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# Slice Rank of Block Tensors and Irreversibility of Structure Tensors of Algebras


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## Abstract

Determining the exponent of matrix multiplication $\omega$ is one of the central open problems in algebraic complexity theory. All approaches to design fast matrix multiplication algorithms follow the following general pattern: We start with one "efficient" tensor $T$ of fixed size and then we use a way to get a large matrix multiplication out of a large tensor power of $T$. In the recent years, several so-called barrier results have been established. A barrier result shows a lower bound on the best upper bound for the exponent of matrix multiplication that can be obtained by a certain restriction starting with a certain tensor.

We prove the following barrier over $\mathbb{C}$: Starting with a tensor of minimal border rank satisfying a certain genericity condition, except for the diagonal tensor, it is impossible to prove $\omega = 2$ using arbitrary restrictions. This is astonishing since the tensors of minimal border rank look like the most natural candidates for designing fast matrix multiplication algorithms. We prove this by showing that all of these tensors are irreversible, using a structural characterisation of these tensors. To obtain our result, we relate irreversibility to asymptotic slice rank and instability of tensors and prove that the instability of block tensors can often be decided by looking only on the sizes of nonzero blocks.



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## 1 Introduction

Determining the exponent $\omega$ of matrix multiplication is one of the central open problems in algebraic complexity theory. Since Strassen's seminal paper [24], who showed that matrices can be multiplied in time $O(n^{\log_2 7}) = O(n^{2.81})$, more and more refined methods have been proposed, see e.g. [5, 23, 25, 16, 17, 28, 19], yielding the currently best upper bound $O(n^{2.373})$. All of these approaches follow the following general pattern: We start with one "efficient" tensor $T$ of fixed size and then we use a way to get a large matrix multiplication out of a large tensor power $T^{\otimes N}$ of $T$. Efficient here means that $T$ has small (border) rank and that the structure of $T$ is "close to a matrix multiplication". Getting a large matrix multiplication out of $T^{\otimes N}$ means that we define an appropriate restriction. In Strassen's algorithm, the tensor $T$ is simply the tensor $\langle 2, 2, 2\rangle$ of $2 \times 2$-matrix multiplication. It becomes efficient, since Strassen showed that its rank is 7 (opposed to the trivial upper bound of 8). Then



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Strassen observes that $\langle 2,2,2\rangle^{\otimes N}$ is a matrix multiplication of size $2^N \times 2^N$, so the restriction is trivial. Coppersmith and Winograd [16] start with two tensors $\mathrm{cw}_q$ and $\mathrm{CW}_q$, which are now called the small and the big Coppersmith–Winograd tensors. These tensors depend on a parameter $q$ and have size $q+1$ and $q+2$, respectively. Their border ranks are both $q+2$. In particular, $\mathrm{CW}_q$ is a tensor of minimal border rank, since for any concise tensor of size $s$, $s$ is a lower bound for the border rank. The tensors $\mathrm{cw}_q$ and $\mathrm{CW}_q$ have a block structure and consist of three and six matrix multiplication tensors, respectively. To get a matrix multiplication out of a high power of the Coppersmith–Winograd tensors, we first degenerate a large diagonal out of the outer structure using the laser method and then apply Schönhage's asymptotic sum inequality [23]. The recent approaches [17, 28, 19] to fast matrix multiplication all start with the big Coppersmith–Winograd tensor $\mathrm{CW}_q$, but find better ways to to get a large matrix multiplication out of $\mathrm{CW}_q^{\otimes N}$ by analysing small powers of $\mathrm{CW}_q$. For more details how to design fast matrix multiplication algorithms, we refer to [6].

## 1.1 Barrier results

As mentioned above, all previous approaches follow the following general pattern: They start with a base tensor $T$. This tensor is raised to a high power $T^{\otimes N}$. Then a restriction $r$ is constructed such that $r$ applied to $T^{\otimes N}$ yields a large matrix multiplication, that is, $\langle n,n,n\rangle = r(T^{\otimes N})$. A barrier result shows a lower bound on the best upper bound that can be obtained by a certain restriction starting with a certain tensor. In such results, typical tensors that occur in actual upper bound constructions are considered as starting tensors. The type of restrictions can be arbitrary restrictions or restrictions that come out of upper bound constructions, for instance, restrictions that are constructed via the laser method. There are three ingredients in a barrier result: the class of starting tensors, which we want to be as large as possible, the type of restrictions, which we want to be as general as possible, and the lower bound on $\omega$, which we want to be as big as possible. Maximizing one of the three parameters typically decreases the other two.

