \mathcal{H})$, with all entries in $\mathbb{Z}_2 G$, and $B_{s-1} \leq W_sB_{s-1}X_s := B_s$. Therefore $B_s \in \mathcal{M}^+_P(C, n, \mathcal{H})$. An argument very similar to the proof of claim (b) now shows that $B'_{s-1} \xrightarrow{(W_s,X_s)} B_s$.

**Proof of (d).** The claim (d) follows by induction from the definitions $U_0 = U$, $V_0 = V$, $U_s = P_sU_{s-1}W_s^{-1}$ and $V_s = X_s^{-1}V_{s-1}Q_s$. 

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Proof of (e). Suppose \(1 \leq s \leq r\) and (e) holds at \(s-1\). (At \(s-1 = 0\), (e) is an empty statement.) We have \(U_s = P_s U_{s-1} W_s^{-1}\), with \(P_s\) and \(W_s^{-1}\) equal to \(I\) outside block \(\{i_s, j_s\}\). On account of the zero block structure of matrices in \(\mathcal{U}_P(n, \mathcal{H})\), we have \(U_s = U_{s-1}\) except possibly in blocks \(\{i, j\}\) such that \(i \preceq i_s\) or \(j \preceq j_s\).

At \((i_s, j_s)\), we have

\[
U_s\{i_s, j_s\} = \left( (P_s U_{s-1} W_s^{-1}) \right)\{i_s, j_s\}
= (P_s U_{s-1})\{i_s, j_s\} + W_s^{-1}\{i_s, j_s\}
= (P_s U_{s-1})\{i_s, j_s\} - W_s\{i_s, j_s\} = 0. 
\]

Now suppose \(i < i_s\). Then \(U_s\{i, j\} = U_{s-1}\{i, j\}\) except possibly in the case \(j = j_s\), where

\[
U_s\{i, j\} - U_{s-1}\{i, j\} = U_{s-1}\{i, i_s\} W_s^{-1}\{i, j_s\}. 
\]

The right side of (8.8) can be nonzero only if \(i < i_s < j = j_s\). In this case, \(\rho(i_s, j_s) < \rho(i, j)\), so \((i, j)\) cannot equal \((i_t, j_t)\) for any \(t\) less than \(s\). Thus if \(1 \leq t < s\), then \(U_s\{i_t, j_t\} = U_{s-1}\{i_t, j_t\}\), which is zero by the induction hypothesis.

The analogous argument for the case \(j_s < j\) finishes the proof. \(\square\)

We make contact to the case with \(U, V \in \mathcal{U}_P(\mathcal{H})\) from the general case using the following lemma in combination with a key result from [12].

Lemma 8.9. Suppose \(i \notin \mathcal{C}\), \(E\) is a basic elementary matrix in \(\text{El}_P(\mathcal{H})\); \(E\{j, k\} = I\{j, k\}\) when \((j, k) \neq (i, i)\); \(B, B' \in \mathcal{M}^{++}_P(\mathcal{H})\); and

\[
(E\{i, i\}, I) : B\{i, i\} \to B'\{i, i\}. 
\]

Then there exists \(V\) in \(\mathcal{U}_P(n, \mathcal{H})\) such that

\[
(E, V) : B \to EB'V. 
\]

Similarly, if

\[
(I, E\{i, i\}) : B\{i, i\} \to B'\{i, i\} 
\]

then there exists \(U\) in \(\mathcal{U}_P(n, \mathcal{H})\) such that

\[
(U, E) : B \to UB'E. 
\]

Proof. We will consider the equivalence \((E, I)\); the other case is similar.

Let \(E(s, t) = v\) be the nonzero off-diagonal entry of \(E\). \(E\) acts on \(B\) from the left to add \(v\) times row \(t\) of \(B\) to row \(s\) of \(B\). If each block \(\{i, \ell\}\) of \(EB\) is \(H_{\ell\ell}\)-positive (e.g., if \(v \geq 0\)), then set \(V = I\).

Otherwise, pick \(r\) an index for a column through the \(\{i, i\}\) block. For a positive integer \(L\), let \(V\) be the matrix in \(\mathcal{U}_P(n, \mathcal{H})\) such that (i) if
$i < \ell$, then every entry of $V\{i, \ell\}$ equals $L\delta_{i\ell}$ and (ii) in other entries, $V$ agrees with $I$. Then for $(s,q)$ in block $\{i, \ell\}$,

$$(EBV)(s,q) \geq (EB)(s,q) + (B(s,r) - B(t,r))(L\delta_{i\ell}).$$

Because $(EB)\{i,i\}$ is $H_i$-positive, for sufficiently large $L$ the displayed sum must for each such $\ell$ be $H_{i\ell}$-positive. Then $B \xrightarrow{(I,V)+} BV$ and $EBV \in \mathcal{M}_P^+(\mathcal{C}, n, \mathcal{H})$, so

$$B \xrightarrow{(I,V)+ BV \xrightarrow{(E,I)+} EBV$$

as required. 

\textbf{Proof of Theorem 7.2.} It follows from Observation 7.10 and Proposition 7.11 that to prove Theorem 7.2 we may assume that $B, B' \in \mathcal{M}_P^+(\mathcal{H})$.

Thus let $(U,V): B \to B'$ be the given El$_\mathcal{P}(n, \mathcal{H})$ equivalence, with $B, B' \in \mathcal{M}_P^+(\mathcal{H})$. Set $U' = \oplus_i U\{i,i\}$ and $V' = \oplus_i V\{i,i\}$. If $i \in \mathcal{C}$, then $n_i = 1$ and $U\{i,i\} = V\{i,i\}$. If $i \notin \mathcal{C}$, then by [12, Theorem 6.1] we have that $(U\{i,i\}, V\{i,i\}): B\{i,i\} \to B'\{i,i\}$ is a positive $\mathcal{Z}H_i$-equivalence through matrices which are $H_i$-positive.

So, there is a string $(E_1, F_1), \ldots, (E_T, F_T)$ of elementary El$_\mathcal{P}(n, \mathcal{H})$-equivalences which accomplishes the elementary positive equivalence decomposition inside the diagonal blocks, such that each $E_i$ and $F_i$ equals the identity outside diagonal blocks $\{i,i\}$ with $i \notin \mathcal{C}$. By Lemma 8.3 we may find $(U_1, V_1), \ldots, (U_t, V_t)$ with each $U_s$ and $V_s$ in $U_\mathcal{P}(n, \mathcal{H})$ such that

$$B \xrightarrow{(U_i,F_i)+} \cdots \xrightarrow{(E_i,F_i)+} B^*.$$  

Let $X = E_0U_1 \cdots E_2U_2E_1U_1$. Let $Y = F_1V_1F_2V_2 \cdots F_tV_t$. Then for all $i \in \mathcal{P}$, $X\{i,i\} = U\{i,i\}$ and $Y\{i,i\} = V\{i,i\}$, so $UX^{-1} \in U_\mathcal{P}(n, \mathcal{H})$ and $Y^{-1}V \in U_\mathcal{P}(n, \mathcal{H})$. It then follows from Lemma 8.7 that

$$B^* \xrightarrow{(UX^{-1}, Y^{-1}V)+} B'.$$

Thus $(U,V): B \to B'$ is the composition

$$B \xrightarrow{(X,Y)+} B^* \xrightarrow{(UX^{-1}, Y^{-1}V)+} B'$$

and therefore $(U,V): B \to B'$. 

\hfill $\square$
9. Conclusion of proofs

We begin with the promised proof of Theorem 5.3.

(1) \implies (2): This implication follows directly from Theorem 6.2.

(2) \implies (1): Suppose (2) holds. Applying first Proposition 7.11 if needed, it follows from Theorem 7.2 that there is a positive El\(_P\)(m, \(\mathcal{H}\))-equivalence from \(I - A^{<0}\) to \(I - C^{<0}\). There is therefore a positive \(\mathbb{Z}G\)-equivalence from \(I - A^{<0}\) to \(I - C^{<0}\). Since every positive \(\mathbb{Z}G\)-equivalence induces a \(G\)-flow equivalence (see Section 3), it follows that \(T_{A^{<0}}\) and \(T_{C^{<0}}\) are \(G\)-flow equivalent. Since \(T_{A^{<0}} = T_A\) and \(T_{C^{<0}} = T_C\), we thus have that \(T_A\) and \(T_C\) are \(G\)-flow equivalent. It follows in a similar way from Proposition A.1 and Proposition B.1 that \(T_B\) and \(T_C\) are \(G\)-flow equivalent as wanted. \(\square\)

Next we describe which equivalence classes of matrices arise in the equivalence classes we use as \(G\)-flow equivalence invariants. (The invariance of these classes under stabilization was discussed in Section 3.11.)

**Theorem 9.1.** Given \(G, \mathcal{P}, \mathcal{C}, \mathcal{H},\) and \(n\) with \(n_i = 1\) if and only if \(i \in \mathcal{C}\), suppose \(B\) is a matrix in \(\mathcal{M}_P(n, \mathcal{H})\). Then the following are equivalent.

1. There is a \(k \geq n\), with \(k_i = 1\) if and only if \(i \in \mathcal{C}\), and a matrix \(A\) in \(\mathcal{M}_P^0(\mathcal{C}, k, \mathcal{H})\), such that \(I - A\) is \(\text{El}_P(k, \mathcal{H})\) to \(I - B^{<0}\), where \(B^{<0}\) is the 0-stabilization of \(B\) in \(\mathcal{M}_P(k, \mathcal{H})\).

2. The following hold:
   (a) If \(i \in \mathcal{C}\), then the \(1 \times 1\) \(i\)th diagonal block of \(B\) has the form \([g]\), with \(H_i\) generated by \(g\).
   (b) If \(i < j\), \(D \in \mathcal{R}_{ij}\) and \(\{i, j\} \subset \mathcal{C}\) and the \(1 \times 1\) \(ij\) block of \(B\) is \(\sum_{g \in \mathcal{G}} n_g g\), with each \(n_g\) in \(\mathbb{Z}\), then \(\sum_{g \in D} n_g > 0\).

Moreover, given (2), the matrix \(A\) can be chosen from \(\mathcal{M}_P^{++}(\mathcal{C}, m, \mathcal{H})\), where \(m_i = 1\) if \(i \in \mathcal{C}\) and \(m_i = n_i + 1\) otherwise.

**Proof.** (2) \implies (1): With \(m\) as defined in the “Moreover” statement, let \(B'\) be the stabilization of \(B\) in \(\mathcal{M}_P^0(\mathcal{C}, m, \mathcal{H})\). Let \(M = B' - I\), with diagonal blocks \(M_i\). It suffices to apply \(\text{El}_P(m, \mathcal{H})\) equivalences to \(M\) which produce a matrix in \(\mathcal{M}_P^{++}(\mathcal{C}, m, \mathcal{H})\) (recall Definition 7.8).

By [12, Proposition 8.8], for \(i\) not in \(\mathcal{C}\) the matrix \(M_i\) is \(\text{El}(\mathbb{Z}H_i)\)-equivalent to an \(H_i\)-positive matrix, \(M'_i\). After applying a block diagonal \(\text{El}_P(m, \mathcal{H})\) equivalence, we may assume for \(i \notin \mathcal{C}\) that \(M_i\) is \(H_i\)-positive. For these \(i\), in increasing order: for \(i < j\), as needed multiply from the right by matrices in \(\text{El}_P(m, \mathcal{H})\) zero outside the \(ij\) block to put all entries of the \(ij\) block of \(M\) into \(\mathbb{Z}H_{ij}\), with strictly positive coefficients. Then similarly for \(j\) in decreasing order: for \(i < j\), as needed multiply from the left to achieve this positivity.
At this point, all blocks of $M$ are in form for $\mathcal{M}_P^\ast(\mathcal{C}, \mathbf{m}, \mathcal{H})$ except perhaps the $1 \times 1$ blocks with $\{i,j\} \subset \mathcal{C}$. First, for each $D \in \mathcal{R}_{ij}$, pick an element $x$ from $D$, and multiply from the left and right by basic elementary matrices, of the form $(g_i x g_j)$ in the $ij$ block, to effect the replacement of $\sum_{g \in D} n_g g$ with $(\sum_{g \in D} n_g) x$, which by (2)(b) is positive. For $D$ not in $\mathcal{R}_{ij}$, the coefficients of $\sum_{g \in D} n_g g$ are made positive by elementary multiplications as in the $(i,j) \not\in \mathcal{R}^c$ step in the proof of Proposition [7.11]. We will refrain from reentering the details of this step.

(1) $\implies$ (2): Suppose (1) holds. Condition (2) holds with $A$ in place of $B$, because $A \in \mathcal{M}_P^\ast(\mathcal{C}, k, \mathcal{H})$ with $k_i = 1$ for $i \in \mathcal{C}$. Let $I - B^{<0>} = U(I - A)V$ be the assumed $\text{El}_P(k, \mathcal{H})$-equivalence. For $i \in \mathcal{C}$, letting $A\{i,i\} = (g_i)$, we have

$$
(I - B)\{i,i\} = (I - B^{<0>})\{i,i\} = U\{i,i\}(I - A)\{i,i\}V\{i,i\} = ((1)(1 - g_i)(1)).
$$

Therefore (2a) holds for $B$. Given $\{i,j\} \subset \mathcal{C}$, let $a, b, u, v$ denote the entries of the singleton $\{i,j\}$ subblocks of $A, B, U, V$. For $D \in \mathcal{R}_{ij}$,

$$
\pi_D((1 - b)) = \pi_D((I - B^{<0>})\{i,j\}) = \pi_D(U\{i,i\}(I - A)\{i,i\}V\{i,i\}) = \pi_D((1 - a) + u(1 - g_j) + (1 - g_i)v).
$$

Clearly $\pi_D(u(1 - g_j)) = 0 = \pi_D((1 - g_i)v)$, and therefore $\pi_D(1 - b) = \pi_D(1 - a)$, and therefore (2b) holds for $B$. $\square$

Remark 9.2. In Theorem [9.1] $I - B^{<0>}$ is a 1-stabilization of the matrix $L = I - B$. The realization can be stated in terms of 1-stabilizations of a matrix $L$ by replacing “$B$ has the form $g$” in 2(a) with “$L$ has the form $1 - g$”, and replacing $\sum_{g \in D} n_g > 0$ with $\sum_{g \in D} n_g < 0$ in 2(b).

Lastly, we prove a finiteness result. Given $G$ an abelian group, $\mathcal{P} = \{1, \ldots, N\}$ a poset and $A \in \mathcal{M}_P^\ast(\mathbb{Z}_4, G)$, let $A_k$ be the $k$th diagonal block of $A$ and let $d(A)$ be the $N$-tuple $(\det(I - A_1), \ldots, \det(I - A_N))$. Up to reordering, $d(A)$ is an invariant of $G$-flow equivalence of the $G$-SFT $T_A$ defined by $A$.

Theorem 9.3. Let $G$ be a finite abelian group, $\mathcal{P} = \{1, \ldots, N\}$ a poset, $\mathcal{H}$ a $(G, \mathcal{P})$ coset structure, and $d = (d_1, \ldots, d_N)$ an $N$-tuple of elements of $\mathbb{Z}G$ which are regular. Then there are only finitely many $\text{El}_P(\mathcal{H})$-equivalence classes of matrices in the set $\mathfrak{M}(d) := \{I - A : A \in \mathcal{M}_P^\ast(\mathcal{H}), d(A) = d\}$. Consequently there are only finitely many flow equivalence classes of $G$-SFTs $T_A$ with $d(A) = d$.

