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Sharing sequentially triggered losses

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Abstract
When losses caused by one agent onto another triggers losses to a third, “victim” turns into “injurer” in the chain’s subsequent steps. Should agents be responsible for the direct loss they cause or also bear some of the indirect losses they trigger? Through an axiomatic approach, we characterize the fixed-fraction rules, holding each agent accountable for a fraction of the direct harm she causes while the chain’s initiator is liable for the residual. We also examine contested individual losses. Extending on the well-known principle of equalizing losses, we show that losses can be distributed equidistant to the agents’ ideal allocations.

Keywords: Sequential losses, strict liability, ordered liability, fair allocation, cost allocation

JEL: K12, C7, D6

1. Introduction
Frequently, agents’ actions initiate sequences of unanticipated events that affect other agents. For instance, suppose a construction company made mistakes when building a housing complex which the current owner now, years later, requires to be fixed. One of the tenants is a coffee shop. It seems intuitively appealing that the construction company should cover the costs of its mistake, but what about the losses incurred by the coffee shop due to reduced sales during the renovations? The construction company reckons that its obligations are towards the owners of the housing complex only, while the owners, facing compensation

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demands from the coffee shop, argue that there would be no losses whatsoever had the construction company been thorough to begin with. In this paper, we take an axiomatic approach to identify solutions to this and many other related disputes.

We study a stylized model in which the actions of some agent are directly harmful to another. The incurred loss is further reinforced by unanticipated externalities triggered sequentially to other agents in a finite chain. Formally, there is an “initiator”, agent 1, who causes a loss $\ell_1$ to agent 2. This further causes a loss $\ell_2$ to agent 3 and so forth—but this comes to 1’s attention first after the initial incident. Specifically, we have in mind situations in which the negative externalities diminish throughout the chain, $\ell_1 \geq \ell_2 \geq \ldots$. That is, a larger event trickles down with ever-smaller consequences: a firm is first affected and as a consequence its employees at a smaller scale; events in a state are spread to cities and on to individual households. We postpone a discussion of this assumption and on different types of loss chains, with potential extensions, to Section 5. We abstract away the specific reasons for the losses as well as any efforts taken to prevent them: these are notoriously hard to measure and will, in any case, be contested in court. Indeed, only the initiator is in position to prevent the cascade of losses: once the first harm is caused, the others follow (say due to prior arrangements between the agents). Hence, it is less a question of “Who is to blame?” but rather “How much should the initiator have to take on due to externalities unknown to them from the outset?”

We seek a systematic way to resolve these disputes (a “rule”) that builds on normative foundations which account for the sequential nature of the losses. We acknowledge that, in practice, the participants may have signed contracts detailing liability in various events. To be sure, the construction company and the owners of the housing complex have a bilateral contract—but this need not extend to potential third or fourth parties. Moreover, one cannot contract on every possible contingency and one certainly cannot contract with parties one is unaware of. Hence, there are many cases related to tort law (see e.g. Landes and Posner, 1987) in which our approach is more appropriate than relying on existing contracts or local laws.

Our main contribution singles out a parametric class of rules in which each agent $i$ pays a fixed fraction $\lambda$ of the loss she inflicts on agent $i+1$ while the initiator, agent 1, covers the residual. The unique role of agent 1 is due to 1’s actions being the root cause for the loss chain. Every fraction $\lambda$ corresponds to a different rule and the class of rules spans from agent 1 being liable for all losses ($\lambda = 0$) to each agent being liable for the loss she inflicts herself ($\lambda = 1$). As an intermediate example, when $\lambda = 1/2$ and all losses are equal, one half of the losses are first assigned agent 1 and then the other half is shared equally among everyone. This compromise takes into account 1’s responsibility as the initiator as well as 1’s ignorance of the externalities/relations between the agents further down the chain. It stresses that the loss inflicted by agent
1 is somehow different from the other ones while the losses inflicted by agent 2 through \( n \) all are of “the same type”: the initiator could have done something about the loss chain; the others could not. In this sense, fixed-fraction rules balance fairness and incentives: the initiator is incentivized to avoid starting the cascade by carrying the share \( 1 - \lambda \geq 0 \) of every subsequent loss, but it is also acknowledged that the initiator does not control interdependencies further down the chain by having the share \( \lambda \geq 0 \) fall on the others.

Our first result, Theorem 1, shows that the fixed-fraction rules have a solid normative foundation in the literature on fair allocation. The result characterizes the class through the following properties. First, two-loss initiator-independence asserts that, in two-loss problems, agent 2’s liability for the loss she inflicts is independent of the size of the loss that agent 1 inflicts on her. Second, merging independence states that, if the first two agents internalize the losses they inflict, then this should have no effect on the others. Third, extension requires that, if the loss chain is extended by an agent, then no intermediate agent—who is neither the root cause nor aware of the existence of the added agent—should be affected. Theorem 1 shows that, in conjunction with scale invariance, the properties characterize the fixed-fraction rules. An alternative characterization that builds on conditional additivity is provided in Theorem 2.

To put the fixed-fraction rules in perspective, we contrast them with the well-known “serial principle”, which fails both merging independence and extension. The “serial principle” holds agent 1 responsible for \( \ell_1 \), shares the responsibility of \( \ell_2 \) between agents 1 and 2, and, in general, holds agents 1 through \( i \) equally responsible for the loss \( \ell_i \) that agent \( i \) causes. In contrast, fixed-fraction rules center responsibility for \( \ell_i \) on the agents causing the loss chain and the particular loss, agents 1 and \( i \). Still, there are similarities between the solutions and we uncover a result to that effect. Specifically, in many linear network structures, the “serial rule” coincides with the Shapley value of an associated cooperative game (e.g. Littlechild and Owen, 1973; Gilles et al., 1992; Ni and Wang, 2007). We find a corresponding result in Theorem 3: our intermediate rule \( (\lambda = 1/2) \) assigns responsibility in a way that coincides with the Shapley value and the nucleolus of an associated liability game. In particular, this game is an instance of a “big boss” game (Muto et al., 1988).