The work by Coppersmith and Winograd [15] can be viewed as the first barrier result: They prove that it is impossible to prove $\omega = 2$ by one application of Schönhage's asymptotic sum inequality to a sum of matrix multiplication tensors. Ambainis et al. [4] proved a barrier result for the big Coppersmith–Winograd tensor and the laser method, the method used by Coppersmith and Winograd to get their upper bound. The laser method applied to the big Coppersmith–Winograd tensor cannot give $\omega = 2$, and in fact not even $\omega \le 2.30$. Alman and Vassilevska Williams [2, 3] showed barrier results for more general restrictions, namely so-called monomial restrictions, for a certain classes of tensors generalizing the big Coppersmith–Winograd tensors. Blasiak et al. [8, 9] studied the group theoretic framework by Umans and Cohn for fast matrix multiplication and showed that this approach cannot prove $\omega = 2$ using any fixed Abelian group and certain non-Abelian groups. Christandl, Vrana, and Zuiddam [14] introduced a parameter called *irreversibility*. They proved that with any starting tensor that is irreversible, one cannot prove $\omega = 2$ using an *arbitrary* restriction. Alman [1] proves a similar result formulated in terms of asymptotic slice rank of tensors. In [14, 1], these barriers are applied to Coppersmith-Winograd tensors, generalized Coppersmith-Winograd tensors (a class of tensors which have the same combinatorial structure as Coppersmith-Winograd tensors), and truncated polynomial multiplication tensors.

## 1.2 Our results

Previous barrier results use the notion of slice rank of tensors or related notions like subrank. Another property of tensors was recently related to slice rank is their instability in terms of geometric invariant theory (see [8, 12]). The first result we prove is a barrier result that applies to the class of all unstable tensors over $\mathbb{C}$. We prove that $\omega = 2$ cannot be proven using powers of unstable tensors of bounded dimension.

We also consider the class of structure tensors of non-semisimple algebras. These tensors have a natural block structure. Using this block structure, we prove that these tensors are unstable and give bounds on their asymptotic slice rank to get better barriers.

Particularly interesting are tensors of minimal border rank. These are concise tensors $t \in V_1 \otimes V_2 \otimes V_3$ with $\dim V_1 = \dim V_2 = \dim V_3 = n$ which can be approximated by tensors of rank $n$. A large subclass of these tensors – binding tensors (defined below) – can be related to structure tensors of commutative algebras. Thus, the previous barrier applies to them (except for diagonal tensors of rank $n$) and it is impossible to prove $\omega = 2$ using arbitrary restrictions from powers of one of these tensors. This is astonishing since the tensors of minimal border rank look like the most natural candidates for designing fast matrix multiplication algorithms, since the smaller the rank of the starting tensor, the smaller is the possible bound on $\omega$.

Our barriers hold true for a large class of tensors, much larger than in previous barrier results. Furthermore, they are true for arbitrary restrictions. For example, Coppersmith-Winograd tensors and truncated polynomial multiplication tensors belong to the smallest class we consider – the binding tensors of minimal border rank. Of course, we have to pay a price: the lower bounds on provable values of $\omega$ that we get are really small. Our results do not mean that tensors of minimal border rank are useless for proving $\omega = 2$, but that to do this, we need to consider a sequence of tensors with increasing size.

Our proof uses the concept of irreversibility introduced by Christandl, Vrana, and Zuiddam [14] and upper bounds on slice rank proved by Alman [1]. We prove that all binding tensors of minimal border rank are irreversible, except for the diagonal tensors. It follows from [7], that we can describe these tensors, which are defined by a complexity theoretical property, in terms of their structure. Binding tensors of minimal border rank are equivalent to the structure tensors of so-called smoothable algebras. Smoothable algebras are commutative, therefore, the only semisimple smoothable algebras over $\mathbb{C}$ are $\mathbb{C}^n$ and are given by the diagonal tensors $\langle n \rangle$. Thus it suffices to prove that structure tensors of algebras with nonzero radical are irreversible.

