Binary–binary scattering in the secular limit

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ABSTRACT

Binary–binary interactions are important in a number of astrophysical contexts including dense stellar systems such as globular clusters. Although less frequent than binary–single encounters, binary–binary interactions lead to a much richer range of possibilities such as the formation of stable triple systems. Here, we focus on the regime of distant binary–binary encounters, i.e. two binaries approaching each other on an unbound orbit with a periapsis distance $Q$ much larger than the internal binary separations. This ‘secular’ regime gives rise to changes in the orbital eccentricities and orientations, which we study using analytic considerations and numerical integrations. We show that ‘direct’ interactions between the three orbits only occur starting at a high expansion order of the Hamiltonian (hexadecupole order), and that the backreaction of the outer orbit on the inner two orbits at lower expansion orders is weak. Therefore, to good approximation, one can obtain the changes of each orbit by using previously known analytic results for binary–single interactions, and replacing the mass of the third body with the total mass of the companion binary. Nevertheless, we find some dependence of the ‘binarity’ of the companion binary, and derive explicit analytic expressions for the secular changes that are consistent with numerical integrations. In particular, the eccentricity and inclination changes of orbit 1 due to orbit 2 scale as $\epsilon_{SA,1}(a_2/Q)^2[m_3m_4/(m_3+m_4)^2]$, where $\epsilon_{SA,1}$ is the approximate quadrupole-order change, and $a_2$ and $(m_3, m_4)$ are the companion binary orbital semimajor axis and component masses, respectively. Our results are implemented in several PYTHON scripts that are freely available.

Key words: gravitation – celestial mechanics – stars: kinematics and dynamics – globular clusters: general.

1 INTRODUCTION

Dense stellar systems such as open and globular clusters are host to a wide range of dynamical interactions involving bound objects such as binaries-single scattering, as well as scattering involving higher order systems, e.g. binary–binary scattering. Since such interactions are believed to lead to mergers of black holes (BHs) and neutron stars (NSs) (e.g. Sigurdsson & Hernquist 1993; Portegies Zwart & McMillan 2000; O’Leary et al. 2006; Ziosi et al. 2014; Rodriguez et al. 2015; Kimpson et al. 2016; Mapelli 2016; Rodriguez, Chatterjee & Rasio 2016; Samsing & Ramirez-Ruiz 2017; Rodriguez et al. 2018; Samsing, MacLeod & Ramirez-Ruiz 2018b; Samsing, Askar & Giersz 2018c; Samsing, Hamers & Tylés 2019, interest in them has recently surged with the direct detection of gravitational waves (GWs) from merging BHs and NSs (e.g. Abbott et al. 2016a, b, 2017a, d, b, c).

The topic of binary–single scattering has received a great deal of attention in the past decades (e.g. Hut 1983; Hut & Bahcall 1983; Davies, Benz & Hills 1993; Goodman & Hut 1993; Heggie & Hut 1993; Hut 1993; Sigurdsson & Phinney 1993; Heggie, Hut & McMillan 1996; McMillan & Hut 1996; Kocsis & Levin 2012; Samsing et al. 2018a). Binary–binary encounters have been studied as well, although perhaps with less intensity given its greater complexity. Nevertheless, in star clusters with binary fractions $\gtrsim 10$ per cent, binary–binary interactions dominate over binary–single interactions (Sigurdsson & Phinney 1993; Leigh & Sills 2011). Furthermore, even if the overall binary fraction of a stellar cluster is low, the binary fraction in the core can be much higher (Leonard 1989; Hut, McMillan & Romani 1992;...
2.1 Setup

We consider two bound binary systems (their orbits indicated with ‘1’ and ‘2’) that approach each other on an unbound orbit (the latter is referred to as orbit ‘3’, or the ‘outer’ orbit). See Fig. 1 for a sketch. Let the masses of the components in orbit 1 be $m_1$ and $m_2$, respectively, and $m_3$ and $m_4$ for orbit 2. For convenience, we introduce the quantities $M_1 \equiv m_1 + m_2$, $M_2 \equiv m_3 + m_4$, and $M \equiv M_1 + M_2$. The (initial) semimajor axes and eccentricities of all orbits are denoted with $a_i$ and $e_i$, where it should noted that $a_1 < 0$, and $e_3 \geq 1$. Also, to further distinguish between the bound and unbound orbits, we introduce the notation $E \equiv e_1 \geq 1$ for the (initial) outer orbit eccentricity, and $Q \equiv a_3(1 - e_3) > 0$ for the outer orbit periapsis distance. Note that in defining $Q$ and $E$, we neglect the extended nature of the bound orbits (i.e. approximate the latter as point masses). Evidently, since we are dealing with two binaries instead of two point particles, the latter approximation breaks down as $Q \to 0$.

Let the relative separation between the two bodies in orbit 1 be denoted with $r_1$; similarly, the relative separation vector between bodies 3 and 4 in orbit 2 is $r_2$. The outer orbit has a separation vector $r_3$ between the two centres of mass of orbits 1 and 2. The instantaneous eccentricity or Laplace–Runge–Lenz vector $e_i$ is given by $e_i = [1/(GM_i)]\dot{r}_i \times (r_i \times \dot{r}_i) - \dot{r}_i$, where dots denote derivatives with respect to time, and hats denote unit vectors. For orbits 1 and 2, the normalized angular-momentum vectors are $j_i = r_i \times \dot{r}_i$, with magnitudes $j_i = \sqrt{1 - e_i^2}$.
arise from an effect of the companion binarity on the outer orbit, i.e. on the expanded Hamiltonian is fully described in terms of pairwise interactions only: the interaction between orbit 1 and its outer orbit, and between orbit 2 and its outer orbit. The other terms in equation (4) give rise to changes to the Keplerian orbits. Note that, as shown by Hamers and Portegies Zwart (2016), the periapsis pointing along the z-axis, and the periapsis pointing along the x-axis, in this case, and neglecting the backreaction of the outer orbit due to the quadrupole moment of the inner orbits (see Section 2.4.1 below), the outer orbit is described according to

\[ r_3 = \frac{Q(1 + E)}{1 + E \cos \theta} \left[ \cos \theta \dot{x} + \sin \theta \dot{y} \right]; \]  

\[ \dot{r}_3 = \sqrt{\frac{GM}{Q(1 + E)}} \left[ -\sin \theta \dot{x} + (E + \cos \theta) \dot{y} \right]. \]

The outer orbit true anomaly \( \theta \) is related to the physical time \( t \) according to

\[ dr = \frac{1}{n_3} \left( \frac{E^2 - 1}{1 + E \cos \theta} \right) d\theta, \]

where \( n_3 = \sqrt{GM/|a_3|^3} \) is the hyperbolic mean motion, and \( |a_3| = Q/(E - 1) \). The true anomaly \( \theta \) ranges between \(-L \) and \( L \) corresponding to \(-\infty < t < \infty \), where

\[ L \equiv \arccos \left( -\frac{1}{E} \right). \]