In Section 4, we extend our study to the case in which individual losses are unknown or contested among the agents. We seek to assign losses in the event of, say, an accident in which the total loss is known but there is contention on how to attribute it to individual agents. A universally agreed-upon order, say due to a legal responsibility structure or a hierarchical power structure, orders the agents from most to least liable. Our objective is to assign liability fairly but now under the restriction that liabilities are ordered. In this setting, we identify each agent \( i \)’s ideal allocation of liability, which minimizes \( i \)’s assigned liability and maximizes the liability assigned \( i \)’s injurer. While any distribution of liability will be a weighted average
of these ideal allocations, we argue that all distributions but one mistreat some agent in favor of another. Specifically, Theorem 4 establishes that there is an allocation equidistant to every agent’s ideal allocation. That is to say, any deviation from this allocation moves us closer to one agent’s ideal allocation and further from another’s. The structure of the equidistant allocation is identical to that discussed above for $\lambda = 1/2$ with equal individual losses: it assigns half the blame to the most liable agent 1 and shares the remainder equally among all agents. In addition to its theoretical appeal, we note also that a famous court case in China was settled in line with this principle (Li, 2000).

Our study relates to several strands of literature. Economically optimal assignment of liability has been analyzed within the law and economics literature at least since Coase (1960). This literature studies agents’ (here injurers’ and victims’) incentives to prevent losses from occurring (e.g. Brown, 1973; Marchand and Russell, 1973; Diamond and Mirrlees, 1975; Green, 1976; Emons and Sobel, 1991). A typical aim of these studies is to analyze how different types of liability rules, in different economic environments, affect socially optimal resource allocation in terms of accident avoidance. In particular, the main concerns of liability rules are whether the injurer or the victim should be held responsible as well as whether, and how, negligence on both sides should be taken into account (see e.g. Shavell, 1980; Landes and Posner, 1987; Shavell, 2007).

We, on the other hand, focus on distributional fairness which places our contribution squarely into the literature on fair division (see e.g. Moulin, 1988, 2004; Thomson, 2016). There is a large literature on cost allocation in networks (surveyed in Hougaard, 2018), a considerable share focused on chain structures. This includes the airport problem (Littlechild and Owen, 1973), river sharing (Ambec and Sprumont, 2002; Ni and Wang, 2007), games with permission structures (Gilles et al., 1992), peer-group games (Brânzei et al., 2002), and revenue sharing in hierarchies (Hougaard et al., 2017). While similar in terms of mathematical structure, there are important conceptual differences between our framework and this literature. This is nicely illustrated by comparing with the model of Dehez and Ferey (2013). They also analyze liability sharing using cooperative game theory, but with a different interpretation than ours.

In Dehez and Ferey (2013), a sequence of agents cause injury to a lone victim; in our model, each intermediate agent takes on both the role as injurer and victim. This leads to a difference in the cooperative games that best capture the problems: while we in Subsection 3.3 derive an instance of a “big boss” game, Dehez and Ferey (2013) study a dual airport problem. When applying the Shapley value, this difference leads them to a variation on the “serial principle” and us to the intermediate fixed-fraction rule. However, the “serial principle” is less appealing in our case due to the special role of intermediate agents as expressed by extension. Oishi et al. (2016) generalize the linear structure of Dehez and Ferey (2013) to “rooted trees”
in which an agent can harm multiple agents. They consider a different cooperative game (akin to a game
with conjunctive permission structure Gilles et al., 1992) and axiomatically characterize the nucleolus on
this class of games. Recently, Juarez et al. (2018) also analyze a model of sharing sequentially generated
values but their focus is on how to select a value-generating path within a network as well as how to share
value along the path. Moreover, our axiomatic approach that involves conditioning the liabilities on the
different characteristics the agents have (distinguishing the initiator from the others) resembles that taken

A third strand of related literature concerns contagion in financial networks (e.g. Elliott et al., 2014;
Acemoglu et al., 2015; Demange, 2018; Csoka and Herings, 2018). Typically, these papers study how network
topology influences cascades of failures among interdependent financial organizations based on a general
liability matrix. We also study cascades through interdependency but in the simple linear structure of a
chain of inflicted losses. Like Csoka and Herings (2018), we focus on the outcome of cascades in terms of
payments and agents claims on each other. A somewhat similar type of application concerns rare events and
their contagion effect among countries in dynamic global game model as in Chen and Suen (2016).

Finally, there is a straightforward, dual interpretation of our cost sharing model as one of revenue sharing.
That is, even though we predominantly discuss negative effects, our conclusions apply equally well in a more
positive setting. For instance, imagine that the invention of a small chip increases the revenue also for a
larger chip built upon it, which further triggers increased revenue for a phone that includes the larger chip.
Our solutions suggest how the increased revenues (or the R&D costs) should be shared along the supply
chain (see e.g. Frascatore, 2011; Song and Gao, 2018).

The paper is structured as follows. In Section 2, we introduce the model with individual losses. The
axiomatic analysis and the characterization of the fixed-fraction rules is in Section 3. In Section 4, we study
the model with contested individual losses. We close with a discussion of model extensions in Section 5.