# 2 Preliminaries

## 2.1 Notations

Binary logarithms are denoted by log, natural logarithms by ln. For a probability distribution $p$ on a finite set $I$ its entropy is denoted by $H(p) = \sum_{i \in I} -p(i) \log p(i)$, and for two probability distributions $p$, $q$ their relative entropy is denoted by $D(p\|q) = \sum_{i \in I} p(i) \log \frac{p(i)}{q(i)}$. For a finite set $S$, the set of all probability distributions on $S$ is denoted by $\mathcal{P}(S)$. The tensor product $F^{N_1} \otimes F^{N_2} \otimes F^{N_3}$ is denoted by $F^{N_1 \times N_2 \times N_3}$.

## 2.2 Tensors and algebras

Let $V_1$, $V_2$, and $V_3$ be finite dimensional vector spaces. Every tensor $t \in V_1 \otimes V_2 \otimes V_3$ can be presented as a sum of *rank one tensors* – tensors of the form $v_1 \otimes v_2 \otimes v_3$ with $v_k \in V_k$. The *rank* $R(t)$ of a tensor $t$ is the minimum $r$ such that $t$ can be written as the sum of $r$ rank one tensors.

For a tensor $t \in V_1 \otimes V_2 \otimes V_3$ and a linear form $x \in V_1^*$, the *contraction* $t \cdot x$ is defined as $(v_1 \otimes v_2 \otimes v_3) \cdot x = x(v_1)(v_2 \otimes v_3)$ for rank one tensors and extended to arbitrary tensors by linearity. Thus, a tensor $t \in V_1 \otimes V_2 \otimes V_3$ defines a map $V_1^* \to V_2 \otimes V_3$ sending $x$ to $t \cdot x$. Two other maps $V_2^* \to V_1 \otimes V_3$ and $V_3^* \to V_1 \otimes V_2$ can be defined similarly. These maps are called *flattenings* of the tensor $t$. A tensor is called *concise* if all its flattenings are injective. Such tensor does not lie in any nontrivial subspace $V_1' \otimes V_2' \otimes V_3'$ with $V_k' \subset V_k$. We denote the maximum of the three ranks of the flattenings by $N(t)$. For a concise tensor, the ranks of the flattenings are the dimensions of $V_k$, and $N(t) = \max\{\dim V_1, \dim V_2, \dim V_3\}$.

A tensor $t \in V_1 \otimes V_2 \otimes V_3$ is called $1_{V_1}$*-generic* if $\dim V_2 = \dim V_3$ and there exists $x \in V_1^*$ such that the matrix $t \cdot x \in V_2 \otimes V_3$ has full rank. 1-genericity for the other indices is defined analogously. We call a tensor *binding* if it is $1_{V_1}$- and $1_{V_2}$-generic.

The slice rank of a tensor is defined similarly to the rank. Here the basic building blocks are tensors which have rank one flattenings, that is, tensors of the form $s \otimes v_3$ with $s \in V_1 \otimes V_2$ and $v_3 \in V_3$ and the two other symmetric forms. The *slice rank* $SR(t)$ is now the minimum number $r$ such that $t$ can be written as the sum of $r$ such basic building blocks. Note that the slice rank of a tensor in $F^{N_1 \times N_2 \times N_3}$ is at most $\min\{N_1, N_2, N_3\}$.

To design asymptotically fast matrix multiplication algorithms, it suffices to bound the asymptotic rank $\tilde{R}(t)$, which is defined by $\lim_{n\to\infty} R(t^{\otimes n})^{1/n}$. The asymptotic slice rank $\widetilde{SR}(t)$ is defined as $\lim_{n\to\infty} SR(t^{\otimes n})^{1/n}$. Unlike the limit in the definition of the asymptotic rank, the existence of this limit is a nontrivial fact, proven for tensors over $\mathbb{C}$ by Christandl, Vrana and Zuiddam [13].