Proof. For $A$ in $\mathfrak{M}(d)$, the set $\mathcal{C}$ of cycle components must be empty, and for each $i$, the matrix $I - A_i$ is injective and the $\mathbb{Z}H_i$-module $\text{cok}(I - A_i)$ has finite size, determined by $\det(I - A_i)$. We will use
some facts from \cite{12} Section 9, which contains more detail. A theorem of Fitting shows that if \( I - A \) and \( I - B \) are injective matrices over \( \mathbb{Z}H_i \) with isomorphic cokernels, then there are \( m, n \) such that \( (I - A) \oplus I_m \) and \( (I - B) \oplus I_n \) are \( \text{GL}(\mathbb{Z}H_i) \)-equivalent \cite{12} Lemma 9.1]. Because \( H_i \) is finite abelian, the group \( \text{SK}_1(\mathbb{Z}H_i) \) is finite \cite{22}; then by \cite{12} Corollary 9.9], there are only finitely many \( \text{El}(\mathbb{Z}H_i) \)-equivalence classes of matrices with determinant the regular element \( d_i \). Given such choices for \( 1 \leq i \leq N \), fix \( A \) in \( \mathcal{M}_p^{\mathbb{Z}}(n, \mathcal{H}) \) with diagonal blocks \( I - A_i \) in the given \( \text{El}(\mathbb{Z}H_i) \) classes.

Suppose \( B \in \mathcal{M}_p^{\mathbb{Z}}(\mathcal{H}) \) with \( I - A_i \) and \( I - B_i \) \( \text{El}(H_i) \)-equivalent for each \( i \). We first claim that \( I - B \) is \( \text{El}(\mathcal{H}) \)-equivalent to a matrix in \( \mathcal{M}_p^{\mathbb{Z}}(n, \mathcal{H}) \) with the same diagonal blocks as \( I - A \). To show this, for each \( i \) let \( k(i), \ell(i), m(i) \) be nonnegative integers such that there are \( U_i, V_i \) in \( \text{El}(m_i, H_i) \) such that

\[
(I - A_i) \oplus I_{k(i)} = U_i ((I - B_i) \oplus I_{\ell(i)}) V_i .
\]

Let \( m = (m_1, \ldots, m_N) \) and let \( A', B' \) be the stabilizations of \( A, B \) in \( \mathcal{M}_p^{\mathbb{Z}}(m, \mathcal{H}) \). Let \( U = U_1 \oplus \cdots \oplus U_N \) and \( V = V_1 \oplus \cdots \oplus V_N \). Set \( I - C = U(I - B') V \). Then \( I - C \) and \( I - B' \) are \( \text{El}(m, \mathcal{H}) \)-equivalent and the \( i \)th diagonal block of \( (I - C) \) equals \( (I - A_i) \oplus I_{k(i)} \). After adding multiples of rows and columns from the \( I_{k(i)} \), we may produce a matrix \( I - D \), \( \text{El}(m, \mathcal{H}) \)-equivalent to \( I - B \), such that \( D \) is zero outside its principal submatrix \( (P) \) on the indices used to define \( A \). Now \( I - P \) is \( \text{El}(\mathcal{H}) \)-equivalent to \( I - B \) and its diagonal blocks equal those of \( I - A \).

To finish, it suffices to show \( I - P \) is \( \text{El}(n, \mathcal{H}) \)-equivalent to a matrix with bounded entries. The \( i \)th diagonal block of \( I - P \) is the \( n_i \times n_i \) matrix \( I - A_i \). Let \( \mathcal{R}_i \) be the image of the space of row vectors \( (\mathbb{Z}H_i)^{n_i} \) under the map \( v \mapsto v(I - A_i) \). Let \( \kappa_i \in \mathbb{N} \) be the index of \( \mathcal{R}_i \) in \( (\mathbb{Z}H_i)^{n_i} \). Then \( \mathcal{R}_i \) contains \( \kappa_i(\mathbb{Z}H_i)^{n_i} \). In the order \( j = 2, 3, \ldots, N \) do the following: for \( i < j \), as needed, multiply \( I - P \) from the left by matrices of \( \text{El}(n, \mathcal{H}) \) which are equal to \( I \) outside the \( ij \)th block to reduce all \( \mathbb{Z} \) coefficients in that block to lie in the interval \( [0, \kappa_j) \). This shows \( I - P \) is \( \text{El}(n, \mathcal{H}) \)-equivalent to one of a bounded set of matrices, as required.

\section*{Appendix A. Cohomology as positive equivalence}

The next proposition was proved in \cite{12}, with (much) worse control over \( m \), using the positive K-theory polynomial strong shift equivalence equations from \cite{13}. The elementary argument below gives a better bound on \( m \); and for the proof of Theorem 6.2 we use the case where \( m_i \) is controlled to be \( n_i \). The identity element of \( G \) is denoted \( e \).

\textbf{Proposition A.1.} Suppose \( D \) is an \( n \times n \) diagonal matrix over \( \mathbb{Z}_+ G \) such that for each \( s \), \( D(s, s) = g_s \in G \). Suppose \( A \) is an \( n \times n \) matrix
over \( \mathbb{Z}_+ \) and \( B = D^{-1}AD \). Then there is an \( m \leq n + 1 \) and \( m \times m \) stabilizations \( A', B' \) of \( A, B \) such that there is a positive \( \mathbb{Z}_G \) equivalence \( (I - A') \rightarrow (I - B') \).

Now suppose in addition that \( A \in \mathcal{M}_{\mathcal{P}}(C, n, \mathcal{H}) \) and for all \( i \) in \( \mathcal{P} \) that \( g_s \in H_i \) whenever \( s \in I_i \). Then \( B \in \mathcal{M}_{\mathcal{P}}(C, n, \mathcal{H}) \), and there are \( m \) and stabilizations \( A', B' \) of \( A, B \) in \( \mathcal{M}_{\mathcal{P}}(C, m, \mathcal{H}) \) such that there is a positive \( \text{El}_P(m, \mathcal{H}) \)-equivalence \( (I - A') \rightarrow (I - B') \). The vector \( m \) can be chosen such that for all \( i \) in \( \mathcal{P} \),

1. \( m_i \leq n_i + 1 \), and
2. if \( g_s = e \) for all \( s \in I_i \) then \( m_i = n_i \).

**Proof.** In the second case, \( B \) will be in \( \mathcal{M}_{\mathcal{P}}(C, n, \mathcal{H}) \) because \( \mathcal{H} \) is a coset structure.

We will describe given \( s \) a positive \( \mathbb{Z}_G \)-equivalence which has the effect of multiplying row \( s \) from the left by \( g_s^{-1} \) and multiplying column \( s \) from the right by \( g_s \). The equivalence will satisfy the stabilization bounds and in the second case be a positive \( \text{El}_P(\mathcal{H}) \)-equivalence. Applying such an equivalence for each \( s \) proves the proposition. For concreteness, suppose \( s = 1 \) and \( g_1 = g \).

We will describe the equivalence as a finite sequence of the row and column cuts from Section 3. We first consider the special case that \( A(1, 1) = 0 \). The target matrix will be named \( A' \). To lighten the notation (avoiding no technical difficulty), we will suppose a nonzero entry of \( A \) is a single element of \( G \); e.g., \( A(s, 1) = a \) means an edge from vertex \( s \) to vertex 1 is labeled by \( a \), as in the graph I below.

\[
(A.2) \quad \begin{array}{ccc}
    s & t & t \\
    a & b & b \\
    1 & 1 & 1
\end{array}
\]

Multiplying \( I - A \) from the left by \( E_{s1}(a) \) effects the row cut of the edge from \( s \) to 1 labeled \( a \) in I and produces the change I \( \rightarrow \) II. Do this for every edge into 1, producing a graph in which 1 has no incoming edge. Then column cut every edge out of 1; the effect is to remove those edges, as in III, leaving 1 an isolated vertex. Let \( A'' \) be the matrix produced from \( A \) by these moves. Note that applying this procedure to the matrix \( A' \) produces the same matrix \( A'' \):

\[
\begin{array}{ccc}
    s & t & t \\
    ag^{-1} & gb & gb \\
    1 & 1 & 1
\end{array}
\]
The positive equivalence for $A \rightarrow A''$ postcomposed with the inverse of the positive equivalence for $A' \rightarrow A''$ gives the required positive equivalence $A \rightarrow A'$ for this case.

For the case that $A(1, 1) = c \neq 0$, we introduce an additional isolated vertex, named $v$. (If for some vertex $s$ there is no edge from $s$ to itself, then by applying the move corresponding to I $\rightarrow$ III in (A.2) we could isolate $s$ and avoid increasing the number of vertices.) The argument again is described by a finite sequence of evolving graphs.

Here is a list of the corresponding positive equivalences.

- II $\rightarrow$ I. Column cut the edge $v \rightarrow 1$.
- III $\rightarrow$ II. Row cut the edge $1 \rightarrow v$.
- III $\rightarrow$ IV. Row cut the edge $v \rightarrow 1$.
- IV $\rightarrow$ V. Row cut all incoming edges to $1$.
- V $\rightarrow$ VI. Column cut all outgoing edges from $1$.
- VII $\rightarrow$ VI. Row cut each outgoing edge from $v$ to a different vertex.

At this point, the move from I to VII in (A.3) has replaced the given matrix $A$ with a matrix $A''$ which satisfies our conditions, except that the vertex $v$ is playing in $A''$ the role we require for vertex $1$. To remedy
this, apply the procedure $I \to V I I$ above to $A''$, but with $(s, t, v, 1, e)$ in place of $(s, t, 1, v, g)$. We end up with the required matrix $A'$, with the additional isolated vertex $v$ (i.e., row $v$ and column $v$ of $A'$ are zero).

\[\square\]

**Appendix B. Permutation similarity as positive equivalence**

Suppose $A$ is an $n \times n$ matrix over $\mathbb{Z}_+G$ and $P$ is an $n \times n$ permutation matrix and $B = P^{-1}AP$. Then $A, B$ are elementary strong shift equivalent over $\mathbb{Z}_+G$, as $B = (P^{-1})(AP)$ and $A = (AP)(P^{-1})$, and therefore $A$ and $B$ are $\mathbb{Z}G$-equivalent positive equivalence \[13\]. In the next proposition we show that we can obtain this positive equivalence through $(n + 1) \times (n + 1)$ matrices. We also show that if $A \in \mathcal{M}_p^o(n, \mathcal{H})$ and $P \in \mathcal{M}_P(n, \mathbb{Z}_+G)$, then we get a positive $\text{El}_P(\mathcal{H})$-equivalence $(I - A) \to (I - B)$.

**Proposition B.1.** Let $A, B, P, G$ be as above. Suppose there is an index $s$ with $A(s, s) = 0$. Then there is a positive $\mathbb{Z}G$-equivalence from $A$ to $B$ through $n \times n$ matrices. In any case there are stabilizations $A', B'$ of $A, B$ which are positive $\mathbb{Z}G$-equivalent through $(n + 1) \times (n + 1)$ matrices.

Now suppose in addition that $A \in \mathcal{M}_p^o(C, n, \mathcal{H})$ and $P \in \mathcal{M}_P(n, \mathbb{Z}_+G)$. Then $B \in \mathcal{M}_p^r(C, n, \mathcal{H})$, and there are $m$ and stabilizations $A', B'$ of $A, B$ in $\mathcal{M}_p^o(C, m, \mathcal{H})$ such that there is a positive $\text{El}_P(m, \mathcal{H})$-equivalence $(U, V) : (I - A') \to (I - B')$. The vector $m$ can be chosen such that $m_i \leq n_i + 1$ for all $i \in P$. If $P \{i, i\} = I$ where $i \in P$, then $U$ and $V$ can be chosen such that $U \{i, i\}$ and $V \{i, i\}$ are the identity matrix.

**Proof.** Assume first that there is an index $s$ with $A(s, s) = 0$. Let $t$ be an index different from $s$. We will describe a positive $\mathbb{Z}G$-equivalence which has the effect of permuting $s$ and $t$. If $t_1, t_2$ are arbitrary indexes, then we get a positive $\mathbb{Z}G$-equivalence which has the effect of permuting $t_1$ and $t_2$ by first permuting $s$ and $t_1$, then permuting $t_1$ and $t_2$, and then finally permuting $t_2$ and $s$. Since every permutation of $\{1, \ldots, n\}$ is the product of transpositions, it will follow that there is a positive $\mathbb{Z}G$-equivalence $(I - A) \to (I - B)$.

The procedure described in [A.2] shows that there is a positive $\mathbb{Z}G$-equivalence $(I - A) \to (I - A_1)$ such that $s$ is an isolated index in $G_{A_1}$. If also $A(t, t) = 0$, then there is a positive $\mathbb{Z}G$-equivalence $(I - A) \to (I - A_2)$ such that $t$ is an isolated index in $G_{A_2}$, and if we then postcompose the equivalence $(I - A) \to (I - A_2)$ with its inverse but with the role of $s$ and $t$ interchanged, then we get a positive $\mathbb{Z}G$-equivalence $(I - A) \to (I - A_3)$ where $A_3$ is the matrix obtained from $A$ by permuting $s$ and $t$. If $A(t, t) \neq 0$, then the procedure described in [A.3] with $g = e$ shows that there is a positive $\mathbb{Z}G$-equivalence $(I - A_1) \to (I - A'_2)$ where $A'_2$ is obtained from $A_2$ by permuting $s$ and $t$. By postcomposing with the
in inverse of the equivalence \((I - A) \to (I - A_1)\) but with the role of \(s\) and \(t\) interchanged, we get a positive \(\mathbb{Z}G\)-equivalence \((I - A) \to (I - A_3)\) where \(A_3'\) is the matrix obtained from \(A\) by permuting \(s\) and \(t\).

If there is no index \(s \in \mathcal{G}_A\) with \(A(s, s) = 0\), then we add a zero row and a zero column to \(A\) and \(B\) to obtain matrices \(A'\) and \(B'\), and then it follows from the argument above that there is a positive \(\mathbb{Z}G\)-equivalence \((I - A') \to (I - B')\).

Now suppose in addition that \(A \in M^s_P(C, n, H)\) and \(P \in M_P(n, \mathbb{Z}G)\). Then \(P\{i, j\} = 0\) if \(i \neq j\). It follows that \(B \in M^s_P(C, n, H)\). We let \(P_i^s\) denote the matrix in \(M_P(n, \mathbb{Z}G)\) such that \(P_i^s\{i, i\} = P\{i, i\}\), \(P_i^s\{j, j\} = I\) for \(j \neq i\), and \(P_i^s\{i', j'\} = 0\) for \(i' \neq j'\). Then \(P = P_{i_1}^s P_{i_2}^s \cdots P_{i_n}^s\). Let \(A_1 = (P_{i_1}^s)^{-1} A P_{i_1}^s\), \(A_2 = (P_{i_2}^s)^{-1} A_1 P_{i_2}^s\), \ldots, \(A_N = (P_{i_N}^s)^{-1} A_{N-1} P_{i_N}^s = B\). By the first half of the proposition, there are stabilizations \(A', A_1', A_2', \ldots, A'_N = B'\) of \(A, A_1, A_2, \ldots, A_N = B\) and positive \(\mathbb{Z}G\)-equivalences

\[(I - A') \to (I - A'_1) \to (I - A'_1) \to (I - A'_2) \to \cdots \to (I - A'_N) = (I - B')\]

and that \(B'\) can be chosen such that \(B' \in M^s_P(C, n, H)\) with \(m_i \leq n_i + 1\) for all \(i \in \mathcal{P}\). It is not difficult to check that the described equivalence \((U, V) : (I - A') \to (I - B')\) is a positive \(El_P(n, H)\)-equivalence, and that if \(P\{i, i\} = I\) where \(i \in \mathcal{P}\), then \(U\) and \(V\) can be chosen such that \((U\{i, i\}\) and \(V\{i, i\}\) are the identity matrix. \(\square\)

**Appendix C. Resolving extensions**

**Proposition C.1.** Suppose \(A\) is a matrix in \(M^s_P(C, n, H)\) and \(A'\) is a matrix obtained from \(A\) by splitting a row \(s\) into two rows. Let the rows of \(A'\) be in the same order as corresponding rows of \(A\), with the interpolation of a new row \(s'\) directly following \(s\). Let \(A'\) have the natural \(\mathcal{P}\) blocking: \(s'\) is in the block of \(s\), and every other index is in the block of the row from which it was copied. Let \(\tilde{A}\) be the matrix of size and blocking from \(n'\) obtained by interpolating a zero \(s'\) row and column into \(A\).

