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Spectral tail processes and max-stable approximations of multivariate regularly varying time series

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Abstract

A regularly varying time series as introduced in [Basrak and Segers (2009)] is a (multivariate) time series such that all finite dimensional distributions are multivariate regularly varying. The extremal behavior of such a process can then be described by the index of regular variation and the so-called spectral tail process, which is the limiting distribution of the rescaled process, given an extreme event at time 0. As shown in [Basrak and Segers (2009)], the stationarity of the underlying time series implies a certain structure of the spectral tail process, informally known as the “time change formula”. In this article, we show that on the other hand, every process which satisfies this property is in fact the spectral tail process of an underlying stationary max-stable process. The spectral tail process and the corresponding max-stable process then provide two complementary views on the extremal behavior of a multivariate regularly varying stationary time series.

Keywords and phrases: max-stable processes; regularly varying time series; spectral tail process; stationary processes

AMS 2010 Classification: 60G70 (60G10; 60G55).

1 Introduction

The concept of regular variation has become a standard tool for the extremal analysis of multivariate time series. Roughly speaking, the assumption of regular variation of a time series implies that extremal episodes from this time series can be modelled as the product of a heavy-tailed radial component which determines the magnitude of the extremal event and an independent and normalized random vector which determines the dependence structure between the different components of one observation and over time. There are different ways of defining this second random component, depending on the specification of an “extremal event”. In the case of a stationary time series, one often looks at the limiting distribution of the rescaled process conditioned on the event that the observation at time 0 exceeds in norm a threshold which tends to infinity.

In the following, let $\| \cdot \|$ be an arbitrary but fixed norm on $\mathbb{R}^d$. Write $\mathcal{L}(X)$ for the distribution of a random quantity $X$ and $\mathcal{L}(X|A)$ or $\mathcal{L}(X|Y)$ for the distribution of $X$ conditioned on the event $A$ or the $\sigma$-algebra which is generated from the random quantity $Y$. We denote weak convergence by $\Rightarrow$.  

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Definition 1.1. Let \((X_t)_{t \in \mathbb{Z}}\) be a stationary time series with values in \(\mathbb{R}^d\). If there exists a time series \((\Theta_t)_{t \in \mathbb{Z}}\), such that
\[
\mathcal{L}\left(\frac{X_s}{\|X_0\|}, \ldots, \frac{X_t}{\|X_0\|} \mid \|X_0\| > x\right) \xrightarrow{w} \mathcal{L}(\Theta_s, \ldots, \Theta_t), \quad x \to \infty,
\]
for all \(s, t \in \mathbb{Z}\), then we call \((\Theta_t)_{t \in \mathbb{Z}}\) the spectral tail process of \((X_t)_{t \in \mathbb{Z}}\). The process \((\Theta_t)_{t \in \mathbb{N}_0}\) is called the forward spectral tail process and \((\Theta_{-t})_{t \in \mathbb{N}_0}\) is called the backward spectral tail process. If there exists a time series \((Y_t)_{t \in \mathbb{Z}}\) such that
\[
\mathcal{L}\left(\frac{X_s}{x}, \ldots, \frac{X_t}{x} \mid \|X_0\| > x\right) \xrightarrow{w} \mathcal{L}(Y_s, \ldots, Y_t), \quad x \to \infty,
\]
for all \(s, t \in \mathbb{Z}\), then we call \((Y_t)_{t \in \mathbb{Z}}\) the tail process of \((X_t)_{t \in \mathbb{Z}}\).

Remark 1.2. As shown in Basrak and Segers (2009), Theorems 2.1 and 3.1, for a stationary time series \((X_t)_{t \in \mathbb{Z}}\) the two convergences \((1.1)\) and \((1.2)\) are equivalent and we have
\[
(Y_t)_{t \in \mathbb{Z}} \overset{d}{=} (Y \cdot \Theta_t)_{t \in \mathbb{Z}},
\]
for a Pareto(\(\alpha\))-distributed r.v. \(Y\) independent of \((\Theta_t)_{t \in \mathbb{Z}}\) (where \(\overset{d}{=}\) denotes equality in distribution). The value of \(\alpha\) is called the index of regular variation of \((X_t)_{t \in \mathbb{Z}}\). Furthermore, any of these two convergences is equivalent to that the time series is regularly varying, i.e. all its finite dimensional distributions are multivariate regularly varying.

In the following we will always deal with a stationary underlying process \((X_t)_{t \in \mathbb{Z}}\). It is important to note that the spectral tail process \((\Theta_t)_{t \in \mathbb{Z}}\) of an underlying stationary process is in general not a stationary time series, since we condition on a particular event that happens at time 0. However, one can show that the resulting spectral tail process of a stationary underlying time series satisfies a different property instead. In the following we write 0 for the real number or a vector or sequence consisting of all zeros. The meaning should be clear from the context.

Property (TCF). We say that a time series \((\Theta_t)_{t \in \mathbb{Z}}\) with values in \(\mathbb{R}^d\) satisfies Property (TCF) if \(P(\|\Theta_0\| = 1) = 1\) and the so-called “time-change formula” (Segers et al. (2012)) holds, i.e. if for \(\alpha > 0\)
\[
E(f(\Theta_{s-i}, \ldots, \Theta_{t-i})) = E\left(f\left(\frac{\Theta_s}{\|\Theta_s\|}, \ldots, \frac{\Theta_t}{\|\Theta_t\|}\right) \mathbb{1}_{\{\|\Theta_s\| > 0\}} \|\Theta_s\|^\alpha\right)
\]
for all \(s \leq 0 \leq t, i \in \mathbb{Z}\), and all bounded and continuous functions \(f : (\mathbb{R}^d)^{t-s+1} \to \mathbb{R}\) such that \(f(\theta_s, \ldots, \theta_t) = 0\) whenever \(\theta_0 = 0\).

Note that the above property depends on the parameter \(\alpha > 0\), so it would be more precise to speak of Property (TCF(\(\alpha\))) instead, but for reasons of brevity we omit the parameter which we assume to be fixed throughout.

Remark 1.4. (i) We have added the indicator function in \((1.3)\) in order to make clear that the expression in the expected value is to be interpreted as 0 on the set where \(\|\Theta_i\| = 0\) (and thus the argument of the function \(f\) is not defined). In the following we will in most cases omit the indicator function for the sake of brevity.
(ii) Similar as in Janßen and Segers (2014) one can show that Property (TCF) implies that even holds for all bounded and measurable functions $f$ such that $f(\theta_s, \ldots, \theta_t) = 0$ whenever $\theta_0 = 0$.

(iii) As the law of $(\Theta_t)_{t \in \mathbb{Z}}$ is completely determined by its finite-dimensional distributions, Property (TCF) implies that

$$E(f((\Theta_t - i)_{t \in \mathbb{Z}})) = E\left( f\left( \left( \frac{\Theta_t}{\|\Theta_i\|} \right)_{t \in \mathbb{Z}} \right) \|\Theta_i\|^\alpha \right), \quad i \in \mathbb{Z},$$

(1.4)

for all bounded functions $f : (\mathbb{R}^d)^{\mathbb{Z}} \to \mathbb{R}$ such that $f((\theta_t)_{t \in \mathbb{Z}}) = 0$ whenever $\theta_0 = 0$ and $f$ is $\mathcal{B}((\mathbb{R}^d)^{\mathbb{Z}})-\mathcal{B}(\mathbb{R})$-measurable, where $\mathcal{B}(\cdot)$ stands for the corresponding Borel $\sigma$-algebra. □

As shown in Basrak and Segers (2009), Theorem 3.1, Property (TCF) (with $\alpha$ being equal to the index of regular variation) is always satisfied for a spectral tail process of an underlying stationary regularly varying process. See also Segers et al. (2017) for the more general case of $X_t$ taking values in a “star-shaped” metric space.

