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Maninska, Laura; Pivotto, Irene; Roberson, David E.; Royle, Gordon F.

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Cores of Cubelike Graphs

Laura Mančinska
mancinska@math.ku.dk

Irene Pivotto
piv8irene@gmail.com

David E. Roberson
dero@dtu.dk

Gordon F. Royle
gordon.royle@uwa.edu.au

Abstract

A graph is cubelike if it is a Cayley graph for some elementary abelian 2-group \( \mathbb{Z}_2^n \). The core of a graph is its smallest subgraph to which it admits a homomorphism. More than ten years ago, Nešetřil and Šámal (On tension-continuous mappings. European J. Combin., 29(4):1025–1054, 2008) asked whether the core of a cubelike graph is cubelike, but since then very little progress has been made towards resolving the question. Here we investigate the structure of the core of a cubelike graph, deducing a variety of structural, spectral and group-theoretical properties that the core “inherits” from the host cubelike graph. These properties constrain the structure of the core quite severely — even if the core of a cubelike graph is not actually cubelike, it must bear a very close resemblance to a cubelike graph. Moreover we prove the much stronger result that not only are these properties inherited by the core of a cubelike graph, but also by the orbital graphs of the core. Even though the core and its orbital graphs look very much like cubelike graphs, we are unable to show that this is sufficient to characterise cubelike graphs. However, our results are strong enough to eliminate all non-cubelike graphs on up to 32 vertices as potential cores of cubelike graphs (of any size). Thus, if one exists at all, a cubelike graph with a non-cubelike core has at least 128 vertices and its core has at least 64 vertices.

1 Introduction

A homomorphism from a graph \( X \) to a graph \( Y \) is an adjacency-preserving, but not necessarily injective, function from \( V(X) \) to \( V(Y) \). If such a function exists, then we say that \( X \) maps to \( Y \), and write \( X \to Y \). As the pre-image of any vertex of \( Y \) is a coclique of \( X \) it is easy to see that a homomorphism \( X \to K_k \) is equivalent to a \( k \)-colouring of \( X \). Therefore the chromatic number of \( X \) can be defined as the smallest \( k \) such that \( X \to K_k \). As a result, homomorphisms are often viewed as a generalization of colourings, and indeed a number of variants of graph colouring can be succinctly expressed in terms of the existence of homomorphisms to particular families of graphs.

Two graphs \( X \) and \( Y \) are homomorphically equivalent, denoted \( X \leftrightarrow Y \), if \( X \) maps to \( Y \) and \( Y \) maps to \( X \). Justifying its name, homomorphic equivalence is an equivalence relation on the set of all (unlabelled) graphs. As far as any graphical parameter or property related to colouring is concerned, the graphs in each equivalence class are “the same”. A graph \( X \) is called a core if it has no homomorphisms to any subgraph of itself with fewer vertices (the name suggests some sort of
incompressibility”). Each homomorphic equivalence class contains a unique core, which is thus a natural canonical representative for the equivalence class. If $X$ is a graph, then $X^\bullet$ denotes the core in the equivalence class containing $X$, and will be called the core of $X$. The core of a graph $X$ is always an induced subgraph of $X$. A key tool used repeatedly in what follows is that if $Y$ is a copy of $X^\bullet$ contained in $X$, then there exists at least one special homomorphism, called a retraction, that maps $X$ onto $Y$ and fixes $Y$ vertex-wise.

From the definitions, it follows that in general $X$ and its core will have the same values for any graph parameter that can be viewed as “colouring-related”, such as the chromatic number or its variants like the fractional chromatic number. More surprising however, is that various symmetry-related properties of $X$ are shared by $X^\bullet$. Hahn and Tardif [?] showed that if $X$ is vertex-transitive, then $X^\bullet$ is also vertex-transitive, and similarly for a number of stronger symmetry properties. These connections are surprising because although the composition of an automorphism of $X$ and a retraction onto $X^\bullet$ is a mapping from $\text{Aut}(X)$ to $\text{Aut}(X^\bullet)$, it is not a group homomorphism. Although having a vertex-transitive automorphism group is a symmetry property inherited by the core of a graph, the stronger property of having a vertex-regular automorphism group is not, for there are Cayley graphs whose core is not a Cayley graph. In fact, any vertex-transitive core is the core of some Cayley graph, and so any non-Cayley vertex-transitive core provides an immediate example: the Petersen graph is an obvious choice here.

Although the class of all Cayley graphs is not closed under taking cores, there are particular sub-families of Cayley graphs that might be. In this paper we consider cubelike graphs, which are the Cayley graphs for an elementary abelian group $\mathbb{Z}_2^n$, originally studied and named by Lovász [?] as a particular family of graphs with only integer eigenvalues. Nešetřil and Šámal [?] asked whether or not the core of a cubelike graph is cubelike, but given the supporting evidence, and the length of time that this question has been open, we feel that their question should be upgraded to a conjecture.

**Conjecture 1.1.** The core of a cubelike graph is cubelike.

In this paper, we consider the structure of the core of a cubelike graph, showing that such a graph is heavily constrained with respect to a variety of graphical, spectral and group-theoretical properties. For graph-theoretical properties it is well-known that if $Z$ is a cubelike graph with core $X$, then $X$ inherits the clique number and chromatic numbers of $Z$. If $Z$ is a cubelike graph of valency $d$, then there is a covering map (a locally-bijective surjective homomorphism) from the $d$-cube $Q_d \rightarrow Z$, which then implies that the eigenvalues of $Z$ are a sub-multiset of the eigenvalues of $Q_d$. We show that the core of a cubelike graph of valency $d$ is also covered by $Q_d$ and so its eigenvalues are also constrained. For group-theoretical properties, it is well-known that the core of a cubelike graph has a generously-transitive automorphism group, which again is a property shared by all cubelike graphs.

Much more strongly we show that if $Z$ is a cubelike graph with core $X$, then it is not only $X$ that must have all of these properties, but all of the orbital graphs of $X$ (that is, graphs with vertex set $V(X)$ whose automorphism group contains $\text{Aut}(X)$) must also share them. Given a particular core $X$, it can often be ruled out as a possible core for a cubelike graph by showing that it, or any one of its orbital graphs, does not have one or more of these properties.

However, despite these quite severe restrictions, we have not been able to show that they characterise cubelike graphs in general. To give some indication of how restrictive these conditions are, we apply the battery of tests to the 677116 connected vertex-transitive graphs on 32 vertices and demonstrate that the only graphs that meet all of our necessary conditions to be the core of a cubelike graph are themselves cubelike. As a result, if a cubelike graph with a non-cubelike core exists, then it has at least 128 vertices and its core has at least 64 vertices. We emphasize here
that our results are “looking upwards” – showing that a particular (non-cubelike) vertex-transitive graph cannot be the core of any cubelike graph of any size, rather than the much easier situation of “looking downwards” and showing that a particular cubelike graph does in fact have a cubelike core. At least in principle, this approach could resolve the conjecture completely if enough properties of the core of a cubelike graph could be found that only a cubelike graph could satisfy them.

In the study of homomorphisms of vertex-transitive graphs, a particular family of Cayley graphs seems to play a special role. We say that the Cayley graph Cay(Γ,S) is normal if S is closed under conjugation by elements of Γ. Normal Cayley graphs have a number of attractive properties of particular relevance to colouring and other graph homomorphisms. In particular, Larose, Laviolette and Tardif [?] showed that a graph X is hom-idempotent, i.e., homomorphically equivalent to its Cartesian square X □ X, if and only if X is homomorphically equivalent to a normal Cayley graph, and raised the following question.

**Question 1.2.** Is the core of a normal Cayley graph itself a normal Cayley graph?

Every Cayley graph of an abelian group is a normal Cayley graph, but there are many vertex-transitive graphs that are not normal Cayley graphs.

## 2 Preliminaries

We refer to the reader to Godsil and Royle [?] for most of the algebraic graph theory background we need, but use this section to highlight terminology, notation and results that will be of particular relevance to the rest of this paper.

### Covers and Eigenvalues

A homomorphism ϕ from X to Y is a covering map if it is surjective and locally bijective. The latter means that for every u ∈ V(X), the restriction of the map ϕ to the neighborhood N(u) of u is a bijection between N(u) and N(ϕ(u)). This implies that u and ϕ(u) have the same degree, and that ϕ also acts surjectively on edges. If there is a covering map from X to Y, then we say that X covers Y or that X is a cover of Y.