2. Model

In this section, we introduce our model of sequentially triggered losses with interdependencies following
a chain structure. Let $N = \{1,\ldots,n\}$ represent a loss chain in which agent 1 initiates the chain and agents
2 through $n-1$ are “intermediate” agents. Agent $n$ is the final agent: the loss she causes to agent $n+1$
triggers no further losses. An $n$-loss problem is a vector $\ell = (\ell_1,\ldots,\ell_n)$ in which agent $i$ causes loss $\ell_i \geq 0$
to agent $i+1$ and where losses diminish, in the weak sense, throughout the chain (this is discussed further
in Section 5). Let $\mathcal{L}^n = \{\ell \in \mathbb{R}^n : \ell_1 \geq \cdots \geq \ell_n \geq 0\}$ denote the set of loss vectors of length $n$. 
An allocation \( x = (x_1,\ldots,x_n) \) specifies each agent \( i \)'s liability \( x_i \geq 0 \) and is such that \( x_1 \geq \ell_1, x_1 + x_2 \geq \ell_1 + \ell_2, \) and so on until \( x_1 + \cdots + x_n = \ell_1 + \cdots + \ell_n \equiv L. \) That is, agents “later in the chain” bear no responsibility for losses caused by “earlier” agents and there is balance between liabilities \( \sum x_i \) and losses \( \sum \ell_i. \) Let \( X(\ell) \) denote the set of allocations. A rule \( \varphi \) maps to each problem \( \ell \in L^n \) an allocation \( \varphi(\ell) = (\varphi_1(\ell),\ldots,\varphi_n(\ell)) \in X(\ell).^{1} \)

We will devise rules that build on the notion of strict liability—that an injurer is responsible for the losses she causes—from the literature on law and economics (Shavell, 2007). Here, in particular, agent \( n+1 \) incurs a loss but causes none. Therefore, we take as given that she is free of liability and leave her out of the analysis (for instance, we do not include her share \( x_{n+1} = 0 \) in the allocation). For the other agents, however, strict liability leaves open many interpretations: is the agent liable only for the direct loss she causes or also for parts of the indirect losses triggered further down the chain?

Taken to their respective extremes, these interpretations lead us in two opposite directions. On the one hand, the direct liability rule \( \varphi^1 \) assigns full liability to each agent for the loss she causes, \( \varphi^1(\ell) = \ell. \) On the other, if an agent can shift blame for the loss she causes on those who appear before her in the loss chain, then responsibility ultimately falls squarely on the initiator. This is the indirect liability rule \( \varphi^0 \) for which \( \varphi^0(\ell) = (L,0,\ldots,0).^{2} \)

3. Fixed-fraction rules

In this section, we single out the particular class of fixed-fraction rules, which represent a compromise between the direct and the indirect liability rules. Each member of the class is associated with a parameter \( \lambda. \) In particular, every agent is held accountable for the fraction \( \lambda \) of the loss she causes. Agent 1, whose actions are the root cause as they initiated the loss chain, holds a special position and is held liable for the residual \( (1-\lambda)L. \) To ensure non-negative liabilities, the solutions are defined only for \( \lambda \in [0,1]. \) Formally,

---

1 As is apparent, the number of agents may vary. Strictly speaking, we assume that there is an infinite set of potential agents out of which \( N \) is a generic finite subset.

2 These extreme cases have been used in related contexts as well. See for instance “local responsibility sharing” (Ni and Wang, 2007), the “full transfer rule” (Hougaard et al., 2017), and “top value” (van den Brink et al., 2017).
Definition 1 (The fixed-fraction rule parameterized by $\lambda$). For each $\ell \in L^n$,

\[
\begin{align*}
\phi_1^\lambda(\ell) &= \lambda \ell_1 + (1 - \lambda)L \\
\phi_2^\lambda(\ell) &= \lambda \ell_2 \\
&\vdots \\
\phi_n^\lambda(\ell) &= \lambda \ell_n.
\end{align*}
\]

The class includes, as its extreme members, the direct and the indirect liability rule: for $\lambda = 1$, we obtain the direct liability rule; for $\lambda = 0$, we obtain the indirect liability rule.

3.1. Main characterization

Inspired by the literature on fair allocation, we introduce several normatively compelling properties of rules. While the underlying principles are well-known with a documented, solid foundation, they are here adapted to the present context; as such, most of the properties as well as the resulting characterizations are novel.

We take as given that agent 1 is fully liable for the loss $\ell_1$ inflicted on agent 2, but, due to the sequential nature of the loss chain, she may in addition be required to cover a fraction of the later triggered losses. Since $\ell_1$ is the largest loss and there is no other correlation between the losses, it is reasonable that the fraction of $\ell_i$ ($i \neq 1$) covered by agent 1 is independent of the size of $\ell_1$. We consider a minimal form of this principle in which there are only two agents causing harm. That is to say, there are only two losses, $\ell_1$ and $\ell_2$, and agent 2 should be unaffected if we were to change the harm caused by the initiator to $\ell'_1$. The axiom is closely related to highest rank revenue independence in Hougaard et al. (2017) and independence of upstream costs in Ni and Wang (2007), but applies only to a small set of problems.

Definition 2 (Two-loss initiator-independence). For each $\ell \in L^2$ and $\ell'_1 \geq \ell_2$,

\[\phi_2(\ell_1, \ell_2) = \phi_2(\ell'_1, \ell_2).\]

If a group of agents choose to internalize their losses, it is beneficial if this does not affect agents outside the group. This allows the group to settle their liabilities in private without changing the remaining agents’ liabilities; the group can come to an agreement without having to cross-check with the “outsiders” whether they, also, agree to it. From a strategic point of view, it may also serve to prohibit groups from colluding to
reduce their (joint) liability. This is particularly desirable when agents are companies that often are able to file claims on various aggregation levels.

Again, we consider a minimal form of this principle: only the first two agents merge and internalize their losses. A well-known variation of this principle is, for instance, no advantageous merging in de Frutos (1999). While two-loss initiator-independence only applies to two-loss problems, merging independence and the properties that follow require the statement to hold for all appropriate problem sizes n.

