The Mystery of ten Wooden Blocks: Hjelmslev’s Geometry of Reality.

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Abstract.

I shall argue that a collection of wood models at the University of Copenhagen was made by Johannes Hjelmslev as an aid in his elementary introduction to his empiric geometry of reality.

Keywords: Mathematical models, Johannes Hjelmslev, Geometry of reality, Didactics.

Introduction. The mystery.

At the Department of Mathematical Sciences at the University of Copenhagen we have a collection of mathematical models. The most spectacular models are on display in the library and the less spectacular or partly ruined ones are kept in boxes in the archive in the basement of the department. Many of the models are easily identifiable. This holds for the many plaster models of surfaces and other models that were acquired from the editor Schilling. However, there are a couple of models that have been more difficult to identify, in particular a metal surface and ten wooden blocks.

![Figure 1](image-url)

The present paper deals with the wooden blocks that do not immediately reveal their maker or their purpose. Each of the blocks (Figure 1) measures about 18 cm in length and together they weigh almost 6 kg. They are all made of the same wood and fit together in a way that clearly indicates that they belong together. It is also clear that their purpose cannot be simply to display the solid figures or their surfaces as

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1 On the department homepage [https://www.math.ku.dk/bibliotek/arkivet/Models_at_the_Department_of_Mathematical_Sciences.pdf](https://www.math.ku.dk/bibliotek/arkivet/Models_at_the_Department_of_Mathematical_Sciences.pdf) one can find a list of the models. For the Schilling models there is a reference to the places in the Schilling Catalogue (Schilling 1911) where one can read about the model.
geometrical forms. Still, being a part of the department’s model collection it is obvious that they must have been produced to illustrate something mathematical. But what? And who designed them?

Johannes Hjelmslev

Until recently, I had no clue about the answer to these questions. The answer came by accident. Evelyn Barbin, Marta Menghini and Klaus Volkert had asked me to write a chapter (Lützen 2019) about the history of descriptive geometry in Denmark for a book dealing with the worldwide spread of this polytechnic art (Barbin, Menghini and Volkert 2019). In this connection, I was led to study the works of Johannes Hjelmslev (1873-1950) who taught descriptive geometry at the Polytechnic College in Copenhagen from 1903 until 1917 when he got a professorship at the University². He was the penultimate professor of descriptive geometry in Denmark and the last author of textbooks on that subject in Danish (Hjelmslev 1904, 1918).

When Hjelmslev was born in 1873 in the village Bjertrup in Hørning parish in a region (herred) of Denmark called Hjelmslev, he was given the name Johannes Trolle Petersen. However, after having studied mathematics under Hieronymus Georg Zeuthen (1839-1920) at the University of Copenhagen and having published his first papers³, he realized that his name J. Petersen was easily confused with the name of the more famous professor at the University of Copenhagen Julius Petersen (1839-1910). For that reason, he changed his last name in 1904 to Hjelmslev. At the same time his four year old son Louis who grew up to be a famous linguist, also had his name changed to Hjelmslev.

Johannes Hjelmslev acquired international fame in 1907 when he contributed to Hilbert’s foundational program in geometry. Following a hint by Hessenberg, he succeeded in proving Pascal’s theorem using only the plane axioms in a finite drawing plane without the use of the parallel axiom or the continuity axioms (Hjelmslev 1907). He even conjectured that the order axioms are not needed either, a conjecture he proved in 1929 (Hjelmslev 1929-49, 2. Mitteilung 1929). His proof was based on a theory of reflections and a new transformation of the plane that he called a half turn⁴.

Hjelmslev used some of these foundational ideas when a few years later he entered a didactical debate among Danish schoolteachers and university professors on the best way to teach geometry in middle and high schools (Hansen 2002, 106-125). One group argued for a continuation of the idealized Euclidean axiomatic deductive approach and another opted for a more intuitive, realist and experimental approach. Hjelmslev sided with the reformers arguing that Euclid’s axioms provided a bad model for the geometry of real objects. His job as professor of descriptive geometry had made it clear to him that real geometric constructions had to take into account that the drawing paper is only finite, and that the intersection point between two lines that intersect in a small angle is badly determined. For that reason, he rejected both the uniqueness axiom stating that two points determine one unique line and the usual theory of parallels. He published his first paper on his new “Geometry of reality” in 1913 (Hjelmslev 1913) and later developed his ideas in many papers of which some were published in German and French (Hjelmslev 1914, 1923). He believed that his geometry of reality was superior to the Euclidean approach both didactically, scientifically and in practice.