Let $t' \in V_1' \otimes V_2' \otimes V_3'$ for vector spaces $V_1'$, $V_2'$, and $V_3'$. Let $A_i : V_i \to V_i'$ be linear maps. The linear map $A_1 \otimes A_2 \otimes A_3 : V_1 \otimes V_2 \otimes V_3 \to V_1' \otimes V_2' \otimes V_3'$ is defined on the rank one elements by $A_1 \otimes A_2 \otimes A_3(v_1 \otimes v_2 \otimes v_3) = A_1(v_1) \otimes A_2(v_2) \otimes A_3(v_3)$ and extends to arbitrary tensors by linear continuation. We call $t'$ a *restriction* of $t$ if there are linear maps $A_k : V_k \to V_k'$, $k = 1, 2, 3$ such that $t' = (A_1 \otimes A_2 \otimes A_3)t$. We write $t' \leq t$ in this case. Two tensors $t$ and $t'$ are called *equivalent* if $t' \leq t$ and $t \leq t'$ A tensor $t'$ is a *degeneration* of $t$ if $t'$ lies in the (Zariski) closure of the set of all restrictions of $t$. We write $t' \trianglelefteq t$ in this case.

It is well known that $R(t) \leq r$ iff $t \leq \langle r \rangle$. Here $\langle r \rangle$ denotes the diagonal tensor of size $r$ given by $\sum_{i=1}^r e_i \otimes e_i \otimes e_i$. We define the *subrank* $Q(t)$ to be the maximum $r$ such that $\langle r \rangle \leq t$. The asymptotic subrank $\tilde{Q}(t)$ is $\lim_{n\to\infty} Q(t^{\otimes n})^{1/n}$. The border rank $\underline{R}(t)$ is defined as the minimal $r$ such that $t \trianglelefteq \langle r \rangle$, that is, $t$ can be approximated by tensors of rank $r$. A tensor $t$ is called a tensor of minimal border rank if $\underline{R}(t) = N(t)$.

The slice rank of the diagonal tensor $\langle r \rangle$ is $r$ [26]. Given a slice rank decomposition of $t$, we can write down a decomposition with the same number of summands for every tensor $t' \leq t$ by applying a restriction operator $A_1 \otimes A_2 \otimes A_3$ to both sides. Thus $SR$ is monotone with respect to restriction of tensors. If $q = Q(t)$ and $r = R(t)$, then $\langle q \rangle \leq t \leq \langle r \rangle$. Therefore, $Q(t) \leq SR(t) \leq R(t)$ and, consequently, $\tilde{Q}(t) \leq \widetilde{SR}(t) \leq \tilde{R}(t)$.

In this work, the term *algebra* always means a finite dimensional associative algebra with unity 1 over some field $k$. If we speak of a basis of an algebra, we always mean a basis of the underlying vector space. A left ideal $I$ (and in the same way, a right ideal or twosided ideal) is called nilpotent, if $I^n = \{0\}$ for some positive integer $n$. For all finite dimensional algebras $A$, the sum of all nilpotent left ideals of $A$ is a nilpotent twosided ideal, which contains every nilpotent right ideal of $A$. This twosided ideal is called the *radical* of $A$. We call an algebra $A$ semisimple, if $\operatorname{rad} A = \{0\}$. The quotient algebra $A/\operatorname{rad} A$ is semisimple. An algebra $A$ is called simple, if there are no twosided ideals in $A$ except for the zero ideal and $A$ itself. Wedderburn's theorem states that every semisimple algebra $A$ is isomorphic to a product of simple algebras and every simple algebra is of the form $D^{n\times n}$ for some division algebra $D$ over $k$. If $k$ is algebraically closed, then the only such division algebra is $k$ itself. For more background on associative algebras, the reader is referred to [18].

The multiplication map $A \times A \to A$ is bilinear map. With a bilinear map, we can associate its structure tensor $t_A \in A^* \otimes A^* \otimes A$ in a canonical way – a rank one tensor $f \otimes g \otimes w$ corresponds to a bilinear map $(x, y) \mapsto f(x)g(y)w$ and a sum of rank one tensors corresponds to a sum of these elementary maps. We define the rank $R(A)$ of $A$ as $R(t_A)$. The same is done for the other measures above. If $A = k^{n\times n}$, then $R(A)$ is simply the rank of the matrix multiplication map.