We will subsequently deal with the question whether in turn Property (TCF) implies that the corresponding process is a spectral tail process of some underlying process. We show that the answer is yes and we will construct corresponding underlying max-stable processes, see Theorem 4.2. It will turn out that the case where additionally a summability or short-range dependence condition (see Section 2 for details) is satisfied allows for a particularly simple construction which is based on multivariate mixed moving maxima processes, see Theorem 3.2 and Corollary 3.3. This construction allows us to connect the extremal behavior of the process conditioned on the specific event of an extremal exceedance at time 0 with the overall extremal behavior over time as modelled by a max-stable process, see Proposition 3.5. The spectral tail process and the approximation of extremal events by a max-stable process thus provide two complementary views on the extremal behavior of a regularly varying time series. Additionally, this point of view also gives rise to a theoretical motivation (Proposition 3.6) for the POT method for dependent observations which uses clusterwise maxima for estimation of extremal parameters, see Davison and Smith (1990).

The very recent works Planinić and Soulier (2017) and Dombry et al. (2017) address similar questions, where properties of the spectral tail process and max-stable representations are analyzed with the help of the so-called tail measure, which is an extension of the limit measure in multivariate regular variation to the sequence space. Our approach here is more focussed on the distribution of the spectral tail process itself. It allows for a new interpretation of the property (TCF), see Theorem 2.4 and representations of corresponding underlying max-stables processes which are generated from i.i.d. copies of the spectral tail process, thereby facilitating for example simulation from these processes.

There are also some links between our work and Engelke et al. (2014), who explore connections between different representations of univariate max-stable processes. However, our approach differs substantially by starting solely from Property (TCF) and focussing on the particular properties implied by stationarity of processes. Finally, some connections exist between the topics studied here and theoretical properties of stationary max- and sum-stable processes which were analyzed in (among others) Rosinski (1995), Roy and Samorodnitsky (2008), Wang and Stoev (2010) and Dombry and Kabluchko (2017). But there the analysis starts from a stable process and is focussed on suitable decompositions of this process, while our approach is somehow the other way round by starting with the spectral process and finding a suitable composition of copies from this process.
in order to generate a max-stable process with given properties. Furthermore, the aforementioned works rely heavily on tools from dynamical systems theory, while our approach is solely based on the description of finite-dimensional distributions of stochastic processes and point process techniques, which are two standard tools of extreme value theory for time series.

The rest of the paper is organized as follows: Section 2 introduces and analyzes a short-range dependence condition which, if satisfied in addition to Property (TCF), allows us to construct an underlying mixed moving maxima process with given spectral tail process, as shown in Section 3. At the end of this section, we also discuss implications of our results for statistical analysis. Section 4 finally treats the general case where we show that again an underlying stationary process exists but has a more complicated representation than in the short-range dependence setting.

2 A shift invariance property derived from Property (TCF)

We start by showing that Property (TCF), together with a summability assumption, implies a certain shift invariance of the process \((\Theta_t)_{t \in \mathbb{Z}}\).

Property (SC). We say that a time series \((\Theta_t)_{t \in \mathbb{Z}} \) with values in \(\mathbb{R}^d\) satisfies the summability condition \((SC)\) for \(\alpha > 0\) if

\[
0 < \sum_{t \in \mathbb{Z}} \|\Theta_t\|^{\alpha} < \infty \quad \text{a.s.}
\]

The same comment as for Property (TCF) about the omission of the parameter \(\alpha\) also applies here. In case that Property (SC) is satisfied, we introduce the following notation

\[
\|\Theta\| = \|(\Theta_t)_{t \in \mathbb{Z}}\| := \left(\sum_{t \in \mathbb{Z}} \|\Theta_t\|^{\alpha}\right)^{1/\alpha}.
\]

For \(\alpha \geq 1\), this is just the \(L_\alpha\)-norm of \((\|\Theta_t\|)_{t \in \mathbb{Z}}\), but remember that \(\alpha < 1\) is also possible.

For further analysis of Property (SC) we also introduce

\[
\|\Theta^*\| = \|\Theta^*\|(\Theta_t)_{t \in \mathbb{Z}} := \sup_{t \in \mathbb{Z}} \|\Theta_t\| \in [0, \infty] \tag{2.1}
\]

and

\[
T^* = T^*(\Theta_t)_{t \in \mathbb{Z}} := \inf\{t \in \mathbb{Z} : \|\Theta_t\| = \|\Theta^*\|\} \in \mathbb{Z} \cup \{-\infty\} \cup \{\infty\}. \tag{2.2}
\]

As usual, we set \(\inf(\mathbb{Z}) = -\infty\) and \(\inf(\emptyset) = \infty\). So, in particular, \(\|\Theta^*\| = \infty\) implies \(T^* = \infty\).

At first, Property (SC) may look rather restrictive, but the following lemma shows that once Property (TCF) is satisfied, we have equivalent assumptions which seem reasonable for a large class of processes, see also Remark 2.3. See also Planinić and Soulier (2017), Corollary 3.3, for this statement and an alternative proof.

Lemma 2.2. Let Property (TCF) hold. Then the following statements are equivalent:

(i) \(\sum_{t \in \mathbb{Z}} \|\Theta_t\|^{\alpha} < \infty \) a.s. (i.e. Property (SC) holds)

(ii) \(\|\Theta_t\| \to 0\) a.s. for \(|t| \to \infty\).

(iii) \(P(T^* \in \mathbb{Z}) = 1\) □
Proof. If (i) holds, then necessarily \( \| \Theta_t \| \to 0 \) a.s. as \( |t| \to \infty \) and thereby (ii) follows. Since furthermore \( \| \Theta_0 \| = 1 \) a.s., statement (ii) implies that both \( \| \Theta^* \| \) and \( T^* \) are finite a.s. and (iii) holds. It is left to show that (iii) implies (i).

Let (iii) hold and assume that \( P(\sum_{k \in \mathbb{Z}} \| \Theta_k \|^\alpha = \infty) > 0 \). Then there exists an \( i^* \in \mathbb{Z} \) such that \( P(\sum_{k \in \mathbb{Z}} \| \Theta_k \|^\alpha = \infty, T^* = i^*) > 0 \) and therefore

\[
\infty = E \left( \mathbb{1}_{\{T^* = i^*\}} \sum_{k \in \mathbb{Z}} \| \Theta_k \|^\alpha \right)
\]

\[
= \sum_{k \in \mathbb{Z}} E \left( \mathbb{1}_{\{T^* = i^*\}} \| \Theta_k \|^\alpha \right)
\]

\[
= \sum_{k \in \mathbb{Z}} E \left( \mathbb{1}_{\{\| \Theta_j \| < \| \Theta_i^* \|, j < i^*, \| \Theta_j \| \leq \| \Theta_i^* \|, j \geq i^* \}} \mathbb{1}_{\{\Theta_0 \neq 0\}} \| \Theta_k \|^\alpha \right)
\]

\[
= \sum_{k \in \mathbb{Z}} E \left( f \left( \frac{\Theta_t}{\| \Theta_k \|} ; t \in \mathbb{Z} \right) \| \Theta_k \|^\alpha \right)
\]

with

\[
f((\theta_t)_{t \in \mathbb{Z}}) = \mathbb{1}_{\{\| \theta_j \| < \| \theta_i^* \|, j < i^*, \| \theta_j \| \leq \| \theta_i^* \|, j \geq i^* \}} \mathbb{1}_{\{\theta_0 \neq 0\}},
\]

which is bounded, measurable and \( f((\theta_t)_{t \in \mathbb{Z}}) = 0 \) whenever \( \theta_0 = 0 \). Therefore, with (1.4) applied to each summand, this expression equals

\[
\sum_{k \in \mathbb{Z}} E \left( f \left( (\Theta_{t-k})_{t \in \mathbb{Z}} \right) \right)
\]

\[
= \sum_{k \in \mathbb{Z}} E \left( \mathbb{1}_{\{\| \Theta_j \| < \| \Theta_i^* - k \|, \| \Theta_j \| \leq \| \Theta_i^* - k \|, j \geq i^* - k \}} \mathbb{1}_{\{\Theta_0 \neq 0\}} \right)
\]

\[
\leq \sum_{k \in \mathbb{Z}} E \left( \mathbb{1}_{\{\| \Theta_j \| < \| \Theta_i^* - k \|, \| \Theta_j \| \leq \| \Theta_i^* - k \|, j \geq i^* - k \}} \right)
\]

\[
= \sum_{k \in \mathbb{Z}} P(T^* = i^* - k) = 1,
\]

which leads to a contradiction to our assumption that \( P(\sum_{k \in \mathbb{Z}} \| \Theta_k \|^\alpha = \infty) > 0 \) and thereby proves the statement.