**Definition 3 (Merging independence).** For each \( n \geq 3, \ell \in \mathcal{L}^n \), and \( i = 3, \ldots, n \),

\[
\varphi_i(\ell) = \varphi_{i-1}(\ell_1 + \ell_2, \ell_3, \ldots, \ell_n).
\]

Next, consider the effect of extending the loss chain by one agent. That is, beyond the previously final agent \( n \) who causes loss \( \ell_n \) to agent \( n + 1 \), now \( n + 1 \) also inflicts a loss \( \ell_{n+1} \leq \ell_n \) on \( n + 2 \). We require that no intermediate agent, who neither has knowledge of \( n + 2 \) nor is to blame for triggering the chain of losses, should be affected by this addition. Extension can be viewed as a strong form of population monotonicity (e.g. Moulin, 2002) or as a solidarity property among intermediate agents (Thomson, 2016).

**Definition 4 (Extension).** For each \( n \geq 2, \ell \in \mathcal{L}^n, \ell_{n+1} \leq \ell_n \), and \( i = 2, \ldots, n \),

\[
\varphi_i(\ell) = \varphi_i(\ell_1, \ldots, \ell_n, \ell_{n+1}).
\]

Finally, we hold as desirable that the same legal principle should apply to every case, that large and small problems are solved alike. The property of scale invariance is well-known in the literature.

**Definition 5 (Scale invariance).** For each \( n \in \mathbb{N}, \ell \in \mathcal{L}^n \), and \( \alpha \geq 0 \),

\[
\alpha \varphi(\ell) = (\alpha \varphi_1(\ell), \ldots, \alpha \varphi_n(\ell)) = \varphi(\alpha \ell_1, \ldots, \alpha \ell_n) = \varphi(\alpha \ell).
\]

Our first result, Theorem 1, establishes that a rule satisfies the four above properties if and only if it is a fixed-fraction rule. Independence of the properties is shown in the Appendix.

**Theorem 1.** A rule \( \varphi \) satisfies two-loss initiator-independence, merging-independence, extension, and scale invariance if and only if there is \( \lambda \in [0, 1] \) such that \( \varphi = \varphi^\lambda \).

**Proof.** It is immediate that each rule \( \varphi^\lambda \) satisfies all four properties. It remains to show the converse, that the rules \( \varphi^\lambda \) are the only ones to do so.
We first show that two-loss initiator-independence and scale invariance together pin down the solution for any two-loss problem (Claim 1). Appealing to merging-independence further settles the solution for the last agent regardless the problem size (Claim 2). Finally, taken in conjunction with extension, we derive the desired conclusion.

**Claim 1.** A rule $\varphi$ satisfies two-loss initiator-independence and scale invariance if and only if there is $\lambda \in [0, 1]$ such that, for each $\ell \in \mathcal{L}^2$, $\varphi_1(\ell) = \lambda \ell_1 + (1 - \lambda)L$ and $\varphi_2(\ell) = \lambda \ell_2$.

**Proof.** Take $\ell \in \mathcal{L}^2$ as given and define $\lambda$ through $\varphi_2(\ell) = \lambda \ell_2$. By definition, $0 \leq \varphi_2(\ell) \leq \ell_2$, so $\lambda \in [0, 1]$. Next, take an arbitrary $\ell' \in \mathcal{L}^2$; it suffices to show that $\varphi_2(\ell') = \lambda \ell'_2$ for the parameter $\lambda$ just defined.

Define $\alpha \geq 0$ such that $\ell'_2 = \alpha \ell_2$. By scale invariance, $\alpha \varphi(\ell) = \varphi(\alpha \ell) = \varphi(\alpha \ell_1, \ell_2)$. As $\ell' \in \mathcal{L}^2$, we have $\ell'_1 \geq \ell_2$, so we can apply two-loss initiator-independence: $\varphi_2(\alpha \ell_1, \ell'_2) = \varphi_2(\ell'_1, \ell'_2) = \varphi_2(\ell')$. Hence, $\varphi_2(\ell') = \alpha \varphi_2(\ell) = \alpha \lambda \ell_2 = \lambda \ell'_2$. By balance, $\varphi_1(\ell) = \lambda \ell_1 + (1 - \lambda)L$. \qed

**Claim 2.** A rule $\varphi$ satisfies two-loss initiator-independence, scale invariance, and merging-independence if and only if there is $\lambda \in [0, 1]$ such that, for each $\ell \in \mathcal{L}^n$, $\varphi_n(\ell) = \lambda \ell_n$.

**Proof.** By Claim 1, there is $\lambda \in [0, 1]$ such that, for each $\ell \in \mathcal{L}^2$, $\varphi_2(\ell) = \lambda \ell_2$. Let $\ell \in \mathcal{L}^n$. Repeatedly merge the agents at the top and apply merging-independence:

$$\varphi_n(\ell) = \varphi_{n-1}(\ell_1 + \ell_2, \ell_3, \ldots, \ell_n) = \cdots = \varphi_2(\ell_1 + \cdots + \ell_{n-1}, \ell_n) = \lambda \ell_n.$$

Finally, we turn to the statement of Theorem 1. By Claim 2, there is $\lambda \in [0, 1]$ such that, for each $\ell \in \mathcal{L}^n$, $\varphi_n(\ell) = \lambda \ell_n$. That is, no matter the problem size $n$, the final agent bears the fraction $\lambda$ of the loss she causes.

Let $\ell \in \mathcal{L}^n$ and $i \in \{2, \ldots, n\}$. Repeatedly appeal to extension and finally apply Claim 2:

$$\varphi_i(\ell) = \varphi_i(\ell_1, \ldots, \ell_{n-1}) = \cdots = \varphi_i(\ell_1, \ldots, \ell_i) = \lambda \ell_i = \varphi_i^\lambda(\ell).$$

Thus, this settles the desired liabilities for agents $i = 2, \ldots, n$. By balance, $\varphi_1(\ell) = \varphi_1^\lambda(\ell)$. \qed

### 3.2. Alternative characterization

To further highlight the strong appeal of the fixed-fraction rules, we next provide an alternative characterization through four different properties.