2 For obituaries of Hjelmslev see (Bohr 1950), (Fog 1950), (Jessen 1950) and (Nielsen 1950). Bohr’s is in German the others are in Danish.
3 In particular his paper on line geometry (Hjelmslev 1898) became famous for its proof of the Hjelmslev-Morley or Petersen-Morley theorem.
4 Hjelmslev’s half turn (Halbdrehung) is not a rotation of 180° around a fix point (see Hjelmslev 1907, 464). A short but precis discussion of Hjelmslev’s ideas (including the definition of a half turn) is to be found in (Karzel and Kroll 1988, 160-166)
“Not only is this system the only one that can give us the geometry that is applied in practice and the best basis for education, but it is also in a scientific sense to be preferred to the Euclidean system because its assumptions are logically less extensive and its range therefore considerably larger than the Euclidean system” (Hjelmslev 1913, 54)

Towards the end of his life, he combined his 1907 approach to Hilbertian axiomatics with his geometry of reality (Hjelmslev 1929-1949).

In 1916 Hjelmslev began publishing a series of textbooks on elementary geometry aimed at middle and high school in which he developed his didactical realist ideas (Hjelmslev 1916a). One day, leafing through the first of these books, pictures of the wooden blocks suddenly stared me in the face (Figure 2). Here was the key to the mystery of the ten wooden blocks.

![Figure 2](image-url)

The solution of the Mystery.

It turned out that the wooden blocks were pedagogical tools aimed for the introduction of the basic concepts of elementary geometry in Hjelmslev’s realist version of geometry. Indeed, the book began with a consideration of various box shaped objects or “blocks” such as bricks, matchboxes and cigar boxes. He thought such considerations would give the pupil “a preliminary idea of what one thinks about when one

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5 Before Hjelmslev, several mathematicians such as Pasch and Klein (see Klein 1928) had advanced empirical realist approaches to geometry, but Hjelmslev went further in this direction. Still his ideas had very limited influence on the development of geometry. I shall soon publish a more extensive historical account of Hjelmslev’s geometry of reality.
speaks about a block. However, the mentioned objects are not manufactured particularly accurately. Yet in geometry we will treat the most accurate forms that it is possible to manufacture.” (Hjelmslev 1916a, 7)

Hjelmslev went on to explain how one manufactures the most accurate forms possible. His explanations were material or even industrial. He started out explaining how planar surfaces are produced industrially:

One begins with three approximately plane iron plates. One places two of the surfaces one on top of the other and notices (using a specified type of ink) where they touch each other. Then one removes the bulges and repeats the procedure with a new pair of plates. Continuing in this manner, one will eventually arrive at three plates that pairwise fit exactly on top of each other. Such industrially produced very exact planes are then used to produce the cruder planes on which one usually draws geometric figures such as paper.

In § 5 Hjelmslev introduced the idea of a normal wedge and this is where our mysterious wooden blocks enter the picture: First he defined a wedge as a “body that has two exactly fashioned plane surfaces meeting along an edge … Two wedges are called neighbor wedges when they can be positioned in such a way that they supplement each other i.e. they both rest on a plane having each a surface lying in that plane while they fit tightly together along the surfaces that do not lie in the plane” (Hjelmslev 1916a, 10). From this description and the accompanying illustration (Fig. 3 in Figure 2) it is obvious that the blocks 1, 2 and 3 in Figure 1 are meant to illustrate wedges and that 1, 2 and 1, 3 illustrate pairs of neighbor wedges.