Then \(\tilde{A}, A' \in M^s_P(C, n', H)\) and there is a positive \(El_P(n', H)\)-equivalence \((U, V) : (I - \tilde{A}) \to (I - A')\). If \(A\) is upper triangular and \(A(s, s) = A'(s, s)\), then \(A'\) is upper triangular and the matrices \(U, V\) can be chosen to be unipotent upper triangular.

Moreover, the same conclusion holds if in the above statements “row” is replaced by “column” and “following” is replaced by “preceding”.

**Proof.** Let us first check that \(A' \in M^s_P(C, n', H)\) (it is obvious that \(\tilde{A} \in M^s_P(C, n', H)\)). It is easy to check that \(A' \in M^s_P(C, n, \mathbb{Z}G) \cap M_P(n, H)\), so we just need to show that \(H\) is a \((G, \mathcal{P})\) coset structure for \(A'\). Since \(H\) is a \((G, \mathcal{P})\) coset structure for \(A\), there is a family of vertices \(\{v(i)\}_{i \in \mathcal{P}}\) such that \(v(i)\) belongs to the irreducible core of \(A\{i, i\}\) for each \(i \in \mathcal{P}\), and \(H_{ij}\) is the set of weights of paths from \(v(i)\)
to \(v(j)\) in \(\mathcal{G}_A\). Let \(i, j \in \mathcal{P}\). We aim to show that the set of weights of paths from \(v(i)\) to \(v(j)\) in \(\mathcal{G}_A'\) is equal to \(H_{ij}\).

Notice that if \(p\) is a path in \(\mathcal{G}_A\) not starting at \(s\), then there is a path in \(\mathcal{G}_A'\) starting and ending at the same vertices as \(p\) and with the same weight as \(p\). Notice also that if \(p\) is a path in \(\mathcal{G}_A'\) starting at \(s\) and ending at the same vertex as \(p\) and with the same weight as \(p\). Similarly, if \(p\) is a path in \(\mathcal{G}_A'\), then there is a path in \(\mathcal{G}_A\) which has the same weight as \(p\) and which starts and ends at the same vertices as \(p\) (except of course if \(p\) starts/ends at \(s'\) in which case the path in \(\mathcal{G}_A\) starts/ends at \(s\) instead).

It follows that if \(v(i) \neq s\), then the set of weights of paths from \(v(i)\) to \(v(j)\) in \(\mathcal{G}_A'\) is equal to \(H_{ij}\).

Suppose that \(v(i) = s\) and that \(p\) is a path in \(\mathcal{G}_A\) starting at \(s\) and that there is a path in \(\mathcal{G}_A'\) starting at \(s'\) and ending at the same vertex as \(p\) and with the same weight as \(p\). Suppose that there is a path from \(s\) to \(s'\) in \(\mathcal{G}_A'\) (if there is no path in \(\mathcal{G}_A\), then there must be a path from \(s'\) to \(s\) because \(s = v(i)\) in the irreducible core of \(A\{i, i\}\), and we just interchange the role of \(s\) and \(s'\)). Let \(\gamma\) be the weight of this path. Since the set of weights of paths in \(\mathcal{G}_A'\) from \(s\) to \(s\) is equal to the set of weights of paths in \(\mathcal{G}_A\) from \(s\) to \(s'\) and is a group (because it is a finite semigroup), it follows that there is a path in \(\mathcal{G}_A'\) starting at \(s\) and ending at the same vertex as \(p\) and with the same weight as \(p\). It follows that the set of weights of paths from \(s = v(i)\) to \(v(j)\) in \(\mathcal{G}_A'\) is equal to \(H_{ij}\). This shows that \(H\) is a \((G, \mathcal{P})\) coset structure for \(A'\) and thus that \(A' \in M_{\mathcal{P}}(\mathcal{C}, n', \mathcal{H})\).

We then show that there is a positive \(\text{El}_P(n', \mathcal{H})\)-equivalence \((U, V) : I - \tilde{A} \to I - A'\). We write \(A\) in a 3 \(\times\) 3 block form, giving

\[
A = \begin{pmatrix}
qu & r & t \\
u & v & w \\
x & y & z
\end{pmatrix}, \quad \tilde{A} = \begin{pmatrix}
qu & r & 0 & t \\
u & v & 0 & w \\
x & y & 0 & z
\end{pmatrix}.
\]

(We omit the easier proof for the case that \(s\) is a first or last index of \(A\), and the block form is smaller.) The central index set of \(A\) is \(\{s\}\), so \(v\) is the 1 \(\times\) 1 matrix \(A(s, s)\). The matrix \(A'\) then has the block form

\[
A' = \begin{pmatrix}
qu & r & r & t \\
u_1 & v_1 & v_1 & w_1 \\
u_2 & v_2 & v_2 & w_2 \\
x & y & y & z
\end{pmatrix}.
\]
with $A'$ nonnegative and $(u_1, v_1, w_1) + (u_2, v_2, w_2) = (u, v, w)$. We then have a string of positive equivalences:

$$I - \tilde{A} \rightarrow E_1(I - \tilde{A}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - q & -r & 0 & -t \\ -u & 1 - v & 0 & -w \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 - z \end{pmatrix}$$

$$= \begin{pmatrix} 1 - q & -r & 0 & -t \\ -u & 1 - v & -1 & -w \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 - z \end{pmatrix} := I - A_1.$$

$$I - A_1 \rightarrow (I - A_1)E_2 = \begin{pmatrix} 1 - q & -r & 0 & -t \\ -u & 1 - v & -1 & -w \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 - z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -u_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - q & -r & 0 & -t \\ -u_1 & 1 - v & -1 & -w \\ -u_2 & 0 & 1 & 0 \\ -x & -y & 0 & 1 - z \end{pmatrix} := I - A_2.$$

$$I - A_2 \rightarrow (I - A_2)E_3 = \begin{pmatrix} 1 - q & -r & 0 & -t \\ -u_1 & 1 - v & -1 & -w \\ -u_2 & 0 & 1 & 0 \\ -x & -y & 0 & 1 - z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -v_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - q & -r & 0 & -t \\ -u_1 & 1 - v_1 & -1 & -w \\ -u_2 & -v_2 & 1 & 0 \\ -x & -y & 0 & 1 - z \end{pmatrix} := I - A_3.$$

$$I - A_3 \rightarrow (I - A_3)E_4 = \begin{pmatrix} 1 - q & -r & 0 & -t \\ -u_1 & 1 - v_1 & -1 & -w \\ -u_2 & -v_2 & 1 & 0 \\ -x & -y & 0 & 1 - z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -w_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - q & -r & 0 & -t \\ -u_1 & 1 - v_1 & -1 & -w_1 \\ -u_2 & -v_2 & 1 & -w_2 \\ -x & -y & 0 & 1 - z \end{pmatrix} := I - A_4.$$

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This exhibits the equivalence \((U, V) : I - \tilde{A} \to I - A'\), with \(U = E_1\) and \(V = E_2E_3E_4E_5\). It is clear that \(E_1, E_2, E_3, E_4, E_5 \in \text{El}_p(\mathbf{n}', \mathcal{H})\), and it is not difficult to check that \(A_1, A_2, A_3, A_4 \in \mathcal{M}_P^0(\mathbf{n}', \mathcal{H})\). It follows that \((U, V) : I - \tilde{A} \to I - A'\) is a positive \(\text{El}_p(\mathbf{n}', \mathcal{H})\)-equivalence.

In general the matrices \(U, V\) will not be upper triangular, because in general \(E_2\) and \(E_3\) are not upper triangular. However, if \(A\) is upper triangular, then \(u = 0\), so \(u_1 = u_2 = 0\); and if \(A'(s, s) = A(s, s)\), then \(v_1 = v\), so \(v_2 = v - v_1 = 0\). Thus under the additional assumptions, \(A'\) is upper triangular and the matrices \(U = E_1\) and \(V = E_2E_3E_4E_5\) are unipotent upper triangular as required.

The argument for the “Moreover” claim is essentially the same, and we omit it. \(\square\)

**Appendix D. G-SFTs following Adler-Kitchens-Marcus**

The purpose of this appendix is to integrate our matrix approach with the classification of \(G\)-SFTs with the group actions framework of Adler-Kitchens-Marcus [2] [4]. This appendix is not necessary for the statements or proofs of the flow equivalence results of earlier sections. Throughout, \(G\) is a finite group.

We make no conceptual advance on [2] (which in turn acknowledges a huge debt to the ergodic-theoretic work [27] of Rudolph). Still, we give a detailed presentation of this framework (with some additional details), as the ideas in [2] are intermingled with that paper’s focus on almost topological conjugacy, and the paper does not isolate all the explicit statements we want. The paper [23] of Parry also covers much, but not all, of this framework.

A matrix \(A\) is \(G\)-\textit{primitive} if its entries lie in \(\mathbb{Z}_+G\) and in addition there is a positive integer \(m\) such that every entry of \(A^m\) is \(G\)-positive. An SFT is \textit{nonwandering} if it has no wandering orbit; equivalently, it is the disjoint union of finitely many irreducible SFTs (its \textit{irreducible components}). A nonwandering/irreducible/mixing \(G\)-SFT is a \(G\)-SFT \((Y, T)\) which as an SFT is nonwandering/irreducible/mixing. A nonwandering \(G\)-SFT was defined to be \(G\)-\textit{transitive} [1] Section 4] if the
$G$ action on irreducible components is transitive. A nonwandering $G$-SFT is $G$-transitive if and only if the canonical factor map collapsing $G$-orbits maps each irreducible component onto the same irreducible SFT. Clearly the classification of nonwandering $G$-SFTs reduces to the classification of $G$-transitive nonwandering $G$-SFTs.

Let $G$ be a finite group and let $(Y, T)$ be a nonwandering $G$-transitive $G$-SFT. We take this left $G$-SFT $(Y, T)$ to be $Y = X \times G$ with $T : (x, g) \mapsto (g\sigma x, g\tau(x))$ with $\sigma : X \to G$ continuous, and left $G$ action by $g : (x, h) \mapsto (x, gh)$, as in Section 3.3.1 (recall our Standing Convention 3.3.1). Let $C$ be an irreducible component of $Y$, with cyclically moving subsets $C^0, \ldots, C^{p-1}$. For $g \in G$, let $gC := \{(x, gh) : (x, h) \in C\}$. Then $gC$ is an irreducible component of $Y$. The map $(x, h) \mapsto (x, gh)$ sending $C$ to $gC$ is a topological conjugacy of SFTs (but not of $G$-SFTs, when $G$ is not abelian). The stabilizer of $C$ is the subgroup $H_C = H = \{g \in G : gC = C\}$. For $g \in G$, we have $H_{gC} = gH_Cg^{-1}$.

The next result explains how to reduce the classification of nonwandering $G$-SFTs to the classification of irreducible $K$-SFTs, for subgroups $K$ of $G$.

**Theorem D.1.** Suppose $G$ is a finite group and $(Y, T)$ and $(Y', T')$ are nonwandering $G$-transitive $G$-SFTs, containing irreducible components $C, C'$ (resp.). Let $H$ denote the stabilizer $H_C$ of $C$. Then the following are equivalent.

1. $(Y, T)$ and $(Y', T')$ are $G$-conjugate.
2. There exists $g \in G$ such that the stabilizer of $gC'$ is $H$ (i.e., the stabilizers of $C$ and $C'$ are conjugate subgroups in $G$) and the irreducible $H$-SFTs $C$ and $gC'$ are $H$-conjugate.

**Proof of Theorem D.1.** (1) $\implies$ (2): A $G$-conjugacy $(Y, T) \to (Y', T')$ will restrict to an $H$-conjugacy from $C$ to some irreducible component $D$ of $Y'$ with stabilizer $H_D = H$. By the $G$-transitivity, there is some $g \in G$ such that $D = gC'$.

(2) $\implies$ (1):

Let $D = gC'$ and let $\varphi_C : C \to D$ be a conjugacy of $H$-SFTs. Pick elements $g_i$ of $G$, $0 \leq i < |G/H|$, such that $g_0 = e$ and the $g_i H$ are the distinct left cosets of $H$ in $G$. Define $\varphi : G \to G$ by setting $\varphi(g,x) = g_i \varphi_C(x)$ for $g_i x$ in $g_i C$. Then for $x \in C$ and $g = gh \in g_i H$, we have

$$\varphi(gx) = \varphi(g_i hx) = g_i \varphi_C(hx) = g_i h \varphi_C(x) = g \varphi(x).$$

For $g_i x \in g_i C$ and $g \in G$, it follows that $\varphi(g(g_i x)) = g_0 \varphi(x) = g \varphi(g_i x)$. Thus, $\varphi g = g \varphi$ for all $g$. Then, $\varphi T = T\varphi$, because for $g_i x \in g_i C$,

$$\varphi(T(g_i x)) = \varphi(g_i (Tx)) = g_i \varphi_C(Tx) = g_i T' \varphi_C(x) = g T'(\varphi_Cx) = T'(g_i (\varphi_Cx)) = T'(\varphi(g_i x)).$$
For a reduction of the classification of irreducible $G$-SFTs to a structure on mixing SFTs, we continue the Adler-Kitchens-Marcus analysis. Suppose $H$ is a finite group and $T : Y \to Y$ is an irreducible $H$-SFT (we use $H$ to match letters in [2]) with period $p > 1$. Let $C^0, \ldots, C^{p-1}$ denote the cyclically moving subsets of $Y$: the $C^i$ are disjoint; $T$ maps $C^i$ onto $C^{i+1}$ (superscripts interpreted mod $p$); and for each $i$, the restriction of $T^p$ to $C^i$ is a mixing SFT. For $0 \leq i < p$, set

$$H^i_C = H^i = \{ g \in G : gC^0 = C^i \} = \{ g \in H : gC^0 \cap C^i \neq \emptyset \} = \{ g \in G : gC^j = C^i \}, \text{ for any } j \in \{0, 1, \ldots, p - 1\}.$$

$H^0$ is a normal subgroup of $H$. The sets $H^i$ are disjoint, with union $H$; each $H^i$, if nonempty, is a left coset of $H^0$. Identifying $H^0$ and $H^p$, define $\kappa_T = \min \{ i \in \mathbb{N} : 0 < i \leq p, H^i \neq \emptyset \}$. Then $\kappa_T$ divides $p$, and for $0 < i < p$, $H^i$ is nonempty if and only if $\kappa_T$ divides $i$.

It can happen that $H^1$ is empty (i.e., $\kappa_T > 1$). For example, if $H = \{ e \}$ and $T$ is a single orbit of length $p$, then $\kappa_T = p$. If $H = \mathbb{Z}_2 = \{ e, g \}$, $T$ is a single orbit of length 6, and $g$ acts by $T^3$, then $H^0 = \{ e \}, H^3 = \{ g \}$ and $\kappa_T = 3$. However, there is a straightforward interpretation of $\kappa_T$, as follows. (Recall, for a property $P$, $T$ is totally $P$ if $T^k$ is $P$ for all $k > 0$.)