Suppose that ϕ is a covering map from X to Y. If f : V(Y) → C is an eigenvector of Y, then f ◦ ϕ : V(X) → C is an eigenvector of X with the same eigenvalue. (Brouwer and Haemers [?, Section 6.4]). It is easy to see that the linear map f ↦ f ◦ ϕ is injective. So if X covers Y, then the eigenvalues of Y are a submultiset of the eigenvalues of X. This will be important for Corollary 7.4.

### Normal Cayley Graphs

Let Γ be a group and let C ⊆ Γ be an inverse-closed subset of the elements of Γ that does not contain the identity element. Then the Cayley graph on Γ with connection set C, denoted Cay(Γ,C), is the graph with vertex set Γ where a and b are adjacent if and only if ba⁻¹ ∈ C. The two constraints on C guarantee that Cay(Γ,C) is loopless and undirected. Right-multiplication by an element of Γ induces an automorphism of Cay(Γ,C), and the set of all such automorphisms is a regular subgroup of Cay(Γ,C).

In contrast, left multiplication by a group element does not necessarily induce an automorphism, since it is possible that ba⁻¹ ∈ C but (gb)(ga)⁻¹ = g(ba⁻¹)g⁻¹ ∉ C. If C is closed under conjugation however, then left-multiplication by any element of Γ is an automorphism of Cay(Γ,C), and conversely. This occurs if and only if C is the union of conjugacy classes of Γ, and in this situation we say that Cay(Γ,C) is a normal Cayley graph. If Γ is abelian then all of the Cayley graphs of Γ are normal. Of particular importance to us is the fact that a normal Cayley graph X is generously

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1 The term normal Cayley graph is also used to refer to Cayley graphs X on a group Γ where the regular subgroup of Aut(X) corresponding to right multiplication by elements of Γ is a normal subgroup of Aut(X).
transitive, meaning that for any pair of distinct vertices $v, w$, there is an element of $\text{Aut}(X)$ that exchanges $v$ and $w$.

**Cliques, cocliques and colourings** As usual, we let $\omega(X)$ denote the size of the largest clique of $X$, $\alpha(X)$ denote the size of the largest coclique of $X$, and $\chi(X)$ denote the chromatic number of $X$. When $X$ is vertex transitive these three parameters are related by the inequalities

$$\alpha(X)\omega(X) \leq |V(X)| \leq \alpha(X)\chi(X).$$

The first inequality is known as the clique-coclique bound for vertex-transitive graphs while the second inequality holds for any graph. If $\omega(X) = \chi(X)$ then all three expressions are equal, and so the clique-coclique bound holds with equality. The converse is not true in general as there are graphs $X$ (even Cayley graphs) for which $\alpha(X)\omega(X) = |V(X)| < \alpha(X)\chi(X)$.

If $X$ is a normal Cayley graph, then the converse does hold (see [?]) and so $\alpha(X)\omega(X) = n$ if and only if $\omega(X) = \chi(X)$ which in turn holds if and only if $X^\bullet$ is the complete graph $K_{\omega(X)}$. Using the notion of homomorphic equivalence, we can extend this slightly to the following:

**Lemma 2.1.** Let $X$ be a vertex-transitive graph that is homomorphically equivalent to a normal Cayley graph. Then $\omega(X) = \chi(X)$ if and only if $\alpha(X)\omega(X) = |V(X)|$.

**Proof.** Let $Y$ be a normal Cayley graph homomorphically equivalent to $X$. Since $\omega(X) = \omega(Y)$ and $\chi(X) = \chi(Y)$, these two parameters are equal for $X$ if and only if they are equal for $Y$. Also, since they are vertex transitive and homomorphically equivalent, it follows that $|V(X)|/\alpha(X) = \chi_f(X) = \chi_f(Y) = |V(Y)|/\alpha(Y)$ (where $\chi_f(X)$ is the fractional chromatic number of $X$). Therefore the clique-coclique bound holds with equality for $X$ if and only if it does so for $Y$. Since the statement of the lemma holds for normal Cayley graphs, the above two equivalences prove the statement for $X$. \qed

In Section 3.1 we will see that the core of a vertex-transitive graph must be vertex transitive. Combining this with the above observations provides a useful tool for ruling out potential cores of normal Cayley graphs.

**Cubelike graphs.** A cubelike graph is a Cayley graph for an elementary abelian 2-group $\mathbb{Z}_2^n$. As one would hope, the $n$-cube $Q_n$ is itself cubelike, because it is isomorphic to $\text{Cay}(\mathbb{Z}_2^n, \{e_1, \ldots, e_n\})$, where $e_i$ is the $i$-th standard basis vector. (Whenever it is convenient to do so, we view $\mathbb{Z}_2^n$ as a vector space, and its elements as binary vectors of length $n$.) A cubelike graph $Z = \text{Cay}(\mathbb{Z}_2^n, C)$ is connected if and only if $C$ contains a basis of $\mathbb{Z}_2^n$, and in this case we may assume that it contains the standard basis. There are two special families of cubelike graphs that we will need later.

- **Folded cube:** For integer $n > 1$, let $C = \{e_1, \ldots, e_{n-1}, e_1 + \ldots + e_{n-1}\}$. The cubelike graph $\text{Cay}(\mathbb{Z}_2^{n-1}, C)$ is known as the folded cube of order $n$, or folded $n$-cube. This graph can be constructed from the $(n - 1)$-cube by adding edges between vertices at maximum distance $n - 1$ from each other, or by identifying pairs of vertices of the $n$-cube at maximum distance from each other. The folded $n$-cube has degree $n$ and is bipartite when $n$ is even and has chromatic number 4 when $n$ is odd. The folded 5-cube is also known as the Clebsch graph.

- **Halved cube:** For integer $n > 1$, let $C = \{e_1, \ldots, e_{n-1}\} \cup \{e_i + e_j : 1 \leq i < j \leq n-1\}$. The cubelike graph $\text{Cay}(\mathbb{Z}_2^{n-1}, C)$ is the halved $n$-cube, denoted $\frac{1}{2}Q_n$. The distance-2 graph of the $n$-cube has two connected components (the even / odd weight vertices respectively), and each is isomorphic to $\frac{1}{2}Q_n$. It is also isomorphic to the graph obtained from $Q_{n-1}$ by
adding edges between vertices at distance two. The clique number of $\frac{1}{2}Q_n$ is $n$ unless $n = 3$ in which case the clique number is 4. The halved 5-cube is also known as the complement of the Clebsch graph.

3 First Observations

Here we discuss some of the basic properties of endomorphisms and cubelike graphs that we will make use of throughout the rest of the paper. We begin by presenting some previously known results about cores of cubelike graphs. Next we consider two ideas for proving Conjecture 1.1 and then show why these ideas do not work. Finally we prove some new results about endomorphisms of graphs that we will use for our later results.

3.1 Previous results

The core of a graph $X$ inherits many of the homomorphic properties of $X$, such as the clique number, chromatic number, and fractional chromatic number. For such parameters, any value that is never attained by a cubelike graph can also never be attained by the core of a cubelike graph. A cubelike graph can never have clique number equal to 3 because any triangle in a cubelike graph is contained in a 4-clique. It follows that the core of a cubelike graph also cannot have clique number equal to 3. Similarly, Payan [?] proved that no cubelike graph can have chromatic number equal to 3, and so the same is true for the core of a cubelike graph. These two results will be useful when ruling out potential cores of cubelike graphs in Section 8.2.

Additionally, as mentioned in Section 1, the core of a graph also inherits some of the symmetry properties of the graph. In particular, if $\text{Aut}(X)$ is transitive, generously transitive, or primitive, then $\text{Aut}(X^\bullet)$ is also transitive, generously transitive, or primitive respectively [?]. As cubelike graphs are normal Cayley graphs and therefore generously transitive, it follows that the core of a cubelike graph is also generously transitive.

Finally, it is a result of Hahn and Tardif [?] that if $X$ is vertex transitive, then $|V(X^\bullet)|$ divides $|V(X)|$. Since cubelike graphs are vertex transitive and have $2^n$ vertices for some $n$, this result implies that the core of a cubelike graph has $2^k$ vertices for some $k$. Summarizing all of the above, we see that if a graph $Z$ is the core of a cubelike graph, then the following hold:

1. $Z$ has $2^k$ vertices for some $k \in \mathbb{N},$
2. $Z$ is generously transitive, and
3. $Z$ does not have clique or chromatic number equal to three.