Remark 2.3. The assumption \( \| \Theta_t \| \to 0 \) as \( |t| \to \infty \) excludes a sort of long-range dependence in extremes and it has been shown in Basrak and Segers (2009), Proposition 4.2, that it is implied by a property introduced in Davis and Hsing (1995): For a sequence \( a_n \to \infty \) with \( nP(|X_0| > a_n) \to c > 0 \), it is satisfied if there exists a sequence \( r_n \to \infty, r_n/n \to 0 \) such that

\[
\lim_{m \to \infty} \limsup_{n \to \infty} P \left( \max_{m \leq |t| \leq r_n} |X_t| > a_n u \right) \rightarrow 0 \quad \text{for all } u \in (0, \infty).
\]  

This property is sometimes called “anti-clustering condition” or “finite mean cluster size condition” and is frequently used in the literature, c.f., e.g., Davis and Mikosch (1998), Basrak et al. (2012) or Mikosch and Wintenberger (2016). It is satisfied for a large variety of time series models such as ARMA models, Max-moving average processes, stochastic volatility models or GARCH(p,q) processes (under mild assumptions about coefficients and innovations, respectively), cf. Basrak et al. (2002) and Mikosch and Zhao (2014).
So, by Lemma 2.2, if $(\Theta_t)_{t \in \mathbb{Z}}$ is a spectral tail process and the underlying process satisfies (2.3), then $(\Theta_t)_{t \in \mathbb{Z}}$ satisfies Property (SC).

Alternatively, a Markovian structure of the spectral tail process as discussed in Janßen and Segers (2014) can simplify the task to check whether $\|\Theta_t\| \to 0$ a.s. as $|t| \to \infty$ and thus Property (SC) is satisfied.

Under the assumption of Property (SC) we will now formulate an equivalent statement for Property (TCF). This equivalence allows for a probabilistic interpretation of the time change formula in form of an invariance property of $(\Theta_t)_{t \in \mathbb{Z}}$ under a specific random shift of time.

**Theorem 2.4.** Let $(\Theta_t)_{t \in \mathbb{Z}}$ with values in $\mathbb{R}^d$ be a time series which satisfies Property (SC). Furthermore, let $(\Theta_t^{RS})_{t \in \mathbb{Z}}$ be a time series such that

$$
(\Theta_t^{RS})_{t \in \mathbb{Z}} \overset{d}{=} \left( \frac{\Theta_{t+K(\Theta)}}{\|\Theta_{K(\Theta)}\|} \right)_{t \in \mathbb{Z}},
$$

where $K(\Theta) = K((\Theta_t)_{t \in \mathbb{Z}})$ is a random integer with conditional probability mass function

$$
P(K(\Theta) = k| (\Theta_t)_{t \in \mathbb{Z}}) = \frac{\|\Theta_k\|^\alpha}{\sum_{t \in \mathbb{Z}} \|\Theta_t\|^\alpha}, \quad k \in \mathbb{Z}. \quad (2.5)
$$

Then

$$
(\Theta_t^{RS})_{t \in \mathbb{Z}} \overset{d}{=} (\Theta_t)_{t \in \mathbb{Z}}
$$

if and only if $(\Theta_t)_{t \in \mathbb{Z}}$ satisfies Property (TCF).

\[ \Box \]

The index $RS$ in the time series above may be interpreted as standing both for “random shift” and “re-scaled”.

**Proof.** 1. “$\Rightarrow$ “: Assume that all assumptions of Theorem 2.4 are met and that (2.6) holds. Since $\|\Theta_0^{RS}\| = 1$ a.s. by (2.4), we have $P(\|\Theta_0\| = 1) = 1$, i.e. the first part of Property (TCF) is satisfied. For $s \leq 0 \leq t$ let $f : (\mathbb{R}^d)^{|s-t-1|} \to \mathbb{R}$ be a bounded and continuous function such that $f(y_s, \ldots, y_t) = 0$ whenever $y_0 = 0$. Then, for $i \in \mathbb{Z},$

$$
E(f(\Theta_{s-i}, \ldots, \Theta_{t-i}))
= E \left( \sum_{k \in \mathbb{Z}} \frac{\|\Theta_k\|^\alpha}{\|\Theta\|^{\alpha}} f(\Theta_{s-i}, \ldots, \Theta_{t-i}) \right)
= E \left( \sum_{k \in \mathbb{Z}} \frac{\|\Theta_k^{RS}\|^\alpha}{\|\Theta^{RS}\|^{\alpha}} f(\Theta^{RS}_{s-i}, \ldots, \Theta^{RS}_{t-i}) \right)
= E \left( E \left( \sum_{k \in \mathbb{Z}} \frac{\|\Theta_{k+K(\Theta)}\|/\|\Theta_{K(\Theta)}\|^{\alpha}}{\|\Theta\|/\|\Theta_{K(\Theta)}\|^{\alpha}} f \left( \frac{\Theta_{s-i+K(\Theta)}}{\|\Theta_{K(\Theta)}\|}, \ldots, \frac{\Theta_{t-i+K(\Theta)}}{\|\Theta_{K(\Theta)}\|} \right) \right) \right)
= E \left( \sum_{k \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \frac{\|\Theta_t\|^\alpha}{\|\Theta\|^{\alpha}} \frac{\|\Theta_{k+1}\|^{\alpha}}{\|\Theta_k\|^\alpha} f \left( \Theta_{s-i+l}, \ldots, \Theta_{t-i+l} \right) \right),
$$

where we used (2.4), (2.5) and the assumption (2.6). The last expression simplifies to

$$
E \left( \sum_{t \in \mathbb{Z}} \frac{\|\Theta_t\|^\alpha}{\|\Theta\|^{\alpha}} f \left( \Theta_{s-i+l}, \ldots, \Theta_{t-i+l} \right) \right)
$$

6
\[ E \left( \sum_{m \in \mathbb{Z}} \frac{\|\Theta_{i+m}\|^\alpha}{\|\Theta\|^\alpha} f \left( \frac{\Theta_{s+m}}{\|\Theta_{i+m}\|}, \ldots, \frac{\Theta_{t+m}}{\|\Theta_{i+m}\|} \right) \right), \quad \text{with } m = l - i, \]

\[ = E \left( \sum_{m \in \mathbb{Z}} \frac{\|\Theta_m\|^\alpha}{\|\Theta\|^\alpha} f \left( \frac{\Theta_{s+m}/\|\Theta_m\|}{\|\Theta_{i+m}/\|\Theta_m\|}, \ldots, \frac{\Theta_{t+m}/\|\Theta_m\|}{\|\Theta_{i+m}/\|\Theta_m\|} \right) \right) \]

\[ = E \left( f \left( \frac{\Theta_s}{\|\Theta\|}, \ldots, \frac{\Theta_t}{\|\Theta\|} \right) \right), \]

where in the last step again (2.5) was used. In the penultimate step we used the assumption about \( f \) which guarantees that the corresponding summand in the expected value vanishes as soon as \( \|\Theta_m\| = 0 \), which allows to expand the fraction by \( \|\Theta_m\|^\alpha \) (again, all expressions are assumed to equal 0 as soon as one of the factors is 0, cf. Remark 1.4 (i)). This proves Property (TCF).

2. \( \Leftarrow \) : Assume now that Property (TCF) holds in addition to Property (SC) and let \((\Theta^\text{RS}_t)_{t \in \mathbb{Z}}\) have a distribution as defined in (2.3) and (2.5). Then it is sufficient to show for all bounded and measurable functions \( f : (\mathbb{R}^d)^\mathbb{Z} \to \mathbb{R} \) such that \( f((\theta_t)_{t \in \mathbb{Z}}) = 0 \) whenever \( \theta_0 = 0 \), that

\[ E(f((\Theta_t)_{t \in \mathbb{Z}})) = E(f((\Theta^\text{RS}_t)_{t \in \mathbb{Z}})). \]

(The above equality has to hold for all bounded and measurable functions in order to have equality in distribution for both processes, but by switching from \( f \) to \( f'((\theta_t)_{t \in \mathbb{Z}}) = f((\theta_t)_{t \in \mathbb{Z}}) \cdot 1_{\{\theta_0 \neq 0\}} \) the above equality is sufficient since \( P(\Theta_0 = 0) = P(\Theta^\text{RS}_0 = 0) = 0 \) by Property (TCF) and the construction of \((\Theta^\text{RS}_t)_{t \in \mathbb{Z}}\).