Note that structure tensors of algebras are concise [25]. Indeed, injectivity of the first flattening means that for each nonzero element $x \in A$ its left multiplication operator $L_x \colon y \mapsto xy$ is nonzero. This always holds (recall that we consider only unital algebras). Injectivity of the second flattening is connected to right multiplication operators in the same way. Injectivity of the third flattening is equivalent to surjectivity of the dual map $x \otimes y \mapsto xy$, which also holds for every algebra.

Smoothable algebras are a subset of commutative algebras defined in algebraic geometry with relation to deformations of 0-dimensional schemes. In [7], these algebras are connected to tensors of minimal border rank.

▶ **Theorem 1** ([7, Cor. 3.6])**.** *A binding tensor has minimal border rank if and only if it is equivalent to the structure tensor of a smoothable algebra.*

## 2.3 Instability and entanglement polytopes

The notion of instability comes from geometric invariant theory, where it can be defined in high generality (the standard reference is [22]). We are interested only in the action of the groups $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$ and $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3)$ on the tensor space $V_1 \otimes V_2 \otimes V_3$ over $\mathbb{C}$, and will give definitions for this case.

▶ **Definition 2.** *A tensor $t \in V_1 \otimes V_2 \otimes V_3$ is called* unstable *if 0 is contained in the (Zariski) closure of the $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3)$ orbit of $t$, and* semistable *otherwise. The set of all unstable tensors in $V_1 \otimes V_2 \otimes V_3$ is a Zariski closed cone. It is called the* nullcone *of the action of $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3)$ on $V_1 \otimes V_2 \otimes V_3$.*

The Hilbert-Mumford criterion says that a tensor $t$ is unstable if and only if there is a 1-parameter subgroup $g$ of $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3)$ (that is, a group homomorphism $g \colon \mathbb{C}^\times \to \mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3)$) such that $\lim_{\varepsilon\to 0} g(\varepsilon)t = 0$. This criterion can be used to give a more combinatorial definition of instability and relate it to slice rank, see [8] for details.

▶ **Theorem 3** ([12, Cor. 6.5])**.** *A tensor $t \in \mathbb{C}^{N\times N\times N}$ is unstable iff $\widetilde{SR}(t) < N$.*

We will also use some notions related to entanglement polytopes. These are special cases of *moment polytopes*, the existence of which for actions of nonabelian groups was established by Kirwan [20].

Entanglement polytope $\Delta(t)$ of a tensor $t \in \mathbb{C}^{N_1\times N_2\times N_3}$ is a convex polytope in $\mathbb{R}^{N_1+N_2+N_3}$ that contains information about representation-theoretic and analytic properties of the orbit closure of this tensor. We refer to [27, 11] for precise definitions. We do not use entanglement polytopes directly, we only need their properties such as connection to slice rank via quantum functionals from [13], the finiteness of the set of moment polytopes, and bounds on the distance to entanglement polytopes from [11].

▶ **Theorem 4** ([13]). *For a tensor $t \in \mathbb{C}^{N_1 \times N_2 \times N_3}$, its asymptotic slice rank can be computed from the entanglement polytope:*

$$\log \widetilde{SR}(t) = \min_{\theta \in \mathcal{P}(\{1,2,3\})} \max_{\substack{p_k \in \mathbb{R}^{N_k} \\ (p_1,p_2,p_3) \in \Delta(t)}} \theta(1)H(p_1) + \theta(2)H(p_2) + \theta(3)H(p_3).$$

▶ **Theorem 5** ([10]). *For each $N_1, N_2, N_3$, the set of all entanglement polytopes of tensors in $\mathbb{C}^{N_1 \times N_2 \times N_3}$ is finite.*

## 2.4 Irreversibility barrier

▶ **Definition 6** ([14]). *The irreversibility of a tensor $t$ is defined as the ratio $\mathbf{i}(t) = \frac{\log \tilde{R}(t)}{\log \tilde{Q}(t)}$. A tensor $t$ is called* reversible *if $\mathbf{i}(t) = 1$ and* irreversible *otherwise.*

▶ **Theorem 7** ([14, Thm. 9]). *Using arbitrary restrictions from powers of a tensor $t$, it is impossible to prove an upper bound on $\omega$ better than $2\mathbf{i}(t)$.*

# 3 Block tensors

In this section, we will introduce some terminology for dealing with block tensors. Block tensors (more specifically, block tensors with tight support) are important for our goals for two reasons. First, this is the class of tensors for which the laser method works (see for example [19, Thm. 4.1]). Second, structure tensors of algebras with radical have a natural block structure, and for graded algebras, this block structure has a tight support. And most tensors of minimal border rank are structure tensors of algebras.

