Decoding the Volatility Smile

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Presentation Overview

- This presentation investigates the relationship between the distribution of the mean integrated variance $\psi(\vartheta)$ where

$$\vartheta = \frac{1}{T} \int_0^T \sigma(t)^2 dt$$

and the corresponding volatility smile.

- Consideration is given to the case of uncorrelated stochastic volatility - much can be discovered here.

- Particular focus is given over to the recovery of the MIV from an asset volatility smile.

- Considering the SABR model in particular, it is concluded that in order to modify the low delta wings of the smile, a negative skew needs to be induced in $\psi(\vartheta)$.

- Following the parallels of effecting skew in an implied asset distribution, local volatility of volatility and stochastic volatility of volatility are considered and the effect on the wings of the volatility smile demonstrated.
Uncorrelated Stochastic Volatility

Consider a lognormal asset process with uncorrelated stochastic volatility i.e. driven by the process

\[ dF(t, T) = \sigma(t) F(t, T) dW(t) \]

for the Weiner process \( W(t) \), volatility \( \sigma(t) \) and asset forward rate \( F(t, T) \). Then the asset process is

\[ F(T, T) = F(0, T) \exp \left( -\frac{1}{2} \int_0^T \sigma(t)^2 dt + \int_0^T \sigma(t) dW(t) \right) \]

When the asset process is uncorrelated to the volatility process, then upon defining the mean integrated variance

\[ \vartheta = \frac{1}{T} \int_0^T \sigma(t)^2 dt \]

then the asset distribution can be represented as

\[ F(T, T) = F(0, T) \exp \left( -\frac{1}{2} \vartheta T + \sqrt{\vartheta} W'(T) \right) \]

for a second Weiner process \( W'(t) \).
Uncorrelated Stochastic Volatility

This leads to an undiscounted (path independent) option representation

\[ V(\{K_i\}, T) = \int_0^\infty V(\{K_i\}, T|\vartheta)\psi(\vartheta) d\vartheta \]

for some option payoff \( V(\ldots) \) with parameterisation set \( \{K_i\} \) and expiry \( T \). The distribution of the mean integrated variance is given by \( \psi(\vartheta) \). For the specific case of a call option

\[ C(K, T) = \int_0^\infty C(K, T|\vartheta)\psi(\vartheta) d\vartheta \]

There are two problems here

- given a volatility process, the determination of \( C(K, T) \)
- the inverse problem of determining \( \psi(\vartheta) \) given a set \( \{C(K, T), \forall K\} \)
Uncorrelated Stochastic Volatility - Hull and White [1987]

The first problem has been studied extensively. In Hull and White [1987] the first few moments of $\psi(\vartheta)$ are determined and used in the Taylor expansion of the option price in the mean integrated variance

$$C(K, T) = C(K, T|\mathbb{E}[\vartheta]) + \frac{1}{2} \left. \frac{\partial^2 C}{\partial \vartheta^2} \right|_{\mathbb{E}[\vartheta]} \mathbb{E} \left[ (\vartheta - \mathbb{E}[\vartheta])^2 \right] + ....$$

The method relies on the calculation of the expected powers of $\vartheta - \mathbb{E}[\vartheta]$, $\mathbb{E}[\vartheta^2]$, ... For their chosen process

$$dV = \mu Vdt + \xi VdW$$

where $V = \sigma(t)^2$ then

$$\mathbb{E}[\vartheta] = \frac{e^{\mu T} - 1}{\mu T} V_0$$

$$\mathbb{E}[\vartheta^2] = \left[ \frac{2 e^{(2\mu+\xi^2)T}}{(\mu + \xi^2)(2\mu + \xi^2)T^2} + \frac{2}{\mu T^2} \left( \frac{1}{2\mu + \xi^2} - e^{\mu T} \mu + \xi^2 \right) \right]$$

The focus of Hull-White is on $C(K, T, \vartheta)$. Although intuitively useful, practically it is not. The attention of Stein and Stein on $\psi(\vartheta)$ is a more powerful approach.
Uncorrelated Stochastic Volatility - Stein and Stein [1991]

Although Stein and Stein is most often referenced in the context of the progression of stochastic volatility processes considered in the 1990’s, the content of their [1991] paper is particularly relevant.