3.2 Two false leads

At first sight, there does not seem to be any strong reason to believe that Conjecture 1.1 should hold. Although the core of a vertex-transitive graph is vertex transitive, there are many Cayley graphs whose cores are not Cayley graphs. Since a graph $X$ is a Cayley graph for a group $\Gamma$ if and only if $\Gamma$ is a regular subgroup of $\text{Aut}(X)$, this means that the property that “$\text{Aut}(X)$ contains a regular subgroup” is not preserved by taking cores. Thus Conjecture 1.1 asserts that something very special happens when the group in question is $\mathbb{Z}_2^n$.

In our experience, researchers encountering this question for the first time almost always quickly arrive at two stronger statements, either of which would imply that cores of cubelike graphs are cubelike. Each of the statements essentially says that there is a copy of the core embedded “nicely” in the host cubelike graph—sufficiently nicely that it can be shown that it is cubelike itself. We
explain these two plausible statements and give counterexamples to each in order to illustrate some
the difficulties involved in precisely identifying a copy of the core, and to justify the lengths to which
we are forced to go to gather information about the core. If this helps dissuade future researchers
from going down these same dead-ends, then that will be a bonus.

False lead 1: There is a copy of the core induced by a subgroup of the vertices Consider
a Cayley graph $X = Cay(\Gamma, C)$. If $Y$ is an induced subgraph of $X$ such that $V(Y) = \Gamma'$ where $\Gamma'$
is a subgroup of $\Gamma$, then it is easy to see that $Y$ is isomorphic to $Cay(\Gamma', C \cap \Gamma')$ and is therefore
a Cayley graph for the group $\Gamma'$. Since any subgroup of $Z_2^n$ is isomorphic to $Z_2^k$ for some $k$, if
a cubelike graph $Z$ contains a copy of its core on a subgroup of vertices, then its core must be
cubelike. Unfortunately this does not always happen for cubelike graphs. Consider the halved
8-cube $\frac{1}{2}Q_8$. The set $\{0, e_1, \ldots, e_7\}$ induces a complete graph $K_8$ in $\frac{1}{2}Q_8$ and as it is known that
$\chi(\frac{1}{2}Q_8) = 8$, this complete graph is its core. A short case analysis shows that the largest subgroup
of vertices forming a clique has size only 4. The same argument can be used to show that for all
$r \geq 3$, the graph $\frac{1}{2}Q_{2r}$ has core $K_{2r}$ but no subgroup of vertices of size $2^r$ forms a clique.

False lead 2: There is a homomorphism to the core whose fibres are the cosets of some
subgroup. In Lemma 7.3, we show that if $Z$ is a Cayley graph on $Z_2^n$ and $\varphi$ is a vertex- and edge-
surjective homomorphism to a graph $X$ such that the fibres of $\varphi$ are cosets of some fixed subgroup
of $Z_2^n$, then $X$ must be a cubelike graph. Any homomorphism from a graph to its core is necessarily
vertex- and edge-surjective, and so Conjecture 1.1 would follow if even one such homomorphism
had suitable fibres. However, as we now show, the complement of the halved 8-cube $\frac{1}{2}Q_8$ provides
an example where no such homomorphism exists.

Let $Z$ denote the complement of the halved 8-cube $\frac{1}{2}Q_8$. Because $\frac{1}{2}Q_8$ meets the clique-coclique
bound with equality, so does $Z$, and therefore by Lemma 2.1 we have

$$\alpha(Z) = 8, \quad \omega(Z) = 16 \quad \text{and} \quad \chi(Z) = 16.$$ 

Any homomorphism from $Z$ to its core is a 16-colouring of $Z$ and so each of its colour classes
is an independent set of size 8 or, equivalently, a clique in $\frac{1}{2}Q_8$. But if even one colour class is
a coset of a subgroup, then some translate of that colour class is a subgroup, contradicting our
previous observation that the maximum size of a subgroup forming a clique in $\frac{1}{2}Q_8$ is 4. The same
argument can be used to show that for $r \geq 3$, the core of the complement of $\frac{1}{2}Q_{2r}$ is $K_{2r-1}$, but
no homomorphism to the core has fibres that are cosets of a fixed subgroup.

3.3 Properties of Endomorphisms

In this section, we prove some preliminary lemmas that we will need later, and that may be useful
more generally. The basic idea is that endomorphisms of a graph $Z$ must act as isomorphisms
between the subgraphs of $Z$ isomorphic to its core. More precisely, we have the following:

Lemma 3.1. Let $Z$ be a graph and $X$ be an induced subgraph of $Z$ isomorphic to $Z^*$. If $\varphi$ is an
endomorphism of $Z$ then, when restricted to $V(X)$, the map $\varphi$ is an isomorphism from $X$ to the
graph induced by $\varphi(V(X))$.

Proof. Let $Y$ be the subgraph induced by $\varphi(V(X))$ and let $\rho$ be a retraction of $Z$ onto $X$. Obviously,
the restriction $\varphi|_X$ is a homomorphism from $X$ to $Y$ and the restriction $\rho|_Y$ is a homomorphism
from $Y$ to $X$. Thus $\rho|_Y \circ \varphi|_X$ is a homomorphism from $X$ to itself and is therefore an automorphism.
since \( X \) is a core. It follows that \( \varphi|_X \) preserves non-edges (thus is injective), and it is surjective by definition. Therefore \( \varphi|_X \) is an isomorphism from \( X \) to \( Y \).

The following result, proved in Hahn & Tardif [?], shows that the distance between two vertices in the core of a graph is equal to their distance in the graph. We use \( \text{dist}_X(u,v) \) to denote the distance in \( X \) (length of a shortest path in \( X \)) between vertices \( u \) and \( v \).

**Lemma 3.2.** Let \( Z \) be a graph and \( X \) be an induced subgraph of \( Z \) isomorphic to \( Z^* \). Then for any \( u,v \in V(X) \), we have \( \text{dist}_X(u,v) = \text{dist}_Z(u,v) \).

Combining the two above lemmas, we arrive at the following:

**Lemma 3.3.** Let \( Z \) be a graph and \( X \) be an induced subgraph of \( Z \) isomorphic to \( Z^* \). If \( \varphi \) is an endomorphism of \( Z \), then for any \( u,v \in V(X) \) we have \( \text{dist}_Z(u,v) = \text{dist}_Z(\varphi(u),\varphi(v)) \).

**Proof.** By Lemma 3.1 the endomorphism \( \varphi \) maps \( X \) isomorphically to some other copy, \( Y \), of the core of \( Z \). Therefore \( \text{dist}_X(u,v) = \text{dist}_Y(\varphi(u),\varphi(v)) \), and together with Lemma 3.2 we have

\[
\text{dist}_Z(u,v) = \text{dist}_X(u,v) = \text{dist}_Y(\varphi(u),\varphi(v)) = \text{dist}_Z(\varphi(u),\varphi(v)).
\]

**Corollary 3.4.** Suppose that \( \rho \) is a retraction of \( Z \) onto a copy \( X \) of its core. If \( u,v \in V(X) \) and there exists \( \sigma \in \text{Aut}(Z) \) such that \( \sigma(u) = u' \) and \( \sigma(v) = v' \), then \( \text{dist}_X(\rho(u'),\rho(v')) = \text{dist}_X(u,v) \).

**Proof.**

\[
\text{dist}_X(\rho(u'),\rho(v')) = \text{dist}_X(\rho \circ \sigma(u),\rho \circ \sigma(v)) = \text{dist}_X(u,v)
\]

by Lemmas 3.2 and 3.3 and the fact that \( \rho \circ \sigma \) is an endomorphism of \( Z \).

Finally, we can apply the above to the special case of cubelike graphs:

**Corollary 3.5.** Let \( Z \) be a cubelike graph, and let \( \rho \) be a retraction of \( Z \) onto a copy \( X \) of its core. If \( a,b \in V(X) \) and \( c + d = a + b \), then

\[
\text{dist}_X(\rho(c),\rho(d)) = \text{dist}_X(a,b) = \text{dist}_Z(a,b).
\]

**Proof.** Adding \( a + c \) to all of the elements of \( \mathbb{Z}_2^n \) is an automorphism of \( Z \) mapping \( a \) to \( c \) and \( b \) to \( d \). Apply above lemmas/corollaries.