When an agent inflicts a loss on another, the loss can sometimes be decomposed into different types of losses. In the case of a polluted river, a factory may leak multiple pollutants; one may consider it as
several different cases, each pertaining to a specific pollutant, or one may combine them into a single case of multiple pollutants. To avoid unnecessary legal complications, it is desirable that the determined liability is independent of the method of decomposition. This is captured by conditional additivity (see e.g. Moulin, 2002), which equates the joint solutions of separate problems to the solution of the joint problem.

**Definition 6** (Conditional additivity). For each \( n \in \mathbb{N} \) and \( \{\ell, \ell'\} \subseteq \mathcal{L}^n \),

\[ \varphi(\ell) + \varphi(\ell') = (\varphi_1(\ell) + \varphi_1(\ell'), \ldots, \varphi_n(\ell) + \varphi_n(\ell')) = \varphi(\ell_1 + \ell'_1, \ldots, \ell_n + \ell'_n) = \varphi(\ell + \ell'). \]

**Conditional additivity** will be imposed together with several properties that apply only to so called unit-loss problems. Formally, the \( n \)-unit-loss problem is \( \mathbf{1}^n = (1, \ldots, 1) \in \mathcal{L}^n \). These particular problems arise naturally when agents are symmetric and losses propagate along the chain due to sequential negligence. This may be the case, for instance, for the spread of fires or epidemics among neighboring provinces. A loss of 1 then indicates that the damage was passed on—that the fire spread, that the disease was transmitted—from one region to the next. We use \( \mathbf{1}^m_n = (1, \ldots, 1, 0, \ldots, 0) \in \mathcal{L}^n \) to denote the \( n \)-loss problem comprised of the \( m \)-unit-loss problem concatenated with \( n - m \) zeroes.

Recall that the interpretation of an \( n \)-loss problem is that agent \( n + 1 \) incurs but does not cause a loss. **Zero truncation** requires the rule to be unaffected when truncating such a trailing zero. It implies that, had we instead modeled the problem as one with \( n + 1 \) agents in which agent \( n + 1 \) causes no loss, \( \ell_{n+1} = 0 \), then the liabilities should be unaffected.

**Definition 7** (Zero truncation). For each \( n \in \mathbb{N} \), \( m \leq n \), and \( i \leq m \),

\[ \varphi_i(\mathbf{1}^n_m) = \varphi_i(\mathbf{1}^m). \]

Next, we introduce **length invariance**: it asserts that, for unit-loss problems, the final agent’s liability is invariant of the chain length. Indeed, consider again the case of sequential negligence among neighboring provinces. As the last negligent province \( m \) only affects \( m + 1 \) who in turn shows no negligence, it appears reasonable that \( m \)'s liability should be independent of the number of negligent provinces before \( m \).

**Definition 8** (Length invariance). For each \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \),

\[ \varphi_m(\mathbf{1}^m) = \varphi_n(\mathbf{1}^n). \]
Lastly, we weaken extension (Definition 4) to apply only to unit-loss problems.

**Definition 9** (Unit extension). For each $n \geq 2$ and $i = 2, \ldots, n$,

$$\phi_i(\mathbf{1}^n) = \phi_i(\mathbf{1}^n, 1) \equiv \phi_i(\mathbf{1}^{n+1}).$$

Theorem 2 shows that the four new properties together characterize the fixed-fraction rules. Again, independence of the properties is shown in the Appendix.

**Theorem 2.** A rule $\phi$ satisfies conditional additivity, zero truncation, length invariance, and unit extension if and only if there is $\lambda \in [0, 1]$ such that $\phi = \phi^\lambda$.

**Proof.** It is immediate that each rule $\phi^\lambda$ satisfies all the listed properties, so we again focus on the converse statement. By length invariance, we can define $\lambda = \phi_n(\mathbf{1}^n)$.

Consider an arbitrary $m \leq n$ and agent $i$. By feasibility, if $i > m$, then $\phi_i(\mathbf{1}_m^n) = 0$. For $i \leq m$, zero truncation yields $\phi_i(\mathbf{1}_m^n) = \phi_i(\mathbf{1}^m)$. Repeatedly applying unit extension, $\phi_i(\mathbf{1}^m) = \phi_i(\mathbf{1}^i)$. Finally, by length invariance, $\phi_i(\mathbf{1}^i) = \lambda$.

We can rewrite any loss vector $\ell \in \mathcal{L}^n$ as a weighted sum of unit-loss problems:

$$\ell = (\ell_1, \ldots, \ell_n) = \sum_{m=1}^{n} (\ell_m - \ell_{m+1}) \cdot \mathbf{1}_m^n$$

$$= (\ell_1 - \ell_2, 0, \ldots, 0) + (\ell_2 - \ell_3, \ell_2 - \ell_3, 0, \ldots, 0) + \cdots + (\ell_n - \ell_{n+1}, \ldots, \ell_n - \ell_{n+1}),$$

in which we use $\ell_{n+1} \equiv 0$. By repeatedly applying conditional additivity,\(^3\)

$$\phi(\ell) = \sum_{m=1}^{n} (\ell_m - \ell_{m+1}) \cdot \phi(\mathbf{1}_m^n).$$

For $i \in \{2, \ldots, n\}$, this simplifies as follows by our above observation on $\phi_i(\mathbf{1}_m^n)$:

$$\phi_i(\ell) = \sum_{m=1}^{i-1} (\ell_m - \ell_{m+1}) \cdot 0 + \sum_{m=i}^{n} (\ell_m - \ell_{m+1}) \cdot \lambda = \lambda \ell_i = \phi_i^\lambda(\ell).$$

Thus, this settles the desired liabilities for agents $i = 2, \ldots, n$. By balance, $\phi_1(\ell) = \phi_1^\lambda(\ell)$. \(\square\)

---

\(^3\)Formally, this applies only when $\ell$ is integer-valued, but the argument can be extended to rational losses through conditional additivity. A technical continuity condition can then be imposed to cover all real values.
3.3. Cooperative games

We continue our analysis of the fixed-fraction rules by highlighting a connection between well-known solution concepts from cooperative game theory (see e.g. Peleg and Sudhölter, 2007) and the intermediate member of the class, namely $\varphi^{1/2}$.