- They consider the mean reverting volatility process

\[ d\sigma(t) = (\theta - \alpha \sigma(t))dt + \xi dW \]

- A closed form solution for the asset distribution, \( \psi(S, t) \) is presented
- They present an approximate solution to \( \phi(S, t) \) as

\[ \psi(S, t) = \int_0^{\infty} L(\sigma) m_t(\sigma) d\sigma \]

where \( L(\sigma) \) is lognormal and \( m_t(\sigma) \) a “mixing distribution”
- They demonstrate that approximating the volatility distribution accurately captures the prices of options.
Stochastic Volatility - Islah [2009] and McGhee [2010]

For the specific case of SABR both Islah [2009] and McGhee [2010] consider the more general case of correlated stochastic volatility, conditioning on the terminal instantaneous volatility and performing the calculation

\[
C(K, T) = \int_0^\infty \left( \int_0^\infty C(K, T, \vartheta) \psi(\vartheta | \sigma(T)) \right) \psi(\sigma(T)) d\sigma(T)
\]

In this approach the issues associated with the SABR approximation of Hagan et al [2002], specifically regions of negative implied probability density and the non-monotonicity of ATM implied volatility in terms of the volatility of volatility parameter, are absent.

In fact, Islah [2009] constructs \( \psi(\vartheta, \sigma(T)) \) following Yor [1992] whereas McGhee [2010] proxies \( \psi(\vartheta | \sigma(T)) \) as a lognormal (in effect representing \( \psi(\vartheta) \) as an integral of lognormals). The latter approach can be implemented highly efficiently and used in production pricing and risk management.
SABR in the wings

Even once the numerical issues associated with the SABR approximation are removed, calibration to a number of markets will show that the low delta behaviour is poor - for example, in the FX markets the low delta volatilities of the SABR approximation will be too high (leading to negative density), and although those of the underlying process (as determined by either Islah[2009] or McGhee[2010]) are lower they are often still appreciably in excess of the market offer rate.
SABR in the wings

Even once the numerical issues associated with the SABR approximation are removed, calibration to a number of markets will show that the low delta behaviour is poor - for example, in the FX markets the low delta volatilities of the SABR approximation will be too high (leading to negative density), and although those of the underlying process (as determined by either Islah[2009] or McGhee[2010]) are lower they are often still appreciably in excess of the market offer rate.
Uncorrelated Stochastic Volatility

- Having corrected for the numerical issues associated with the SABR approximation via the method of Islah [2009] or McGhee [2010] the natural question to ask then is what drives the value of the low delta options?
- Clearly the lognormal volatility process is inappropriate.
- The driver of the high volatility paths, giving rise to the low delta option premium, is the extreme right tail of the MIV density.
- Mean reversion deals with this poorly - it contracts the entire distribution - raising low vol paths as well as lowering high vol paths.
- What would be more desirable is to invoke negative skew in the MIV distribution over and above that induced by lognormal volatility. This can be done with local volatility of volatility or stochastic volatility of volatility.
- The first question is what can be determined from the MIV density from the observed asset smile?

This has received limited attention with the notable exception of Friz and Gatheral [2005] following Carr and Lee [2005].
Fredholm Operator of the First Kind
The inverse problem of determining $\psi(\vartheta)$ given $C(K, T)$ and the kernel function $C(K, \vartheta, T)$ is known as a Fredholm integral equation of the first kind:

$$\int_0^\infty d\vartheta C(K, \vartheta, T)\psi(\vartheta) = C(K, T)$$

Note we could equally write this equation in terms of the digital prices or the implied asset density function). A few pointers to where difficulties may arise:

▶ given the nature of the inverse problem, the solution of the Fredholm 1st kind integral equation is known to be difficult; various strategies exist in the literature (see for example Press et al [2007];

▶ since inverting the integral operator is highly unstable for many problems, a possible choice is to use the Moore-Penrose pseudo inverse;

▶ another possible way out, along the lines of Philips [1962] and Towney [1963], consists in choosing a smoothing parameter that makes the process of inverting the operator $C$ less ill-conditioned

It is important to note that here the asset distribution need not be lognormal - it could be CEV or some other asset process.
Uncorrelated Stochastic Volatility