### 2.2 Hamiltonian

#### 2.2.1 Expansion

In the limit that the two binaries approach each other with a periapsis distance \( Q \gg r_1, r_2 \), it is appropriate to expand the Hamiltonian of the four-body system in terms of the small ratios \( r_1/r_5 \ll 1 \), and \( r_2/r_5 \ll 1 \). The resulting ‘binary–binary’ Hamiltonian is (Hamers et al. 2015, also see Hamers & Portegies Zwart 2016)

\[ H_{bb} = H_{kep}(m_1, m_2, r_1, r_2) + H_{kep}(m_3, m_4, r_3, r_4) + H_{quad}(m_1, m_2, m_3 + m_4, r_3, r_4) + H_{quad}(m_3, m_4, m_1 + m_2, r_3, r_4) + H_{ac}(m_1, m_2, m_3 + m_4, r_3, r_4) + H_{ac}(m_3, m_4, m_1 + m_2, r_3, r_4) + H_{bd}(m_1, m_2, m_3 + m_4, r_3, r_4) + H_{bd}(m_3, m_4, m_1 + m_2, r_3, r_4) + \mathcal{O} \left[ \frac{1}{r_3} \left( \frac{r_1}{r_2} \right)^4 \left( \frac{r_3}{r_4} \right)^4 \right]. \]

(4)

Here, \( i + j + k \geq 5 \). The various ‘universal’ functions in equation (4) are given by

\[ H_{kep}(m, m’, r, \hat{r}) = \frac{1}{2} \frac{m m’}{m + m’} - \frac{Gm m’}{r}; \]

(5a)

\[ H_{quad}(m, m”, m’, r, \hat{r}) = \frac{Gm m”}{m + m’} \left( \frac{1}{r’} \right)^2 \left( \frac{r}{r’} \right)^2 \left[ 3 \left( \hat{r} \cdot \hat{r}’ \right)^2 - 1 \right]; \]

(5b)

\[ H_{ac}(m, m”, m’, r, \hat{r}) = -\frac{Gm m”}{(m + m’)^2} \left( \frac{1}{r} \right)^3 \left( \frac{r}{r’} \right) \left[ 5 \left( \hat{r} \cdot \hat{r}’ \right)^3 - 3 \left( \hat{r} \cdot \hat{r}’ \right)^2 \right]; \]

(5c)

\[ H_{bd}(m, m”, m’, r, \hat{r}) = -\frac{Gm m”}{(m + m’)^3} \left( \frac{r}{r’} \right) \left[ 35 \left( \hat{r} \cdot \hat{r}’ \right)^4 - 30 \left( \hat{r} \cdot \hat{r}’ \right)^2 \right]; \]

(5d)

\[ H_{bd, cross, bb}(m_1, m_2, m_3, m_4, r_1, r_2, r_3, r_4) = -\frac{Gm_1 m_2 m_3 m_4}{(m_1 + m_2)(m_3 + m_4)} \left( \frac{r_1}{r_3} \right)^2 \left( \frac{r_2}{r_4} \right)^2 \times \frac{3}{4} \left[ 1 - 5 \left( \hat{r}_1 \cdot \hat{r}_3 \right)^2 - 5 \left( \hat{r}_2 \cdot \hat{r}_3 \right)^2 + 35 \left( \hat{r}_1 \cdot \hat{r}_3 \right)^2 \left( \hat{r}_2 \cdot \hat{r}_3 \right)^2 + 2 \left( \hat{r}_1 \cdot \hat{r}_3 \right)^2 - 20 \left( \hat{r}_1 \cdot \hat{r}_2 \right) \left( \hat{r}_1 \cdot \hat{r}_3 \right) \left( \hat{r}_2 \cdot \hat{r}_3 \right) \right]. \]

(5e)

The first three terms in equation (4) are the Keplerian terms; in the limit that the orbits are described by Keplerian orbits, these terms individually reduce to the constant terms \(-G(m + m’)/2a_0\), where \( i \) refers to the corresponding orbit and the masses should be replaced appropriately for each orbit. The other terms in equation (4) give rise to changes to the Keplerian orbits. Note that, as expected, \( H_{bb} \) is symmetric with respect to binaries 1 and 2, i.e. it is invariant under the interchange of parameters \( m_1 \leftrightarrow m_3 \), \( m_2 \leftrightarrow m_4 \), and \( r_1 \leftrightarrow r_2 \).

It is immediately clear that, to the lowest expansion orders, the quadrupole and octupole orders, \( \propto (r l r)^2 \) and \( \propto (r l r)^3 \), respectively, the expanded Hamiltonian is fully described in terms of pairwise interactions only: the interaction between orbit 1 and its outer orbit, and between orbit 2 and its outer orbit. This implies that, up to and including octupole order, any effect of the ‘binarity’ of the companion orbit can only arise from an effect of the companion binarity on the outer orbit, i.e. on \( r_3 \). This is discussed further below analytically (Section 2.4.1), as well as numerically (Section 3.1).

Only at the ‘hexadecapole’ order, \( \propto (r l r)^4 \), does there appear a term that involves the properties of three orbits simultaneously, described by \( H_{bd, cross, bb} \). We remark that the latter term contains the factors \( (r_1/r_3)^2 \) and \( (r_2/r_5)^2 \) which, individually considered, might suggest that...
2.2.2 Partial orbit averaging

In the ‘secular’ approximation, one averages the expanded Hamiltonian, equation (4), over some or all orbits. Here, we choose to average over the ‘inner’ orbits, i.e. orbits 1 and 2. This approximation is generally expected to be a good one if $\mathcal{R}_i \ll 1$, where $i \in \{1, 2\}$ refers to orbits 1 and 2, and

\[
\mathcal{R}_i = \left( \frac{1 + M_{3-i}}{M_i} \right) \left( \frac{a_i}{\mathcal{Q}} \right)^3 \left( 1 + E \right)^{1/2}.
\]

If $\mathcal{R}_i \ll 1$ for both $i = 1$ and $i = 2$, the mean motions of both bound orbits are much faster than the angular speed of the para/hyperbolic orbit at periapsis (this consideration is analogous to the binary–single case; see e.g. equation 1 of Hamers & Samsing 2019a).

The result of the ‘inner’ averaging, written explicitly to the same order as in equation (4), is (see e.g. Hamers 2018 for a general derivation of the pairwise averaged expressions to any expansion order; the term $\mathcal{H}_{\text{hex, cross, bb}}$ is derived new here)

\[
\mathcal{H}_{\text{bb}} = \mathcal{H}_{\text{kep}}(m_1, m_2, a_1) + \mathcal{H}_{\text{kep}}(m_3, m_4, a_2) + \mathcal{H}_{\text{quad}}(m_1, m_2, m_3 + m_4, a_1, e_1, J_1, r_1) + \mathcal{H}_{\text{quad}}(m_3, m_4, m_1 + m_2, a_2, e_2, J_2, r_2) + \mathcal{H}_{\text{hex}}(m_1, m_2, m_3 + m_4, a_1, e_1, J_1, r_1) + \mathcal{H}_{\text{hex}}(m_3, m_4, m_1 + m_2, a_2, e_2, J_2, r_2) + \mathcal{H}_{\text{hex, cross, bb}}
\]

\[
\times (m_1, m_2, m_3, m_4, a_1, e_1, J_1, a_2, e_2, J_2, r_3) + \mathcal{O}\left[ \frac{1}{r_3} \left( \frac{a_1}{a_2} \right)^{1/3} \left( \frac{a_2}{r_3} \right)^{1/3} \right].
\]