## 4 Cubelike Hulls

Given a graph \( X \), its cubelike hull \( Z_2[X] \) is the graph whose vertex set is the set of even-weight vectors of \( Z^{V(X)}_2 \), and where two vertices are adjacent if their sum is equal to \( e_u + e_v \) for some edge \( uv \) in \( X \). These even weight vectors form a group, say \( \Gamma' \), that is isomorphic to \( \mathbb{Z}_2^{n-1} \) and as \( Z_2[X] = \text{Cay}(\Gamma',\{e_u + e_v : uv \in E(X)\}) \), it is immediate that \( Z_2[X] \) is cubelike for any \( X \). Our two special families of cubelike graphs arise in this fashion, because \( Z_2[C_n] \) is the folded \( n \)-cube and \( Z_2[K_n] \) is the halved \( n \)-cube.

For any fixed vertex \( u \in V(X) \), the set of vertices \( \{e_u + e_v : v \in V(X)\} \) induces a copy of \( X \) in \( Z_2[X] \) and therefore \( X \to Z_2[X] \) for any graph \( X \). Cubelike hulls were introduced by Beaudou, Naserasr, and Tardif [?], who called them power graphs, and proved the following result, which says that \( Z_2[X] \) is the minimal (in the homomorphism order) cubelike graph admitting a homomorphism from \( X \):
Lemma 4.1 (Beaudou, Naserasr, and Tardif [?]). Let $X$ be a graph and $Z$ a cubelike graph. Then $X \rightarrow Z$ if and only if $Z_2[X] \rightarrow Z$. Moreover, the map from $Z_2[X]$ to $Z$ can be taken to be a group homomorphism.

We omit the details of the proof, but note that if $\varphi : X \rightarrow Z$ is a homomorphism then the linear extension of the map taking $u$ to $\varphi(u)$ is a homomorphism from $Z_2[X]$ to $Z$.

The primary focus of this work is the question of whether the core of a cubelike graph is necessarily cubelike. Given a cubelike graph $Z$, it is straightforward how to determine whether its core is cubelike: first find its core $Z^*$, then decide whether it is cubelike by testing whether its automorphism group contains a regular elementary abelian 2-group. However, it is not so clear how to go in the other direction: given a core $X$, how does one determine whether it is the core of some cubelike graph? Indeed, it is not even clear if this problem should be decidable, since in principle one may have to consider every cubelike graph $Z$ and determine whether $Z^* \cong X$. However, the following two lemmas, based on Lemma 4.1, provide us with an algorithm for this problem.

Lemma 4.2. A graph $X$ is homomorphically equivalent to a cubelike graph if and only if it is homomorphically equivalent to $Z_2[X]$.

Proof. If $X$ is homomorphically equivalent to $Z_2[X]$, then it is homomorphically equivalent to a cubelike graph since the latter is cubelike.

Conversely, suppose that $X$ is homomorphically equivalent to a cubelike graph $Z$. Since $X \rightarrow Z$ and $Z$ is cubelike, we have that $Z_2[X] \rightarrow Z \rightarrow X$ by Lemma 4.1. As noted above, $X \rightarrow Z_2[X]$ is true for any $X$ and so $Z_2[X]$ and $X$ are homomorphically equivalent.

Lemma 4.3. A graph $X$ is the core of a cubelike graph if and only if it is the core of $Z_2[X]$.

Proof. If $X$ is the core of $Z_2[X]$, then it is the core of a cubelike graph since the latter is cubelike.

Conversely, if $X$ is the core of a cubelike graph, then by the above lemma $X$ must be homomorphically equivalent to $Z_2[X]$. But since $X$ is a core, it must be the core of $Z_2[X]$.

By the above, in order to determine whether a particular graph $X$ is the core of some cubelike graph, we can check whether it is a core and whether $Z_2[X] \rightarrow X$. Unfortunately, this is not feasible in practice because if $X$ is any graph with more than 8 vertices that might be the core of a cubelike graph, then $X$ has at least 16 vertices and $Z_2[X]$ has at least $2^{15} = 32768$ vertices. However, our results will allow us to prove by hand that if $X$ is the core of a cubelike graph and $|V(X)| \leq 16$, then $X$ is itself cubelike. Additionally, with the aid of a computer we are able to prove this for $|V(X)| \leq 32$.

On the other hand, Lemma 4.1 does provide us with useful necessary conditions for the core of a cubelike graph, which we will use to prove these results. Indeed we have the following:

Lemma 4.4. If $X$ is the core of a cubelike graph, then $Y \rightarrow X$ if and only if $Z_2[Y] \rightarrow X$.

Proof. Suppose that $X$ is the core of a cubelike graph $Z$. If $Z_2[Y] \rightarrow X$, then $Y \rightarrow X$ since $Y \rightarrow Z_2[Y]$. Conversely, suppose that $Y \rightarrow X$. Since $X$ and $Z$ are homomorphically equivalent, we have that $Y \rightarrow Z$ and thus by Lemma 4.1 we have that $Z_2[Y] \rightarrow Z$ and thus $Z_2[Y] \rightarrow X$, again using the homomorphic equivalence of $X$ and $Z$.

This lemma will be of significant use to us in Section 8. In particular, we will apply Lemma 4.4 when $Y$ is a complete graph (so $Z_2[Y]$ is a halved cube) or when $Y$ is an odd cycle (so $Z_2[Y]$ is a folded cube). The next lemma extends results of Beaudou, Naserasr and Tardif [?].

\[ \text{After arriving at these results independently, we learned that they appear in [?].} \]
Lemma 4.5. Let $X$ be a graph that is homomorphically equivalent to a cubelike graph $Z$. If $X$ has odd girth $g < \infty$, then it contains $Z_2[C_g]$ as an induced subgraph.

Proof. Since $X$ has odd girth $g$, we have that $C_g \to X$. Since $X$ is homomorphically equivalent to a cubelike graph, we have that $Z_2[C_g] \to X$. Let $\varphi$ be a homomorphism from $Z_2[C_g]$ to $X$. Since any two vertices of $Z_2[C_g]$ are contained in a cycle of length $g$, and identifying two vertices of an odd cycle results in a graph containing an odd cycle of strictly shorter length, the map $\varphi$ must be injective, i.e., $Z_2[C_g]$ is a subgraph of $X$. Furthermore, adding an edge between any pair of non-adjacent vertices in an odd cycle also results in a graph containing a shorter odd cycle. Therefore, the map $\varphi$ must be faithful, i.e., $Z_2[C_g]$ must be an induced subgraph of X. □

The above lemma is useful because looking for induced subgraphs isomorphic to a graph $X$ is often easier than looking for homomorphisms from $X$. This will also be of theoretical use, for instance in the proof of Theorem 8.

5 Shift Graphs and Hom-Idempotence

The Cartesian product of graphs $X$ and $Y$, denoted $X \square Y$, has vertex set $V(X) \times V(Y)$ and an edge between $(x,y)$ and $(x',y')$ if and only if $x = x'$ and $y \sim y'$, or $x \sim x'$ and $y = y'$. It is immediate that $X \to X \square Y$ and $Y \to X \square Y$. For any graph $X \to X \square X$, but for some graphs $X \square X \to X$ as well, in which case $X$ is called hom-idempotent. Larose, Lavoie, and Tardif [?] noted that if $X = \text{Cay}(\Gamma, C)$ is a normal Cayley graph, then the map $(g,h) \mapsto gh$ is a homomorphism from $X \square X$ to $X$. Thus normal Cayley graphs are always hom-idempotent. If $X$ and $Y$ are homomorphically equivalent, then so are $X \square X$ and $Y \square Y$, and so if $X$ is homomorphically equivalent to a normal Cayley graph, then $X$ is hom-idempotent. Remarkably, Larose et. al. prove the converse: a graph is hom-idempotent if and only if it is homomorphically equivalent to a normal Cayley graph. Moreover, they showed that $X$ is hom-idempotent if and only if it is homomorphically equivalent to a particular normal Cayley graph.

Definition. Given a graph $X$, an automorphism $\sigma \in \text{Aut}(X)$ is a shift of $X$ if $\sigma(x) \sim x$ for all $x \in V(X)$, i.e., $\sigma$ maps each vertex to one of its own neighbors. The shift graph of $X$, denoted $\text{Sh}(X)$, is the Cayley graph $\text{Cay} \left( \text{Aut}(X), S \right)$ where $S$ is the set of shifts of $X$.