**Proposition D.2.** Suppose $(Y, T)$ is an irreducible $H$-SFT. Let $\overline{T}$ be the irreducible SFT which is the factor of $T$ obtained by collapsing $H$-orbits. Then $\kappa_T$ is the period of $\overline{T}$. The following are equivalent.

1. $\kappa_T = 1$.
2. $T$ is totally $H$-transitive.
3. $\overline{T}$ is mixing.

**Proof.** For (1) $\iff$ (2), note $H$-transitivity of all powers of $T$ is equivalent to transitivity of the $H$-action on cyclically moving subsets, which is equivalent to $\kappa_T = 1$. For (2) $\iff$ (3), note every power of $T$ is $H$-transitive if and only if every power of $\overline{T}$ is irreducible. This happens if and only if the irreducible SFT $\overline{T}$ is mixing.

Now let $k$ be the period of $\overline{T}$. $\kappa_T$ is the smallest positive integer $j$ such that $T^j$ is the disjoint union of $j$ irreducible SFTs, each of which has $\kappa = 1$. For $j < k$, irreducible components of $\overline{T}^j$ are not mixing, so irreducible components of $T^j$ cannot have $\kappa = 1$. On the other hand, $(\overline{T})^k$ is the disjoint union of $k$ mixing SFTs, so $T^k$ is the disjoint union of $k$ $H$-SFTs, each of which has $\kappa = 1$. Thus $\kappa_T = k$. $\square$

\[\text{4}^{4}\text{The statements [2] Part (iii) of the p.22 Lemma and p.23 Corollary] neglect the possibility } H^1 = \emptyset, \text{ but the p.23 proof addresses it.}\]
Reduction to the case $\kappa_T = 1$ clarifies issues and simplifies notation. Note, for $k = \kappa_T$, the restriction of $T^k$ to any of its irreducible components is $H$-invariant and therefore an $H$-SFT.

**Proposition D.3.** Suppose $(Y, T)$ and $(Y', T')$ are irreducible $H$-SFTs with period $p > 1$, with $\kappa_T = \kappa_{T'} := k > 1$. Then the restriction of $T^k$ or $(T')^k$ to any of its irreducible components has $\kappa = 1$. If $C$ is an irreducible component for $T^k$, then a conjugacy of $H$-SFTs $T, T'$ restricts to a conjugacy of $H$-SFTs $T^k|C, (T')^k|C'$, for some irreducible component $C'$ of $T^k$. Conversely, any conjugacy of $H$-SFTs $T^k|C, (T')^k|C'$ extends uniquely to a conjugacy of $H$-SFTs $T, T'$.

**Proof.** We will prove the extension claim. Suppose $\varphi_0 : C \to C'$ is a conjugacy of $H$-SFTs $T^k|C, (T')^k|C'$. The unique extension of $\varphi_0$ to a topological conjugacy of $T, T'$ as SFTs is given by $\varphi : T^i x \mapsto (T')^i (\varphi_0 x)$, for $x \in C$ and $0 \leq i < k$. For $h \in H$, $0 \leq i < k$, $x \in X$ and $y = T^i x$, we then have $\varphi(hy) = \varphi(hT^i x) = \varphi(T^i h x) = (T')^i \varphi_0(hx) = h(T')^i \varphi_0(x) = h \varphi(T^i x) = h \varphi(y)$. □

Now suppose $T$ is an irreducible $H$-SFT of period $p > 1$ with $H^1 = H^1(T)$ nonempty. Pick $c$ in $H^1$. We have $H^0$ a normal subgroup of $H$, $H^i = c H^0$ and the disjoint sets $H^i$ are the cosets of $H^0$. We call the subgroup $H^0$ of $H$ the primitive stabilizer of $C$. We call $H^1$ the stabilizer coset ("primitive stabilizer coset" would be more accurate, but lengthier). Adler-Kitchens-Marcus are not responsible for the terms "primitive stabilizer" and "stabilizer coset".

Given irreducible $H$-SFTs $T, T'$ of period $p$, let $T_0, T'_0$ be mixing SFTs given by restriction of $T^p, (T')^p$ to some cyclically moving subset of $T, T'$. If $p > 1$ and $\kappa_T = 1$, then $T_0$ and $T'_0$ are not $H$-SFTs, but they are $H^0$-SFTs. However, a natural candidate reduction fails badly, as in the following example.

**Example D.4.** Let $H = \mathbb{Z}_2 = \{e, g\}$. We define two irreducible, period 2 $G$-SFTs, $T$ and $T'$, with $H^0 = \{e\}$ and $H^1 = \{g\}$, such that the restrictions $T_0, T'_0$ of $T^2$ and $(T')^2$ to irreducible components are conjugate $H^0$-SFTs, but $T$ and $T'$ are not conjugate as $H$-SFTs. Let $T = T_A$ for $A = g(\frac{1}{2} \frac{1}{2})$, and similarly define $T'$ from $A' = g(\frac{2}{1} \frac{1}{2})$. The SFTs $g^{-1}T$ and $g^{-1}T'$ are not conjugate, having different numbers of fixed points. But, $T_0$ and $T'_0$ are conjugate, because $A^2 = (A')^2$.

To find some reduction of the classification of irreducible $H$-SFTs to a classification defined on mixing SFTs, one is forced to consider the $\alpha$-skew $H$-SFTs. For this, we continue below to follow [2].

Suppose $H$ is a finite group and $\alpha : H \to H$ is a group automorphism. In [2], Adler, Kitchens and Marcus defined a $\mathbb{Z} \otimes_{\alpha} H$ action on an SFT $T$ (we will call this pair an $\alpha$-skew $H$-SFT) to be an embedding of $H$ as a group of homeomorphisms such that $Th = \alpha(h) T$ (i.e. $Th(x) = \alpha(h) T(x)$ for all $x$ and $h$). (The $H$-SFT case is the case that $\alpha$ is the
identity.) They showed (see \cite{2}, Observation 1, p.4) that an \( \alpha \)-skew \( H \)-SFT can be presented very concretely, as a one-step SFT with an embedding \( h \mapsto \pi_h \) of \( H \) into the group of permutations of the alphabet, such that for \( x = (x_n) \) and \( h \in H \), \( hx \) is defined by \( (hx)_n = \pi_{\alpha \circ h}(x_n) \).

They also showed (see \cite{2}, Theorem 1) these skew SFTs are abundant: for every \( H \) and \( \alpha \), every positive entropy irreducible SFT admits a (not necessarily free) \( \alpha \)-skew \( H \)-SFT, which (by \cite{2}, Theorem 3) is then a 1-1 a.e. factor of an \( \alpha \)-skew \( H \)-SFT which is free (the \( H \)-orbit of \( x \) has cardinality \( |H| \), for every \( x \)).

Let \( T \) be an irreducible \( H \)-SFT of period \( p > 1 \), with \( H^1(T) \) nonempty. Fix \( c \) in \( H^1(T) \), and define \( S = c^{-1}T \) (i.e., \( S(x) = c^{-1}T(x) \)). If \( c \) is in the center of \( H \), then \( S \) is an \( H \)-SFT; otherwise, it is not. Two \( \alpha \)-skew \( H \)-SFTs are by definition topologically conjugate if they are conjugate as SFTs by a conjugacy intertwining the \( H \) actions.

**Theorem D.5.** Suppose \( T \) is an irreducible \( H \)-SFT of period \( p > 1 \), with nonempty coset \( H^1(T) \). Fix \( c \) in \( H^1(T) \), and define \( S = c^{-1}T \). Let the cyclically moving subsets of \( T \) be \( C^i \), \( 0 < i < p \). Let \( S_i \) be the restriction of \( S \) to \( C^i \). Define \( \alpha : h \mapsto c^{-1}hc \) (with domain given by context). Then the following hold.

1. With the given \( H \) action, \( S \) is a free \( \alpha \)-skew \( H \)-SFT (and by restriction of the action, \( S \) is an \( \alpha \)-skew \( H_0 \)-SFT).
2. Each \( S_i : C^i \to C^i \) is a free mixing \( \alpha \)-skew \( H_0 \)-SFT.
3. Suppose \( T' \) is another irreducible \( H \)-SFT. Assume the period \( T' \) is also \( p \), and \( H^1(T') = H^1(T) = H^1 \neq \emptyset \). (These are necessary conditions for conjugacy of \( T, T' \) as \( H \)-SFTs.) Let \( C'^i \), \( 0 < i < p \), be the cyclically moving subsets of \( T' \). Define \( S'^i = c^{-1}T' \) and \( S'_i = S'^i|C'^0 \). Then the following hold.
   a. A conjugacy \( \varphi \) of \( H \)-SFTs \( T, T' \) restricts to a conjugacy of mixing \( \alpha \)-skew \( H_0 \) SFTs \( S_0, S'_0 \), for some \( i \). Then \( \varphi \circ T'^{-i} \) is a conjugacy of the mixing \( \alpha \)-skew \( H_0 \)-SFTs \( S_0, S'_0 \).
   b. Given a conjugacy \( \varphi_0 : C^0 \to C'^0 \) of the \( \alpha \)-skew \( H_0 \)-SFTs \( S_0, S'_0 \), there is a unique conjugacy \( \varphi \) of \( H \)-SFTs \( T, T' \) such that \( \varphi = \varphi_0 \) on \( C^0 \).

**Proof.** (1) Suppose \( c^r = e \). Because \( cS = Sc \), it follows that \( S^r = (c^{-1}T)^r = T^r \), an SFT. As a root of an SFT, \( S \) must be an SFT. Next, given \( h \in H \), we have \( S(hx) = c^{-1}T(hx) = c^{-1}hT(x) = c^{-1}hc^{-1}T(x) = \alpha(h)(Sx) \). \( S \) is free because an \( H \)-SFT is by definition (in this paper) free.

(2) Clearly \( C^i \) is mapped to \( C^i \) by \( S_i \) and \( H_0 \). Also, \( (S_i)^{pr} = (c^{-1}T)^{pr}|_{C^i} = (T^p)^{pr}|_{C^i} \). Roots and powers of mixing SFTs are mixing SFTs, so \( S_i \) is a mixing \( \alpha \)-skew \( H_0 \)-SFT.

(3) The claim (a) follows from parts (1) and (2). Now suppose \( \varphi_0 \) is given as in (b). For \( x \in C^0 \) and \( 0 \leq i < p \), define \( \varphi(c^ix) = c^i\varphi_0(x) \).
This φ is the only possible extension of φ₀ to a conjugacy of the $H$-SFTs $T, T'$.

We claim $S'φ = φS$. For this, suppose $y = c^i x$, with $x \in C^0$ and $0 \leq i < p$. Note $cS = Sc$ and $cS' = S'c$. So, we have $φ(Sy) = φ(S(c^i x)) = φ(c^iφ₀(Sx)) = c^iφ₀(Sx) = S'(c^iφ₀(x)) = S'φ(y)$.

Next, we claim $cφ = φc$. For $x \in C^i$ with $0 \leq i < p - 1$, this is clear. For $y = c^{p-1}x \in C^{p-1}$, we have $φ(cy) = φ₀(c^p x) = c^pφ₀(x) = c(c^{p-1}φ₀(x)) = cφ(c^{p-1}x) = cφ(y)$.

We now have $φT = T'φ$, because $φT = φcS = cφS = cS'φ = T'φ$. For $h$ in $H₀$ and $y \in Y$, it remains to check $φ(hy) = hφy$. Let $y = c^i x \in C^i, 0 \leq i < p$. Then $φ(hy) = φ(hc^i x) = φ(c^i (c^{-i}hc^i x)) = c^i φ₀(c^{-i}hc^i x) = c^i (c^{-i}hc^i φ₀x) = hc^i φ₀x = hφ(y)$.

Theorem D.5 reduces the problem of classifying irreducible $H$-SFTs, for all $H$, to the problem of classifying mixing α-skew $H$-SFTs (with $H$ acting freely), for all $α$ and $H$. For mixing $H$-SFTs, there is a reasonably satisfactory framework of invariants arising from a theory of strong shift equivalence of matrices over $\mathbb{Z}H$ (or elementary equivalence of matrices over $\mathbb{Z}[t]$). One naturally has an analogous (somewhat ill-defined) problem:

**Problem D.6.** Find a satisfactory classification scheme for mixing α-skew $H$-SFTs.

Finally, we turn to relating matrix properties to the Adler-Kitchens-Marcus setting. Let $A$ be a square matrix over $\mathbb{Z}_+G$ with augmentation $\overline{A}$ over $\mathbb{Z}_+$ as in Section 3. Let $a_{ij} = A(i, j)$ and define nonnegative integers $a_{ijkg}$ by $A^k(i, j) = \sum_{g \in G} a_{ijkg}g$; let $a_{ijg}$ denote $a_{ij1g}$.

We recall some terminology. $A$ is irreducible/primitive if $\overline{A}$ is irreducible/primitive. $A$ is nondegenerate if it has no zero row and no zero column. The nondegenerate core of $A$ is its maximum nondegenerate principal submatrix. For a property $P$, $A$ is essentially $P$ if its nondegenerate core is $P$. $A$ is $G$-primitive if there exists $k > 0$ such that $a_{ijkg} > 0$ for all $i, j, g$.

**Definition D.7.** Let $A$ be a square matrix over $\mathbb{Z}_+G$. For an index $i$ of $A$, the *weights group* (at $i$) is

$$W_i(A) = \{g \in G : \exists k > 0 \text{ such that } a_{ikg} > 0\}$$

and the *ratio group* (at $i$) is

$$Δ_i(A) = \{gh^{-1} : \exists k \text{ such that } a_{ijkg} > 0 \text{ and } a_{ijkh} > 0\} = \{gh^{-1} : \exists k \text{ such that } a_{ikg} > 0 \text{ and } a_{ikh} > 0\}.$$ 

$W_i(A)$ is clearly a finite semigroup, and hence a group. $Δ_i(A)$ is a group, because given $(g_2)^j = e$, we have

$$g_1h_1^{-1}g_2h_2^{-1} = (g_1h_1^{-1}g_2)(h_2h_1^{-1}).$$
\(\Delta_i(A)\) is named after a “ratio group” which plays an analogous role in
the theory of Markov shifts \[21, 25\].

Our next task is to compute the stabilizer data for \(T_A\) from the
matrix \(A\). First we recall a standard reduction; see the citation for a
proof.

**Proposition D.8.** \[12\] Proposition 4.4] Let \(A\) be an irreducible matrix
over \(\mathbb{Z}_+ G\). Let \(i\) be an index of \(A\) and let \(H = W_i(A)\). Then
there is a diagonal matrix \(D\) over \(\mathbb{Z}_+ G\) with each diagonal entry in \(G\) (i.e.,
\(D = I\)) such that every entry of \(DAD^{-1}\) lies in \(\mathbb{Z}_+ H\). (It suffices for
each \(j\) to set \(D(j,j) = g\) for some \(g\) such that for some \(k\), \(a_{ijk}g > 0\).)

An important technical point for proofs below is the following obser-
vation.

**Remark D.9.** Suppose \(A\) is an essentially irreducible square matrix
over \(\mathbb{Z}_+ G\), \((w,g), (x,h) \in X_\mathbb{X} G\), \(g = h\), and the initial vertices of
\(x_0\) and \(w_0\) are equal. Then \((w,g)\) and \((x,h)\) are in the same cyclically
moving subset of the same irreducible component of \(X_\mathbb{X} G\).

**Proposition D.10.** Let \(A\) be an irreducible matrix over \(\mathbb{Z}_+ G\). Let \(x\)
be a point in \(X_\mathbb{X}\) with \(x_0\) beginning at index \(i\). Let \(C\) be an irreducible
component of \(T_A\) containing \((x,e)\) and choose \(C^0, \ldots, C^{n-1}\) such that
\((x,e) \in C^0\). Then the following hold.