Set

\[ \tilde{f}((\theta_t)_{t \in \mathbb{Z}}) := f \left( \left( \frac{\theta_t}{\|\Theta_t\|} \right)_{t \in \mathbb{Z}} \right) \frac{\theta_0^\alpha}{\|\Theta_t\|_\alpha^\alpha} 1_{\{\theta_0 \neq 0\}}, \]

such that, by using the boundedness of \( f \) in order to interchange sum and expectation,

\[ E(f((\Theta^\text{RS}_t)_{t \in \mathbb{Z}})) = E \left( \sum_{k \in \mathbb{Z}} \frac{\|\Theta_k\|^\alpha}{\|\Theta\|^\alpha} f \left( \frac{\Theta_{t-k}}{\|\Theta_{k}\|} \right) _{t \in \mathbb{Z}} \right) \]

\[ = \sum_{k \in \mathbb{Z}} E \left( \tilde{f}((\Theta_{t-k})_{t \in \mathbb{Z}}) \right) \]

\[ = \sum_{k \in \mathbb{Z}} E \left( \tilde{f} \left( \left( \frac{\Theta_t}{\|\Theta_k\|} \right) _{t \in \mathbb{Z}} \right) \right. \frac{\|\Theta_k\|^\alpha}{\|\Theta\|^\alpha} \]

\[ = \sum_{k \in \mathbb{Z}} E \left( \tilde{f} \left( \left( \frac{\Theta_t/\|\Theta_k\|}{\|\Theta_0/\|\Theta_k\|} \right) _{t \in \mathbb{Z}} \right) \right. \frac{\|\Theta_0/\|\Theta_k\|\}^\alpha}{\|\Theta/\|\Theta_k\|^\alpha} \]

\[ = \sum_{k \in \mathbb{Z}} E \left( \tilde{f}((\Theta_0)_{t \in \mathbb{Z}}) \frac{\|\Theta_k\|^\alpha}{\|\Theta\|^\alpha} \right) = E \left( \tilde{f}((\Theta_0)_{t \in \mathbb{Z}}) \right) = E(f((\Theta_t)_{t \in \mathbb{Z}})) \]

where we have used (1.4). This concludes the proof.

3 Construction of a max-stable process with given spectral tail process under Property (SC)

In this section, we will show that for each process \((\Theta_t)_{t \in \mathbb{Z}}\) which satisfies Properties (TCF) and (SC) there exists an underlying max-stable process which has \((\Theta_t)_{t \in \mathbb{Z}}\) as corresponding spectral
tail process. The general case where Property (SC) does not need to hold will be treated later in Section 3. For simplicity, we restrict ourselves to the case that all components of $\Theta_t$ are non-negative, but see Remark 3.4 for the general case. Since the term “max-stable” has different interpretations in the context of multivariate stochastic processes in the literature, we start with a definition.

**Definition 3.1.** We call a stochastic process $(X_t)_{t \in \mathbb{Z}}$ with values in $[0, \infty)^d$ max-stable with index $\alpha$, if for all $k \in \mathbb{N}, s \leq t, x_s, \ldots, x_t \in [0, \infty)^d$

$$P(X_s \leq x_s, \ldots, X_t \leq x_t)^k = P(X_s \leq k^{-1/\alpha}x_s, \ldots, X_t \leq k^{-1/\alpha}x_t),$$

(3.1)

where all inequalities are to be interpreted componentwise.

The next theorem has some similarities to Theorem 4.1 in Engelke et al. (2014). Compared to their result we restrict ourselves to the integers as the index set but as an extension we allow both for multivariate observations and also for $\Theta_t = 0$ for some $t \in \mathbb{Z}$.

**Theorem 3.2.** Let $(\Theta_t)_{t \in \mathbb{Z}}$ with values in $[0, \infty)^d$ be a stochastic process which satisfies Property (SC). Furthermore, let $(U_i, T_i, (\Theta_{t}^{(i)})_{t \in \mathbb{Z}})_{i \in \mathbb{N}}$ be an enumeration of points from a Poisson point process on $(0, \infty) \times \mathbb{Z} \times ([0, \infty)^d)^\mathbb{Z}$ with intensity $\alpha u^{-\alpha} \otimes \lambda(dt) \otimes P(\Theta_t \in \mathbb{Z}(d\theta))$, where $\lambda$ denotes the counting measure on $\mathbb{Z}$, i.e. $\lambda(B) = |B|$ for $B \subset \mathbb{Z}$. Then the stochastic process

$$(Z_t)_{t \in \mathbb{Z}} = \left( \bigvee_{i \in \mathbb{N}} U_i \frac{\Theta_{t+h+T_i}^{(i)}}{\|\Theta_{t}^{(i)}\|_{\alpha}} \right)_{t \in \mathbb{Z}}$$

(3.2)

is an almost surely finite, stationary and max-stable process with index $\alpha$ (in (3.2) and in the following, all maxima are meant to be taken componentwise). The process is furthermore regularly varying with corresponding spectral tail process $(\Theta_t^{RS})_{t \in \mathbb{Z}}$ as defined in Theorem 2.4.

**Proof.** The process defined in (3.2) is stationary, since

$$(Z_{t+h})_{t \in \mathbb{Z}} = \left( \bigvee_{i \in \mathbb{N}} U_i \frac{\Theta_{t+h+T_i}^{(i)}}{\|\Theta_{t}^{(i)}\|_{\alpha}} \right)_{t \in \mathbb{Z}} = \left( \bigvee_{i \in \mathbb{N}} U_i \frac{\Theta_{t+T_i}^{(i)}}{\|\Theta_{t}^{(i)}\|_{\alpha}} \right)_{t \in \mathbb{Z}} = (Z_t)_{t \in \mathbb{Z}}$$

for all $h \in \mathbb{Z}$, because the point processes $(T_i)_{i \in \mathbb{N}}$ and $(T_i + h)_{i \in \mathbb{N}}$ have the same intensity $\lambda$ and are both independent of all other random variables. Similarly as in de Haan and Ferreira (2006), Chapter 9, it follows that for $s \leq t, x_n = (x_n^1, \ldots, x_n^d) \in [0, \infty)^d, s \leq n \leq t$ (and with $[0, x_n] := [0, x_n^1] \times \ldots \times [0, x_n^d]$)

$$P(Z_s \leq x_s, \ldots, Z_t \leq x_t) = \exp \left( -\int \int \int \mathds{1}_{([0, x_s] \times \ldots \times [0, x_t])} \left( \frac{u_{\|\theta\|_{\alpha}}(\theta_{n+z})_{s \leq n \leq t}}{\|\theta\|_{\alpha}} \right) \nu_{\alpha}(du)\lambda(dz)P(\Theta_t \in \mathbb{Z}(d\theta)) \right)$$

$$= \exp \left( -\int \|\theta\|_{\alpha}^{-\alpha} \sum_{z \in \mathbb{Z}} \left( \min_{z \leq n \leq t} \frac{x_n^i}{\theta_{n+z}^i} \right)^{-\alpha} \right) P(\Theta_t \in \mathbb{Z}(d\theta))$$