Given a set of strikes \( \{ K_i, i = 1, \ldots, N \} \) and associated call option prices \( \{ C(K_i, T), i = 1, \ldots, N \} \) with a corresponding discretization of the \( \vartheta \)-space, \( \{ \vartheta_j; j = 1, \ldots, N \} \) with equal spacing \( \Delta \vartheta \) then each option can be expressed as

\[
C(K_i, T) = \sum_{j=1}^{N} C(K_i, T|\vartheta_j) \psi(\vartheta_j) \Delta \vartheta
\]

leading to the matrix representation

\[
C = C_{\text{cond}} \times \Psi
\]

with \( C(i) = C(K_i, T) \), \( C_{\text{cond}}(i, j) = C(K_i, T|\vartheta_j) \) and \( \Psi(j) = \psi(\vartheta_j) \Delta \vartheta \). The recovery of \( \Psi \) is via the apparently simple inversion

\[
\Psi = C_{\text{cond}}^{-1} \times C
\]

However, the construction of \( C_{\text{cond}}^{-1} \) is far from straightforward and is in fact a well studied numerical problem.
Uncorrelated Stochastic Volatility

By way of an example, now consider the simple case where $\psi(\vartheta)$ is lognormal. Here $\mathbb{E}[\vartheta] = 0.01$ and standard deviation 90%. Given a forward rate of $F(0, T) = 120.00$ and time horizon $T = 1 \text{yr}$ then the associated smile is

The strikes are chosen to correspond to $\{1P, 5P, 10P, ..., 10C, 5C, 1C\}$ forward deltas.
Uncorrelated Stochastic Volatility

The MIV distribution is discretized into 21 levels corresponding to \([-5.0, -4.5, -4.0, \ldots, +4.0, +4.5, +5.0]\) standard deviations (from 0.86\% to 77.49\% asset implied volatility). The individual asset densities are as shown

The observed implied asset densities is a weighted sum of the above.
Uncorrelated Stochastic Volatility

Now perform the calculation

$$\Psi = C^{-1}_{\text{cond}} \times C$$

with the known set of variance levels \(\{\vartheta_j; j = 1, \ldots, N\}\) and a chosen set of strikes. The intention is to recover \(\psi(\vartheta)\)
Uncorrelated Stochastic Volatility

Now perform the calculation

$$\Psi = C_{\text{cond}}^{-1} \times C$$

with the \textit{known} set of variance levels \{\vartheta_j; j = 1, \ldots, N\} and a chosen set of strikes. Choosing the original smile strikes is a disaster.
Uncorrelated Stochastic Volatility

Now perform the calculation

\[ \Psi = C_{\text{cond}}^{-1} \times C \]

with the known set of variance levels \( \{\vartheta_j; j = 1, ..., N\} \) and a chosen set of strikes. Here \( \{K_i = 120 \times 1.05^i; i = 0, ..., 20\} \). Note the upper part of \( \psi(\vartheta) \) is matched well.
Uncorrelated Stochastic Volatility

Now perform the calculation

$$\Psi = C_{\text{cond}}^{-1} \times C$$

with the known set of variance levels \( \{ \vartheta_j; j = 1, \ldots, N \} \) and a chosen set of strikes. Here the strikes are chosen as 3 asset standard deviations at each variance level. The entirety of \( \psi(\vartheta) \) is now matched well.
Uncorrelated Stochastic Volatility

- Even in this idealised case where only the weights of the conditional Gaussians were needed to be calculated it was far from straightforward
- Where a smile is given and the underlying distribution is unknown the problem is even more difficult but not across its entire range
- The right hand tail where the densities are less tightly packed is still accessible
- Further, it is this region that drives the value of the low delta options and so is the region of primary concern
- The standard academic techniques present in the literature need to be employed as well as conditioning from our knowledge of the underlying being a probability density (so constraints such as $\psi(\vartheta) \to 0$ as $\vartheta \to 0$, $+\infty$ can be imposed as well as the density having a single peak - though the latter assumes no jumps in the volatility)
- More details to be presented in due course.
Uncorrelated Stochastic Volatility

Taking SABR as an example,

- it would be desirable to attain more effective control over the wings of the volatility smile via a modification of the density of the mean integrated variance - in particular to introduce more negative skew.

- This is somewhat analogous to the requirement of modification of the asset distribution to effect smile and skew.

- There are two standard methods of approach - local volatility and stochastic volatility.

- By analogue, what is achieved via local volatility of volatility and stochastic volatility of volatility?
Introducing skew to $\psi(\vartheta)$

Specifically,

- **local volatility of volatility (ZABR)** - see Andreasen and Huge [2012]

\[
dF(t, T) = \sigma(0)\lambda(t)F(t, T)dW_F(t) \quad d\lambda(t) = \xi\lambda(t)^\beta dW_\lambda
\]

where $\lambda(0) = 1$.