Here, we defined

\[
\mathcal{H}_{\text{kep}}(m, m', a) = -\frac{Gmm'}{2a};
\]

\[
\mathcal{H}_{\text{quad}}(m, m', m'', a, e, J, r) = -\frac{Gmm'm''}{m + m'} \left( \frac{a}{r} \right)^{3/2} \left[ 1 + 6e^2 + 15 (e \cdot \hat{r})^2 - 3 (J \cdot \hat{r})^2 \right];
\]

\[
\mathcal{H}_{\text{hex}}(m, m', m'', a, e, J, r) = -\frac{Gmm'm''}{m + m'} \left( \frac{a}{r} \right)^{3/2} \left[ 1 + 3 \left( 1 - 8e^2 \right) + 35 (e \cdot \hat{r})^2 - 15 (J \cdot \hat{r})^2 \right];
\]

\[
\mathcal{H}_{\text{hex, cross, bb}}(m_1, m_2, m_3, m_4, a_1, e_1, J_1, a_2, e_2, J_2, r_3) = -\frac{Gm_1m_2m_3m_4}{(m_1 + m_2)(m_3 + m_4)} \left( \frac{a_1}{r} \right)^{2} \left( \frac{a_2}{r} \right)^{2} \left[ 1 - 6e_1^2 - 6e_2^2 + 36e_1^2e_2^2 + 50(e_1 \cdot e_2)^2 
\]

\[-10(e_1 \cdot J_1)^2 - 10(J_1 \cdot e_2)^2 + 2(J_1 \cdot J_2)^2 + 25(e_1 \cdot \hat{r})^2 + 25(e_2 \cdot \hat{r})^2 + 5(6e_1^2 - 1)(J_1 \cdot \hat{r})^2 + 5(6e_2^2 - 1)(J_2 \cdot \hat{r})^2 
\]

\[-150e_1^2(e_1 \cdot \hat{r})^2 - 150e_2^2(e_2 \cdot \hat{r})^2 - 500(e_1 \cdot e_2)(e_1 \cdot \hat{r})(e_2 \cdot \hat{r}) + 100(J_1 \cdot e_2)(J_1 \cdot \hat{r})(e_2 \cdot \hat{r}) + 100(J_2 \cdot e_2)(e_1 \cdot \hat{r})(J_2 \cdot \hat{r}) 
\]

\[-20(J_1 \cdot J_2)(J_1 \cdot \hat{r})(J_2 \cdot \hat{r}) - 175(J_1 \cdot \hat{r})^3(e_2 \cdot \hat{r})^2 - 175(e_1 \cdot \hat{r})^2(J_2 \cdot \hat{r})^2 + 35(J_1 \cdot \hat{r})^2(J_2 \cdot \hat{r})^2 + 875(e_1 \cdot \hat{r})^2(e_2 \cdot \hat{r})^2 \right].
\]

As should be, the inner-averaged Hamiltonian, equation (7), is still symmetric with respect to orbits 1 and 2.

2.3 Inner-averaged equations of motion

Hamilton’s equations applied to the inner-averaged Hamiltonian equation (2.2.2) imply the following set of equations of motion for the eccentricity $e_i$ and angular momentum $J_i$ vectors of orbits 1 and 2, as well as the equation of motion for the outer orbital separation $r_3$

\[
\frac{de_i}{d\theta} = \epsilon_{\text{SA}}(1 + E \cos \theta) f_{e, \text{quad}}(e_i, J_i, \hat{r}_3) + \epsilon_{\text{SA}, \epsilon_{\text{hex, cross}}}(1 + E \cos \theta)^3 f_{e, \text{hex}}(e_i, J_i, \hat{r}_3) + \epsilon_{\text{SA}, \epsilon_{\text{hex, cross}}}(1 + E \cos \theta)^3 f_{e, \text{hex, cross}}(e_i, J_i, e_{3-i}, J_{3-i}, \hat{r}_3) + \ldots;
\]

\[
\frac{dJ_i}{d\theta} = \epsilon_{\text{SA}}(1 + E \cos \theta) f_{J, \text{quad}}(e_i, J_i, \hat{r}_3) + \epsilon_{\text{SA}, \epsilon_{\text{hex, cross}}}(1 + E \cos \theta)^3 f_{J, \text{hex}}(e_i, J_i, \hat{r}_3) + \epsilon_{\text{SA}, \epsilon_{\text{hex, cross}}}(1 + E \cos \theta)^3 f_{J, \text{hex, cross}}(e_i, J_i, e_{3-i}, J_{3-i}, \hat{r}_3) + \ldots;
\]

\[
\frac{dr_3}{d\theta} = -\frac{GM}{r_3^2} r_3 + \frac{GM}{r_3^3} f_{r_3}(m_1, m_2, a_1, e_1, J_1, r_3) + \frac{GM}{r_3^3} f_{r_3}(m_3, m_4, a_2, e_2, J_2, r_3) + \ldots.
\]
Here, ‘...’ denotes higher order expansion terms, and the auxiliary functions are defined according to

\[ f_{\text{quad}}(\boldsymbol{e}, \hat{r}_3) = \left[ -3 (\boldsymbol{e} \times r) - \frac{3}{2} (\boldsymbol{e} \cdot \hat{r}_3) (\boldsymbol{e} \times \hat{r}_3) + \frac{15}{2} (\boldsymbol{e} \cdot \hat{r}_3) (\boldsymbol{e} \times \hat{r}_3) \right]; \]

\[ f_{\text{j,quad}}(\boldsymbol{e}, \hat{r}_3) = \left[ -3 (\boldsymbol{e} \cdot \hat{r}_3) (\boldsymbol{e} \times \hat{r}_3) + \frac{15}{2} (\boldsymbol{e} \cdot \hat{r}_3) (\boldsymbol{e} \times \hat{r}_3) \right]; \]

\[ f_{\text{e,oct}}(\boldsymbol{e}, \hat{r}_3) = \frac{15}{16} \left[ -16 (\boldsymbol{e} \cdot \hat{r}_3)(\boldsymbol{e} \times \hat{r}_3) + (1 - 8e^2)(\boldsymbol{e} \times \hat{r}_3) - 10 (\boldsymbol{e} \cdot \hat{r}_3)(\boldsymbol{e} \times \hat{r}_3) - 5 (\boldsymbol{e} \cdot \hat{r}_3)^2(\boldsymbol{e} \times \hat{r}_3) + 35 (\boldsymbol{e} \cdot \hat{r}_3)^2(\boldsymbol{e} \times \hat{r}_3) \right]; \]

\[ f_{\text{j,oct}}(\boldsymbol{e}, \hat{r}_3) = \frac{15}{16} \left[ (1 - 8e^2)(\boldsymbol{e} \times \hat{r}_3) - 10 (\boldsymbol{e} \cdot \hat{r}_3)(\boldsymbol{e} \times \hat{r}_3) - 5 (\boldsymbol{e} \cdot \hat{r}_3)^2(\boldsymbol{e} \times \hat{r}_3) + 35 (\boldsymbol{e} \cdot \hat{r}_3)^2(\boldsymbol{e} \times \hat{r}_3) \right]; \]

\[ f_{\text{e,hex}}(\boldsymbol{e}, \hat{r}_3) = \frac{15}{16} \left[ 7 (21 (\boldsymbol{e} \cdot \hat{r}_3)^3(\boldsymbol{e} \times \hat{r}_3) - 7 (\boldsymbol{e} \cdot \hat{r}_3)^2(\boldsymbol{e} \times \hat{r}_3) - (3 - 10e^2)(\boldsymbol{e} \cdot \hat{r}_3)(\boldsymbol{e} \times \hat{r}_3) + 10 (\boldsymbol{e} \cdot \hat{r}_3)^2(\boldsymbol{e} \times \hat{r}_3) - 2 (1 - 8e^2)(\boldsymbol{e} \times \hat{r}_3) \right]; \]

\[ f_{\text{j,hex}}(\boldsymbol{e}, \hat{r}_3) = \frac{15}{16} \left[ 7 \left( 1 - 10e^2 + 21 (\boldsymbol{e} \cdot \hat{r}_3)^2 - 7 (\boldsymbol{e} \cdot \hat{r}_3)^2 \right)(\boldsymbol{e} \cdot \hat{r}_3)(\boldsymbol{e} \times \hat{r}_3) + \left( -3 + 10e^2 - 49 (\boldsymbol{e} \cdot \hat{r}_3)^2 + 7 (\boldsymbol{e} \cdot \hat{r}_3)^2 \right)(\boldsymbol{e} \cdot \hat{r}_3)(\boldsymbol{e} \times \hat{r}_3) \right]; \]