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\[
\begin{align*}
\mu(A) &= \int \int \int \mathbf{1}_A \left( \frac{v(\theta_{n+z})_{s \leq n \leq t}}{\|\theta\|_\alpha} \right) \nu_\alpha(\theta) d\Theta_t \in \mathbb{Z} \\
&= \sum_{z \in \mathbb{Z}} \int_0^\infty \mathbf{1}_{A \cap B} \left( \frac{u_z(\theta_{n+z})_{s \leq n \leq t}}{\|\theta\|_\alpha} \right) \nu_\alpha(\theta) d\Theta_t \in \mathbb{Z} \\
&= \sum_{z \in \mathbb{Z}} \int_0^\infty \mathbf{1}_{A \cap B} \left( \frac{u_z(\theta_{n+z})_{s \leq n \leq t}}{\|\theta\|_\alpha} \right) \frac{\|\theta\|_\alpha^\alpha \nu_\alpha(\theta_z)}{d\Theta_t} d\Theta_t \in \mathbb{Z} \\
&= \int \sum_{z \in \mathbb{Z}} \int_1^\infty \mathbf{1}_A \left( \frac{u_z(\theta_{n+z})_{s \leq n \leq t}}{\|\theta\|_\alpha} \right) \frac{\|\theta\|_\alpha^\alpha \nu_\alpha(\theta_z)}{d\Theta_t} d\Theta_t \in \mathbb{Z},
\end{align*}
\]
for Borel sets \(A \subset ([0, \infty)^d)^{t-s-1}\) bounded away from 0. Thus, for a Borel set \(A \subset ([0, \infty)^d)^{t-s+1-\delta} \times \{x_1, \ldots, x_d\} \in [0, \infty)^d : \|x_1, \ldots, x_d\| > 1\} \times ([0, \infty)^d)^{t'}\) with \(\mu(\partial(A \cap B)) = 0\), we have

\[
\lim_{x \to \infty} P\left(\frac{Z_s}{x}, \ldots, \frac{Z_t}{x} \in A \| Z_0 \| > x\right) = \frac{\mu(A \cap B)}{\mu(B)}. 
\]

Using (3.4) and substituting \(u_z = v\|\theta_z\|/\|\theta\|_\alpha, z \in \mathbb{Z}\), we get

\[
\mu(A \cap B) = \sum_{z \in \mathbb{Z}} \int_0^\infty \mathbf{1}_{A \cap B} \left( \frac{u_z(\theta_{n+z})_{s \leq n \leq t}}{\|\theta\|_\alpha} \right) \nu_\alpha(\theta) d\Theta_t \in \mathbb{Z} \\
= \sum_{z \in \mathbb{Z}} \int_0^\infty \mathbf{1}_{A \cap B} \left( \frac{u_z(\theta_{n+z})_{s \leq n \leq t}}{\|\theta\|_\alpha} \right) \frac{\|\theta\|_\alpha^\alpha \nu_\alpha(\theta_z)}{d\Theta_t} d\Theta_t \in \mathbb{Z} \\
= \int \sum_{z \in \mathbb{Z}} \int_1^\infty \mathbf{1}_A \left( \frac{u_z(\theta_{n+z})_{s \leq n \leq t}}{\|\theta\|_\alpha} \right) \frac{\|\theta\|_\alpha^\alpha \nu_\alpha(\theta_z)}{d\Theta_t} d\Theta_t \in \mathbb{Z},
\]
where we used for the substitution that, due to the definition of \(B\), the integrand in the first line is 0 if \(\|\theta_z\| = 0\) and we used in the last equation that \(u_z(\theta_{n+z})_{s \leq n \leq t}/\|\theta\|_\alpha \in B\) if and only if \(u_z \geq 1\). But the last expression is equal to

\[
P((Y \cdot \Theta_s^{RS}, \ldots, Y \cdot \Theta_t^{RS}) \in A)
\]
for a random variable $Y$ with Pareto($\alpha$) distribution independent of $(\Theta_s^{RS}, \ldots, \Theta_t^{RS})$. Analogously, one can show that $\mu(B) = 1$ and thus, by Remark 1, the process $(Z_t)_{t \in \mathbb{Z}}$ has spectral tail process $(\Theta_t^{RS})_{t \in \mathbb{Z}}$, which completes the proof.

The following corollary addresses the same question as Theorem 5.1 in Planinić and Soulier (2017) but gives an alternative construction of the underlying max-stable process.

**Corollary 3.3.** If $(\Theta_t)_{t \in \mathbb{Z}}$ satisfies both Property (SC) and Property (TCF), then the spectral tail process of $(Z_t)_{t \in \mathbb{Z}}$ as defined in (3.2) is given by $(\Theta_t)_{t \in \mathbb{Z}}$.

**Proof.** This follows immediately from Theorem 3.2 in connection with Theorem 2.4.

**Remark 3.4.** The construction of a max-stable process as in Theorem 3.2 only works if all components of the spectral tail process are non-negative. To overcome this and to describe the general dependence structure between observations that are extremely large or small (i.e., smaller than $-c$ for $c \to \infty$) we can look instead at the 2d-dimensional, non-negative process $(\Theta_t^+)_{t \in \mathbb{Z}} := ((\Theta_t^+)_+), (\Theta_t^-)_-, \ldots, (\Theta_t^+)_+,(\Theta_t^-)_-\in \mathbb{Z}$ with $x_+ = \max(x,0), x_- = \max(-x,0)$.

Next, we show that the max-stable process constructed in Theorem 3.2 is actually the “maximum attractor” (i.e., the limiting distribution for maxima) for the process underlying $(\Theta_t)_{t \in \mathbb{Z}}$. All maxima are meant to be taken componentwise.

**Proposition 3.5.** Let $(X_t)_{t \in \mathbb{Z}}$ with values in $\mathbb{R}^d$ be a stationary regularly varying time series with index $\alpha > 0$ and spectral tail process $(\Theta_t)_{t \in \mathbb{Z}}$ with values in $[0, \infty)^d$ that satisfies Property (SC). Then, for each $t \in \mathbb{N}_0$, there exists a sequence $b_n > 0, n \in \mathbb{N},$ of normalizing constants such that

$$\lim_{n \to \infty} \frac{(X_0, \ldots, X_t)^i}{b_n} \overset{w}{\Rightarrow} (Z_0, \ldots, Z_t),$$

where $(X_0, \ldots, X_t)^i, i \in \mathbb{N},$ are i.i.d. copies of $(X_0, \ldots, X_t)$ and $(Z_0, \ldots, Z_t)$ has the distribution as defined in Theorem 3.2.

**Proof.** Note first that since $(X_t)_{t \in \mathbb{Z}}$ is stationary, the process $(\Theta_t)_{t \in \mathbb{Z}}$ satisfies Property (TCF) and thus all assumptions of Theorem 3.2 and Corollary 3.3 are satisfied. The stationary time series $(X_t)_{t \in \mathbb{Z}}$ is multivariate regularly varying if and only if all $(X_0, \ldots, X_t), t \in \mathbb{N}_0,$ are multivariate regularly varying. By Proposition 7.1 in Resnick (2007) this is equivalent to the existence of a normalizing sequence $b_n > 0, n \in \mathbb{N},$ such that

$$\lim_{n \to \infty} \frac{(X_0, \ldots, X_t)^i}{b_n} \overset{w}{\Rightarrow} (U_0, \ldots, U_t),$$

for some random vector $(U_0, \ldots, U_t)$ with a non-degenerate distribution. This distribution is (up to a scaling factor which only depends on $b_n, n \in \mathbb{N}$) determined by the limit law of regular variation of $(X_0, \ldots, X_t)$ and this is in turn (up to a constant) determined by the spectral tail process $(\Theta_t)_{t \in \mathbb{Z}}$ and the index $\alpha$ of regular variation, see the proof of Theorem 2.1 in Basrak and Segers (2009). Therefore, since the processes $(Z_t)_{t \in \mathbb{Z}}$ from Theorem 3.2 and $(X_t)_{t \in \mathbb{Z}}$ have the same spectral tail

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process and index of regular variation, there exist $\tilde{b}_n > 0, n \in \mathbb{N}$, such that

$$\bigvee_{i=1}^{n} \frac{(Z_0, \ldots, Z_t)^i}{\tilde{b}_n^n} \xrightarrow{w} (U_0, \ldots, U_t), \quad n \to \infty,$$

where $(Z_0, \ldots, Z_t)^i, i \in \mathbb{N}$, are i.i.d. copies of $(Z_0, \ldots, Z_t)$. But we also know from Definition 3.1 and Theorem 3.2 that

$$\bigvee_{i=1}^{n} \frac{(Z_0, \ldots, Z_t)^i}{n^{1/\alpha}} \xrightarrow{d} (Z_0, \ldots, Z_t),$$

which implies by the convergence to types theorem that

$$(U_0, \ldots, U_t) \xrightarrow{d} c \cdot (Z_0, \ldots, Z_t)$$

for some $c > 0$. Setting $b_n = \tilde{\tilde{c}} b_n$ then leads to the result.