For each $n$-loss problem $\ell \in \mathcal{L}^n$, the initiator (agent 1) plays a special role. One way to model this role is by defining an associated liability game $v_\ell$ as follows. For every coalition $S \subseteq N$, let

$$v_\ell(S) = \begin{cases} 
\sum_{i \in S} \ell_i & \text{if } 1 \in S \\
0 & \text{otherwise.}
\end{cases}$$

That is, only coalitions that include the initiator are considered as liable, and their liability equals the total losses caused by its members. Coalitions that do not include the initiator are not considered liable for any losses: none of its members would have caused any losses had it not been for the actions of the initiator. In particular, $N$ is liable for the total loss, $v_\ell(N) = L$. Theorem 3 shows that the intermediate fixed-fraction rule coincides with the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969) of the associated liability game.

**Theorem 3.** For each $\ell \in \mathcal{L}^n$, $\varphi^{1/2}(\ell)$ coincides with the Shapley value and the nucleolus of the associated liability game $v_\ell$.

**Proof.** Liability games are a special case of “big boss” games and the result is derived from Muto et al. (1988) as follows.

For any coalition $S$ such that $1 \not\in S$, we have $v_\ell(S) = 0$ and $v_\ell(S \cup \{1\}) = \sum_{i \in S} \ell_i + \ell_1$. As each $\ell_i \geq 0$, for each $S \subseteq T$, we have $v_\ell(S \cup \{1\}) - v_\ell(S) \leq v_\ell(T \cup \{1\}) - v_\ell(T)$. Moreover, for each agent $i \neq 1$, we have $v_\ell(S \cup \{i\}) - v_\ell(S) = v_\ell(T \cup \{i\}) - v_\ell(T)$ if $1 \in S$ or $1 \not\in T$, while $v_\ell(S \cup \{i\}) - v_\ell(S) \leq v_\ell(T \cup \{i\}) - v_\ell(T)$ otherwise. Hence, $v_\ell$ is convex.

The agents’ contributions to the grand coalition are $M_1(v_\ell) = v_\ell(N) - v_\ell(N \setminus \{1\}) = L$ and, for each $i \neq 1$, $M_i(v_\ell) = v_\ell(N) - v_\ell(N \setminus \{i\}) = \ell_i$. By Muto et al. (1988, Theorem 4.2), the nucleolus, $\nu(v_\ell)$, is such that, for each $i \neq 1$, $\nu_i(v_\ell) = M_i(v_\ell)/2 = \ell_i/2 = \varphi^{1/2}_i(\ell)$. By balance, we also have $\nu_1(v_\ell) = \varphi^{1/2}_1(\ell)$. Finally, by Muto et al. (1988, Theorem 4.5), as $v_\ell$ is convex, the Shapley value and nucleolus of $v_\ell$ coincide.

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4 Chun and Hokari (2007) also show coincidence between the Shapley value and the nucleolus but in the context of queuing problems.
4. Contested individual losses

We proceed to consider the scenario in which the individual losses $\ell_i$ are unknown/unobservable or, as will be our main interpretation, contested among the agents. That is to say, we relax the assumption imposed in Section 2, that there is implicit agreement on the non-increasing individual losses, and assume instead that there is agreement only on the total loss $L$ and the order of individual liability. Hence, all agree that agents 1 and $n$ are the most and least liable ones—but they disagree on how liable everyone is. Such an ordering may arise due to a legal responsibility structure or a hierarchical power structure.

Thus, the problem now is to allocate the loss for the group as a whole while respecting the liability order. Simplifying further, we restrict to scale-invariant solutions and normalize the loss to $L = 1$. A problem is then described simply by its number of agents $n$. An allocation $x = (x_1, \ldots, x_n)$ again balances losses and liabilities, $x_1 + \cdots + x_n = L = 1$, while ordering the liabilities accordingly, $x_1 \geq \cdots \geq x_n \geq 0$. Let $X(n)$ denote the set of allocations in the $n$-loss problem.

Without access to individual losses, the rules developed in Section 3 cannot be applied. Moreover, as the problem now is described by a single integer, there is very little base for axiomatic analysis, and thus it is difficult to produce meaningful characterizations in line with Theorems 1 and 2. We therefore instead appeal to a principle that has been suggested in the literature on bankruptcy, taxation, and bargaining problems (Aumann and Maschler, 1985; Young, 1987; Chun, 1988; Ju and Moreno-Ternero, 2017). Namely, the principle of equalizing losses among agents or, more generally, equalizing distances to agents' ideal payoffs.

In the literature referred to above, an agent’s ideal payoffs is simply a number representing the most the agent can hope to be awarded in the specific problem. In the present context, we submit that agents’ “ideal payoffs” are naturally extended to ideal allocations of liability due to the dual role of injurer and victim. In particular, in the role as injurer, the agent wants to minimize her liability (given the order of liabilities), whereas in the role of victim, the agent wants to maximize the loss incurred by maximizing the blame of her direct injurer.\(^5\) Agent 1 is special as she only is an injurer. Therefore, agent 1’s ideal allocation is the equal split $c_n = (1/n, \ldots, 1/n)$; this is the best that agent 1 can hope for given the underlying order of liabilities.