▶ **Definition 8.** *A* block tensor *is specified by the following data:*
- *a triple of finite-dimensional vector spaces $V_1, V_2, V_3$;*
- *a triple of* index sets *$I_1, I_2, I_3$;*
- *a triple of direct sum decompositions*

$$V_1 = \bigoplus_{i \in I_1} V_{1,i}, \;\; V_2 = \bigoplus_{i \in I_2} V_{2,i}, \;\; V_3 = \bigoplus_{i \in I_3} V_{3,i};$$

- *a tensor $t \in V_1 \otimes V_2 \otimes V_3$.*

We will often say that $t$ is a block tensor, assuming that all the other data are implicit or given by context.

The decompositions of each $V_k$ induce a decomposition of the tensor space

$$V_1 \otimes V_2 \otimes V_3 = \bigoplus_{(i_1,i_2,i_3) \in I_1 \times I_2 \times I_3} V_{1,i_1} \otimes V_{2,i_2} \otimes V_{3,i_3}.$$

For a block tensor $t$, we denote by $t_{i_1 i_2 i_3}$ its projection onto $V_{1,i_1} \otimes V_{2,i_2} \otimes V_{3,i_3}$.

▶ **Definition 9.** *The* support *of a block tensor $t$ is defined as*

$$\operatorname{supp} t = \{(i_1,i_2,i_3) \in I_1 \times I_2 \times I_3 \mid t_{i_1 i_2 i_3} \neq 0\}.$$

▶ **Definition 10.** *The* block format *of a block tensor is a triple* $(n_1, n_2, n_3)$ *of maps* $n_k \colon I_k \to \mathbb{N}$ *defined as* $n_k(i) = \dim V_{k,i}$, $k = 1, 2, 3$. *The* relative block format *is a triple* $(f_1, f_2, f_3)$ *of probability distributions* $f_k$ *on* $I_k$ *defined as* $f_k(i) = \frac{n_k(i)}{N_k}$ *where* $N_k = \dim V_k$.

The following bound on the asymptotic subrank of block tensors was proven by Alman.

▶ **Theorem 11** ([1, Thm. 18]). *Let* $t$ *be a block tensor with block format* $(n_1, n_2, n_3)$. *Then*

$$\widetilde{SR}(t) \leq \sup_{p \in \mathcal{P}(\operatorname{supp} t)} \min_{k \in \{1,2,3\}} \prod_{i \in I_k} \left( \frac{n_k(i)}{\mu_k[p](i)} \right)^{\mu_k[p](i)}$$

*where* $\mu_k[p]$ *denotes the marginal distribution of* $p$ *on* $I_k$.

We will need this statement in a slightly different form.

▶ **Corollary 12.** *For every block tensor* $t \in \mathbb{C}^{N_1 \times N_2 \times N_3}$ *with relative block format* $(f_1, f_2, f_3)$ *the following bounds hold:*

$$\log \widetilde{SR}(t) \leq \max_{p \in \mathcal{P}(\operatorname{supp} t)} \min_{k \in \{1,2,3\}} (\log N_k - D(\mu_k[p] \| f_k))$$

*and*

$$\log \widetilde{SR}(t) \leq \max_{p \in \mathcal{P}(\operatorname{supp} t)} \sum_{k=1}^{3} \theta(k) \left(\log N_k - D(\mu_k[p] \| f_k)\right)$$

*for every* $\theta \in \mathcal{P}(\{1, 2, 3\})$.

**Proof.** The first bound is Alman's bound from the previous theorem after taking logarithms. The supremum can be changed to maximum since we optimize a continuous function over the set of probability distributions, which is compact. The second bound follows from the fact that $\theta_1 m_1 + \theta_2 m_2 + \theta_3 m_3 \geq \min\{m_1, m_2, m_3\}$ for $\theta \in \mathcal{P}(\{1, 2, 3\})$. ◀

The second bound of Corollary 12 can be seen as a generalization to block tensors of Theorem 2.11 from [13], which relates asymptotic slice rank and Strassen's support functionals.