- **stochastic volatility of volatility**

\[
dF(t, T) = \sigma(t)F(t, T)dW_F \quad d\sigma(t) = \xi(t)\sigma(t)dW_\sigma \quad d\xi(t) = \pi\xi(t)dW_\xi
\]

where

\[
< dW_F, dW_\sigma > = 0, \quad < dW_F, dW_\xi > = 0 \quad \text{and} \quad < dW_\sigma, dW_\xi > = \rho_{\xi,\pi} dt
\]

will allow indirect control over the skew of $\psi(\vartheta)$ and $\pi$ the volatility of the vol-of-vol control over the kurtosis. The case may appear overly complex.
Introducing skew to $\psi(\vartheta)$ - local volatility of volatility

Here the CEV exponent on the volatility process is $\{60\%, 80\%, 100\%, 120\%, 140\%\}$.
Introducing skew to $\psi(\vartheta)$ - local volatility of volatility

Here the CEV exponent on the volatility process is \{60\%, 80\%, 100\%\} so restricted to generative negative MIV skew.
Calculating under stochastic volatility of volatility

There are a number of ways in which to calculate option prices under this model

**Method 1 - Full Monte Carlo**

**Method 2 - Analytic Black-Scholes**
Generate the $j^{th}$ $\xi(t)$-path and in turn the $j^{th}$ $\sigma(t)$-paths. From this calculate the $j^{th}$ mean integrated variance $\vartheta_j$ and so the $j^{th}$ option price

$$C_j(K, T) = F(0, T)N \left[ \frac{\ln(F(0, T)/K) + \vartheta_j T/2}{\sqrt{\vartheta_j}} \right] - KN \left[ \frac{\ln(F(0, T)/K) - \vartheta_j T/2}{\sqrt{\vartheta_j}} \right]$$

The advantage of this method is every path has non-zero value - so convergence is relatively fast (this is nothing other than Willard [1997]). Compare to method 1 where if $DG_c(K, T)$ is the probability of the option being ITM then $1 - DG_c(K, T)$ percent of paths will have no value. So as the lower delta region is explored fewer and fewer paths will have non-zero value.
Calculating under stochastic volatility of volatility

**Method 3 - Pseudo-analytic SABR Process Extension**

Here we return to the earlier work of McGhee [2010]. For the uncorrelated SABR process then an option price can be calculated as

\[
C(K, T) = \int_{0}^{\infty} C(K, T | \vartheta) \psi(\vartheta) d\vartheta 
\]

\[
= \int_{0}^{\infty} \left( \int_{0}^{\infty} C(K, T | \vartheta) \psi(\vartheta | \sigma(T)) \right) \psi(\sigma(T)) d\sigma(T) 
\]

\[
\sim \int_{0}^{\infty} \left( \int_{0}^{\infty} C(K, T | \vartheta) \psi_{ln}(\vartheta | \sigma(T)) \right) \psi(\sigma(T)) d\sigma(T) 
\]

with \(\psi_{ln}(\vartheta | \sigma(T))\) being determined by \(E[\vartheta | \sigma(T)]\) and \(E[\vartheta^2 | \sigma(T)]\).
Calculating under stochastic volatility of volatility

Method 3 - Pseudo-analytic SABR Process Extension (cont)
For the SABR lognormal volatility process, then

$$\mathbb{E} \left[ \frac{1}{T} \int_0^T \sigma(t)^2 dt \mid \sigma(T) \right] = \frac{\sigma(0)^2}{T} \frac{\sqrt{2\pi T}}{2\xi} \exp \left( \frac{1}{2} q^2 \frac{\xi^2}{T} \right) \left[ N(\phi(T)) - N(\phi(0)) \right]$$

where

$$\phi(t) = \frac{2\xi}{\sqrt{T}} \left( t - \frac{1}{2} q \right) \quad \text{and} \quad q = T + \frac{1}{\xi^2} \log \left( \frac{\sigma(T)}{\sigma(0)} \right)$$

and

$$\mathbb{E} \left[ \left( \frac{1}{T} \int_0^T \sigma(t)^2 dt \right)^2 \mid \sigma(T) \right] = \frac{2}{T^2} \int_0^T dt_1 \int_{t_1}^T dt_2 \mathbb{E} \left[ \sigma(t_1)^2 \sigma(t_2)^2 \mid \sigma(T) \right]$$