Proposition 3.5 and Theorem 3.2 combined with Corollary 3.3 provide a view on the extremal behavior of a random process that is somehow complementary to the description given by the spectral tail process. The latter one only describes the behavior given that we have seen an extremal event at a specific time. The process constructed in Theorem 3.2 shares the extremal behavior of a process with given spectral tail process in the sense of Proposition 3.5 and thus gives us an impression about when the extremal events will happen over time. “Extremal episodes” of the process in (3.2) will typically occur for an extremely large value of $U_i$ and the corresponding “epicenters” $T_i$ are uniformly distributed over time. Due to Property (SC), the influence of a large value of $U_i$ will only be visible in a certain neighborhood around the corresponding time $T_i$ and, roughly speaking, extremal events will become more and more independent if the elapsed time between them grows.

From a statistical point of view, it is important to connect the extremal behavior that we observe in the “extremal episodes” of a time series to the extremal behavior of the stationary distribution, i.e. the distribution at a fixed point in time (or maybe even the joint distribution at specific lags). The extremal behavior of the stationary distribution can be described by

$$\lim_{x \to \infty} \mathcal{L}(X_0/\|X_0\| | \|X_0\| > x) = \mathcal{L}(\Theta_0).$$

Surely, the extremal realizations of a time series are the ones to look at for this task, but if we have an extremal cluster of events we look at the process at random points in time. One may then ask which of those extremal observations from an observed cluster best represents a “typical” extremal observation, i.e. an extremal observation from the stationary distribution. This question is even more important if one wants to decrease dependence between used observations and therefore chooses only one observation per cluster for inference. This is for example done in the declustering approach of the POT method as introduced in Davison and Smith (1990), and has by now become a standard tool for the extremal analysis of time series. Here one chooses (in a univariate setting) the cluster maximum as a representative for the whole cluster and treats the resulting observations as extremal outcomes of the stationary distribution which are nearly i.i.d.

As Proposition 3.5 shows, the extremal clusters can be seen as realizations of $(\Theta_t)_{t \in \mathbb{Z}}$ with random scaling and under an unobservable shift in time, $T_i$. So we will usually not observe $(\Theta_t)_{t \in \mathbb{Z}}$ but, with the notation from Section 2, only $(\Theta_{T_i+t}/\|\Theta_t\|)_{t \in \mathbb{Z}}$, that is the observed “pattern” in form of the self-standardized process that has maximum norm 1 and is shifted in a way such that the maximal norm is first attained at time zero. Alternatively, the observation could also be seen as a realization in the quotient space of double-sided sequences with respect to the shift operator, cf. the space $\tilde{\tilde{L}}_0$ in Basrak et al. (2018+). The conditional distribution of $(\Theta_t)_{t \in \mathbb{Z}}$, given the shifted and rescaled observation as just described, is found in the next proposition.
**Proposition 3.6.** Let \((\Theta_t)_{t \in \mathbb{Z}}\) be a time series which satisfies Property (TCF) and Property (SC). Then, with the notation from (2.1), (2.2),

\[
\mathcal{L} \left( (\Theta_t)_{t \in \mathbb{Z}} \left| \left( \frac{\Theta_{t+k}^*}{\|\Theta^*\|} \right)_{t \in \mathbb{Z}} \right. \right) = \sum_{k \in \mathbb{Z}} \frac{\|\Theta_{t+k}^*\|_{\alpha}^\alpha}{\|\Theta_t^*\|_{\alpha}^\alpha} \delta_{\left( \frac{\Theta_{t+k}^*}{\|\Theta^*\|} \right)_{t \in \mathbb{Z}}},
\]

where \(\delta_x\) denotes the Dirac measure in \(x \in (\mathbb{R}^d)^{\mathbb{Z}}\).

**Proof.** Note first that the random probability measure on the r.h.s. in (3.5), applied to some \(A \subset \mathcal{B}((\mathbb{R}^d)^{\mathbb{Z}})\), is equal to

\[
\sum_{k \in \mathbb{Z}} \frac{\|\Theta_{t+k}^*\|_{\alpha}^\alpha}{\|\Theta_t^*\|_{\alpha}^\alpha} \mathbf{1}_A \left( \left( \frac{\Theta_{t+k}^*}{\|\Theta^*\|} \right)_{t \in \mathbb{Z}} \right)
\]

and is thus a measurable function of the conditioning expression. Let now \(A, B \subset \mathcal{B}((\mathbb{R}^d)^{\mathbb{Z}})\). Then, from Theorem 2.4

\[
P \left( (\Theta_t)_{t \in \mathbb{Z}} \in A \left| \left( \frac{\Theta_{t+k}^*}{\|\Theta^*\|} \right)_{t \in \mathbb{Z}} \right. \in B \right)
\]

\[
= E \left( \sum_{t \in \mathbb{Z}} \frac{\|\Theta_t^*\|_{\alpha}^\alpha}{\|\Theta_t^*\|_{\alpha}^\alpha} \mathbf{1}_A \left( \left( \frac{\Theta_{t+k}^*}{\|\Theta^*\|} \right)_{t \in \mathbb{Z}} \right) \mathbf{1}_B \left( \left( \frac{\Theta_{t+k}^*}{\|\Theta^*\|} \right)_{t \in \mathbb{Z}} \right) \right)
\]

\[
= E \left( \sum_{t \in \mathbb{Z}} \frac{\|\Theta_t^*\|_{\alpha}^\alpha}{\|\Theta_t^*\|_{\alpha}^\alpha} \mathbf{1}_A \left( \left( \frac{\Theta_t^*}{\|\Theta^*\|} \right)_{t \in \mathbb{Z}} \right) \mathbf{1}_B \left( \left( \frac{\Theta_t^*}{\|\Theta^*\|} \right)_{t \in \mathbb{Z}} \right) \right)
\]

which finishes the proof. \(\square\)

A way of interpreting Proposition 3.6 is that given an observed extremal “pattern” \((\Theta_{t+k}^*/\|\Theta^*\|)_{t \in \mathbb{Z}}\) from the realization of a time series, the resulting conditional distribution of the underlying spectral tail process is a random shift of this pattern (where the probability that an observation is set to be at time 0 is proportional to the norm of that observation to the power of \(\alpha\)), and scaled in a way to ensure that \(\|\Theta_0^*\| = 1\). Furthermore, a closer look at (3.5) shows that given an observation of \((\Theta_{t+k}^*/\|\Theta^*\|)_{t \in \mathbb{Z}}\) (and if the maximum norm is attained only once) this very sequence itself is also the most likely of all possible underlying sequences \((\Theta_t)_{t \in \mathbb{Z}}\), because the weight in (3.5) is largest for \(k = 0\). Therefore, \(\Theta_{t}/\|\Theta^*\|\) can be seen as the best approximation of \(\Theta_{0}\) from the observed sequence, which makes the representational choice of the observation with maximal norm from a cluster as in the POT method reasonable.

### 4 Construction of a max-stable process with given spectral tail process in the general case

So far, we have focussed on processes \((\Theta_t)_{t \in \mathbb{Z}}\) which satisfy both Property (TCF) and Property (SC). Corollary 3.3 shows that this ensures the existence of a max-stable underlying process which
realizes \((\Theta_t)_{t \in \mathbb{Z}}\) as a spectral tail process. In fact, this max-stable process is of a specific form which is called a *mixed moving maxima (M3) process*, cf. for example Definition 7 in Dombry and Kabluchko (2017) for the univariate case where \(\alpha = 1\). Dombry and Kabluchko (2017) show that a (univariate) max-stable process has a representation as a M3 process of the above form if and only if the process is (purely) dissipative, cf. Theorem 8 in Dombry and Kabluchko (2017) and Theorem 5.4 in Wang and Stoev (2010), where also the case for general \(\alpha\) is covered explicitly. 