Other (dimensionless) parameters appearing in equation (9) are defined according to

\[ \epsilon_{\text{SA,i}} = \left[ \frac{M_{\text{e,i}}}{M} \right]^2 \left( \frac{a_i}{r_3} \right)^2 \left( 1 + \frac{a_i}{r_3} \right)^{-\frac{3}{2}} \left( 1 + E \right)^{-\frac{3}{2}}; \]

\[ \epsilon_{\text{oct,i}} = \left[ \frac{M_{\text{e,i}} - m_{i,B}}{M_i} \right] \frac{1}{Q} \left( 1 + \frac{a_i}{r_3} \right)^{-\frac{3}{2}} \left( 1 + E \right)^{-\frac{3}{2}}; \]

\[ \epsilon_{\text{hex,i}} = \left[ \frac{m_{i,A} - m_{i,B}}{M_i} \right]^2 \left( \frac{a_i}{r_3} \right)^2 \left( 1 + \frac{a_i}{r_3} \right)^{-\frac{3}{2}} \left( 1 + E \right)^{-\frac{3}{2}}; \]

\[ \epsilon_{\text{hex,cross,i}} = \left[ \frac{m_{i,A} - m_{i,B}}{M_i} \right]^2 \left( \frac{a_i}{r_3} \right)^2 \left( 1 + \frac{a_i}{r_3} \right)^{-\frac{3}{2}} \left( 1 + E \right)^{-\frac{3}{2}}; \]

Here, \( m_{i,A} = m_1 \) and \( m_{i,B} = m_2 \) if \( i = 1 \), and \( m_{i,A} = m_3 \) and \( m_{i,B} = m_4 \) if \( i = 2 \). For future convenience (Section 2.4 below), we formulated the equations of motion for \( \boldsymbol{e}_i \) and \( \boldsymbol{j}_i \) in terms of \( \theta \), the true anomaly of the outer orbit, which is related to the physical time according to equation (2). Note that in the latter equation and in the inner-averaged approximation, \( E \) and \( a_3 \) are allowed to vary and are determined by the equation for \( r_3 \), equation (9c).
2.4 Approximate analytic expressions for the eccentricity and angular-momentum changes

2.4.1 Outer orbit

We first consider the backreaction of the outer orbit on the quadrupole moment of the inner two binaries. This effect is described by equation (9c) to quadrupole expansion order (since the backreaction effect turns out to be small even at lowest order, we will not consider it at higher orders). We can get an approximate expression for the outer orbital changes \( \Delta a_3, \Delta e_3, \) and \( \Delta i_3 \) by substituting the solution to equation (9c) in the absence of the perturbation terms \( \propto f_3 \) (i.e. the solution if \( r_3 = -GM/r_3 \), resulting in purely Keplerian motion), into the perturbation terms and integrating the subsequent expressions over the outer orbit. Let the perturbation term to the Keplerian acceleration be denoted as

\[
f_3 = \frac{GM}{r_3^3} f_{4s}(m_1, m_2, a_1, e_1, j_1, r_3) + \frac{GM}{r_3^3} f_{4s}(m_3, m_4, a_2, e_2, j_2, r_3). \tag{12}
\]

The changes to the outer semimajor axis, eccentricity vector, and (specific) angular-momentum vector \( (h_3 = r_3 \times \dot{r}_3) \) can then be found according to (e.g. Eggleton 2006, appendix C)

\[
\frac{\Delta a_3}{a_3} = \int_{-L}^{L} \frac{d\theta}{GM} \frac{-2a_3}{M_1 M_2} (-r_3 \cdot f_3) \frac{dt}{d\theta} = 0; \tag{13a}
\]

\[
\Delta e_3 = \int_{-L}^{L} \frac{d\theta}{GM} \left(2r_3 (r_3 \cdot f_3) - f_3 (r_3 \cdot r_3) - r_3 (r_3 \cdot f_3)\right) \frac{dt}{d\theta} = f_{\Delta e_3}(m_1, m_2, a_1, e_1, j_1, E) + f_{\Delta e_3}(m_3, m_4, a_2, e_2, j_2, E); \tag{13b}
\]

\[
\Delta h_3/h_3 = \int_{-L}^{L} \frac{d\theta}{h_3} \left(r_3 \times f_3 \right) \frac{dt}{d\theta} = f_{\Delta h_3}(m_1, m_2, a_1, e_1, j_1, E) + f_{\Delta h_3}(m_3, m_4, a_2, e_2, j_2, E). \tag{13c}
\]

Here, we defined the additional expressions

\[
f_{\Delta e_3}(m, m', a, e, j, E) = \frac{mm'}{(m+m')^2} \left(\frac{a}{Q}\right)^2 \left(\frac{E-1}{E+1} \right)^{3/2} \frac{\sqrt{E+1}}{E^3} (5e_j e_j - j_1 j_1) \ddot{x} + \frac{3}{32(E+1)^2} \left(4E^2 - 1 \right) \left(1 + 4e_j^2 - e_j^2 - 6e_j^2 E \right) \right]
\]

\[
-2j_1^2 - j_2^2 + E^2 \left(2 - 17e_j^2 + 23e_j^2 - 12e_j^2 - j_2^2 \right) + 2 \left(5e_j^2 - 5e_j^2 - j_1^2 \right) + 8EL \left(2 + 3e_j^2 + 3e_j^2 - 12e_j^2 - 3j_2^2 - 3j_2^2 \right) \right] \ddot{y}
\]

\[
- \sqrt{E^2 - 1} \left(2E^2 + 1 \right) + 3EL \left(5e_j e_j - j_1 j_1 \right) \ddot{z} \right]
\]

\[
= \frac{mm'}{(m+m')^2} \left(\frac{a}{Q}\right)^2 \left(\frac{1}{2E+1} \right)^{3/2} \frac{\sqrt{E+1}}{E^3} (5e_j e_j - j_1 j_1) \ddot{x} - \sqrt{1 - \frac{1}{2E+1} \left(4E^2 - 1 \right) + 3EL \left(5e_j e_j - j_1 j_1 \right) \ddot{y}
\]

\[
+ \left(\frac{E-1}{E+1} \right)^{3/2} (5e_j e_j - j_1 j_1) \ddot{z} \right]. \tag{14a}
\]

From \( \ddot{e_3} = \ddot{x} \) initially, for small perturbations, the scalar eccentricity change is given by

\[
\Delta e_3 \approx \ddot{e}_3 \cdot \Delta e_3 = \frac{(E - 1)^{3/2} \sqrt{E + 1}}{E^3} \left[ \frac{m_1 m_2}{M_1^2} \left(\frac{a_1}{Q}\right)^2 \left(5e_{1,1,1} e_{1,1,1} - j_{1,1,1} j_{1,1,1} \right) + \frac{m_3 m_4}{M_2^2} \left(\frac{a_2}{Q}\right)^2 \left(5e_{2,2,2} e_{2,2,2} - j_{2,2,2} j_{2,2,2} \right) \right]. \tag{15}
\]

Note that \( \Delta e_3 = 0 \) if \( E = 1 \) (parabolic orbits), and becomes independent of \( E \) as \( E \ll 1 \).