Here the inner integral can be determined analytically leaving at most just one time numerical integration.
Calculating under stochastic volatility of volatility

Method 3 - Pseudo-analytic SABR Process Extension (cont)
Now, given a vol-of-vol path \( \xi(0, T) = \{ \xi(t); 0 \leq t \leq T \} \) then the conditional volatility process is

\[
d\sigma(t) | \xi(0, T) = \rho_{\sigma, \xi} \xi(t) \sigma(t) dW_\xi(t) + \bar{\rho}_{\sigma, \xi} \xi(t) \sigma(t) dW'_\sigma(t)
\]

using the standard resolution \( dW_\sigma(t) = \rho_{\sigma, \xi} dW_\xi + \bar{\rho}_{\sigma, \xi} dW'_\sigma(t) \) with \( \bar{\rho}_{\sigma, \xi} = (1 - \rho^2_{\sigma, \xi})^{1/2} \) then

\[
d\sigma(t) | \xi(0, T) = \bar{\mu}(t) \sigma(t) dt + \bar{\xi}(t) \sigma(t) d\bar{W}_\sigma(t)
\]

where

\[
\bar{\mu}(t) = \frac{\rho_{\sigma, \xi}}{\xi} \frac{d\xi(t)}{dt} \quad \text{and} \quad \bar{\xi}(t) = \bar{\rho}_{\sigma, \xi} \xi(t)
\]
Calculating under stochastic volatility of volatility

Method 3 - Pseudo-analytic SABR Process Extension (cont)

It is therefore necessary to extend the analysis of McGhee[2010] from the process

\[ d\sigma(t) = \xi\sigma(t)dW_\sigma(t) \]

\[ \text{to} \quad d\sigma(t) = \mu(t)\sigma(t)dt + \xi(t)\sigma(t)dW_\sigma(t) \]

Upon integrating gives

\[ \ln \sigma(T) = \ln \sigma(0) + \int_0^T \mu(t)dt - \frac{1}{2} \int_0^T \xi^2(t)dt + \int_0^T \xi(t)dW_\sigma(t) \]

Defining

\[ \bar{\mu} = \frac{1}{T} \int_0^T \mu(t)dt \quad \text{and} \quad \bar{\xi} = \left( \frac{1}{T} \int_0^T \xi^2(t)dt \right)^{1/2} \]

then the asset is represented as

\[ \sigma(T) = \sigma(0) \exp \left( \bar{\mu}T - \frac{1}{2} \bar{\xi}(t)^2 T + \bar{\xi}W(T) \right) \]
Calculating under stochastic volatility of volatility

Method 3 - Pseudo-analytic SABR Process Extension (cont)

It is straightforward - though detailed - to construct $\sigma(t)|\sigma(T)$ and in turn calculate both

$$
E \left[ \frac{1}{T} \int_0^T \sigma(t)^2 \, dt \bigg| \sigma(T) \right] \quad \text{and} \quad E \left[ \left( \frac{1}{T} \int_0^T \sigma(t)^2 \, dt \right)^2 \bigg| \sigma(T) \right]
$$

as before thereby providing $\psi_{ln}(\vartheta|\sigma(T), \xi(0, T))$ and in turn the option price given a vol of vol path $C(K, \xi(0, T), T)$. The option price is therefore,

$$
C_{mc}(K, T) = \frac{1}{N} \sum_{\xi(0, T)} \int_0^\infty \left( \int_0^\infty d\vartheta \, C(K, \vartheta, T) \psi(\vartheta|\xi(0, T)) \right) \, d\sigma(T) \psi(\sigma(T))
$$

$$
= \frac{1}{N} \sum_{\xi(0, T)} C(K, \xi(0, T), T)
$$

This calculation can be improved yet further.
Calculating under stochastic volatility of volatility

The last equation is the Monte Carlo form of the integral

\[ C(K, T) = \int_{\xi(0,T)} C(K, \xi(0, T), T) d\xi(0, T) \]

The final step to implementing this most efficiently is to condition on the terminal \( \xi(T) \) which is Gaussian

\[ C(K, T) = \int_0^\infty \left( \int_{\xi(0,T)|\xi(T)} C(K, \xi(0, T), T) d\xi(0, T) \right) \psi(\xi(T)) d\xi(T) \]