Intuitively, this case implies that the impact of an extremal event at time 0 may in principle last forever (since \(\|\Theta_t\| > 0\) for all \(t \in \mathbb{Z}\) is possible) but that it diminishes over time since \(\|\Theta_t\| \to 0\) as \(|t| \to \infty\) almost surely.

We shall now look at the case where \((\Theta_t)_{t \in \mathbb{Z}}\) satisfies Property (TCF) but not necessarily Property (SC) for a given value of \(\alpha > 0\). If Property (SC) is not satisfied, then this implies by Lemma 2.2 that \(P(T^* \notin \mathbb{Z}) > 0\) and, roughly speaking, this corresponds to the case where an extremal event at time 0 will actually “return” infinitely often, for example in a periodic manner, like the following example shows.

**Example 4.1 (Simple periodic spectral tail process).** Let \(d = 1\) and

\[
\Theta_t = \begin{cases} 
1 & \text{if } t \in 2\mathbb{Z}, \\
0 & \text{if } t \in 2\mathbb{Z} + 1.
\end{cases}
\]

One easily checks that the process \((\Theta_t)_{t \in \mathbb{Z}}\) satisfies Property (TCF) for any \(\alpha > 0\) but there exists no \(\alpha > 0\) such that \((\Theta_t)_{t \in \mathbb{Z}}\) also satisfies Property (SC). Define now

\[
(X_t)_{t \in \mathbb{Z}} = \left( \bigvee_{j=0}^{1} \left( \bigvee_{i \in \mathbb{N}} \mathcal{U}_i^{(j)} \Theta_{t+j} \right) \right)_{t \in \mathbb{Z}},
\]

where \((\mathcal{U}_i^{(j)}, (\Theta_{t+j})_{t \in \mathbb{Z}})_{i \in \mathbb{N}}\) for \(j = 0, 1\) is an enumeration of points from a Poisson point process on \((0, \infty) \times ([0, \infty])^\mathbb{Z}\) with intensity \(\alpha u^{-\alpha-1} du \otimes P(\Theta_t)_{t \in \mathbb{Z}}(d\theta)\) and let those two Poisson point processes be independent for \(j = 0\) and \(j = 1\). It is easily seen that \((X_t)_{t \in \mathbb{Z}}\) is a stationary max-stable process and that

\[
(X_t)_{t \in \mathbb{Z}} \overset{d}{=} (Z_0 \Theta_t \mathbbm{1}_{2\mathbb{Z}}(t) + Z_1 \Theta_{t+1} \mathbbm{1}_{2\mathbb{Z}+1}(t))_{t \in \mathbb{Z}} = (Z_0 \mathbbm{1}_{2\mathbb{Z}}(t) + Z_1 \mathbbm{1}_{2\mathbb{Z}+1}(t))_{t \in \mathbb{Z}}
\]

for \(Z_0, Z_1\) being independent and Fréchet(\(\alpha\))-distributed. Therefore, the resulting spectral tail process of \((X_t)_{t \in \mathbb{Z}}\) is given by \((\Theta_t)_{t \in \mathbb{Z}}\).

The following theorem shows how the construction principle from the above example can be generalized in order to construct a corresponding stationary max-stable process for a general process \((\Theta_t)_{t \in \mathbb{Z}}\) which satisfies Property (TCF). We restrict ourselves again to the non-negative case, but note that an analogue of Remark 3.4 holds in this case as well. See Dombry et al. (2017), Theorem 2.9, for a different approach that shows the existence of an underlying process based on the concept of the tail measure.

**Theorem 4.2.** Let \((\Theta_t)_{t \in \mathbb{Z}}\) with values in \([0, \infty]^d\) be a stochastic process which satisfies Property (TCF). Furthermore, for \(j \in \mathbb{Z}\) let \((\mathcal{U}_i^{(j)}, (\Theta_{t+j})_{t \in \mathbb{Z}})_{i \in \mathbb{N}}\) be an enumeration of points from a Poisson point process on \((0, \infty) \times ([0, \infty]^d)^\mathbb{Z}\) with intensity \(\alpha u^{-\alpha-1} du \otimes P(\Theta_t)_{t \in \mathbb{Z}}(d\theta)\) and let those Poisson point processes be independent for different values of \(j\). Define for \(k > 0\) the sets

\[
Q_k = \{ (\theta_t)_{t \in \mathbb{Z}} : \theta_0 \neq 0, \theta_1 = \theta_2 = \ldots = \theta_{2k} = 0 \}.
\]
Proof. Note first that for all \( n, k \in \mathbb{N}_0 \) and \( x = (x_0, \ldots, x_n) \in ([0, \infty)^d)^{n+1} \) we have, analogously to the proof of Theorem 3.2 that

\[
P(Z_{-k} \leq x_0, \ldots, Z_{n-k} \leq x_n)
= \exp \left( -\sum_{j \in \mathbb{Z}} \int \int \mathbb{1}_{[0,x]^d} \left( y(\theta_{t+j})_{0 \leq t \leq n-k} \mathbb{1}_{\{(\theta_t)_{t \in \mathbb{Z}} \in Q_j\}} \right) \nu_\alpha(dy) P(\Theta_{t \in \mathbb{Z}})(d\theta) \right)
= \exp \left( -\sum_{j \in \mathbb{Z}} \int \int \mathbb{1}_{[0,x]^d} \left( y(\theta_{t+j})_{0 \leq t \leq n} \mathbb{1}_{\{(\theta_t)_{t \in \mathbb{Z}} \in Q_j\}} \right) \nu_\alpha(dy) P(\Theta_{t \in \mathbb{Z}})(d\theta) \right),
\]

where again \( \nu_\alpha((x, \infty)) = x^{-\alpha}, x > 0 \). Due to the definition of the sets \( Q_j \), only finitely many summands in the exponent in (4.2) are different from 0. Therefore, if all components of \( x_0, \ldots, x_n \) go to infinity, the expression (4.2) converges to 1, which shows that \( (Z_t)_{t \in \mathbb{Z}} \) is an almost surely finite process.

The stationarity now follows from (4.2) if we can show that

\[
\sum_{j \in \mathbb{Z}} \int \int \mathbb{1}_A \left( y(\theta_{t+j})_{0 \leq t \leq n} \mathbb{1}_{\{(\theta_t)_{t \in \mathbb{Z}} \in Q_j\}} \right) \nu_\alpha(dy) P(\Theta_{t \in \mathbb{Z}})(d\theta)
= \sum_{j \in \mathbb{Z}} \int \int \mathbb{1}_A \left( y(\theta_{t+j})_{0 \leq t \leq n} \mathbb{1}_{\{(\theta_t)_{t \in \mathbb{Z}} \in Q_j\}} \right) \nu_\alpha(dy) P(\Theta_{t \in \mathbb{Z}})(d\theta).
\]

for all \( k, n \in \mathbb{N}_0 \) and all Borel sets \( A \subset ([0, \infty)^d)^{n+1} \) which are bounded away from 0. Note that (4.3) follows for all \( A \) bounded away from 0 as soon as we can show it for all \( A \) such that \( A \cap \{(x_0, \ldots, x_n) : x_0 = 0\} = \emptyset \), which can be seen as follows: For \( n = 0 \) all sets \( A \) which are bounded away from 0 already satisfy \( A \cap \{(x_0) : x_0 = 0\} = \emptyset \). For \( n \geq 1 \) any \( A \) bounded away from 0 is the disjoint union of the sets

\[
A \cap \{(x_0, \ldots, x_n) : x_0 \neq 0\}
\]

and

\[
A \cap \{(x_0, \ldots, x_n) : x_0 = 0\} = \{(x_0, x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in A'\} \setminus \{(x_0, x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in A', x_0 \neq 0\}
\]

for a set \( A' \subset ([0, \infty)^d)^n \) bounded away from 0. By induction, the statement then follows.