Agent 2 wants to minimize her own liability and maximize the blame on agent 1, making $c_1 = (1, 0, \ldots, 0)$ her ideal allocation. Agent 3 has ideal allocation $c_2 = (1/2, 1/2, 0, \ldots, 0)$ and, in general, agent $i + 1$ has ideal allocation $c_i = (1/i, \ldots, 1/i, 0, \ldots, 0)$. Taken together, the points $c_1$ through $c_n$ form the extreme points of

---

\(^5\)See for instance the example in the introduction: the coffee shop files their complaints to the owners of the housing complex, not to the construction company. In similar spirit, when your flight is cancelled due to a pandemic, you seek compensation from the airline, not from the mayor in the city in which the pandemic originated.
the set of allocations $X(n)$, that is, $X(n)$ is the convex hull of $\{c_1, \ldots, c_n\}$.

With this generalized notion of ideal payoffs, the question is whether it is still possible to apply the principle of equalizing distances to ideal allocations. Theorem 4 answers this in the affirmative: there is a unique allocation $x^* \in X(n)$ at equal (Euclidean) distance to every ideal allocation. We define this allocation next.

**Definition 10** (The $1/2$-equal-split allocation $x^*$). For each problem $n$,

\[
x_1^* = \frac{1}{2n} + \frac{1}{2},
\]

\[
x_2^* = \cdots = x_n^* = \frac{1}{2n}.
\]

Analogous to how $\varphi^{1/2}$ operates on unit-loss problems in Section 3, $x^*$ first puts half the responsibility on agent 1, the root cause of the loss chain, and then shares the other half equally among the agents. Figure 1 provides a graphical illustration for $n = 3$.

![Figure 1: Illustration of Theorem 4 for $n = 3$. The gray area in the simplex is the set of allocations $X(3)$, the convex hull of $c_1 = (1, 0, 0)$, $c_2 = (1/2, 1/2, 0)$, and $c_3 = (1/3, 1/3, 1/3)$. The allocation $x^* = (4/6, 1/6, 1/6)$ is equidistant to $c_1$, $c_2$, and $c_3$.](image)

**Theorem 4.** For any problem $n$, $x^*$ is at equal (Euclidean) distance from the ideal allocation $c_1, \ldots, c_n$.

**Proof.** The (squared) distance between $x \in X(n)$ and an arbitrary ideal allocation $c_k = (1/k, \ldots, 1/k, 0, \ldots, 0)$ is as follows:

\[
\sum_{i=1}^{k} (x_i - 1/k)^2 + \sum_{i=k+1}^{n} (x_i - 0)^2 = \sum_{i=1}^{k} x_i^2 - \frac{2}{k} \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 - \frac{2}{k} \sum_{i=1}^{k} x_i + \frac{1}{k}.
\]

We claim that there is an allocation $x$ equidistant to all ideal allocation $c_k$, that is, an allocation $x$ for which the above expression is constant in $k$. This is already satisfied for the first terms (the sum of squares), so we
can ignore these henceforth. Hence, we wish to find \( \alpha \in \mathbb{R} \) such that, for \( k = 1, \ldots, n \),

\[
\frac{2}{k} \sum_{i=1}^{k} x_i + \frac{1}{k} = 2\alpha \quad \text{or, equivalently,} \quad \sum_{i=1}^{k} x_i - \frac{1}{2} = \alpha k.
\]

For \( k = 1 \), we immediately derive \( x_1 = \alpha + 1/2 \). For \( k = 2, \ldots, n \), we find that \( x_2 = \cdots = x_n = \alpha \):

\[
\sum_{i=1}^{k} x_i - \frac{1}{2} = \alpha k \iff x_k = \alpha k + \frac{1}{2} - \sum_{i=1}^{k-1} x_i \implies x_{k+1} - x_k = \alpha - x_k \implies x_{k+1} = \alpha.
\]

That is, \( x_2 = \cdots = x_n = \alpha \). We then determine \( \alpha \) through \( x_1 + \cdots + x_n = 1 \):

\[
n\alpha + \frac{1}{2} = 1 \iff \alpha = \frac{1}{2n} \implies x_1 = \frac{1}{2n} + \frac{1}{2} = \frac{n+1}{2n} = x_1^*, \; x_2 = \cdots = x_n = \frac{1}{2n} = x_2^* = \cdots = x_n^*.
\]

We next propose an alternative rationale for settling on \( x^* \). Suppose that a benevolent arbiter knows that the true individual losses \( \ell = (\ell_1, \ldots, \ell_n) \) fall somewhere in \( X(n) \)—on this there is agreement among all agents—but not exactly where. Without more information, there is no reason to believe that one allocation \( x \) lies closer to the true losses than another allocation \( y \). The arbiter may then take a cautious approach and look to each liability distribution \( x \)'s worst case \( w \); she may seek to minimize how incorrect her recommendation may be. Corollary 1 shows that \( x^* \) is the optimal choice under these considerations when using squared deviations, an approach familiar from, for instance, inequality measurement (Atkinson, 1970) and citation indices (Perry and Reny, 2016).

**Corollary 1.** For any problem \( n \), the allocation \( x^* \in X(n) \) solves

\[
\min_{x \in X(n)} \max_{w \in X(n)} \sum (x_i - w_i)^2.
\]

Finally, sharing a common loss may also be interpreted as settling the legal responsibility for a group of ordered liable parties together with a lone victim (as in Dehez and Ferey, 2013). For instance, when a reckless driver crashes into another whose airbag fails to deploy, to what extent should the reckless driver be held responsible for the injuries that were aggravated due to the airbag? Dehez and Ferey (2013) solve this problem by estimating the damage associated with every coalition of liable agents and make each agent pay the average marginal damage they inflict (as if using the serial principle). However, coalitional damage is counterfactual and will typically be contested in court. We submit that, without having to estimate the damage of every coalition of liable agents, an impartial judge may alternatively apply the solution \( x^* \) when
the law induces a decreasing order of liabilities; compare for instance the “tiger case” of Li (2000).