1. The stabilizer \(H_C\) is the weights group \(W_i(A)\).
2. The matrix \(B = DAD^{-1}\) over \(\mathbb{Z}_+ H\) from Theorem D.8 defines
   an irreducible H-SFT \(T_B\) which is G-cohomologous to the H-
   SFT \(T|C\).
3. The primitive stabilizer \(H^0_C\) is the ratio group \(\Delta_i(A)\).

**Proof.** (1): Suppose \(g \in H_C\); then \((x,g) \in C\). By irreducibility of \(C\),
there must then be some path \(z_0 \cdots z_{k-1}\) in \(X_\mathbb{X}\) from \(i\) to \(i\) with weight
\(g\). Therefore \(g \in W_i(A)\).

Conversely, suppose \(g \in W_i(A)\). Then there is \(k > 0\) and a periodic
point \(w\) in \(X_\mathbb{X}\) with \(\ell(w_0)\ell(w_1)\cdots\ell(w_{k-1}) = g\) (here \(\ell(w_n)\) denotes the
label of the edge \(w_n\) in \(G_A\), see Section 3) such that \(i\) equals the initial
vertex of \(w_0\) and the terminal vertex of \(w_{k-1}\). The point \((w,e)\) must be
in \(C\). Therefore \((w,g) \in C\). Thus \(gC \cap C \neq \emptyset\), so \(gC = C\) and \(g \in H_C\).

(2): The diagonal matrix \(D\) gives the \(G\)-cohomology. \(T_B\) is irre-
ducible because \(C\) is irreducible.

(3): Given \(g, h, k\) as in the definition of \(\Delta_i(A)\), using Remark D.9
one can see \(g\) and \(h\) are in \(H^0_C\), and therefore \(gh^{-1} \in H^0_C\). Conversely,
suppose \(g \in H^0_C\). Then \(g\) there is \(k > 0\) and a path \(w_0 \cdots w_{kp-1}\) from \(i\)
to \(i\) with weight \(g\). Then for arbitrarily large \(j > 0\) there is a path (by
concatenations) of length \(jp\) from \(i\) to \(i\) with weight \(g\). For sufficiently
large \(j\), there is also a path of length \(jp\) from \(i\) to \(i\) with weight \(e\),
because the \(H^0\)-SFT \((T_A)p|_{C^0}\) is mixing. \(\square\)
Proposition D.11. Let $G$ be a finite group and $A$ a square matrix over $\mathbb{Z}_+G$, defining a $G$-SFT $T_A$ as in Section 6. The following hold.

1. $A$ is essentially $G$-primitive $\iff$ $T_A$ is mixing.
2. $A$ is essentially irreducible with a weights group $W_i(A) = G$ $\iff$ the $G$-SFT $T_A$ is irreducible.
3. $A$ is essentially irreducible $\implies$ $T_A$ is $G$-transitive and non-wandering.
4. $A$ is essentially irreducible, with a weights group $W_i(A) = G$ and with $\overline{A}$ essentially primitive $\iff$ $T_A$ is irreducible with $\kappa = 1$.

Proof. (1) is [10, Cor. B.7]. (2) is clear. (3) follows from (2) and part (1) of Proposition D.10. Then (4) follows from Proposition D.2.

Finally, we give algorithmically a presentation (essentially following [25]) for the classifying $\alpha$-skew $H_0$-SFT, from a given presenting matrix $A$ over $\mathbb{Z}_+H$. For an example, let $\mathbb{Z}_2 = \{1, c\}$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $W_i(A) = \mathbb{Z}_2$, $\Delta_i(A) = \{1\} = H^0$ and $H^1 = \{c\}$; with $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\overline{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, we have and $DAD^{-1} = cA$.

Proposition D.12. Suppose $A$ is an irreducible matrix over $\mathbb{Z}_+H$, with $\overline{A}$ primitive, and $H$ equal to a weights group $W_i(A)$ such that $W_i(A) \neq \Delta(A)$ (i.e., $H \neq H^0$, so $T_A$ is not mixing). Let $\beta$ be the weight of a point of period $n + 1$ from $i$ to $i$; let $\gamma$ be the weight of a point of period $n$ from $i$ to $i$; let $c = \gamma^{-1}\beta$. Pick $N$ such that for each $j$, there is a path of length $N$ from $i$ to $j$; choose $d_j$ the weight of some such path. Let $D$ be the diagonal matrix with $D(j, j) = d_j$.

Then $DAD^{-1} = cA$, where $A$ is a $H_0$-primitive matrix.

$c^{-1}T$ acts on $C^0$ in $X \times G$ by the rule $(x, g) \mapsto (\sigma x, (c^{-1}gc)\tau_A(x))$, where $\tau_A(x)$ is the $A$ label of the edge $x_0$.

For $c$ in the center of $H$, the $H_0$-SFT $c^{-1}T_A$ is conjugate to $T_A$. If $H^1$ contains an element of the center of $H$, then this element can be chosen as $c$.

Proof. If $z$ is a point of period $j$ from $i$ to $i$ in $X\overline{T}$, with weight $h$, then $h$ must be in $H^j$ ($j$ interpreted mod $p$). This is because $T^j(z, e) = (z, h)$, and $h$ takes $(z, e)$ to $(z, h)$. It follows that the chosen $c$ lies in $H^1$. Also, let $g$ be the weight of a path from $k$ to $i$; then

$$DAD^{-1}(j, k) = (d_ja_{jk})(d_k)^{-1} = (d_ja_{jk}g)(d_kg)^{-1}.$$ Interpreting $(d_ja_{jk}g)$ and $(d_kg)$ as weights of paths from $i$ to $i$, we see every entry of $DAD^{-1}$ lies in $H^1$. Likewise, every entry of $c^{-1}DAD^{-1}$ lies in $H^0$. Finally, given another element $c'$ of $H^1$, let $c' = ch$ with $h \in H^0$; then we may pass from $cA$ to $(ch)(h^{-1}A)$. Finally, $A$ being $H_0$-primitive follows from $c^{-1}T$ being mixing, as the latter means that for a large $n$, $T^n$ maps some point $(x, e)$ with initial vertex $i$ to a point with $(z, h)$ where $z$ has initial vertex $j$. Here, $h = c^{-n}eg$ such that $a_{ijng} > 0$. This forces $A$ to be $H$-primitive.
The image of \((x, g) \in X \times G\) under \(c^{-1}T_A\) computes to be
\[
(\sigma x, c^{-1}g \tau_A(x)) = (\sigma x, (c^{-1}gc)c^{-1}\tau_A(x)) = (\sigma x, (c^{-1}gc)\tau_A(x)).
\]
The final claim follows from
\[
e^{-1}T_A : (x, g) \mapsto (\sigma x, c^{-1}g \tau_A(x)) = (\sigma x, gc^{-1}\tau_A(x)) = (\sigma x, g \tau_A(x)).
\]

**Appendix E. A special case**

Recall, \(G\) denotes a finite group. The general theorems of this paper reduce the classification of \(G\)-SFTs to an algebraic problem which is far beyond the scope of this paper. Still, in this section we study the algebraic invariants of a natural initial class, including a complete solution in a meaningful (though very special) case. The argument points to algebraic challenges for the general case.

Throughout, \(\mathcal{P}\) is the poset \(\mathcal{P}_2\), and \(\mathcal{H}\) is the coset structure for which \(H_{11} = H_{12} = H_{22} = G\). To facilitate a quicker overview, proofs of some results do not immediately follow statements.

Given \(p, q, x\) in \(\mathbb{Z}G\), let \(M(p, q, x) = \left( \begin{smallmatrix} p & x \\ 0 & q \end{smallmatrix} \right)\). Let \(\mathcal{M}^{++}(p, q, x)\) be the set of matrices \(A\) in \(\mathcal{M}_p^n(\mathbb{Z}, \mathbb{Z}G)\) with coset structure \(\mathcal{H}\) such that \(I - A\) and \(M(p, q, x)\) are \(\text{El}_p(\mathbb{Z}G)\)-equivalent (i.e., they have 1-stabilizations which for some \(n\) are \(\text{El}_p(n, \mathbb{Z}G)\)-equivalent). When \(G\) is abelian, this forces \(\det(I - A\{1, 1\}) = p\) and \(\det(I - A\{2, 2\}) = q\).

Let \(\mathcal{M}^{++}(p, q) = \cup_{x} \mathcal{M}^{++}(p, q, x)\). In this appendix, we study algebraic invariants of \(G\)-flow equivalence for the \(G\)-SFTs defined by \(\mathcal{M}^{++}(p, q) = \cup_{x} \mathcal{M}^{++}(p, q, x)\): for general \(G\), then abelian \(G\) satisfying a \(K\)-theory constraint, and finally for \(G = \mathbb{Z}_2\), where we give a solution which is complete and algorithmically practical. For example, when \(G = \mathbb{Z}_2\) and \(\mathcal{M}^{++}(p, q)\) consists of finitely many \(G\)-FE classes, we can count them (Theorem [E.16]).

Before turning to the algebra, we note the following consequence of Proposition [E.11](or a simple exercise), which shows the algebraic study corresponds to actual \(G\)-SFTs.

**Proposition E.1.** Suppose \(G\) is a finite group. For all \(p, q, x\) in \(\mathbb{Z}G\), \(\mathcal{M}^{++}(p, q, x)\) is nonempty.

Now we consider an arbitrary finite \(G\). Given \(p\) in \(\mathbb{Z}G\) we define \(\tilde{U}(p)\) to be the set of \(y\) in \(\mathbb{Z}G\) such that \(\mu_y : v \mapsto yv\) induces an automorphism of \(\mathbb{Z}G/\langle p \mathbb{Z}G \rangle\). The induced map is a right \(G\)-module homomorphism; it is a right \(G\)-module automorphism if and only if it an abelian group automorphism. The map \(\mu_y\) induces an automorphism if there exist \(v, w, x\) in \(\mathbb{Z}G\) such that \(vy = 1 + pw\) and \(yv = 1 + px\). When \(G\) is abelian, this means simply that \([y]\) is a unit in the quotient ring \(\mathbb{Z}G/\langle p \mathbb{Z}G \rangle\). Similarly, given \(q \in \mathbb{Z}G\) we define \(\tilde{W}(q)\) to be the set of \(z\) in \(\mathbb{Z}G\) such that \(v \mapsto vz\) defines a group automorphism (equivalently,
a left $G$-module automorphism) of $\mathbb{Z}G/(\mathbb{Z}G)q$. This means there exist $v,w,x$ in $\mathbb{Z}G$ such that $vy = 1 + wq$ and $vy = 1 + xq$.

We need a little more. Let $M_k(p,q,x)$ be the 1-stabilization of $M(p,q,x)$ in $\mathcal{M}_G(n,\mathbb{Z}G)$ with $n = (k,k)$. So, for $M = M_k(p,q,x)$, $M = I$ except that $M_{1,1} = p$, $M_{k+1,k+1} = q$ and $M_{1,k+1} = x$. A matrix $A$ has a 1-stabilization $\text{El}(\mathbb{Z}G)$-equivalent to a 1-stabilization of $M(p,q,x)$ if and only if for all/any sufficiently large $k$, $A$ has a 1-stabilization $\text{El}(\mathbb{Z}G)$-equivalent to $M_k(p,q,x)$. Now we define $\tilde{U}^eq(p)$ to be the set of $y$ in $\tilde{U}(p)$ such that for some $k$, there is an $\text{El}(k+1,\mathbb{Z}G)$ equivalence $U(p \oplus I_k)V = (p \oplus I_k)$ such that $U(1,1) = y$. It will be convenient to write this equivalence in the form $U(p \oplus I_k) = (p \oplus I_k)W (W^{-1})$. Similarly, we define $\tilde{W}^eq(q)$ to be the set of $z$ in $\tilde{W}(q)$ such that for some $k$, there is an $\text{El}(k+1,\mathbb{Z}G)$ equivalence $U(q \oplus I_k) = (q \oplus I_k)W$ such that $W(1,1) = z$.

Finally, given $p,q$ in $\mathbb{Z}G$ we define the abelian group $L(p,q) = \{pc + dq \in \mathbb{Z}G : c \in \mathbb{Z}G, d \in \mathbb{Z}G\}$. When $G$ is abelian, $L(p,q)$ is also an ideal in $\mathbb{Z}G$. For $G$ not abelian, $L(p,q)$ need not be even a onesided ideal.

**Theorem E.2.** Suppose $G$ is a finite group, $p,q,x \in \mathbb{Z}G$, and $A,A'$ are matrices in $\mathcal{M}^{++}(p,q,x)$, $\mathcal{M}^{++}(p,q,x')$ respectively. Then the following are equivalent.

1. The $G$-SFTs $T_A$, $T_{A'}$ are $G$-flow equivalent.
2. $M(p,q,x)$ and $M(p,q,x')$ have 1-stabilizations which are $\text{El}_F(\mathbb{Z}G)$-equivalent.
3. There exist $y \in \tilde{U}^eq(p)$ and $z \in \tilde{W}^eq(q)$ such that $yx - x'z \in L(p,q)$.

Suppose $G$ is abelian, and consider quotient groups as rings. Then $\tilde{U}(p)$ is the set of $x$ in $\mathbb{Z}G$ such that $[x]$ is a unit in $\mathbb{Z}G/(p)$ (and similarly for $\tilde{U}(q)$). Condition (3) holds if and only if there exist elements $y \in \tilde{U}^eq(p)$, $z \in \tilde{U}^eq(q)$ such that $[y][x] = [z][x']$ in $\mathbb{Z}G/L(p,q)$.

Given $p,q$, with $L = L(p,q)$ Theorem E.2 tells us\(^{5}\) that the $G$-flow equivalence classes of $G$-SFTs defined from matrices in $\mathcal{M}^{++}(p,q)$ are in bijective correspondence with the set of full orbits of points in the abelian group $\mathbb{Z}G/L$ under the action of the semigroup of homomorphisms $v + L \mapsto yv + L$ and $v + L \mapsto vz + L$ coming from $y \in \tilde{U}^eq(p)$, $z \in \tilde{W}^eq(q)$. Often the group $\mathbb{Z}G/L(p,q)$ is finite, and in this case, with $G$ abelian, that orbit relation on $\mathbb{Z}G/L(p,q)$ can be computed mechanically.

We now turn to some proofs.

---

\(^{5}\)The relation (3) in the statement of Theorem E.2 is a special case version of an adaptation to $\mathbb{Z}G$ of the invariant introduced by Huang in [13] for flow equivalence of reducible SFTs with two irreducible components.
Proof of Theorem E.2. (1) \(\iff\) (2): This follows from the classifying Theorem [4.1] by our choice of \(H\), because the only permutation of \(\{1, 2\}\) respecting the \(P_2\) relation \(\leq\) is the identity.

(2) \(\implies\) (3): For some \(k\), we have an equivalence \(UM_k(p, q, x) = M_k(p, q, x')W\) of \(2k \times 2k\) matrices, which we can write in block form as

\[
\begin{pmatrix}
U\{1, 1\} & U\{1, 2\} \\
0 & U\{2, 2\}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix} p & 0 \\
0 & I_{k-1} \end{pmatrix} & \begin{pmatrix} x & 0 \\
0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix} & \begin{pmatrix} q & 0 \\
0 & I_{k-1} \end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
W\{1, 1\} & W\{1, 2\} \\
0 & W\{2, 2\}
\end{pmatrix}.
\]

Computation of the 1, \(k + 1\) entry of (E.3) produces

\[
U_{1,1}x + U_{1,k+1}q = pW_{1,k+1} + x'W_{k+1,k+1}.
\]

Set \(y = U_{1,1}\) and \(z = W_{k+1,k+1}\). If follows that \(yx - x'z \in L(p, q)\).