The inclination change, \( \Delta i_3 \), is obtained from the new \( h_3 = h_3 e_3 + \Delta h_3 \) and noting that the inclination is measured with respect to the z-axis, giving

\[
\cos \Delta i_3 = \left[ \frac{\ddot{h}_3}{h_3} + \frac{\Delta h_3}{h_3} \right]
\]

\[
= 2E \left[ a_1^2 (E - 1)^{3/2} m_1 m_2 M_1^2 (5e_{1,1,1} e_{1,1,1} - j_{1,1,1}) + a_2^2 (E - 1)^{3/2} m_3 m_4 M_2^2 (5e_{2,2,2} e_{2,2,2} - j_{2,2,2}) + 2E \sqrt{E + 1} M_1^2 M_2^2 Q^2 \right]
\]

\[
\times (1 - E^2)^{3/2} \left[ \frac{Q^2}{(E - 1)^3} \left( \frac{(E - 1)^3}{E^3} \right)^{3/2} \right]
\]

\[
+ E^3 M_1^2 M_2^2 \sqrt{(E + 1)Q} \right] + (1 - E^4) \left[ (1 - E^2)^{3/2} \right]
\]

\[
\left( 1 - \frac{1}{E^2} \right) \left( 4E^2 - 1 \right) + 3EL \right] \left( a_1^2 m_1 m_2 M_1^2 (5e_{1,1,1} e_{1,1,1} - j_{1,1,1}) + a_2^2 m_3 m_4 M_2^2 (5e_{2,2,2} e_{2,2,2} - j_{2,2,2}) \right)
\]

\[
+ a_1^2 m_1 m_2 M_1^2 (5e_{1,1,1} e_{1,1,1} - j_{1,1,1}) + a_2^2 m_3 m_4 M_2^2 (5e_{2,2,2} e_{2,2,2} - j_{2,2,2}) \right) \right]^{-1/2} \right]. \tag{16}
\]
Here, we used that the initial $h_2 = \sqrt{GMQ(1 + E)}$.

From these equations for $\Delta e_3$ and $\Delta e_1$, it is clear that the backreaction effects scale with $(a_i Q)^2$ and so are typically small. This is also borne out by numerical simulations below (Section 3).

### 2.4.2 Inner orbits

We can obtain approximate expressions for the scalar eccentricity change of orbit $i$ ($i \in \{1, 2\}$) by integrating the equations of motion, equation (9), over $\theta$ assuming that all orbits (including the outer orbit) are static (i.e. constant $e_i$ and $J_i$). The result is

$$\Delta e_i = \Delta e_{i,\text{quad}} + \Delta e_{i,\text{oct}} + \Delta e_{i,\text{hex}} + \Delta e_{i,\text{hex, cross}},$$

(17)

where

$$\Delta e_{i,\text{quad}} = \epsilon_{SA,i} \frac{5}{2e_i E} \left( \sqrt{1 - \frac{1}{E^2}} \left( 2e_i \epsilon_{i,x} (E^2 - 1) J_{i,z} + e_i \epsilon_{i,z} (1 - 4E^2) J_{i,x} + \epsilon_{i,x} \epsilon_{i,z} (2E^2 + 1) J_{i,x} + 3e_i \epsilon_i E L (\epsilon_{i,y} J_{i,x} - e_i J_{i,y}) \right) \right);$$

(18a)

$$\Delta e_{i,\text{oct}} = \epsilon_{SA,i} \epsilon_{o,\text{oct, i}} \frac{5}{2e_i E} \left[ 3E^3 L \left( \epsilon_{i,x}^2 (3e_i J_{i,z} - 73e_i J_{i,y}) + 10e_i \epsilon_{i,x} (7e_i \epsilon_{i,z} + J_{i,y}) \right)
+ e_i J_{i,y} \left( -3e_i J_{i,z} + 5J_{i,x}^2 + 4 \right) + e_i J_{i,z} \left( 3e_i^2 - 15J_{i,x}^2 - 5J_{i,y}^2 + 4 \right) - 32e_i \epsilon_i J_{i,x}^2 + 32e_i \epsilon_i J_{i,y} \right]
+ \sqrt{1 - \frac{1}{E^2}} \left( -e_i \epsilon_{i,x} (160E^4 + 45E^2 + 14) J_{i,y} - 3e_i \epsilon_{i,y} (16E^4 - 27E^2 + 14) J_{i,z} \right)
+ 2e_i \epsilon_{i,x} (8E^4 + 9E^2 - 2) J_{i,z} \left( 7e_i \epsilon_{i,x} + J_{i,y} \right) + e_i \epsilon_{i,y} \left( -8E^4 + 31E^2 - 14 \right) J_{i,x} + \epsilon_{i,x} \epsilon_{i,y} (8E^4 - 31E^2 + 14) J_{i,y}
- e_i \epsilon_{i,y} (8E^4 (8e_i^2 + 4J_{i,x}^2 + J_{i,y}^2 - 1) + E^2 (32e_i^2 + 11J_{i,x}^2 + 9J_{i,y}^2 - 4) + 2J_{i,x}^2 - J_{i,y}^2 \right)
+ e_i \epsilon_{i,x} \epsilon_{i,y} (8E^4 (8e_i^2 + 2J_{i,x}^2 + J_{i,y}^2 - 1) + E^2 (32e_i^2 - 7J_{i,x}^2 + 9J_{i,y}^2 - 4) + 6J_{i,x}^2 - 2J_{i,y}^2 \right) \right];$$

(18b)

$$\Delta e_{i,\text{hex}} = \epsilon_{SA,i} \epsilon_{\text{hex, i}} \frac{7}{128e_i E} \left[ 15E^3 L \epsilon_{i,x} \left( e_i J_{i,z} (e_i^2 (129E^2 + 46) + E^2 (21J_{i,x}^2 + 21J_{i,y}^2 + 6) - 2 (7J_{i,x}^2 + 7J_{i,y}^2 - 4)) \right)
+ 2e_i \epsilon_{i,x} J_{i,z} \left( 3E^2 (e_i^2 - 14J_{i,x}^2 + 2) + 14 (J_{i,x} - J_{i,y}) \right) - 12e_i \epsilon_i J_{i,x} \left( 3E^2 + 4 \right) J_{i,y} \right)
+ e_i \epsilon_{i,y} \left( e_i^2 \epsilon_{i,z} \left( -135E^2 + 46 \right) + E^2 (21J_{i,x}^2 + 7J_{i,y}^2 - 6) + 2 (7J_{i,x}^2 + 7J_{i,y}^2 - 4) \right) + 14e_i \epsilon_{i,z} \left( 3E^2 + 2 \right) J_{i,x} \left( 7J_{i,x} - 7J_{i,y} \right) \right)
+ 20e_i^3 (9E^2 + 4) + e_i \epsilon_{i,z} \left( 6e_i E^2 J_{i,x} + e_i \left( 3E^2 + 46 \right) J_{i,x} \right) - e_i \epsilon_{i,z} \epsilon_{i,y} \left( e_i J_{i,z} + 9E^2 + 46 \right) + 14 \left( 3E^2 + 2 \right) J_{i,x} \left( J_{i,x} J_{i,y} \right) \right)
+ \sqrt{1 - \frac{1}{E^2}} \left( e_i \epsilon_{i,y} \left( 32e_i^2 - 63E^2 + 70E^2 - 24 \right) J_{i,z} + e_i \epsilon_{i,z} \left( -1024E^6 - 1751E^4 + 24E^2 + 36 \right) J_{i,y} \right)
+ e_i \epsilon_{i,z} \left( e_i J_{i,z} (832E^6 + 2129E^4 - 444E^2 + 108) + 2 \left( 128E^6 + 421E^4 - 36E^2 + 12 \right) J_{i,y} \right)
+ e_i \epsilon_{i,x} \left( -6e_i J_{i,y} \left( 16E^6 - 81E^4 + 74E^2 - 24 \right) J_{i,z} + e_i J_{i,z} \left( 128E^6 - 1049E^4 + 204E^2 - 108 \right) J_{i,y} \right)
- 4e_i \epsilon_{i,y} J_{i,z} \left( 8E^6 (30e_i^2 + 20J_{i,x}^2 + 7J_{i,y} - 3) + E^4 (270e_i^2 + 269J_{i,x}^2 - 80J_{i,y} - 27) - 3E^2 (20e_i^2 + J_{i,x}^2 + 13J_{i,y} - 2) - 6J_{i,x}^2 + 6J_{i,y}^2 \right)
+ e_i J_{i,y} \left( 128E^6 \left( 10e_i^2 + 4J_{i,x}^2 + 2J_{i,y} - 1 \right) + E^4 (2740e_i^2 + 655J_{i,x}^2 + 421J_{i,y} - 274) - 12E^2 \left( 10e_i^2 - 2J_{i,x}^2 + 3J_{i,y} - 1 \right) \right)
+ 12 \left( J_{i,y} - 3J_{i,y} \right) \right) + e_i \epsilon_{i,z} \left( e_i \epsilon_{i,y} \left( 32E^6 - 563E^4 - 240E^2 + 36 \right) + 2e_i \epsilon_{i,y} (128E^6 + 421E^4 - 36E^2 + 12) J_{i,y} \right)
+ e_i \left( 32E^6 (10e_i^2 + 4J_{i,x}^2 + 5J_{i,y} - 1) + E^4 (1660e_i^2 + 421J_{i,x}^2 - 101J_{i,y} - 166) + 12E^2 \left( 10e_i^2 - 3J_{i,x}^2 + 16J_{i,y} - 1 \right) \right)
+ 12 \left( J_{i,x} - 3J_{i,y} \right) \right];$$