5. Concluding remarks

Our analysis of sequentially triggered losses has been centered on allocational fairness and various forms of characterizations of normatively desirable allocation rules. The solutions that we derive both in the baseline model (Sections 2–3) and its extension with contested individual losses (Section 4) can be applied also in more general settings. We conclude with some remarks on possible extensions and avenues for future research.

It is immediate that fixed-fraction rules can be applied in any setting that distinguishes the initiator from the other agents. For instance, we can extend the model to a tree structure in which one agent may harm many (compare e.g. Oishi et al., 2016). The axioms in Section 3 generalize readily and we obtain an analogue characterization of the fixed-fraction rules. Similarly, the strict order of Section 4 can be relaxed to a partial order. On the other hand, more challenging may be to permit multiple initiators; we then require a tie-breaking rule among the initiators to specify how they share the residuals. See Hougaard et al. (2017) for a similar extension when sharing revenues in hierarchical organizations.

In practice, we may encounter mixed scenarios in which both individual losses and a common loss have to be shared. For example, consider a river that flows through several regions or provinces and say that some pollution is created most upstream at its source (see e.g. Ni and Wang, 2007; Alcalde-Unzu et al., 2015; van den Brink et al., 2018; Sun et al., 2019). The polluted water flows downstream to the neighboring provinces, causing a local loss in terms of environmental damages and potential clean-up costs. However, river pollution not only leads to local damage along the river but also causes global harm to the entire ecosystem that affects, for instance, the common fish stock in the river. How these effects are balanced in a fair manner among the liable agents remains an interesting open question.

We have consistently focused on problems in which losses are non-increasing. Indeed, this is the domain that we find most important, but an interesting way forward is to extend to general losses that, at some steps, may increase down the chain. Widening the scope of the model in this way opens up the possibility

\textsuperscript{6} Li (2000) describes an accident in which a tour bus driver was killed by tigers in a wildlife park. The owners of the park argued that the driver’s death was caused by his own negligence as he had gotten out of the bus to fix a mechanical problem despite being advised not to, but the court found the park primarily liable. Compensating the drivers family, the park had to pay half of the estimated damages while the remainder was shared equally between the employer (the bus company) and the driver’s family, recognizing the fault of the driver himself.

\textsuperscript{7} In the proof of Theorem 1, merging-independence helps to settle the solution for the last agent. With a tree structure, there are instead multiple “leaves”. Still, we arrive at the same solution for each leaf, namely $\lambda_i$, by first merging all other branches into the root and then merging the leaf’s branch until there are only two agents. The notation is more cumbersome, but the intuition is unchanged.
for many more solutions—indeed, there is room for many rules that on the domain of non-increasing losses coincide with a fixed-fraction rule but elsewhere behave very differently.

For instance, once losses are increasing, it may be appropriate to limit the liability assigned to “small” agents. Consider, for instance, \( \ell = (10, 100, 1000) \), in which the initiator only does minor harm. Applying the intermediate fixed-fraction rule \( \varphi^{1/2} \) outright results in \( \varphi^{1/2}(\ell) = (560, 50, 500) \), which one may argue assigns excessive liability to the initiator. A simple way to limit consequences for those who do little harm is to impose \( \varphi_i(\ell) \leq n\ell_i \); an agent can at most be liable for \( n \) times the loss she causes (compare individual upper bounds in Oishi et al., 2016). While on the domain of non-increasing losses this condition is vacuous, it may have a significant effect once losses increase. Indeed, there is plenty of room for other axioms as well beyond those we impose in Section 3 once we extend the scope of the model. However, this has to be done with care. For instance, the particular approach above may lead to strategic complications: with some measures in place to protect smaller agents, some may attempt to aggregate at a different level to appear as many smaller rather than one large agent (compare the discussion around Definition 3). Still, it is an interesting question that we leave for future research.

**Appendix: Independence of properties**

We show that the properties imposed in Theorems 1 and 2 are independent. For each theorem, we design a rule that satisfies all properties except one. Below, by balance, the initiator’s liability can always be inferred through \( \varphi_1(\ell) + \cdots + \varphi_n(\ell) = L \). For each problem \( \ell \in \mathbb{L}^n \) and agent \( i = 2, \ldots, n \),

- \( \varphi^A_i(\ell) = \frac{\ell_i}{\ell_1 + \cdots + \ell_i} \cdot \ell_i \) **Loss-dependent fractions**
- \( \varphi^B_i(\ell) = \lambda_i \ell_i \) **Varied-fraction rules with parameters \((\lambda_z)_{z \in \mathbb{N}}\)**
- \( \varphi^C_i(\ell) = \ell_n \) **Constant liabilities**
- \( \varphi^D_i(\ell) = \lambda \left( \ell_i + (1 - \lambda)\ell_{i+1} + \cdots + (1 - \lambda)^{n-1}\ell_n \right) \) **Geometric rule with parameter \( \lambda \in [0, 1] \)**
- \( \varphi^E_i(\ell) = \min\{\ell_i, z\} \) **Liabilities truncated at fixed \( z \geq 0 \)**

The table shows which of the five rules satisfy the various properties of Theorems 1 and 2. Specifically, a ‘+’ indicates that the rule satisfies the property.
### Table

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<th>Property</th>
<th>$\varphi^A$</th>
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<th>$\varphi^C$</th>
<th>$\varphi^D$</th>
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### References