Note that the inverse of a shift is a shift $(x \sim \sigma(x) \Rightarrow \sigma^{-1}(x) \sim \sigma^{-1}(\sigma(x)) = x)$, and therefore the shift graph of $X$ is indeed a Cayley graph. Moreover, if $\sigma$ is a shift and $\pi$ is an automorphism, then $\pi \circ \sigma \circ \pi^{-1}(\pi(x)) = \pi(\sigma(x)) \sim \pi(x)$ since $\sigma(x) \sim x$ and $\pi$ is an automorphism. By varying $x$ over all of $V(X)$, we also vary $\pi(x)$ over all of $V(X)$, and thus $\pi \circ \sigma \circ \pi^{-1}$ is a shift. Thus $\text{Sh}(X)$ is a normal Cayley graph. Larose et. al. proved the following [?]:

Lemma 5.1. Let $X$ be a graph. Then the following are equivalent:

1. $X$ is hom-idempotent;
2. $X$ is homomorphically equivalent to a normal Cayley graph;
3. $X$ is homomorphically equivalent to $\text{Sh}(X^*)$. □

For any graph $X$ and vertex $x \in V(X)$, the map $\sigma \mapsto \sigma^{-1}(x)$ is a homomorphism from $\text{Sh}(X)$ to $X$. Therefore, if $X$ is a core then it is homomorphically equivalent to its shift graph if and only if the latter contains the former as an induced subgraph.

The above lemma provides two algorithms for checking whether a given graph $X$ is homomorphically equivalent to some normal Cayley graph: either check if $X$ is hom-idempotent, or check
whether $X^\bullet$ is a subgraph of $\text{Sh}(X^\bullet)$. It is not obvious from the definition that hom-idempotence should have anything to do with normal Cayley graphs. Thus part of the significance of the above lemma is that it hints at a deep connection between homomorphisms and normal Cayley graphs, thus providing some justification for Question 1.2.

6 Orbital Graphs

Given a permutation group $\Gamma$ acting on a set $V$, the orbit of an element $x \in V$ is $\Gamma(x) = \{\gamma(x) : \gamma \in \Gamma\}$. The action of $\Gamma$ on $V$ also induces a natural action on $V \times V$, whose orbits are referred to as the orbitals of $\Gamma$. If $O$ is an orbital, then the set $O^* = \{(y, x) : (x, y) \in O\}$ is also an orbital, called the paired orbital of $O$. If $O^* = O$ then $O$ is a self-paired orbital. Any orbital that contains only pairs of the form $(x, x)$ is called a diagonal orbital.

Each non-diagonal orbital can be viewed as the set of arcs of a directed graph with vertex set $V$. More generally, we define an orbital digraph of $\Gamma$ to be any directed graph with vertex set $V$ whose arc set is the union of non-diagonal orbitals. If the collection of orbitals is closed under taking paired orbitals, then the arcs of the orbital digraph all occur in oppositely-directed pairs, yielding an undirected orbital graph. In our case, the groups under consideration are mostly generously transitive, so every orbital is self-paired, and only orbital graphs arise.

The next theorem relates the orbital digraphs of a graph $X$ to those of its core $X^\bullet$. First some notation: if $E \subseteq V \times V$, then $\Gamma(E) = \{(\gamma(x), \gamma(y)) : (x, y) \in E, \gamma \in \Gamma\}$, and $X(E)$ is the (di)graph with vertex set $V(X)$ and arc/edge set $E$.

**Theorem 6.1.** Let $X$ be a graph and let $\rho$ be a retraction of $X$ onto a copy $Y$ of its core. Suppose that $E$ is a union of orbitals of $\text{Aut}(Y)$ and that $\Gamma \leq \text{Aut}(X)$. Then $\rho : X(\Gamma(E)) \to Y(E)$ is a homomorphism, and so $X(\Gamma(E))$ and $Y(E)$ are homomorphically equivalent.

**Proof.** Suppose that $xy$ is an arc of $X(\Gamma(E))$. We must show that $\rho(x)\rho(y)$ is an arc of $Y(E)$. By the definition of $X(\Gamma(E))$, there exists $ab \in E$ and $\gamma \in \Gamma$ such that $\gamma(ab) = xy$. As $\gamma \in \text{Aut}(X)$, the map $\rho \circ \gamma$ is an endomorphism of $X$ with image $Y$. Thus the restriction $\rho \circ \gamma|_Y$ is an automorphism of $Y$, and as $ab \in E$, it follows that $\rho(x)\rho(y) = \rho \circ \gamma(ab) \in E$. Therefore $X(\Gamma(E)) \to Y(E)$ and conversely $Y(E) \to X(\Gamma(E))$ since the former is a subgraph of the latter. \qed

If we apply the above theorem to Cayley graphs we obtain the following:

**Corollary 6.2.** Let $X$ be a Cayley (di)graph for a group $\Gamma$. Then every orbital (di)graph of $X^\bullet$ is homomorphically equivalent to a Cayley (di)graph for $\Gamma$.

**Proof.** The group $\Gamma$ acts regularly on $V(X)$, so if $E$ is any union of orbitals of $\text{Aut}(X^\bullet)$, then $\Gamma$ acts regularly on $X(\Gamma(E))$. By Theorem 6.1, $X(\Gamma(E))$ is homomorphically equivalent to $X^\bullet(\Gamma(E))$. \qed

If $X$ is the core of a cubelike graph, then each of its orbital graphs is homomorphically equivalent to some cubelike graph, so cannot have clique or chromatic number equal to three. As we will see in Section 8.2, this is a very useful tool for ruling out potential counterexamples to Conjecture 1.1.

7 The Covering Cube Theorem and the Degree Bound

In this section we will show that if $Z$ is a cubelike graph with core $X$, then there is a cubelike subgraph of $Z$ that covers $X$, and that if $X$ has degree $d$, then it is covered by the $d$-cube $Q_d$.

**Lemma 7.1.** Let $Z$ be a connected cubelike graph of degree $d$. Then $Z$ is covered by the $d$-cube $Q_d$. 

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Proof. By assumption $Z = \text{Cay}(\mathbb{Z}_2^n, C)$ for some $C = \{c_1, \ldots, c_d\}$. Let $f : \mathbb{Z}_2^d \to Z$ be the unique linear function such that $f(e_i) = c_i$. As $Z$ is connected, $C$ spans $\mathbb{Z}_2^n$ and so $f$ is surjective. Consider a vertex $x \in V(Q_d)$ and two of its neighbours $x + e_i$, $x + e_j$ for some $i \neq j$. By linearity $f(x + e_i) = f(x) + f(e_i) = f(x) + c_i$ and so $f$ is a homomorphism from $Q_d$ to $Z$. Similarly, $f(x + e_j) = f(x) + c_j$ and because $f(x) + c_i \neq f(x) + c_j$ it follows that $f$ is locally injective. \qed

**Theorem 7.2.** Let $Z$ be a Cayley graph for $\mathbb{Z}_2^n$ and let $\rho$ be a retraction onto a core, $X^\bullet$, which contains 0. Let $X$ be an orbital graph of $X^\bullet$ and let $a_1, \ldots, a_d$ be the neighbors of 0 in $X$. Let $Y = \text{Cay}(\langle a_1, \ldots, a_d \rangle, \{a_1, \ldots, a_d\})$. Then the restriction, $\rho|_Y$, is a covering map from $Y$ to the connected component of $X$ containing 0. Moreover, this implies that this connected component is covered by the $d$-cube.

Proof. Let $O$ be the union of orbitals of $X^\bullet$ such that $X = X^\bullet(O)$, and let $X'$ be the connected component of $X$ containing the vertex 0. Since $\mathbb{Z}_2^n \leq \text{Aut}(Z)$ and $(0, a_i) \in O$ for $i \in [d]$, we have that $Y$ is a subgraph of the orbital graph of $Z$ with edge set $\mathbb{Z}_2^n(O)$. Therefore, by Theorem 6.1 the map $\rho|_Y$ is a homomorphism to $X$. However, $Y$ is connected and thus its image under $\rho$ must be connected. Since $Y$ contains the vertex 0 and $\rho$ is a retraction, we see that $\rho|_Y$ is a homomorphism to $X'$. It remains to show that $\rho|_Y$ is surjective and locally bijective. We first prove the latter.