So assume in the following that \( A \cap \{(x_0, \ldots, x_n) : x_0 = 0\} = \emptyset \), such that

\[
\sum_{j \in \mathbb{Z}} \int \int \mathbb{1}_A \left( y(\theta_{t+j})_{0 \leq t \leq n} \mathbb{1}_{\{(\theta_t)_{t \in \mathbb{Z}} \in Q_j\}} \right) \nu_\alpha(dy) P(\Theta_{t \in \mathbb{Z}})(d\theta)
\]
and so the sum of the indicator functions of these disjoint events is equal to 
for any equals 1 almost surely due to since 
where we used that \( (\Theta_{\theta+1}) = \) that is bounded away from 0. Now, the events in the indicator functions above are equivalent to 
\[
\sum_{j \in \mathbb{Z}} \int \mathbb{I}_A \left( y(\theta_{t+j-k}) \right) \mathbb{I}_{\|\Theta-k\|} \nu_\alpha(dy) P(\Theta_t) dx(d\theta) 
\]
\[
= \sum_{j \in \mathbb{Z}} E \left( \int \mathbb{I}_A \left( y(\Theta_{t+j-k}) \right) \mathbb{I}_{\|\Theta-k\|} \nu_\alpha(dy) 
\right) 
\]
\[
= \sum_{j \in \mathbb{Z}} E \left( \int \mathbb{I}_A \left( y \left( \frac{\theta_{t+j-k}}{\|\Theta-k\|} \right) \right) \mathbb{I}_{\|\Theta-k\|} \nu_\alpha(dy) 
\right) 
\]
\[
= \sum_{j \in \mathbb{Z}} \int \mathbb{I}_A \left( y \left( \frac{\theta_{t+j-k}}{\|\Theta-k\|} \right) \right) \mathbb{I}_{\|\Theta-k\|} \nu_\alpha(dy) P(\Theta_t) dx(d\theta) 
\]
where we used that \( (\Theta_t)_{t \in \mathbb{Z}} \) satisfies property (TCF). Since \( (\theta_t)_{t \in \mathbb{Z}} \in Q_j \) if and only if \( (c\theta_t)_{t \in \mathbb{Z}} \in Q_j \) for any \( c > 0 \), and by substituting \( u \) for \( y/\|\Theta-k\| \) in each summand, this equals 
\[
\sum_{j \in \mathbb{Z}} \int \mathbb{I}_A \left( u(\theta_t) \mathbb{I}_{\|\Theta-k\|} \right) \mathbb{I}_{\|\Theta-k\|} \nu_\alpha(du) P(\Theta_t) dx(d\theta) 
\]
\[
= \sum_{j \in \mathbb{Z}} \int \mathbb{I}_A \left( u(\theta_t) \mathbb{I}_{\|\Theta-k\|} \right) \mathbb{I}_{\|\Theta-k\|} \nu_\alpha(du) P(\Theta_t) dx(d\theta), 
\]
since \( A \) is bounded away from 0. Now, the events in the indicator functions above are equivalent to 
\[
\begin{align*}
\theta_k & \neq 0, & j = 0, \\
\theta_{k+1} & \neq 0, \theta_k = 0, & j = -1, \\
\theta_{k-1} & \neq 0, \theta_k = \theta_{k+1} = 0, & j = 1, \\
\theta_{k+2} & \neq 0, \theta_{k-1} = \theta_k = \theta_{k+1} = 0, & j = -2 \\
& \vdots
\end{align*}
\]
and so the sum of the indicator functions of these disjoint events is equal to \( \mathbb{I}_{\{\theta_t \neq 0\}} \), which equals 1 almost surely due to \( P(\|\Theta_0\| = 1) = 1 \). Therefore,
\[
\begin{align*}
\sum_{j \in \mathbb{Z}} \int \mathbb{I}_A \left( y(\theta_{t+j-k}) \mathbb{I}_{\|\Theta-k\|} \right) \nu_\alpha(dy) P(\Theta_t) dx(d\theta) 
\end{align*}
\]
\[
= \int \int \mathbb{I}_A \left( u(\theta_t) \mathbb{I}_{\|\Theta-k\|} \right) \nu_\alpha(du) P(\Theta_t) dx(d\theta) 
\]
\[
= \sum_{j \in \mathbb{Z}} \int \mathbb{I}_A \left( u(\theta_{t+j}) \mathbb{I}_{\|\Theta-k\|} \right) \nu_\alpha(du) P(\Theta_t) dx(d\theta), 
\]
where in the last equality we used that \( (\theta_t)_{t \in \mathbb{Z}} \in Q_j \) implies that \( \theta_j = 0 \) for all \( j \neq 0 \) and therefore \( u(\theta_{t+j}) \mathbb{I}_{\|\Theta-k\|} \notin A \) for \( j \neq 0 \). This shows \( (4.3) \) and thereby the stationarity of \( (Z_t)_{t \in \mathbb{R}} \).

For any \( x = (x_0, \ldots, x_n) \in ([0, \infty)^d)^{n+1} \) and \( k \in \mathbb{N} \) we have furthermore (by substituting \( u \) for \( k^{1/\alpha} y \)) that 
\[
P(\text{Truncated}_k x, \ldots, \text{Truncated}_k x_n) 
\]
\[
= \exp \left( -\sum_{j \in \mathbb{Z}} \int \mathbb{I}_{[0,k^{1/\alpha} x]} \left( y(\theta_{t+j}) \mathbb{I}_{\|\Theta-k\|} \right) \nu_\alpha(dy) P(\Theta_t) dx(d\theta) \right) 
\]
which shows that \((Z_n)_{n \geq 0}\) is regularly varying with limit measure

\[
\mu(A) = \int \int \mathbb{1}_A \left( y(\theta_{t+j}) - n \leq \theta_t \leq \theta_{t+j} \right) \nu_\alpha(dy) P(\Theta_{t+j} \in \mathbb{E}) (d\theta)
\]

for sets \(A\) bounded away from \((0, \ldots, 0)\). Note that, again due to the definition of the sets \(Q_j, j \neq 0,\)

\[
\mu(A \cap \{(x_1, \ldots, x_n) : \|x_0\| > 1\}) = \int \int \mathbb{1}_A \left( y(\theta_{t+j}) - n \leq \theta_t \leq \theta_{t+j} \right) \nu_\alpha(dy) P(\Theta_{t+j} \in \mathbb{E}) (d\theta)
\]

and

\[
\mu(\{(x_1, \ldots, x_n) : \|x_0\| > 1\}) = \int \int \mathbb{1}_{\{(x_1, \ldots, x_n) : \|x_0\| > 1\}} y(\theta_{t+j}) - n \leq \theta_t \leq \theta_{t+j} \nu_\alpha(dy) P(\Theta_{t+j} \in \mathbb{E}) (d\theta)
\]

such that for Borel sets \(A\) with \(\mu(\partial(A \cap \{(x_1, \ldots, x_n) : \|x_0\| > 1\})) = 0,\)

\[
P((Z_n)_{n \geq 0} ; \ldots, Z_n) \in A \mid \|Z_0\| > z) = \frac{\mu(A \cap \{(x_1, \ldots, x_n) : \|x_0\| > 1\})}{\mu(\{(x_1, \ldots, x_n) : \|x_0\| > 1\}), \ z \to \infty,}
\]

\[
= \int \int \mathbb{1}_A \left( y(\theta_{t+j}) - n \leq \theta_t \leq \theta_{t+j} \right) \nu_\alpha(dy) P(\Theta_{t+j} \in \mathbb{E}) (d\theta)
\]

\[
= P((Y \cdot \Theta_{t+j} ; \ldots, Y \cdot \Theta_0) \in A),
\]

for a Pareto(\(\alpha\)) distributed random variable \(Y\) independent of \((\Theta_t)_{t \in \mathbb{N}_0}\). By Remark \(1.2\) this proves the statement.

An analogue to Proposition 3.5 holds in this case as well, i.e. the process \((Z_t)_{t \in \mathbb{N}_0}\) from Theorem \(1.2\) is the maximum attractor of the process which underlies \((\Theta_t)_{t \in \mathbb{E}}\). The proof of this result is completely along the lines of the proof of Proposition 3.5. \(\Box\)
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References