(18c)
\[
\Delta e_{\text{hex, cross}} = \epsilon_{S, A, t} e_{\text{hex, cross}} \frac{5}{64e E} \left[ 3 \left( \epsilon_{S, A, t} 2e_{\text{hex, cross}} (11E^2 + 2) J_{\text{y}, i}^+ + e_{\text{hex, cross}} (3E^2 + 2) J_{\text{r}, j}^+ \right) - J_{\text{r}, j}^+ \right] + \sum_{\text{cross}} \epsilon_{S, A, t} e_{\text{hex, cross}} \frac{5}{64e E} \left[ 3 \left( \epsilon_{S, A, t} 2e_{\text{hex, cross}} (11E^2 + 2) J_{\text{y}, i}^+ + e_{\text{hex, cross}} (3E^2 + 2) J_{\text{r}, j}^+ \right) - J_{\text{r}, j}^+ \right],
\]

where the corresponding expression was derived in Hamers & Samsing (2019a). We include the quadrupole-order term \( \propto \epsilon_{S, A, t}^2 \), where the correct expression was derived in Hamers & Samsing (2019a).

### 3 NUMERICAL INTEGRATIONS

In this section, we carry out several numerical integrations to illustrate orbital changes in the two binaries for various parameters, and compare to the analytic expressions of Section 2.4. In Section 3.1, we focus on the backreaction of the outer orbit; in Section 3.2, we consider series of integrations with varying properties of binary 2. An overview of the initial conditions adopted in these sections is given in Table 1. We choose to restrict to systems with equal masses in binary 1, which is motivated by the fact that this eliminates the octupole-order terms (see equation 11b), which would otherwise dominate the hexadecupole-order terms and thus decrease the importance of the hexadecupole-order cross term even further.

Our numerical integrations are based on four-body calculations, as well as calculations based on the equations of motion averaged over the inner orbits (see Section 2.3). The four-body integrations were carried out using the IAS15 integrator within the REBOUND package (Rein & Liu 2012; Rein & Spiegel 2015). We integrated the inner-averaged equations of motion using ODEINT from the PYTHON SCIPY library, with the relative and absolute tolerances set to 10^{-13}. In both cases of the four-body and inner-averaged integrations, the integration time was...
set to $t_{\text{end}}$ with periastron passage (ignoring backreaction) occurring at $t_{\text{end}}/2$, where

$$
t_{\text{end}} = \frac{1}{n_3} \left[ -4 \arctanh \left( \frac{(E - 1) \tan(\beta/2)}{\sqrt{E^2 - 1}} \right) + \frac{2E\sqrt{E^2 - 1} \sin \beta}{1 + E \cos \beta} \right].$$

(19)

Here, $\beta = f_0 \arccos(-1/E)$ indicates the fraction of the outer orbit true anomaly $\theta$ in the integrations compared to integrating from $t \to -\infty$ to $t \to \infty$. Specifically, $f_0$ corresponds to integrating over true anomaly $\theta$ from $-f_0 \arccos(-1/E)$ to $f_0 \arccos(-1/E)$, with $f_0 = 1$ corresponding to integrating from $t \to -\infty$ to $t \to \infty$. We have checked our results for convergence with respect to $f_0$.

Several PYTHON scripts to carry out the four-body and inner-averaged integrations and to compute the analytic expressions are freely available.\(^1\)

### 3.1 Changes of the outer orbit

As discussed in Sections 2.2.1 and 2.4.1, both binaries can affect the outer orbit and cause the latter to deviate from purely Keplerian motion. Consequently, this can affect the eccentricity and angular-momentum changes of the inner orbits, which we refer to as ‘backreaction’. In Fig. 2, we show the time evolution of the orbital elements (semimajor axes, eccentricities, and inclinations) of the three orbits. The top (bottom) nine panels correspond to the situation in which the backreaction terms to quadrupole order (see the expression for $P_1$ in equation 9) were included (excluded).

In each set of nine panels, the top row shows the semimajor axes. The four-body integrations (solid green lines) show tiny fluctuations in the semimajor axes near periastron (note that in the top-left panel, $+1$ should be added in the $y$-axes). The inner-averaged integrations (black dashed lines) show no change in $a_1$ and $a_2$, as an immediate consequence of orbit averaging. When backreaction is included, the inner-averaged integrations give a fluctuation in $a_1$ near periastron with no net change, and which agrees with the four-body integrations. The fact that the semimajor axes are conserved is expected for this system, which is well within the secular regime.

The middle and bottom rows in each set of nine panels show the eccentricities and inclinations, respectively. Without backreaction, $e_3$ and $i_3$ in the inner-averaged integrations remain constant by construction, whereas the four-body integrations show that there is a net change in these quantities – the net change in $e_3$ is tiny, whereas it is more significant (but still very small) in $i_3$, with $\Delta i_3 \simeq 0.06^\circ$ in this case. With backreaction included, the inner-averaged integrations agree with the four-body integrations in terms of $e_3$ and $i_3$. Also, the analytic prediction for $\Delta i_3$ agrees with the numerical results.

Moreover, in terms of the inner orbit eccentricities and inclinations from the inner-averaged integrations and comparing the top and bottom set of nine panels, it is clear that the backreaction terms have no appreciable effect (the only noticeable effect is a slight different in $i_2$ of $\simeq 0.002^\circ$, as shown in the bottom-middle panel of the low set of nine panels).

### 3.2 The impact of the ‘binarity’ of the companion

Here, we carry out several series of integrations to investigate the effect that the ‘binarity’ of orbit 2 has on the eccentricity change of orbit 1. All initial conditions can be found in Table 1.