Since $Y$ has the same degree as $X'$ by construction, it suffices to show that it is locally injective. Consider $y \in V(Y)$ and its neighbors $y + a_1, \ldots, y + a_d$ in $Y$. For distinct $i, j \in [d]$, we have that $(y + a_i) + (y + a_j) = a_i + a_j$. Since $a_i$ and $a_j$ appear in $X^\bullet$ at nonzero distance from each other, we have that $\rho(y + a_i) \neq \rho(y + a_j)$ by Corollary 3.5. Therefore $\rho|_Y$ is locally injective.

Since $\rho|_Y$ is locally injective, the image of $Y$ under this map is a subgraph of $X'$ of minimum degree at least $d$. However, since $X'$ is connected, this implies that the image must be all of $X'$. Therefore, the map $\rho|_Y$ is surjective and thus we have shown it is a covering map.

Finally, by Lemma 7.1 the graph $Y$ is covered by the $d$-cube. Since the composition of covering maps is a covering map, we have that $X'$ is covered by the $d$-cube. \qed

The covering cube theorem has two immediate consequences. The first gives a bound on the number of vertices in the core of a cubelike graph in terms of the degree of the core. We refer to this as the degree bound:

**Corollary 7.3.** Let $X$ be the core of a cubelike graph. If $X'$ is a connected component of an orbital graph of $X$ with degree $d$, then $X'$ has at most $2^d$ vertices. If $X$ has degree $d$, then it has at most $2^{d-1}$ vertices unless $d = 1$ and $X = K_2$.

Proof. Since $X$ is the core of a cubelike graph, it is vertex transitive and thus so are all of its orbital graphs. Thus the connected components of any fixed orbital graph are all isomorphic. If an orbital graph $X'$ of $X$ has degree $d$, then each of its connected components are covered by the $d$-cube by Theorem 7.2 above. The $d$-cube has $2^d$ vertices, and thus each component of $X'$ has at most this many vertices.

In the case of $X$ itself, if it has $2^d$ vertices then the covering map from the $d$-cube must in fact be an isomorphism. However, this implies that $X$ is bipartite and therefore not a core unless $X = K_2$, in which case $d = 1$. Therefore, $X$ has at most $2^{d-1}$ vertices unless $d = 1$ and $X = K_2$. \qed

Note that if we apply the above to the case of degree 3 cores of cubelike graphs, we immediately obtain that such a core has at most 4 vertices. Since it has degree 3 it must also have at least 4 vertices and it must be $K_4$. So we get that the only degree 3 core of a cubelike graph is $K_4$.

Recall from Section 2 that if a graph $X$ covers a graph $Y$, then the eigenvalues of $Y$ are a submultiset of the eigenvalues of $X$, so by the Covering Cube Theorem we have the following:
Corollary 7.4. Let $X$ be the core of a cubelike graph. If $X'$ is a connected component of an orbital graph of $X$ with degree $d$, then its eigenvalues are a sub-multiset of the eigenvalues of the $d$-cube.

The eigenvalues of the $d$-cube are $d - 2i$ for $i = 0, 1, \ldots, d$ with respective multiplicities $\binom{d}{i}$. Thus the core of a cubelike graph, and its orbital graphs, must all have integer eigenvalues.

We remark that there are cores that are covered by a cube but are not the core of any cubelike graph. For instance, let $S$ be the Shrikhande graph and $R_{4,4}$ be the $4 \times 4$ rook graph (the Cartesian product $K_4 \square K_4$). Each is a $(16,6,2,2)$ strongly regular graph, hence they have the same spectrum despite being non-isomorphic. The Shrikhande graph is a core, but it has clique number three, so it cannot be the core of any cubelike graph. The $4 \times 4$ rook graph is cubelike (though not a core). Since $R_{4,4}$ is cubelike, so is its bipartite double cover, and so they are both covered by the 6-cube. However the bipartite double cover of the Shrikhande graph is isomorphic to that of $R_{4,4}$. Therefore, $S$ is also covered by the 6-cube, even though it is not the core of any cubelike graph.

7.1 The equality case of the degree bound

As with many bounds, it is of interest to consider what can be said in the case where the degree bound is met with equality. It is easy to see that if an orbital graph of the core of a cubelike graph has degree $d$ and $2^d$ vertices, then the covering map from the $d$-cube to the orbital graph must be an isomorphism. However, in the case of the core itself, where the bound is $2^{d-1}$ vertices, equality does not imply that the covering map is an isomorphism. In fact, in this case the map will have fibres of size two. We will see that this can still be handled, though with significantly more work. We will make use of the following result which is generally known within the research community, though we were unable to find a reference and thus we include a proof.

Lemma 7.5. Let $\varphi$ be a vertex- and edge-surjective homomorphism from a cubelike graph $Z = \text{Cay}(\mathbb{Z}_2^n, C)$ to a graph $X$, whose fibres are the cosets of a subgroup $\Gamma$ of $\mathbb{Z}_2^n$. Then $X$ is cubelike.

Proof. Since the fibres of $\varphi$ are the cosets of $\Gamma$, we can label the vertices of $X$ with these cosets, which form the quotient group $\mathbb{Z}_2^n / \Gamma$. Define $C' = \{y + \Gamma : \exists g \in \Gamma \text{ s.t. } y + g \in C\} \subseteq \mathbb{Z}_2^n / \Gamma$. Note that $0 + \Gamma \not\subseteq C'$ since $\varphi$ is a homomorphism and therefore its fibres are independent sets.

Since $\varphi$ is edge surjective, we have that $a + \Gamma \sim b + \Gamma$ in $X$ if and only if there exists $g, g' \in \Gamma$ such that $a + g \sim b + g'$ in $Y$ if and only if $(a+b) + (g+g') \in C$ if and only if $C' \ni (a+b)+\Gamma = (a+\Gamma)+(b+\Gamma)$. Therefore, $X$ is the Cayley graph of $\mathbb{Z}_2^n / \Gamma$ with connection set $C'$. Since $\mathbb{Z}_2^n / \Gamma$ must be isomorphic to some power of $\mathbb{Z}_2$, we have proven the lemma.

To characterize the graphs which meet the degree bound, we first need to show that the only cubelike cores that meet it are the folded cubes of odd order.

Lemma 7.6. Let $X$ be a cubelike graph that is a core, has degree $d$, and has $2^{d-1}$ vertices. Then $d$ is odd and $X$ is the folded cube of order $d$.

Proof. Since $X$ is a core it is connected and therefore its connection set $C$ contains a basis. Without loss of generality we may assume that $C$ contains the $d-1$ standard basis vectors and one other element $c$. If $c$ is not the all-ones vector, then we may assume that $c = \sum_{i=1}^{m} e_i$ for some $m < d-1$. Define an endomorphism $\varphi$ of $X$ as the linear extension of the map fixing $e_i$ for $i \leq m$ and sending $e_i$ to $c$ for $i > m$. Note that this maps every element of $C$ to an element of $C$. Suppose that $x \sim y$, i.e., $x + y \in C$. Then $\varphi(x) + \varphi(y) = \varphi(x + y) \in C$, and thus $\varphi(x) \sim \varphi(y)$. Therefore $\varphi$ is a proper endomorphism (nothing is mapped to $e_i$ for $i > m$), contradicting the assumption that $X$ is a core. Therefore $c = e_1 + \cdots + e_{d-1}$ and $X$ is the folded $d$-cube. If $d$ is odd then this graph is indeed a core, but if $d$ is even then $X$ is bipartite, contradicting the assumption that $X$ is a core.
We can now characterize the graphs meeting the degree bound with equality:

**Theorem 7.7.** Suppose that $X$ is the core of a cubelike graph and that $X$ has valency $d$ and $2^{d-1}$ vertices. Then $d$ is odd and $X$ is the folded cube of order $d$.

**Proof.** Let $e_1, \ldots, e_d$ be the standard basis vectors of $\mathbb{Z}^d$, and let $\varphi$ be the covering map from the $d$-cube to $X$ guaranteed by Theorem 7.2. First we will show that $X$ is cubelike by proving that the fibres of $\varphi$ are the cosets of a two-element subgroup and applying Lemma 7.3.

