A Complete Characterization of Infinitely Repeated Two-Player Games having Computable Strategies with no Computable Best Response under Limit-of-Means Payoff

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A Complete Characterization of Infinitely Repeated Two-Player Games having Computable Strategies with no Computable Best Response under Limit-of-Means Payoff

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It is well-known that for infinitely repeated games, there are computable strategies that have best responses, but no computable best responses. These results were originally proved for either specific games (e.g., Prisoner’s dilemma), or for classes of games satisfying certain conditions not known to be both necessary and sufficient.

We derive a complete characterization in the form of simple necessary and sufficient conditions for the existence of a computable strategy without a computable best response under limit-of-means payoff. We further refine the characterization by requiring the strategy profiles to be Nash equilibria or subgame-perfect equilibria, and we show how the characterizations entail that it is efficiently decidable whether an infinitely repeated game has a computable strategy without a computable best response.


CCS Concepts: • Theory of computation → Algorithmic game theory; Computability; Exact and approximate computation of equilibria.

Additional Key Words and Phrases: repeated game, subgame-perfect equilibrium, computable strategy

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1 INTRODUCTION

We consider two-player games \( G \) with simultaneous moves and perfect information. In a repeated game (or supergame), \( G \) is played repeatedly with both players aware of all moves played by each player in all previous games. The payoff of each player in such a game is a function of the payoffs obtained in the repetitions of \( G \), for example the limit-of-means payoff is the limit inferior of the undiscounted averages of the payoff for each finite sequence of repetitions. A computable strategy for infinitely repeated games is one where an algorithm computes the next action based on the finite history of previous repetitions of the game. Classic results show that infinitely repeated games admit computable strategies that have a best response, but no computable best response [1, 2]: some algorithm will play a strategy such that there will exist a counterstrategy for the other player that will achieve maximum payoff among all strategies, but no such counterstrategy is computable. For infinitely repeated games with limit-of-means payoff, results are known solely for Prisoner’s dilemma, and the computable strategy involved is not known to form a Nash equilibrium [1].
Contributions: For infinitely repeated games with limit of means payoff, we extend previous results in two directions: First, we identify necessary and sufficient conditions for games to have computable strategies that have no computable best response, even though a best response exists; as a consequence of our techniques, we also provide necessary and sufficient conditions for strategies (computable or otherwise) to have no best response at all. Second, we obtain necessary and sufficient conditions for games to have such strategies in the case where the only strategies allowed are those that form Nash equilibrium, respectively a subgame-perfect equilibrium. In both cases, it is efficiently decidable whether a game satisfies the conditions.

2 MAIN RESULTS

Let \( G = (\{1, 2\}, A, u) \) be a 2-player normal-form game and \( G^\infty \) its infinitely repeated version with limit of means payoff. Then, \( G \) is said to be trivial for Player \( i \in \{1, 2\} \) if 
\[
\max_{a_i \in A_i} u_i(a) = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_{-i}),
\]
and non-trivial for Player \( i \) if it is not trivial for Player \( i \). For example, Prisoner’s Dilemma is non-trivial for any player, and Rock-Paper-Scissors is trivial for any player.

Our first main result is:

**Theorem.** Let \( G \) be a 2-player normal-form game. The following are equivalent:

- (a) \( G \) is non-trivial for Player 1.
- (b) There is a strategy for Player 2 in \( G^\infty \) that has no best response.
- (c) There is a computable strategy for Player 2 in \( G^\infty \) that has no best response.
- (d) There is a strategy profile \( s = (s_1, s_2) \) in \( G^\infty \) satisfying
  
  1. \( s_1 \) is a best response to \( s_2 \),
  2. \( s_2 \) is computable,
  3. \( s_2 \) does not have a computable best response.

In the above theorem, the strategy profile \( (s_1, s_2) \) is not required to be a Nash or subgame-perfect equilibrium. If we add either of these as requirements, we derive the following surprising result:

**Theorem.** Let \( G \) be a 2-player normal-form game. The following are equivalent:

- (a) There is a strategy profile \( s = (s_1, s_2) \) in \( G^\infty \) satisfying
  
  1. \( s \) is a Nash equilibrium of \( G^\infty \),
  2. \( s_2 \) is computable,
  3. \( s_2 \) does not have a computable best response.
- (b) There is a strategy profile \( s = (s_1, s_2) \) in \( G^\infty \) satisfying
  
  1. \( s \) is a subgame-perfect equilibrium of \( G^\infty \),
  2. \( s_2 \) is computable,
  3. \( s_2 \) does not have a computable best response.
- (c) \( |A_1| \geq 2 \), and there is a Nash equilibrium \( s' \) of \( G^\infty \) that is strictly individually rational for Player 1, that is, the limit-of-means payoff \( v_1(s') \) of Player 1 when playing \( s' \) satisfies:
  
  \[
  v_1(s') > \min_{a_2 \in A_2} \max_{a_1 \in A_1} u_1(a_1, a_2).
  \]

By using Folk Theorem-style reasoning, it is not hard to see that for a game \( G \) with payoff matrix of dimension \( n \times m \), deciding whether condition (c) holds above can be done in time \( O(nm \log nm) \).

REFERENCES