Gauss-Hermite quadrature can then be used to express the integral as

\[ C(K, T) = \sum_{p=1}^{M} \omega_p C(K, \xi_p(T), T) \]

where \( M \) is typically small (32 would be a large number of points). For a standard 8- or 16- core machine (GPU’s may be more sexy, but are less practical here) which is now commonplace, each quadrature sub-calculation can be performed in parallel at the core level. The resulting calculation is highly optimal.
Introducing skew to $\psi(\vartheta)$ - correlated stochastic $\xi(t)$

Here $\pi = 20\%$ and $\rho_{\sigma,\xi} \in \{-75\%, -50\%, -25\%, 0\%\}$. 
Introducing skew to $\psi(\vartheta)$ - correlated stochastic $\xi(t)$

Here $\pi = 40\%$ and $\rho_{\sigma,\xi} \in \{-75\%, -50\%, -25\%, 0\%\}$. 
Introducing skew to $\psi(\vartheta)$ - correlated stochastic $\xi(t)$

Here $\pi = 60\%$ and $\rho_{\sigma,\xi} \in \{-75\%, -50\%, -25\%, 0\%\}$.
What about the asset skew?

- The above calculations all carry over to the case where the asset process is governed by CEV i.e.

\[
C_{cev}(K, T) = \int_0^\infty C_{cev}(K, \vartheta, T)\psi(\vartheta) d\vartheta
\]

where the underlying asset and volatility processes are still uncoupled.

- So the inverse problem of recovering \( \psi(\vartheta) \) given \( C_{cev}(K, T) \) as well as pricing in the presence of either local or stochastic volatility of volatility are both carried over.

- The CEV calculation is more computationally intensive, making the multithreaded calculation all the more important.
What about the asset skew?

Return to the lognormal asset process with correlated lognormal volatility (constant $\xi$)

$$dF(t, T) = \sigma(t)F(t, T)dW_F \quad \text{and} \quad d\sigma(t) = \xi\sigma(t)dW_\sigma$$

where $<dW_F, dW_\sigma> = \rho_{F,\sigma}dt$. Decomposing $dW_F = \rho_{F,\sigma}dW_\sigma + \bar{\rho}_{F,\sigma}dW'_F$ then

$$F(T, T) = F(0, T) \exp \left( -\frac{1}{2}\vartheta T + \rho_{F,\sigma} \int_0^T \sigma(t)dW_\sigma + \bar{\rho}_{F,\sigma} \int_0^T \sigma(t)dW'_F \right)$$

Then given a volatility path, the asset is still lognormal. The correlation 1) shifts the conditional asset forward and 2) reduces the conditional asset volatility. The correlated case therefore also relies on

$$\int_0^T \sigma(t)dW_\sigma = \int_0^T \frac{d\sigma(t)}{\xi} = \frac{1}{\xi} (\sigma(T) - \sigma(0))$$

The calculation then proceeds as before under conditioning on $\sigma(T)$. 
What about the asset skew?

So in the presence of correlation the option price can be represented as

$$C(K, T) = \int_0^\infty \left( \int_{-\infty}^{\infty} C(K, \vartheta, I_{\sigma dW}, T) \psi(I_{\sigma dW} | \vartheta) \right) \psi(\vartheta) d\vartheta$$

where

$$I_{\sigma dW} = \int_0^T \sigma(t) dW_\sigma$$

Therefore,

- the inverse problem should seek to recover $\psi(I_{\sigma dW}, \vartheta)$ though this is “technically challenging”.

- in the case of SABR with stochastic $\xi$ then $I_{\sigma dW}$ becomes path dependent. It can however be proxied by a most likely path approach and so allow the calculation to proceed as before.
Conclusions

- The market standard volatility model - SABR - has a number of well known issues;
- These issues can be removed following the work of Islah [2009] or McGhee [2010];
- Even with the improvements these models bring, the underlying process is still not appropriate to capture the market smile;
- The market smile yields information about the distribution of the underlying mean integrated variance;
- The low delta wings of the volatility smile are governed by the shape of the right tail of the mean integrated variance distribution - suggesting it should be more negatively skewed;
- This skew can be invoked by a riched volatility process - either local (CEV) volatility of volatility or stochastic volatility of volatility
- Both give generous control over the smile wings
- The latter can be implemented efficiently.
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