Let $x$ be a vertex of $X$. Then there exists a $y \in \mathbb{Z}^d$ such that $x = \varphi(y)$. Since $X$ is not bipartite, it contains an odd cycle, and as it is vertex transitive, the vertex $\varphi(y)$ must be contained in some shortest odd cycle. Since $\varphi$ is a covering map, the vertices of this cycle can be written as

$$
\varphi(y), \varphi(y + e_{i(1)}), \varphi(y + e_{i(1)} + e_{i(2)}), \ldots, \varphi(y + e_{i(1)} + \cdots + e_{i(g)}) = \varphi(y),
$$

for some odd $g$ and $i(j) \in [d]$ for all $j \in [g]$. However, if $i(j) = i(j')$ for any $j \neq j'$, then removing $e_{i(j)}$ and $e_{i(j')}$ in all of the above terms in which they appear, and then removing the two redundant terms, would result in a shorter odd cycle of $X$, a contradiction. Therefore the $e_{i(j)}$ are distinct and we may assume without loss of generality that the vertices of the odd cycle have the form

$$
\varphi(y), \varphi(y + e_1), \varphi(y + e_1 + e_2), \ldots, \varphi(y + e_1 + \cdots + e_g) = \varphi(y).
$$

Let $\Delta = \sum_{i=1}^g e_i \neq 0$. We have that $y + \Delta \neq y$, but $\varphi(y) = \varphi(y + \Delta)$. So $y$ and $y + \Delta$ are the two vertices in the preimage of $x$. We aim to show that the sum of two vertices in a fibre of $\varphi$ is always equal to $\Delta$, i.e. every fibre is a coset of the subgroup $\{0, \Delta\}$. We will show this for the neighbors of $\varphi(y)$ and thus it will hold by connectivity of $X$.

Since $x$ was arbitrary, the above implies that the sum of two vertices in a fibre of $\varphi$ must have weight equal to the length of the shortest odd cycle of $X$, which is $g$. The neighbors of $\varphi(y)$ in $X$ can be written in the form $\varphi(y + e_i)$ for $i \in [d]$. Moreover, there is a bijection $f : [d] \to [d]$ such that $\varphi(y + e_i) = \varphi(y + \Delta + e_f(i))$ for all $i \in [d]$. We want to show that $f(i) = i$. The vertices in the preimage of $\varphi(y + e_i)$ have sum $\Delta + e_i + e_{f(i)}$. If $i \in [g]$ and $i \neq f(i) \in [g]$, then $\Delta + e_i + e_{f(i)}$ has weight $g - 2$, a contradiction. Similarly, if $i \in [d] \setminus [g]$ and $i \neq f(i) \in [d] \setminus [g]$, then $\Delta + e_i + e_{f(i)}$ has weight $g + 2$, a contradiction. So $f$ either fixes an element $i \in [d]$ or maps it to whichever of $[g]$ and $[d] \setminus [g]$ that does not contain $i$. Therefore, showing that $f$ fixes the elements of $[g]$ implies that it fixes all of the elements of $[d]$.

Now consider moving around the same shortest odd cycle we considered above, but this time we write the vertices as

$$
\varphi(y + \Delta), \varphi(y + \Delta + b_1), \varphi(y + \Delta + b_1 + b_2), \ldots, \varphi(y + \Delta + b_1 + \cdots + b_g) = \varphi(y + \Delta),
$$

where the $b_j$ are some distinct elements of $\{e_1, \ldots, e_d\}$. As above, we have that $y + \Delta + b_1 + \ldots + b_g \neq y + \Delta$, but they have the same image under $\varphi$. This implies that

$$
y = y + \Delta + b_1 + \ldots + b_g = y + \sum_{i=1}^g e_i + \sum_{i=1}^g b_i.
$$

From this we see that $\sum_{i=1}^g e_i = \sum_{i=1}^g b_i$, which implies that $\{b_1, \ldots, b_g\} = \{e_1, \ldots, e_g\}$. Since $\varphi(y + e_1) = \varphi(y + \Delta + b_1)$ we have that $e_{f(1)} = b_1$, and since $b_1 \in \{e_1, \ldots, e_g\}$, we have that $f(1) \in [g]$. By the above this implies that $f(1) = 1$. By permuting the elements of $\{e_1, \ldots, e_g\}$, we can obtain different shortest odd cycles containing $\varphi(y)$ which will similarly imply that $f(i) = i$ for all $i \in [g]$. From this and the above arguments we see that $f$ fixes every element of $[d]$ and therefore...
the sum of two vertices in the preimage of any neighbor of \( \varphi(y) \) is equal to \( \Delta \), and by connectivity this is true for every fibre of \( \varphi \).

By the above and Lemma \ref{lem6.3}, we have that \( X \) is cubelike, and then by Lemma \ref{lem6.6} we have proven the theorem.

Together, Corollary \ref{cor6.3} and Theorem \ref{thm7.7} immediately give the following corollary:

**Corollary 7.8.** If \( X \) is the core of a cubelike graph and has even degree \( d \), then \( |V(X)| \leq 2^{d-2} \).

Letting \( d = 4 \) in the above corollary, we see that any degree 4 core of a cubelike graph has at most 4 vertices, but no such (simple) graph exists. It is also not too difficult to use Theorem \ref{thm7.7} to show that the only degree 5 core of a cubelike graph is the Clebsch graph.

## 8 Small Cores

In this section we combine all of our tools from previous sections to show that if \( X \) is the core of a cubelike graph and \( |V(X)| \leq 32 \), then \( X \) is itself cubelike. In the next subsection, we prove this by hand when \( |V(X)| \leq 16 \), and in the following we use a computer to prove the same for \( |V(X)| = 32 \).

### 8.1 A fistful of vertices . . .

**Theorem 8.1.** Suppose that \( X \) is the core of a cubelike graph and \( |V(X)| \leq 16 \). Then \( X \) is either \( K_2, K_4, K_8, K_{16} \), or the Clebsch graph or its complement.

**Proof.** We will start with the \( |V(X)| = 16 \) case. We know that the core of an abelian Cayley graph is complete if and only if the clique-coclique bound holds with equality. Furthermore, the clique-coclique bound holds with equality for an abelian Cayley graph if and only if it holds for its core. Therefore, the clique-coclique bound does not hold for the core of a cubelike graph unless that core is a complete graph.

Recall that a cubelike graph cannot have clique number equal to three, and therefore neither can the core of a cubelike graph. Suppose that \( X \) is the core of a cubelike graph with \( |V(X)| = 16 \). If \( X \) and its complement contain a \( K_3 \), then they both contain a \( K_4 \) and thus the clique-coclique bound holds with equality, a contradiction. So either \( X \) or its complement is triangle-free.

Suppose that \( X \) is triangle-free. Since it is a core, it must not be bipartite. It therefore has odd girth \( 2g + 1 \geq 5 \) and thus must contain \( \mathbb{Z}_2[C_{2g+1}] \) as an induced subgraph by Lemma \ref{lem4.5}. If \( 2g + 1 \geq 7 \) this graph has more than 16 vertices, a contradiction. Therefore, \( 2g + 1 = 5 \) and \( X \) contains an induced Clebsch graph which has 16 vertices and so \( X \) must be the Clebsch graph.

Now suppose that the complement of \( X \) is triangle-free. If the complement is empty, then \( X = K_{16} \). If the complement is a non-empty bipartite graph, then it must have a perfect matching since it is vertex transitive, and therefore we can colour \( X \) with 8 colours. Furthermore, one side of the bipartition of the complement of \( X \) will induce a \( K_8 \) in \( X \). This implies that \( X \) is homomorphically equivalent to \( K_8 \), a contradiction. Otherwise the complement has an odd cycle and since it is homomorphically equivalent to a cubelike graph by Corollary \ref{cor6.2} it must contain a folded cube as an induced subgraph. By a similar argument to the previous case this folded cube must be exactly the Clebsch graph. Therefore \( X \) is the complement of the Clebsch graph. These are the only possibilities for \( |V(X)| = 16 \).

Suppose \( |V(X)| = 8 \). If \( X \) contains a triangle then it contains a \( K_4 \). If its complement is not empty the clique-coclique bound will hold with equality which is a contradiction. Therefore if \( X \)
has a triangle it must be \( K_8 \). If \( X \) is triangle-free it must contain an induced folded cube with at least 16 vertices, a contradiction. Therefore \( K_8 \) is the only core of a cubelike graph on 8 vertices.