Next, view \((ZG)^k\) as a set of column vectors, and let \(v = (v_1, \ldots, v_k)^T\) denote an element of \((ZG)^k\). Because \(U\{1, 1\} \begin{pmatrix} p & 0 \\
0 & I_{k} \end{pmatrix} = \begin{pmatrix} p & 0 \\
0 & I_{k} \end{pmatrix} W\{1, 1\}\), the map \(v \mapsto U\{1, 1\}v\) induces a group automorphism (also a right \(ZG\)-module automorphism) of \(\text{cok}(\begin{pmatrix} p & 0 \\
0 & I_{k} \end{pmatrix}) = (ZG)^{k+1}/(\begin{pmatrix} p & 0 \\
0 & I_{k} \end{pmatrix})(ZG)^{k+1}\).

The map \(\pi : v \mapsto v_1\) induces an isomorphism \(\text{cok}(\begin{pmatrix} p & 0 \\
0 & I_{k} \end{pmatrix}) \rightarrow \text{cok}(p) = ZG/p(ZG)\). Here \(\pi\) pushes the automorphism of \(\text{cok}(\begin{pmatrix} p & 0 \\
0 & I_{k} \end{pmatrix})\) down to the automorphism of \(ZG/p(ZG)\) induced by \(v_1 \mapsto yv_1\). A similar argument, using the equivalence \(U\{2, 2\} \begin{pmatrix} q & 0 \\
0 & I_{k} \end{pmatrix} = \begin{pmatrix} q & 0 \\
0 & I_{k} \end{pmatrix} W\{2, 2\}\) and considering the cokernel of \(\begin{pmatrix} q & 0 \\
0 & I_{k} \end{pmatrix} = \begin{pmatrix} q & 0 \\
0 & I_{k} \end{pmatrix} W\{2, 2\}\) with respect to the action on row vectors, shows that the rule \(v_{k+1} \mapsto v_{k+1}z\) induces an automorphism of \(ZG/(ZG)q\). This finishes the proof that (2) \(\implies\) (3).

(3) \(\implies\) (2): By assumption, for large enough \(k\) we have an elementary equivalence \(G\begin{pmatrix} p & 0 \\
0 & I_{k-1} \end{pmatrix}H = \begin{pmatrix} p & 0 \\
0 & I_{k-1} \end{pmatrix}\) such that \(G_{1,1} = y\).

Then

\[
\begin{pmatrix}
G & 0 \\
0 & I_k
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix} p & 0 \\
0 & I_{k-1} \end{pmatrix} & \begin{pmatrix} z & 0 \\
0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix} & \begin{pmatrix} q & 0 \\
0 & I_{k-1} \end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
H & 0 \\
0 & I_k
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix} p & 0 \\
0 & I_{k-1} \end{pmatrix} & \begin{pmatrix} yx & 0 \\
w & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix} & \begin{pmatrix} q & 0 \\
0 & I_{k-1} \end{pmatrix}
\end{pmatrix}
\]

where \(w\) is a size \(k - 1\) column vector. Adding suitable multiples of columns 2, \ldots, \(k\) to column \(k + 1\) to zero out \(w\) implements a unipotent (hence elementary) equivalence to \(M(p, q, yx)\), which therefore is elementary equivalent to \(M(p, q, x)\). Similarly \(M(p, q, x')\) is elementary equivalent to \(M(p, q, x'z)\). Because \(yx - x'z \in L(p, q)\), say \(yx + pr = x'z + sq\), another application of unipotent equivalence shows \(M(p, q, yx)\) and \(M(p, q, x'z)\) are \(El_P(ZG)\)-equivalent.

The final claims, for abelian \(G\), are easily checked. \(\square\)

We recall some facts from the book [22] pages 1-4] of Oliver. Suppose the finite group \(G\) is abelian. Then \(SK_1(ZG)\) is the subgroup of
\(K_1(\mathbb{Z}G)\) represented by matrices with determinant 1; it is the torsion subgroup of \(K_1(\mathbb{Z}G)\), and it is finite. The meaning of \(SK_1(\mathbb{Z}G)\) being trivial is precisely that every matrix over \(\mathbb{Z}G\) with determinant 1 has a 1-stabilization which is an elementary matrix over \(\mathbb{Z}G\). Being abelian, \(G\) is the direct sum of its Sylow \(p\)-subgroups, and \(SK_1(\mathbb{Z}G)\) is trivial if and only if one of the following hold\(^6\) (in which \(C_k\) denotes the cyclic group of order \(k\)).

1. \(G\) is a direct sum of copies of \(C_2\).
2. Each Sylow \(p\)-subgroup of \(G\) has the form \(C_p^n\) or \(C_p \oplus C_{p^n}\).

**Proposition E.4.** Suppose \(G\) is abelian and \(SK_1(\mathbb{Z}G)\) is trivial. Then \(\widetilde{U}^{eq}(p) = \widetilde{U}(p)\) and \(\widetilde{W}^{eq}(q) = \widetilde{W}(q)\).

*Proof.* Because \(G\) is abelian, obviously it is enough to prove \(\widetilde{U}^{eq}(p) = \widetilde{U}(p)\). So, suppose \(y \in \widetilde{U}(p)\); we will show \(y \in \widetilde{U}^{eq}(p)\). Because \(x \mapsto yx\) induces an automorphism of \(ZG/p(\mathbb{Z}G)\) there exist \(a\) in \(\mathbb{Z}G\) (implementing the inverse automorphism) and \(b\) in \(\mathbb{Z}G\) such that \(ay = 1 + bp\). Thus there is an SL(2, \(\mathbb{Z}G\))-equivalence

\[
\begin{pmatrix} y & p \\ b & a \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & -bp -1 \\ -bp & y \end{pmatrix} = \begin{pmatrix} ypa -bp^2 & -bp + ay \\ 0 & -bp + ay \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}
\]

and so, for arbitrary \(k > 0\) there is also an SL(\(k+2\), \(\mathbb{Z}G\))-equivalence

\[
\begin{pmatrix} y & p & 0 \\ b & a & 0 \\ 0 & 0 & I_k \end{pmatrix} \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_k \end{pmatrix} \begin{pmatrix} a & -1 & 0 \\ -bp & y & 0 \\ 0 & 0 & I_k \end{pmatrix} = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_k \end{pmatrix}.
\]

Because \(SK_1(\mathbb{Z}G)\) is trivial, for some \(k\) this SL(\(k+2\), \(\mathbb{Z}G\))-equivalence is an El(\(k+2\), \(\mathbb{Z}G\))-equivalence. This shows \(y \in \widetilde{U}^{eq}\). \(\square\)

To finish, we work out the classification in complete detail for the case \(G = \mathbb{Z}_2\). From here, \(G\) denotes \(\mathbb{Z}_2\). We use \(\mathbb{Z}_n\) to denote \(\mathbb{Z}/n\mathbb{Z}\) and \(G = \{e, g\}\).

**The ring \(R \cong \mathbb{Z}G\) and its ideals.** Let \(R\) denote the subring of \(\mathbb{Z}^2\) consisting of all \((\alpha, \beta)\) with \(\alpha \equiv \beta \mod 2\). The map \(\mathbb{Z}G \rightarrow R\) given by \(ae + bg \mapsto (a + b, a - b)\) is a well known ring isomorphism (with inverse \((\alpha, \beta) \mapsto \frac{\alpha + \beta}{2}e + \frac{\alpha - \beta}{2}g\)). We will work with \(R\) rather than \(\mathbb{Z}G\). We define \(E\) to be the even elements of \(R\), i.e. the \((\alpha, \beta)\) with \(\alpha\) and \(\beta\) even integers. The set \(R \setminus E\) is the set of odd elements. Let \(E_+ = \{(\alpha, 0) \in R\} = \langle(2, 0)\rangle\) and \(E_- = \{(0, \beta) \in R\} = \langle(0, 2)\rangle\). \(J\) will always refer to an ideal in \(R\). Given \(J\), let \(J_+\), \(J_-\) be the ideals and \(j_+, j_-\) the nonnegative integers such that \(J_+ = E_+ \cap J = j_+E_+ = \langle(2j_+, 0)\rangle\) and \(J_- = E_- \cap J = j_-E_- = \langle(0, 2j_-)\rangle\). If \((\alpha, \beta) \in J\), then \((2\alpha, 0) \in J_+\) and \((0, 2\beta) \in J_-\). From this we deduce that either \(J = J_+ \oplus J_-\) or \(|J/(J_+ \oplus J_-)| = 2\). \(J\) has rank 2 (as an abelian group) if and only if \(j_+ \neq 0 \neq j_-\). If \(|J/(J_+ \oplus J_-)| = 2\), then \(J\) is a rank two principal ideal, \(J = \langle j\rangle\) with \(j = (j_+, j_-)\). For a nonnegative vector \(v = (a, b)\), \(\gamma_v : R \rightarrow \mathbb{Z}_a \oplus \mathbb{Z}_b\) denotes the obvious map \((\alpha, \beta) \mapsto ([\alpha], [\beta])\).

\(^6\)See [22, pp. 1-19] for an overview of \(K_1(\mathbb{Z}G)\) and its history.
Proposition E.5. For J an ideal of \( R \), by cases the following maps \( \rho \) give a presentation of the abelian group epimorphism \( R \to R/J \) (i.e., they define group epimorphisms with kernel \( J \)).

1. \( J = \langle j \rangle \), \( j \) odd, \( j = (j_+, j_-) \). Then \( \rho : R \to \mathbb{Z}_{j_+} \oplus \mathbb{Z}_{j_-} \).

(a) \( \rho : (\alpha, \beta) \mapsto \gamma_j(\alpha/2, \beta/2) \) if \( (\alpha, \beta) \in E \),

(b) \( \rho : x \mapsto \gamma_j(x - j) \) if \( (\alpha, \beta) \in R \setminus E \).

2. \( J = j_+E_+ \oplus j_-E_- \). Then \( \rho = \gamma_{j_+} : R \to R_{j_+}, \) where \( \gamma_{j_+} = R_{j_+} = \{(a, b) \in \mathbb{Z}_{j_+} \oplus \mathbb{Z}_{j_-} : a \equiv b \mod 2 \} \).

3. \( J = \langle j \rangle \) rank 2 (i.e., \( j_+ \neq 0 \neq j_- \), \( j = (j_+, j_-) \) even. Let \( \gamma_j = \gamma_{j_+} \). Then \( \rho : R \to \gamma_{j_+}R/\{0, \gamma_j \} \). Here \( \rho \) is \( \gamma_{j_+} \) followed by the quotient map from \( \gamma_{j_+}R \) with kernel the two-point subgroup \( \{0, \gamma_j \} \). Moreover, \( \rho(x) = \rho(x') \) if and only if \( \gamma_{j_+}(x) = \gamma_{j_+}(x') \) or \( \gamma_{j_+}(x - j) = \gamma_{j_+}(x') \).

\textbf{Proof.} Let \( J' = \langle (2j_+, 0), (0, 2j_-) \rangle \).

\textbf{Case 1.} If \( x \) is even, then \( x \in J \) if and only if \( x \in J' \). If \( x \) is odd, then \( x \in J \) if and only if \( x - r \in J' \). Let \( \rho'(x) = \gamma_{j_+}(x) \) if \( x \in E \), and \( \rho'(x) = \gamma_{j_+}(x - j) \) if \( x \in R \setminus E \). Then \( \rho' : R \to \mathbb{Z}_{j_+} \oplus \mathbb{Z}_{j_-} \) is a homomorphism and \( \ker(\rho') = J \). The image of \( \rho' \) is the subgroup \( H \) of \( ([\alpha], [\beta]) \) with \( \alpha \) and \( \beta \) even. The map \( ([\alpha], [\beta]) \mapsto ([\alpha/2], [\beta/2]) \) is a group isomorphism \( H \to \mathbb{Z}_{j_+} \oplus \mathbb{Z}_{j_-} \).

\textbf{Case 2.} This is clear, because \( J = J' \).

\textbf{Case 3.} \( \{0, \gamma_j \} \) is a group, because \( 2\gamma_j = 0 \) in \( \mathbb{Z}_{j_+} \oplus \mathbb{Z}_{j_-} \). Then \( \ker(\rho) = \gamma_{j_+}^{-1} \{0, \gamma_j \} = J' \cup (j + J') = J \).

The final “Moreover” statement follows. □

In Case 1 above, on account of the final map \( H \to \mathbb{Z}_{j_+} \oplus \mathbb{Z}_{j_-} \), the group epimorphism \( \rho \) is not a ring homomorphism (unless \( J = R \)).

Given \( v \in R \), we let \( \mu_v \) denote the map \( R \to R \) given by \( w \mapsto vw \), and also (for a lighter notation) the homomorphism induced on a quotient group of \( R \). For an ideal \( J \), let \( \tilde{U}(J) \) be the set of \( v \) in \( R \) such that \( \mu_v \) is an automorphism of \( R/J \). For elements \( v_1, \ldots, v_k \) of \( R \), \( \tilde{U}(v_1, \ldots, v_k) \) denotes \( \tilde{U}(J) \) for \( J = \langle v_1, \ldots, v_k \rangle \).

Theorem E.6. \( \tilde{U}(J) \) is characterized by cases.

1. \( J = \langle j \rangle \), \( j = (j_+, j_-) \) odd. Then \( (\alpha, \beta) \in \tilde{U}(J) \) if and only if \( \gcd(\alpha, j_+) = 1 = \gcd(\beta, j_-) \).

2. \( J \subseteq E \). Then \( (\alpha, \beta) \in \tilde{U}(J) \) if and only if \( (\alpha, \beta) \) is odd with \( \gcd(\alpha, j_+) = 1 = \gcd(\beta, j_-) \).

In Case 1, \( \langle 2^k, 2^\ell \rangle \in \tilde{U}(J) \) if \( k \) and \( \ell \) are positive. In Case 2, if \( J = \langle j \rangle \), then the requirement that \( (\alpha, \beta) \) must be odd is redundant; if \( J = j_+E_+ \oplus j_-E_- \), then \( J \subseteq E \), and when \( (j_+, j_-) \) is odd the requirement is not redundant. For \( k \) nonzero in \( \mathbb{Z} \), \( \gcd(k, 0) = |k| \). So, if \( j_+ \neq 0 \neq j_- \), Theorem E.6 gives \( \tilde{U}(J) = \{(\alpha, \pm 1) \in R : \gcd(\alpha, j_+) = \gcd(\beta, j_-) = 1 \} \).
1}. Likewise, if \( j_+ = 0 \neq j_- \), then \( \tilde{U}(J) = \{ (\pm 1, \beta) \in R : \gcd(\alpha, j_+) = 1 \} \). If \( J = \langle 0 \rangle \), then \( \tilde{U}(J) = \{ \pm 1, \pm 1 \} \), the units of \( R \). If \( J = R \), then \( \tilde{U}(J) = R \).

**Proof of Theorem E.6**  Let \( v = (a, b) \in R \). We use the maps \( \rho \) from Proposition E.5 which present \( R \to R/J \).

**Case 1.** Here \( \rho : R \to \mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-} \). For \( x = (\alpha, \beta) \in E \) the induced map \( \rho(x) \mapsto \rho(vx) \) is

\[
((a/2), [\beta/2]) \mapsto ([\alpha a/2], [b\beta/2]) = (a[\alpha/2], b[\beta/2]),
\]

and for \( x \in R \setminus E \) the map is

\[
\left( [(\alpha - j_+)/2], [(\beta - j_-)/2] \right) \mapsto \left( [(\alpha \alpha - j_+)/2], [(\beta - j_-)/2] \right)
\]

where the last equality holds because \((j_+, j_-)\) is odd. It follows that the induced map \( \mu_v \) is an automorphism if and only the gcd conditions hold.