In Fig. 3, we vary $a_2$, keeping all other parameters fixed. Evidently, as $a_2 \to 0$, we recover the limit of an encounter of a binary with a single point mass. First of all, note that the eccentricity changes in orbit 1 are very weakly dependent on $a_2$: varying $a_2$ between 0.5 and 3 au affects $\Delta e_1$ by only $\sim 10^{-2}$. Even for the largest value of $a_2$ considered, it is still a good approximation to consider orbit 2 as a point mass in the computation of $\Delta e_1$ (see the red horizontal dot–dashed line in Fig. 3, which shows the corresponding analytic value assuming binary 2 is a point mass). Note that if $a_2$ is much larger than 3 au, the system would no longer be in the secular regime (cf. equation 6). The eccentricity

\(^1\)https://github.com/hamers/lybyhs_bin
Figure 2. Evolution of the orbital elements of the three orbits as a function of time (normalized to the integration time, $t_{\text{end}}$, see equation 19). See Table 1 for the initial conditions. The backreaction terms were included in the top nine panels, and excluded in the bottom nine panels. In each set of nine panels, the top row shows the semimajor axes, the middle row shows the eccentricities, and the bottom row shows the inclinations. Note that, initially, $i_3 = 0$ by the choice of the coordinate system. Solid green lines correspond to four-body integrations and black dashed lines to integrations averaged over the inner orbit (but not the outer orbit). In the third column, red dotted lines show analytic results for the net changes in the outer orbit (see Section 2.4.1).
Figure 3. Scalar eccentricity changes in orbits 1 (top panel) and 2 (bottom panel) as a function of $a_2$. See Table 1 for the initial conditions. Green dots correspond to four-body integrations, solid black lines to inner-averaged integrations (‘inner averaged’ in the legend), and red lines to analytic expressions (see Section 2.4.2). For the red dotted lines (‘analytic’ in the legend), the hexadecupole-order cross term is included, whereas it is not for the horizontal red dot–dashed line (‘analytic repl.’, i.e. ‘analytic replaced’ in the legend). In other words, orbit 2 is considered to be a point mass in the ‘analytic repl.’ horizontal red dot–dashed lines.

of orbit 2 changes much more appreciably and according to a power law, which is expected given that $a_2$ is varied in this series of integrations (e.g. Heggie & Rasio 1996).

The inner-averaged integrations generally agree with the four-body results, i.e. $\Delta e_1$ increases with increasing $a_2$ with our choice of initial conditions. Some deviations are apparent at specific values of $a_2$, as well as for larger values of $a_2$. The latter can be explained by the fact that $\epsilon_{SA, 2}$ is approaching unity as $a_2$ increases, with $\epsilon_{SA, 2} \simeq 0.017$ if $a_2 = 3$ au, i.e. the system gradually becomes less secular. The discrepancies at smaller values of $a_2$ are likely due to mean-motion resonances (MMRs) between the two inner orbits. This is suggested by their occurring locations in $a_2$, which correspond to various MMRs and which are indicated with vertical black dashed lines. Specifically, we show the $1: \alpha$ resonances between orbits 1 and 2, where $\alpha \in \{1, 2, 3, 4\}$; setting $P_1 = \alpha P_2$, where $P_i$ denotes orbital period and $\alpha$ is a dimensionless factor, implies

$$a_2 = a_1 \left[ a^2 (M_2/M_1)^{1/3} \right].$$

(20)

The red dotted lines in Fig. 3 show the analytic results from Section 2.4.2 with the inclusion of terms up to and including hexadecupole order (and including the cross term), as well as the quadrupole-order term that is second order in $\epsilon_{SA, 1}$ (see Hamers & Samsing 2019a). Overall, these analytic expressions agree with the numerical results, although some deviation can be seen, especially for larger $a_2$. This can be attributed to the fact that the analytic expressions do not fully take into account the changing $e_i$ during the encounter (only to second order...
in $\epsilon_{SA,i}$, and at quadrupole expansion order). Although it is possible in principle to derive more accurate expressions, they are excessively long and so are not practical (see e.g. table 1 of Hamers & Samsing 2019b).

In Fig. 4, we consider the same series as in Fig. 3, but include only inner-averaged numerical integrations, and compare the cases including backreaction on the outer orbit (black solid lines), and without (black dashed lines). As shown, there are only very small differences between the two cases, again illustrating that the backreaction of the inner two orbits on the outer orbit can be neglected.

We show a similar figure to Fig. 3 in Fig. 5, but now with a higher initial value of $e_1$. The eccentricity changes are now slightly larger, and the relative importance of MMRs appears to be lower. In Fig. 6, we choose different values of $\omega_2$ and $i_2$. With this different choice of relative orbital orientations, the changes in $\Delta e_1$ with increasing $a_2$ are even smaller, of the order of $\sim 10^{-6}$. The different relative orientation between the orbits in this example leads to a decrease in $\Delta e_1$ with increasing $a_2$, instead of increasing in Figs 3 and 5. Also, MMRs appear to have a larger impact on $\Delta e_1$.

In Fig. 7, we fix $a_2$ but vary $q_2 = m_4/m_3$ instead, keeping the total mass of binary 2, $M_2 = m_3 + m_4$, fixed to $M_2 = 10 M_\odot$. The point-mass limit is approached as $q_2 \to 0$, and $\Delta e_1$ indeed approaches the point-mass value (red horizontal dot–dashed line) as $q_2$ decreases. The dependence on $q_2$, like $a_2$, is very weak, with changes in $\Delta e_1$ of the order of $10^{-5}$. The analytic results including the hexadecupole-order cross term (red dotted lines) agree with the numerical results (both four body and inner averaged), except for $\Delta e_2$. This may be related to the omission of higher order terms in $\epsilon_{SA,i}$. In addition, the four-body integrations do not agree well with the inner-averaged integrations with respect to $\Delta e_2$, which may be due to a breakdown of the inner-averaging approximation.

Lastly, in Fig. 8, we consider the dependence on relative orientation by varying $i_2$ and fixing the other parameters. The changes in $e_1$ are again very small, and $\Delta e_1$ decreases with increasing $i_2$. The analytic expressions agree reasonably with the numerical results.
4 DISCUSSION

4.1 Importance of the cross term

As shown in the above sections, in the expansion of the Hamiltonian of the system, the hexadecupole order is the lowest expansion order at which a term appears that explicitly depends on all three orbits simultaneously (the inner two bound orbits and the outer unbound orbit). This ‘cross term’ gives rise to the largest changes of the secular changes of one binary due to the ‘binarity’ of the other binary. Given that the cross term appears at a high expansion order, the ‘binarity’ effect of the companion binary is typically small and, in most cases, it is well justified to simply apply the known expressions for the secular changes for binary–single interactions (Heggie & Rasio 1996; Spurzem et al. 2009; Hamers 2018; Geller et al. 2019; Hamers & Samsing 2019a, b) with the ‘third body’ mass replaced with the total mass of the companion binary.