“If the relative position of the two planes (in a pair of neighbor wedges) is the same, we call them right wedges or normal wedges” (Hjelmslev 1916a, 10). This description of a normal wedge, however, was not sufficient for Hjelmslev. It was important for him to explain how one can test if a wedge is normal. To this end, he used a method similar to the method for manufacturing plane surfaces. Having manufactured two neighbor wedges one must “test if the two wedges fit into the same space, e.g. if they can be neighbor wedges to one and the same third wedge. In this way neighbor wedges can be manufactured simultaneously three at a time in such a way that any two of the wedges can supplement each other” (Hjelmslev 1916a, 10). The three wedges 1, 2, and 3 in Figure 1 do not satisfy this requirement. To be sure, the wedges 1 and 2 as well as the wedges 1 and 3 supplement each other, but as one can see in Figure 3 the wedges 2 and 3 do not supplement each other. Thus, the wedges 1, 2, and 3 are not normal wedges. However, any pair of the wedges 3, 4, and 5 in Figure 1 do in fact supplement each other and so they are normal. Fig. 4 in Figure 2 shows two normal wedges in Hjelmslev’s book.
Having now explained the meaning of right angles between planes, Hjelmslev went on to explain the concept of right angles between a plane and a line and between two lines. The key object here is the so-called normal corner: “A right corner or normal corner is a body with 3 exactly manufactured plane surfaces that meet two and two under a right angle along 3 edges that meet at a vertex also called a point.” (Hjelmslev 1916a, 11) (see Fig. 5 in Figure 2). In this connection, Hjelmslev also mentioned that an edge is called a straight line. Clearly, the blocks 7-9 in Figure 1 are models of normal corners. Hjelmslev could then explain what it means for two lines to be normal to each other and what it means for a line to be normal to a plane. Moreover, he claimed that it is an empirical fact that two normal corners with a common vertex and each having a plane surface in one and the same plane will have a common edge. One can easily test it using the wooden normal corners. From this, he deduced that the perpendicular line to a plane in a given point of the plane is unique. Other experiments with normal corners show that from a point outside a plane one can draw precisely one perpendicular to the plane and that one can draw exactly one plane orthogonal to a given line and through a given point.

Finally, Hjelmslev introduced the normal block:

“By a normal block one understands a block that is bounded by exact planes and where the planes along each edge make up a normal wedge. Where 3 edges meet they form a normal corner, so that the block has 8 normal corners in total.” (Hjelmslev 1916a, 13). So clearly, block 10 in figure 1 is a normal block. Hjelmslev explained how one can manufacture very accurate normal blocks and concluded: “Experience in the machine factories has shown that all these conditions [for the normal block] can be satisfied so that the normal blocks produced in this way exactly satisfy the conditions that we have already established to be approximatively satisfied for match boxes and bricks. It has turned out that one can manufacture mutually equal (congruent) normal blocks so accurately that when they are placed together forming one block they fit so tightly together that it is only possible to separate them again if one applies considerable force “ (Hjelmslev 1916a, 13).

From the last empirical fact, Hjelmslev concluded that one can produce squared drawing paper (Hjelmslev 1916a, 23). This was one of the fundamental aids in his further development of plane school geometry and provided him with a local replacement of the parallel postulate.

Thus, it is quite evident that the wooden blocks found among the models at the Department of Mathematical Sciences were meant to illustrate the different material figures that lie at the heart of Hjelmslev’s new realist introduction to geometry. The need to produce such models was emphasized by Hjelmslev himself in a paper that was published the same year his first schoolbook was published. In this paper where he explained the ideas behind his new realist approach to geometry, he wrote about the new definitions of planes, lines, right angles and rectangles that “they are not definitions in words but definitions in action. For this introductory part of the teaching, it will naturally be necessary that one has models of blocks, wedges and corners at hand. ... For the manufacture of wedges and corners, one can use plainly cut pieces of firewood. Cardboard models can also be produced. If the pupils could get the chance to model the mentioned objects in clay, applying the control tests mentioned in the book, it would of course be far the best. Eventually, exactly manufactured models in wood or plaster could be put on the market if it turns out to be desirable.” (Hjelmslev 1916b, 14)

Thus, Hjelmslev had imagined that the models could be produced commercially if his system of textbooks became a success. However, I have not found any trace of a commercial production of the models. In fact, Hjelmslev’s system of textbooks seems to have had a limited circulation (Hansen 2002, 120), (Jessen 1950, 241) so the market for models probably turned out to be too small.
The wooden models at the University of Copenhagen may very well have been Hjelmslev’s own models. That would explain why they ended up among the other mathematical models of the Department of Mathematics where Hjelmslev worked until he retired in 1942. Moreover, according to Jakob Nielsen’s obituary of Hjelmslev (Nielsen 1950, 7), from 1916 (when he published his first school book on geometry) to 1926 Hjelmslev taught mathematics for 10 hours a week at the State’s College for Teacher Training (Statens Lærehøjskole). Nielsen conjectured (reasonably) that Hjelmslev took on this heavy extra teaching load in order to disseminate his new didactical methods to the teachers in the elementary schools. We may further conjecture that the wooden blocks were manufactured for the use in Hjelmslev’s geometry lectures to the schoolteachers.

Literature.


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