**Case 2.** First suppose \( J = j_+ E_{j_+} \oplus j_- E_{j_-} \). Here \( \rho = \gamma_{j_2} : R \to \mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-} \). The map \( \rho(x) \mapsto \rho(vx) \) is \( ([\alpha], [\beta]) \mapsto ([\alpha a], [b\beta]) \), so it is an automorphism iff \( \gcd(a, 2j_+) = 1 = \gcd(b, 2j_-) \).

Lastly, suppose \( J = \langle j \rangle \), with \( j = (j_+, j_-) \) even and \( j_+ \neq 0 \neq j_- \). Here \( \rho : R \to (\mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-})/\{0, j\} \) presents \( R \to R/J \). If \( v \) is even or the gcd condition fails, then \( \mu_v \) as an endomorphism of \((\mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-})/\{0, j\}\) has a nontrivial kernel. If \( v \) is odd, then \( \mu_v \) as an endomorphism of \((\mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-})\) fixes \( j \), so it defines an automorphism of \((\mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-})/\{0, j\}\) if and only if defines an automorphism of \((\mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-})\), which is equivalent to the gcd conditions.

**Corollary E.7.** For elements \( p, q \) of \( R \), let \( J = \langle p, q \rangle \). Then

\[
\tilde{U}(p) \cup \tilde{U}(q) \subset \tilde{U}(p, q) = \{ vw : v \in \tilde{U}(p), w \in \tilde{U}(q) \}.
\]

**Proof.** For an ideal \( J \) and element \( x \) of \( R, x \in \tilde{U}(J) \) iff its image in \( R/J \), considered as a ring, is a unit. As a unit in \( R/\langle p \rangle \) pushes down to a unit in \( R/\langle p, q \rangle \), we have \( \tilde{U}(p) \subset \tilde{U}(p, q) \), and likewise \( \tilde{U}(q) \subset \tilde{U}(p, q) \).

It remains to show \( \tilde{U}(p, q) \subset \{ vw : v \in \tilde{U}(p), w \in \tilde{U}(q) \} \). We appeal to Theorem E.6 by cases. Define \( d_+ = \gcd(\alpha_p, \alpha_q) \) if at least one of \( \alpha_p, \alpha_q \) is nonzero, and \( d_- = \gcd(\beta_p, \beta_q) \) if at least one of \( \beta_p, \beta_q \) is nonzero.

**Case 1.** \( J = \langle r \rangle \) with \( r = (\alpha_r, \beta_r) \) odd.

By Lemma E.13 \( r = (d_+, d_-) \). If e.g. \( \gcd(\alpha, \alpha_r) = 1 \) and \( \alpha \) is an odd prime, then \( (\alpha, 1) \in \tilde{U}(p) \cup \tilde{U}(q) \). We see that an odd element \( u \) of \( \tilde{U}(p, q) \) is the product of elements of the form \((\alpha, 1), (1, \beta) \) from \( \tilde{U}(p) \cup \tilde{U}(q) \). If \( k \) and \( \ell \) are positive, then \((2^k, 2^\ell) \in \tilde{U}(p) \cup \tilde{U}(q) \), because at least one of \( p \) and \( q \) must be odd (because \( r \) is odd). Thus \( \tilde{U}(p) \cup \tilde{U}(q) \)
generates \( \tilde{U}(p, q) \). A product of several elements from \( \tilde{U}(p) \cup \tilde{U}(q) \) is a product of a single element of \( \tilde{U}(p) \) and a single element of \( \tilde{U}(q) \).

**Case II.** \( J \subset E \). Here \( p \) and \( q \) must be even, and there can be no odd element in any of \( \tilde{U}(p), \tilde{U}(q), \tilde{U}(J) \). The rest of the argument proceeds as in Case I by considering \( \text{gcd} \) and generating units \((\alpha, 1), (1, \beta)\).

**Definition E.8.** Given elements \( x, x' \) of a ring \( S \), we say \( x \sim x' \) in \( S \) if there is a multiplicative unit \( u \) in \( S \) such that \( ux = x' \).

In the product ring \( \mathbb{Z}_m \oplus \mathbb{Z}_n \), the units are the pairs \([a], [b]\) such that \( \text{gcd}(a, m) = 1 = \text{gcd}(b, n) \), and these are the units defining \( \sim \) in part (3) of the statement of Theorem E.9 below. (The map \( \rho \) in part (3a) is not a ring homomorphism, but the logic of the proof does not need it to be.) Also, \( \mathcal{M}^{++}(p, q, x) \) was defined with \( \{p, q, x\} \subset \mathbb{Z}G \). In Theorem E.9 we use the corresponding elements in \( R \), without introducing more notation. E.g., \( x = ae + bg \) becomes \( x = (a + b, a - b) \).

**Theorem E.9.** Let \( J \) be the ideal \( \langle p, q \rangle \) in \( R \), with \( J \cap E = j_+E_+ \oplus j_-E_- \). Suppose \( x, x' \) are in \( R \). Then the following are equivalent:

1. Matrices in \( \mathcal{M}^{++}(p, q, x) \) and \( \mathcal{M}^{++}(p, q, x') \) define \( G \)-flow equivalent \( G \)-SFTs.
2. \([x] \sim [x'] \) in \( R/J \).
3. The conditions below hold, according to the type of \( J \).
   a. \( J = \langle j \rangle \) with \( j = (j_+ \oplus j_-) \) odd. Then \( \rho(x) \sim \rho(x') \) in \( \mathbb{Z}_j_+ \oplus \mathbb{Z}_j_- \), where
      i. \( \rho(x) = \gamma_j(x/2) \) if \( x \in E \), and
      ii. \( \rho(x) = \gamma_j((x - j)/2) \) if \( x \notin E \).
   b. \( J = j_+E_+ \oplus j_-E_- \). Then \( \gamma_{j_+}(x) \sim \gamma_{j_-}(x') \) in \( \mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-} \).
   c. \( J = \langle j \rangle \), \( j = (j_+ \oplus j_-) \in E \), \( j_+ \neq 0 \neq j_- \).
      Then \( \gamma_{j_+}(x) \sim \gamma_{j_-}(x') \) or \( \gamma_{j_+}(x - j) \sim \gamma_{j_-}(x') \) in \( \mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-} \).

**Proof.** Because \( SK_1(\mathbb{Z}_2) \) is trivial, it follows from Proposition E.4 and Theorem E.2 that (1) holds if and only there are elements \( y \in \tilde{U}(p) \) and \( z \in \tilde{U}(q) \) such that \( [y][x] = [z][x'] \) in \( R/J \). By Corollary E.7 these elements \( y, z \) exist if and only if \([x] \sim [x'] \) in \( R/J \). We have shown (1) \( \iff \) (2).

We next show (1) \( \iff \) (3). Let \( x = (\alpha, \beta) \) and \( x' = (\alpha', \beta') \) be elements of \( R \). We will use the map \( \rho \) of Proposition E.5 which presents \( R \rightarrow R/J \). Then

\[
[x] \sim [x'] \text{ in } R/J \iff (E.10) \exists (a, b) \in \tilde{U}(J), \rho((a \alpha, b \beta)) = \rho((a' \beta', b \beta')).
\]
Case (a): $J = \langle j \rangle$ with $j = (j_+, j_-)$ odd. Here $\rho : R \to \mathbb{Z}_{j_+} \oplus \mathbb{Z}_{j_-}$. By Theorem [E.6] and Corollary [E.7] $(a, b) \in \mathcal{U}(J)$ if and only if $\gcd(a, j_+) = 1 = \gcd(b, j_-)$.

Suppose $(a, b) \in \mathcal{U}(J)$ and $(\alpha, \beta) \in R$. Then

$$\rho(\alpha, \beta) = \begin{cases} 
\gamma_j(\alpha/2, \beta/2) & \text{if } (\alpha, \beta) \in E, \\
\gamma_j((\alpha - j_+)/2, (\beta - j_-)/2) & \text{if } (\alpha, \beta) \in R \setminus E;
\end{cases}$$

$$\rho(\alpha, b\beta) = \begin{cases} 
\gamma_j(\alpha a/2, b\beta/2) & \text{if } (\alpha, \beta) \in E, \\
\gamma_j(\alpha a/2, b\beta/2) & \text{if } (\alpha, \beta) \notin E \text{ and } (a, b) \in E, \\
\gamma_j((\alpha a - j_+)/2, (b\beta - j_-)/2) & \text{if } (\alpha, \beta) \notin E \text{ and } (a, b) \notin E.
\end{cases}$$

In each case, we can check that $\rho(\alpha, b\beta)$ equals the image of $\rho(\alpha, \beta)$ under the automorphism of $\mathbb{Z}_{j_+} \oplus \mathbb{Z}_{j_-}$ defined by $([m], [n]) \mapsto ([am], [bm])$. Therefore (E.10) is equivalent to $\rho(x) \sim \rho(x')$ in $\mathbb{Z}_{j_+} \oplus \mathbb{Z}_{j_-}$.

Case (b): $J = j_+ E_+ \oplus j_- E_- = \langle (2j_+, 0), (0, 2j_-) \rangle$. Here $\rho = \gamma_{2j}$. By Theorem [E.6] and Corollary [E.7] $(a, b) \in \mathcal{U}(J)$ if and only if $\gcd(a, 2j_+) = 1 = \gcd(b, 2j_-)$. Thus (E.10) is equivalent to $(\alpha, \beta) \sim (\alpha', \beta')$ in $\mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-}$.

Case (c): $J = \langle j \rangle$, $j = (j_+, j_-) \in E$, $j_+ \neq 0 \neq j_-$. By Proposition [E.5], here $\rho(x) = \rho(x')$ if and only if $\gamma_{2j}(x) = \gamma_{2j}(x')$ or $\gamma_{2j}(x + j) = \gamma_{2j}(x')$. So, (E.10) holds iff there exists $(a, b) \in \mathcal{U}(J)$ such that

$$\gamma_{2j}((\alpha a, b\beta)) = \gamma_{2j}(\alpha'_a, \beta') \text{ or } \gamma_{2j}((\alpha a, b\beta) + j) = \gamma_{2j}(\alpha'_a, \beta').$$

As in Case (b), $(a, b) \in \mathcal{U}(J)$ if and only if $\gcd(a, 2j_+) = 1 = \gcd(b, 2j_-)$. For $(a, b) \in \mathcal{U}(J)$, $\gamma_{2j}((\alpha a, b\beta) + j) = \gamma_{2j}(\alpha(a + j_+), b(\beta + j_-))$ because $(a, b)$ is odd. Thus (E.11) is equivalent to $\gamma_{2j}(x) \sim \gamma_{2j}(x')$ or $\gamma_{2j}(x + j) \sim \gamma_{2j}(x')$ in $\mathbb{Z}_{2j_+} \oplus \mathbb{Z}_{2j_-}$.

The next two propositions explain how to compute the ideal $J$ in the form of Theorem [E.9] from its generating polynomials $p, q$. The (trivial) Proposition [E.12] is stated to give a complete list of possibilities.

**Proposition E.12.** Let $p = (\alpha_p, \beta_p)$ and $q = (\alpha_q, \beta_q)$. Suppose $J = \langle p, q \rangle$ with $\{p, q\} \subset E_+ \cup E_-$ (i.e., $\alpha p \beta_p = 0 = \alpha q \beta_q$).

1. $J = 0$ if and only if $p = q = 0$.
2. Suppose $\beta_p = 0 = \beta_q$ and at least one of $\alpha_p, \alpha_q$ is nonzero. Let $d_+ = \gcd(\alpha_p, \alpha_q)$. Then $J$ has rank 1, and $J = \langle (d_+, 0) \rangle = j_+ E_+$ with $j_+ = d_+/2$. 

\[\Box\]
(3) Suppose \( \alpha_p = 0 = \alpha_q \) and at least one of \( \beta_p, \beta_q \) is nonzero. Let \( d_- = \gcd(\beta_p, \beta_q) \). Then \( J \) has rank 1, and \( J = \langle (0, d_-) \rangle = j_-E_- \) with \( j_-=d_-/2 \).

(4) Suppose one of \( p, q \) is \((2m,0)\neq0\) and the other is \((0,2n)\neq0\) with \( m \neq 0 \neq n \). Then \( J \) has rank 2, and \( J = j_+E_+ \oplus j_-E_- \) with \( (j_+, j_-) = (m, n) \).

**Proposition E.13.** Suppose elements \( p = (\alpha_p, \beta_p) \) and \( q = (\alpha_q, \beta_q) \) generate a rank 2 ideal \( J = \langle p, q \rangle \), and at least one of \( p, q \) lies outside \( E_+ \cup E_- \). Set \( d_+ = \gcd(\alpha_p, \alpha_q) \) and \( d_- = \gcd(\beta_p, \beta_q) \). Then one of the following holds.

1. \( J = \frac{d_+}{2}E_+ \oplus \frac{d_-}{2}E_- = \langle (d_+, 0), (0, d_-) \rangle \).
2. \( J \) is the principal ideal \( \langle (d_+, d_-) \rangle \).

Write \( \alpha_p = r_p d_+, \quad \beta_p = s_p d_- \), \( \alpha_q = r_q d_+ \) and \( \beta_q = s_q d_- \). Then \( J \) is principal if and only if

\[
\begin{align*}
\alpha_p &\equiv s_p \mod 2 \quad \text{and} \quad r_p \equiv r_q \mod 2 ,
\end{align*}
\]

**Proof.** The claim when \( \{p, q\} \subset E_+ \cup E_- \) is obvious. Now suppose \( p \notin (E_+ \cup E_-) \) (the argument when \( q \notin (E_+ \cup E_-) \) is essentially the same). Then \( \alpha_p \neq 0 \neq \beta_p \). Let \( H \) be the subgroup of \( R \) generated by the groups \( H_p = \alpha_pE_+ \oplus \beta_pE_- \) and \( H_q = \alpha_qE_+ \oplus \beta_qE_- \). Then

\[
\begin{align*}
H &= d_+E_+ \oplus d_-E_- \\
&= \langle (2d_+, 0), (0, 2d_-) \rangle \subset J \subset \langle (d_+, 0), (0, d_-) \rangle .
\end{align*}
\]

There are integers \( \ell_1, \ell_2 \) such that \( d_+ = \ell_1 \alpha_p + \ell_2 \alpha_q \). Let \( v = \ell_1 p + \ell_2 q \). Then

\[
\begin{align*}
v &= \ell_1(\alpha_p, \beta_p) + \ell_2(\alpha_q, \beta_q) \\
&= \ell_1(r_p d_+, s_p d_-) + \ell_2(r_q d_+, s_q d_-) \\
&= \langle (\ell_1 r_p + \ell_2 r_q) d_+, (\ell_1 s_p + \ell_2 s_q) d_- \rangle = (d_+, c_1 d_-)
\end{align*}
\]

for some integer \( c_1 \). Likewise for some \( w \) in \( J \) we have \( w = (c_2 d_+, d_-) \in J \). So, \((d_+, d_-) \in J \) and \((d_+, 0) \in J \) iff \((0, d_-) \in J \).