Nonetheless, it is informative to explore more generally the importance of the hexadecupole-order cross term in relation to the other terms of interest, i.e. the pairwise quadrupole, octupole, and hexadecupole-order terms. In Fig. 9, we estimate (i.e. within approximately an order of magnitude) the changes in eccentricity and inclination of orbit 1, plotting their rough approximations as a function of $Q/a_1$ for fixed $a_1$, and various values of $a_2$. Here, we estimate the eccentricity/inclination changes based on equation (17), ignoring the complex dimensionless functions of $e_i, f_i$, and $E$ and any terms $O(\epsilon_2^{2i})$. Specifically, we set

$$\Delta \epsilon_{1,\text{quad}} \sim \epsilon_{SA,1};$$  \hspace{1cm} (21a)

$$\Delta \epsilon_{1,\text{oct}} \sim \epsilon_{SA,1} \epsilon_{\text{oct},1};$$  \hspace{1cm} (21b)

$$\Delta \epsilon_{1,\text{hex}} \sim \epsilon_{SA,1} \epsilon_{\text{hex},1};$$  \hspace{1cm} (21c)
4.2 Limitations of the analytic expressions and the inner-averaged approach

In Section 2.4.2, we derived analytic expressions for the eccentricity changes taking into account the hexadecupole-order cross term, which is the lowest order term that leads to a direct coupling between the inner two orbits. These expressions agree reasonably with numerical integrations, both four-body integrations and inner-averaged integrations, although the agreement is by no means perfect. Any deviation between the inner-averaged integrations and the analytic expressions arises from the fact that we assumed in Section 2.4.2 that all three orbits are static during the encounter. This approximation can break down, especially when the initial eccentricities are already large (making the inner orbits more susceptible to large secular changes).

Corrections to counter the breakdown of this approximation could be derived to second (and higher) order in \( \epsilon_{SA,i} \), as has been done in Hamers & Samsing (2019a). When comparing to numerical results in Section 3, we also included second-order terms in \( \epsilon_{SA,i} \), but only to the quadrupole order. Similar expressions to higher orders in \( \epsilon_{SA,i} \) give rise to excessively long expressions (see Hamers & Samsing 2019b), which severely reduces their practical usefulness. Moreover, contributions from the second-order terms in \( \epsilon_{SA,i} \) at higher expansion orders
Figure 7. Similar to Fig. 3, but here with fixed $a_2$ and varying $m_3$ and $m_4$, keeping $M_2 \equiv m_3 + m_4 = 10 M_\odot$ fixed and plotting the eccentricity changes as a function of $q_2 \equiv m_4/m_3$ (see also Table 1).

(octupole, hexadecupole, etc.) become increasingly small. Here, we therefore did not derive new expressions for the eccentricity changes taking into account non-static orbits during the encounter for high expansion orders (in particular, for the hexadecupole-order cross term).

In addition, we found discrepancies between the four-body integrations and the inner-averaged integrations (on which the fully analytic expressions are based). This is reflected in Fig. 3 and further, where the inner averaged (black solid lines) and fully averaged (red dotted lines) show disagreement with the four-body integrations near the mean motion resonance locations, as well as for large $a_2$, when the averaging approximation breaks down because $a_2$ is becoming too large. These discrepancies can be attributed to a breakdown of averaging in the inner orbits. Averaging corrections to the inner orbits as well (see e.g. Lei 2019) are beyond the scope of this paper.

4.3 Implications for larger $N$ scattering in the secular limit

We briefly discuss further implications of the main result of this paper, i.e. that, in the secular limit, a binary perturbed by another distant binary is not significantly affected by the quadrupole moment of the companion binary, and that the orbital changes can simply be obtained by applying known results for binary–single encounters and replacing the third body’s mass with the total companion binary mass. With this result in mind, it is clear that an extension to encounters with higher-multiplicity systems in the secular limit can easily be made: for a binary encountering an arbitrary hierarchical system composed of nested orbits one can, to first approximation, apply the secular expressions for binary–single encounters (possibly including higher-order terms in $\epsilon_{SA,1}$) replacing the ‘third body’ mass with the total mass of the encountering hierarchical system. This also implies that any internal evolution of the encountering system does not play any major role, no matter its own evolution time-scale in relation to the encounter time-scale. For example, a binary encountering a triple results in approximately the same secular effects on the binary compared to the case of a binary encountering a single object with the same mass as the triple.
5 CONCLUSIONS

We studied the dynamical evolution of two binaries approaching each other on unbound orbits. We focused on the ‘secular’ regime, in which the binaries approach each other with a sufficiently large periapsis distance such that the semimajor axes of the two bound orbits do not change appreciably after the encounter, but eccentricity and angular-momentum changes are possible. We carried out numerical integrations, as well as derived analytic results. Our main conclusions are given below.

(i) The Hamiltonian, expanded in the small ratios \( x_1 = r_1/r_3 \ll 1 \) and \( x_2 = r_2/r_3 \ll 1 \), where \( r_1 \) and \( r_2 \) are the relative separations of the inner two bound binaries and \( r_3 \) is the separation of the ‘outer’ unbound orbit, consists of pairwise terms at the quadrupole and octupole orders (\( x_i = 2 \) and \( x_i = 3 \), respectively, for \( i \in \{1, 2\} \)). Only at the hexadecupole order (\( x_i = 4 \)) there appears a term, the hexadecupole-order cross term, that explicitly depends on the separations of all three orbits. This implies that any effect of the ‘binarity’ of orbit 2 on orbit 1 (i.e. its quadrupole moment) is only exhibited through (1) a backreaction of the outer orbit, \( r_3 \), and (2) high-order expansion terms, starting at the hexadecupole order. We explicitly derived the expanded Hamiltonian (up to and including hexadecupole order) and averaged over the inner two orbits (Section 2.2.2), as well as the corresponding equations of motion (Section 2.3).

(ii) We derived approximate analytic expressions for the eccentricity and inclination changes of the outer orbit due to the backreaction of orbits 1 and 2 (Section 2.4.1). These expressions show that the backreaction effects are very small, which we confirmed with numerical integrations (Section 3.1).

(iii) We also derived approximate analytic expressions for the secular effects on the inner orbits taking into account the hexadecupole-order cross term. In particular, the quadrupole moment of the companion orbit gives rise to the secular changes which are of the order of \( \epsilon_{SA,1}(a_2/Q)^2[m_3 m_4/(m_3 + m_4)^2] \), where \( \epsilon_{SA,1} \) is the magnitude of the quadrupole-order change (see equation 11a), and \( a_2 \) and \( (m_3, m_4) \) are the companion binary orbital semimajor axis and component masses, respectively. Here, we largely ignored the fact that the inner orbits change

Figure 8. Similar to Fig. 3, but here with fixed \( a_2 \) and varying \( i_2 \) (see also Table 1).
Figure 9. Changes of the eccentricity and inclination of orbit 1 plotted as a function of $Q/a_1$ for a fixed $a_1 = 1$ au and different $a_2$ (the latter being either 0.1, 1.0, or 5.0 au). We show the contributions from various expansion orders in the Hamiltonian to $\Delta e_1$ and $\Delta i_1$, which we roughly estimate (within approximately an order of magnitude) using equation (21a). Each panel corresponds to a certain choice of the masses $m_i$ and the outer orbit eccentricity $E$, indicated in the top. Note: in the left-hand column, the ‘Oct.’ lines are zero and therefore not shown, since $m_1 = m_2$. Also, the ‘Hex.’ lines coincide exactly with the ‘Hex. cross’ lines at $a_2 = 1$ au.

(iv) Most importantly, as shown by our analytic and numerical results, the ‘binarity’ of orbit 2, when considering orbit 1, typically leads to only very small eccentricity and inclination changes. To good approximation, one can obtain the secular changes by using the analytic results for binary–single interactions (Heggie & Rasio 1996; Spurzem et al. 2009; Hamers 2018; Geller et al. 2019; Hamers & Samsing 2019a, b) and replacing the mass of the intruding unbound third body with the total mass of binary 2. In other words, the point-mass approximation works well in this case.

Several PYTHON scripts implementing the two numerical integration methods and the analytical results as well as routines used to make all the plots in this paper, are freely available at the link given in Section 3.

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