It is easy to see that the only cores on four and two vertices are \( K_4 \) and \( K_2 \) respectively. 

\[ \square \]

### 8.2 \( \ldots \) and a few vertices more

For the case where the putative core of a cubelike graph has 32 vertices, we require the aid of a computer, and need to use all of our previous results to rule out every non-cubelike vertex-transitive graph as a possible core. We start with the complete list of the transitive permutation groups of degree 32, which was found by Cannon & Holt [?] and is available in the computer algebra system Magma. From this, it is relatively straightforward to compute all of the 677116 connected vertex-transitive graphs of order 32, and this list forms our initial set of candidates for the core of a (possibly much larger) cubelike graph. From this list, we remove graphs that cannot be the core of any cubelike graph, because they violate one or more of the conditions outlined in the lemmas of the previous sections. In effect each of these lemmas is used as a filter to eliminate unsuitable graphs from the list of candidates. After applying what is (in retrospect) a surprising number and variety of these filters, every possibility is eliminated.

1. **Integer Eigenvalues:** from 677116 graphs to 8648 graphs

   By Theorem 7.2 if the core of a cubelike graph is \( d \)-regular, then it is covered by the \( d \)-cube \( Q_d \), and hence its eigenvalues are a submultiset of the eigenvalues of \( Q_d \) as noted in Corollary 7.4. As \( Q_d \) has all integer eigenvalues, so does the core of any cubelike graph. We used Magma to compute and factor the characteristic polynomial of each of the candidate graphs, retaining only those with integral spectrum.

   This proved to be a very powerful test, eliminating just over 98.72% of the candidate graphs, leaving only 8648 to proceed to the next testing phase.

2. **Clique-Coclique Bound Equality:** from 8648 graphs to 1966 graphs

   By Lemma 2.1 if \( X \) is vertex transitive and homomorphically equivalent to a cubelike graph, then it satisfies the clique-coclique bound with equality if and only if \( X \) is complete. Therefore, if \( X \) is the core of a cubelike graph that satisfies the clique-coclique bound with equality, then it must itself be a complete graph. In this case, \( X \) is cubelike since it must have a power of two vertices. Thus no graph \( X \) with \( \alpha(X)\omega(X) = |V(X)| \) can be a non-cubelike core of a cubelike graph, and so these may be filtered out from the list of candidate graphs.

3. **Generously Transitive:** from 1966 to 318 graphs

   As a cubelike graph is generously transitive, its core must also be generously transitive and so we can remove any graph from the list that does not have a generously transitive automorphism group. After this test, the list now contains only 318 candidate cores.

4. **Not Cubelike:** from 318 to 196 graphs

   There are only 1372 cubelike graphs on 32 vertices, and so it is now easy to decide which of the remaining candidates are cubelike by checking whether they are isomorphic to any graph in the list of 1372 graphs. This test reduced the number of candidate graphs from 318 to 196.

5. **Cliques of Orbital Graphs:** from 196 to 32 graphs

   If \( X \) is the core of a cubelike graph, then by Corollary 6.2 each of its orbital graphs is homomorphically equivalent to a cubelike graph, so cannot have clique number exactly three.
Although $X$ may have many orbital graphs, it is not usually necessary to construct all of them. In particular, if any orbital graph of $X$ has clique number 3, then there is an orbital graph of $X$ whose edge set is the union of at most three orbitals with the same property (just take the union of the orbitals containing each of the three edges of the triangle). This test reduces the list to 32 graphs.

6. Eigenvalues of Orbital Graphs: from 32 to 20 graphs

If $X$ is the core of a cubelike graph, then by Corollary 7.4 the spectrum of a connected component of any $d$-regular orbital graph of $X$ is a submultiset of the spectrum of the cube $Q_d$. In this test, we check both the eigenvalues and their multiplicities.

7. Not a Core: from 20 to 18 graphs

Testing whether a graph is a core is a difficult computational task, mostly because showing that a graph actually is a core involves an exhaustive search to demonstrate that the graph has no non-surjective endomorphisms. However as there are so few remaining graphs and they are of modest size, the digraphs package ([?] in GAP can easily determine that 18 of the 20 are cores, while eliminating 2 non-cores that have passed all previous tests.

8. Hom-Idempotence: from 18 to 4 graphs

Since any cubelike graph is a normal Cayley graph, by Lemma 5.1 we have that the core $X$ of a cubelike graph must be hom-idempotent, and moreover must be homomorphically equivalent to (and therefore the core of) its shift graph $\text{Sh}(X)$. As $\text{Sh}(X) \rightarrow X$ always holds, the latter is equivalent to the existence of a homomorphism from $X$ to $\text{Sh}(X)$.

The 18 remaining graphs have automorphism groups of orders ranging from 256 to 1536 and so it is not difficult to compute the shift graphs. For many of these shift graphs, it is then possible to use the digraphs package to determine whether or not the graphs are hom-idempotent. For some of the graphs though, this computation takes too long. However, employing Corollary 6.2 we see that any orbital graph of the core of a cubelike graph must be hom-idempotent. Thus if a candidate graph has an orbital graph that is not hom-idempotent, it can be removed from the list of candidates.

After this stage, only 4 graphs remain; these have not been eliminated because they are hom-idempotent, as they are all Cayley graphs for the abelian group $Z_4 \times Z_8$.

9. Cubelike Hulls: from 4 to zero graphs

The graph $X = \text{Cay}(Z_4 \times Z_8, \{(1,0),(0,6),(0,3),(0,7),(1,5),(1,1),(1,6),(2,2)\})$ from our list has $\omega(X) = 5$. By Lemma 4.4, if $X$ is homomorphically equivalent to a cubelike graph then $Z_2[K_5] \rightarrow X$. However it is easy to check (using digraphs) that there is no homomorphism from $Z_2[K_5]$ to $X$, and therefore $X$ is not homomorphically equivalent to a cubelike graph. The remaining three graphs all have $X$ as an orbital graph and hence none of them are the core of a cubelike graph by Corollary 6.2.

At this point, we have shown the following:

**Theorem 8.2.** If $X$ is the core of a cubelike graph with $|V(X)| \leq 32$, then $X$ is cubelike.

Applying the Degree Bound, we also obtain the following:

**Corollary 8.3.** If $X$ is the core of a cubelike graph with valency at most 7, then $X$ is cubelike.
9 Conclusion

In this work we have shown that both the core of a cubelike graph, and the orbital graphs of this core, must share a variety of graph-theoretical properties. In particular, we have the following:

**Theorem 9.1.** Suppose that $X$ is the core of a cubelike graph, and let $Y$ be a connected component of an orbital graph of $X$ with valency $d$. Then the following hold:

1. $Y$ is homomorphically equivalent to a cubelike graph. In particular, $\omega(Y), \chi(Y) \neq 3$.
2. $Y$ is generously transitive.
3. There is a covering map from the $d$-cube $Q_d$ to $Y$, and so $Y$ has at most $2^d$ vertices.
4. If $Y$ is a core other than $K_2$, then $|V(Y)| \leq 2^{d-1}$ with equality if and only if $d$ is odd and $Y$ is the folded $d$-cube.
5. The eigenvalues of $Y$ are a sub-multiset of the eigenvalues of $Q_d$ and are thus integers.
6. If $\alpha(Y)\omega(Y) = |V(Y)|$ then the core of $Y$ must be complete.
7. $Y$ is hom-idempotent, and if $Y$ is a core then this is equivalent to $Y$ being an induced subgraph of $Sh(Y)$.
8. If $Z \to Y$, then $\mathbb{Z}_2[Z] \to Y$ for any graph $Z$.
9. If $Y$ has odd girth $g$, then $Y$ contains the folded cube of order $g$ as an induced subgraph.

As we have seen, the combination of these properties is very restrictive, allowing us to rule out all non-cubelike vertex-transitive graphs on up to 32 vertices as possible cores of cubelike graphs. While we are unable to show that this list of properties suffices to characterize cubelike cores, we also know of no non-cubelike core that satisfies all of them, and so we cannot definitely say that the list does not characterise cubelike cores.

The vertex-transitive graphs on 32 vertices are varied and quite numerous, and so we feel that our results provide considerable supporting evidence in favour of the conjecture. Proving the analogous result for vertex-transitive graphs on 64 vertices, which would provide even stronger evidence, is not possible using these techniques. The sheer number of groups and graphs will certainly not be manageable without considerably stronger theoretical results constraining the structure of the core of a cubelike graph. Ultimately, to prove the conjecture in its entirety along these lines, it will be necessary to find such strong structural properties of the core of a cubelike graph that they are sufficient to characterise cubelike graphs.

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