We can now see that \( J \) is principal if and only the following holds for all integers \( \ell_1, \ell_2 \):

\[
(E.15) \quad (\ell_1 r_p + \ell_2 r_q) \text{ is odd } \iff (\ell_1 s_p + \ell_2 s_q) \text{ is odd} .
\]

Clearly \( E.14 \) implies \( E.15 \). For the converse, assume \( E.15 \). If \( r_p, r_q \) are both odd, then \( E.14 \) holds, because \( s_p, s_q \) are not both even, as \( \gcd(s_p, s_q) = 1 \). Suppose \( r_p \) is even; then \( r_q \) is odd. Let integers \( \ell_1, \ell_2 \) give \( \ell_1 r_p + \ell_2 r_q = 1 \). Here, \( \ell_2 \) must be odd, but we may choose \( \ell_1 \) even or odd, because \((\ell_1 + r_q) r_p + (\ell_2 - r_p) r_q = 1 \). Because \( \ell_1 s_p + \ell_2 s_q \) must be odd in either case, it follows that \( s_p \) must be even, which forces \( s_q \) to be odd, and therefore \( E.14 \) holds. The argument for the case that \( r_q \) is even is essentially the same. \qed
To use Theorem [E.9] we recall some elementary number theory. Suppose \( n \) is a nonnegative integer. For integers \( \alpha \) and \( \alpha' \), \([\alpha] \sim [\alpha']\) in \( \mathbb{Z}_n \) means there exists an integer \( a \) such that \( \gcd(a, n) = 1 \) and \([a\alpha] = [\alpha']\). If \( n = 0 \), then \([\alpha] \sim [\alpha']\) means \( a = \pm \alpha' \). If \( n = 1 \) then \([\alpha] \sim [\alpha']\) holds for all \( a, \alpha' \). Suppose \( n = p^m \) with \( p \) prime and \( m > 0 \). Write \( y \) in \( \mathbb{Z}_{p^m} \) as \( y = \sum_{j=0}^{m-1} y_j p^j \) with each \( y_j \) in \( \{0, 1, \ldots, p - 1\} \); \( y \) is a unit if and only if \( y_0 \neq 0 \). Now, \([y] \sim [y']\) in \( \mathbb{Z}_{p^m} \) if and only if \( y = y' \) or \( \min\{j : y_j \neq 0\} = \min\{j : y'_j \neq 0\} \). Equivalently, \( \max\{k : 1 \leq k \leq m, p^k|y\} = \max\{k : 1 \leq k \leq m, p^k|y'\} \). For the equivalence relation \([\alpha] \sim [\alpha']\) in \( \mathbb{Z}_{p^m} \), there are exactly \( m + 1 \) equivalence classes.

Finally, suppose \( n > 1 \) with prime power factorization \( n = \prod_{i=1}^{k} p_i^{m_i} \). As a ring, \( \mathbb{Z}_n \) is isomorphic to \( \prod_{i=1}^{k} \mathbb{Z}_{p_i^{m_i}} \). In this presentation, let \( x = (x_1, \ldots, x_k) \) and \( x' = (x'_1, \ldots, x'_k) \). Then \( x \sim x' \) in \( \prod_{i=1}^{k} \mathbb{Z}_{p_i^{m_i}} \) if and only if \( x_i \sim x'_i \) in \( \mathbb{Z}_{p_i^{m_i}} \), for \( 1 \leq i \leq k \). To express this another way, for \( a \) in \( \mathbb{Z} \) and \( p = p_i \), let \( \delta_{p,n}(a) = \max\{k : 1 \leq k \leq m_i, p^k|a\} \). Then \([\alpha] \sim [\alpha']\) in \( \mathbb{Z}_n \) if and only if \( \delta_{p,n}(a) = \delta_{p,n}(a') \), for each prime \( p \) dividing \( n \). Given \( a, \alpha', n \) this is straightforward to compute.

For \( n = \prod_{i=1}^{k} p_i^{m_i} \) as above, define \( \kappa(n) = \prod_{i=1}^{k} (m_i + 1) \), and also define \( \kappa(1) = 1 \). Then for positive integers \( m, n \) it follows that for the equivalence relation \( \sim \) in the product ring \( \mathbb{Z}_m \oplus \mathbb{Z}_n \), the number of equivalence classes is \( \kappa(m)\kappa(n) \).

We will need to count \( \sim \) classes in \( \mathbb{Z}_m \oplus \mathbb{Z}_n \) represented by elements of \( \gamma_{2j} R = \{([a], [b]) : a + b \equiv 0 \mod 2\} \). Let \( j_+ = 2^{k_+} g_+ \), with \( g_+ \) an odd integer. The \( \sim \) classes in \( \mathbb{Z}_{2j_+} \) arising from odd and even integers are disjoint. The number of classes arising from odd integers is \( \kappa(g) \); the number arising from even integers is \( (k_+ + 1)\kappa(g) \). The analogous statements hold for \( j_- = 2^{k_-} g_- \), with \( g_- \) an odd integer. So, in \( \gamma_{2j} R \), the number of \( \sim \) classes arising from odd elements of \( R \) is \( \kappa_{g_+}\kappa_{g_-} \) and the number arising from even elements of \( R \) is \( (k_+ + 1)(k_- + 1)\kappa_{g_+}\kappa_{g_-} \).

**Theorem E.16.** Let \( J = \langle p, q \rangle \) be rank 2 in \( R \), i.e. \( J \cap E = j_+ E_+ \oplus j_- E_- \) with \( j_+ \neq 0 \neq -j_- \). Write \( j_+ = 2^{k_+} g_+ \), \( j_- = 2^{k_-} g_- \) with \( g_+, g_- \) odd integers. Let \( f_{p,q} \) be the number of distinct \( G \)-flow equivalence classes defined by matrices in \( M(p, q) \). There are three cases:

1. \( J = \langle j \rangle \), with \( j \) odd.
2. \( J = j_+ E_+ \oplus j_- E_- = \langle (2j_+, 0), (0, 2j_-) \rangle \).
3. \( J = \langle j \rangle \), with \( j \) even.

The computation of \( f_{p,q} \) in these cases is

1. \( f_{p,q} = \kappa(g_+)\kappa(g_-) = \kappa(j_+)\kappa(j_-) \).
2. \( f_{p,q} = \kappa(g_+)\kappa(g_-) ((k_+ + 1)(k_- + 1) + 1) \).
3. \( f_{p,q} = \kappa(g_+)\kappa(g_-) (k_+ k_- + 2) \).

**Proof.** The cases (1) and (2) follow immediately from Theorem [E.9] and the discussion preceding the theorem. Now suppose we are in case (3).
For \( x \in R \), let \( C(x) \) be the \( \sim \) class of \( \gamma_{2j}(x) \) in \( \gamma_{2j}R \). By Theorem [E.3], we must compute the number of distinct sets of the form \( C(x) \cup C(x+j) \). For any integer \( \alpha \) and odd prime \( p \), \( \delta_{p,2j+}(\alpha + j) = \delta_{p,2j+}(\alpha) \). If \( \delta_{2,2j+}(\alpha) = i \), then
\[
\delta_{2,2j+}(\alpha + j) = \begin{cases} 
  i & \text{if } i < k_+ \\
  k_+ + 1 & \text{if } i = k_+ \\
  k_+ & \text{if } i = k_+ + 1.
\end{cases}
\]

The analogous statements hold for \( j_-, k_- \). Thus \( C(x) \cup C(x+j) = C(x) \) if \( x = (\alpha, \beta) \) with \( \delta_{2,2j+}(\alpha) < k_+ \) and \( \delta_{2,2j-}(\beta) < k_- \). The number of sets \( C(x) \) of this form is \( \kappa(g_+)\kappa(g_-)((k_+ - 1)(k_- - 1) + 1) \). (Note \( k_+ \geq 1 \) and \( k_- \geq 1 \) because \( j \) is even.)

If \( x = (\alpha, \beta) \) with \( \delta_{2,2j+}(\alpha) \in \{k_+, k_+ + 1\} \), then \( C(x) \) and \( C(x+j) \) are disjoint; also, \( \alpha \) is an even number, so \( (\alpha, \beta) \in R \) iff \( \beta \) is even. Let \( C^+_x \) be the collection of sets \( C(x) \) for which \( x = (\alpha, \beta) \in R \) with \( \delta_{2,2j+}(\alpha) = i \). Then \( |C^+_{k_+}| = |C^+_{k_+ + 1}| = \kappa(g_+)\kappa(g_-)(k_+ - 1) \). The collections \( C^+_{k_+}, C^+_{k_+ + 1} \) are disjoint, and the rule \( C(x) \mapsto C(x + j) \) induces a bijection \( C^+_{k_+} \leftrightarrow C^+_{k_+ + 1} \). So, \( C^+_{k_+} \cup C^+_{k_+ + 1} \) gives rise to \( \kappa(g_+)\kappa(g_-)(k_+ - 1) \) distinct sets of the form \( C(x) \cup C(x + j) \). We can reverse the roles of + and − and say likewise that \( C^-_{k_+} \cup C^-_{k_+ + 1} \) gives rise to \( \kappa(g_-)\kappa(g_+)(k_+ + 1) \) distinct sets of the form \( C(x) \cup C(x + j) \).

Finally, to compute \( f_{p,q} \) we add the three counts above, and correct for the doublecount:
\[
f_{p,q} = \kappa(g_+)\kappa(g_-)((k_+ - 1)(k_- - 1) + 1) + (k_+ + 1) + (k_- + 1) - 2
\]
\[
= \kappa(g_+)\kappa(g_-)(k_+ k_- + 2) .
\]

Example E.17. We will work out the classification of G-FE classes in \( M^{++}(p, q) \) for the example \( p = (12, 96), q = (8, 24) \). (These elements of \( R \) correspond to the elements \( 54e - 42g, 16e - 8g \) in \( \mathbb{Z}G \).) Here \( d_+ = \text{gcd}(12, 8) = 4 \) and \( d_- = \text{gcd}(96, 24) = 24 \). We compute \( (12, 96) = (r_p d_+, s_p d_-) = (3(4), 4(24)) \), and see \( r_p \not\equiv s_p \mod 2 \). By Proposition [E.13] it follows that \( (p, q) = (\langle 2j_+, 0, 0, 2j_- \rangle) \) with \( (2j_+, 2j_-) = (d_+, d_-) = (4, 24) \). By Theorem [E.9] matrices in \( M^{++}(p, q, x) \) define G-FE G-SFTs iff \( \gamma_{4(24)}(x) \sim \gamma_{4(24)}(x) \) in \( \mathbb{Z}_4 \oplus \mathbb{Z}_{24} \). We will write this as \( x \sim x' \). Write \( x = (\alpha, \beta), x' = (\alpha', \beta') \). As \( 4 = 2^2 \) and \( 24 = 2^3 3^1 \), we have \( x \sim x' \) iff the following hold: \( \delta_{2,4}(\alpha) = \delta_{2,4}(\alpha'), \delta_{2,24}(\beta) = \delta_{2,24}(\beta') \), and \( \delta_{3,24}(\beta) = \delta_{3,24}(\beta') \). The allowed combinations of possible values of these invariants is displayed in Table 1, with a few examples.

We can check that e.g. \( (0, 6) \sim (0, 18) \not\sim (0, 12) \) and \( (1, 3) \sim (3, 15) \not\sim (3, 11) \). From Table 1 we see that 14 \( \sim \) classes arise here for elements of \( R \), consistent with Theorem [E.16](2), which in our case
Table 1.

<table>
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<th>(α, β)</th>
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<th>δ_{3,24}(β)</th>
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<td>0</td>
<td>0.1</td>
</tr>
<tr>
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<td>1,2,3</td>
<td>0.1</td>
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<td>(0,18)</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(2,6)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(3,11)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

gives

\[(j_+, j_-) = (2, 12) = (2^{k_+}g_+, 2^{k_-}g_-) = (2^1(1), 2^3(3))\]

\[f_{p,q} = \kappa(g_+)\kappa(g_-) ((k_+ + 1)(k_- + 1) + 1) = (1)(2)((1 + 1)(2 + 1) + 1) = 14.\]

Next, we work to Proposition E.19, which shows how to determine when a matrix \(A\) lies in some \(\mathcal{M}^{++}(p, q, x)\), and then compute \(p, q, x\).

Two matrices are \(El(R)\)-equivalent when there exists \(k\) such that they have 1-stabilizations of size \(k \times k\) which are \(El(k, R)\)-equivalent. If \(B\) is \(El(R)\)-equivalent to a \(1 \times 1\) matrix \((p)\), then \(p\) must be \(\det(B)\).

**Lemma E.18.** Given an \(n \times n\) matrix \(B\) over \(R\), there is an algorithm which determines whether \(B\) is \(El(n, R)\)-equivalent to some \(1 \times 1\) matrix \((p)\), and computes an explicit \(El(n, R)\)-equivalence from \(B\) to \((p \oplus I_{n-1})\), when this is the case.

**Proof.** A matrix over \(R\) (even a \(2 \times 2\) matrix) need not be \(GL(R)\) equivalent to a diagonal matrix, or even a triangular matrix (see e.g. [12, Example 8.7]). But, for the special case we consider here, a modification of the Smith form argument applies to produce the explicit \(El(n, R)\)-equivalence required for (1), or a failure which shows there is no such equivalence. For detail, see [12, Lemma 8.2, Remark 8.3], and the algorithmic proof of [12, Lemma 8.2]. \(\square\)

**Proposition E.19.** Suppose \(B \in \mathcal{M}_R(n, R)\), with \(n = (n_1, n_2)\) and \(\mathcal{P} = \mathcal{P}_2\). There is an algorithm to determine whether there exist \(p, q, x\) such that \(B\) is \(El_{\mathcal{P}}(n, R)\)-equivalent to the matrix \(M(p, q, x) = \begin{pmatrix} \begin{array}{cc} p & x \\ \end{array} \end{pmatrix}\), and in this case to compute \(p, q, x\).

**Proof.** The required elementary equivalence will exist only if there are \(El(R)\)-equivalences of the diagonal blocks of \(B\) to \((p)\) and \((q)\). By Lemma E.18 we can decide whether this occurs, and in the case it does we can construct elementary equivalences, \(UB\{1, 1\}V = (p) \oplus I_{n_1-1}\) and \(U'B\{2, 2\}V' = (q) \oplus I_{n_2-1}\). Given this data, we define an elementary equivalence of \(B\),

\[
\begin{pmatrix} U & 0 \\ 0 & U' \end{pmatrix} \begin{pmatrix} B\{1, 1\} & B\{1, 2\} \\ 0 & B\{2, 2\} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V' \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} p & 0 \\ 0 & I_{n_1-1} \end{pmatrix} & Y \\ 0 & \begin{pmatrix} q & 0 \\ 0 & I_{n_2-1} \end{pmatrix} \end{pmatrix}.
\]
Then for suitable blocks $Z, Z'$,

$$
\begin{pmatrix}
  I & Z \\
  0 & I
\end{pmatrix}
\begin{pmatrix}
  p & 0 \\
  0 & I_{n_1-1}
\end{pmatrix}
\begin{pmatrix}
  Y \\
  0
\end{pmatrix}
\begin{pmatrix}
  I & Z' \\
  0 & I
\end{pmatrix}
= 
\begin{pmatrix}
  p & 0 \\
  0 & I_{n_1-1}
\end{pmatrix}
\begin{pmatrix}
  X \\
  0
\end{pmatrix}
\begin{pmatrix}
  q & 0 \\
  0 & I_{n_2-1}
\end{pmatrix}
$$

where $x$ is the upper left entry of $Y$.

Note, for $B$ in Proposition E.19 if $B$ is not $\text{El}(n, R)$-equivalent to a 1-stabilization of the matrix $(p) \oplus I_{n-1}$, then no 1-stabilization of $B$ is $\text{El}(n, R)$-equivalent to a 1-stabilization of the matrix $(p) \oplus I_{n-1}$.

REFERENCES


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