Of special interest are block tensors with tight and subtight supports, defined as follows.

▶ **Definition 13.** *A subset* $S \subset I_1 \times I_2 \times I_3$ *is called* $s$-tight *(or* $s$-subtight*) with* numbering *given by three maps* $a_k \colon I_k \to \mathbb{Z}$ *if for each* $(i_1, i_2, i_3) \in S$ *we have* $a_1(i_1) + a_2(i_2) + a_3(i_3) = s$ *(or* $a_1(i_1) + a_2(i_2) + a_3(i_3) \leq s$*, respectively).*

▶ **Example 14.** Consider an algebra $A$. It is filtered by the powers of the radical: we have $A = (\operatorname{rad} A)^0 \supset \operatorname{rad} A \supset (\operatorname{rad} A)^2 \supset \dots$. Let $d$ be minimal such that $(\operatorname{rad} A)^d = \{0\}$ and choose for each $i < d$ a subspace $R_i \subset (\operatorname{rad} A)^i$ such that $(\operatorname{rad} A)^i = R_i \oplus (\operatorname{rad} A)^{i+1}$. We have $A = \bigoplus_{i=0}^{d-1} R_i$ and $(\operatorname{rad} A)^j = \bigoplus_{i=j}^{d-1} R_i$. Consider the corresponding decomposition $A^* = \bigoplus_{i=0}^{d-1} R_i^*$.

The structure tensor $t_A \in A^* \otimes A^* \otimes A$ is a block tensor with 0-subtight support with respect to these decompositions and numbering given by $a_1(i) = a_2(i) = i$, $a_3(i) = -i$. Indeed, if $x_1 \in R_{i_1} \subset (\operatorname{rad} A)^{i_1}$ and $x_2 \in R_{i_2} \subset (\operatorname{rad} A)^{i_2}$, then $x_1 x_2 \in (\operatorname{rad} A)^{i_1 + i_2} = \bigoplus_{i \geq i_1 + i_2} R_i$, which means that $t_{A, i_1 i_2 i_3}$ is zero if $i_1 + i_2 > i_3$, that is, $a_1(i_1) + a_2(i_2) + a_3(i_3) > 0$.

Similarly, if an algebra is graded, that is, $A = \bigoplus_{i=0}^{d-1} A_i$ such that $A_i A_j \subset A_{i+j}$, then its structure tensor is a block tensor with tight support.

## 4 Irreversibility and instability

In the definition of irreversibility (Definition 6) both the numerator and the denominator are mysterious quantities: we do not know of any effective method to compute either of them for general tensors.

There are no general lower bounds for the asymptotic rank except for the trivial lower bound $\tilde{R}(t) \geq N(t)$, and upper bounds mostly come from ad hoc degenerations constructed for specific tensors or simple tensor families. In some sense, understanding the asymptotic rank is the main goal of the complexity theory of bilinear maps.

More is known about the asymptotic subrank. It can be computed explicitly for a large class of tensors – the tight tensors. Lower bounds for $\tilde{Q}$ can be obtained for some structured tensors, such as block tensors with tight support. We also have the upper bound $\tilde{Q}(t) \leq \widetilde{SR}(t)$. In fact, slice rank was first introduced by Tao [26] in order to bound the subrank. The asymptotic slice rank $\widetilde{SR}(t)$ can in principle be computed (though not efficiently) for a given tensor $t$ over $\mathbb{C}$ by constructing the entanglement polytope and using Theorem 4.

Taken together, the bounds $\tilde{R}(t) \geq N(t)$ and $\tilde{Q}(t) \leq \widetilde{SR}(t)$ prove the following Proposition.

▶ **Proposition 15.** *For the irreversibility of a tensor $t$, the following bound holds:*

$$\mathbf{i}(t) \geq \frac{\log N(t)}{\log \widetilde{SR}(t)}.$$

*As a consequence, every unstable concise tensor is irreversible.*

This means that powers of one unstable tensor cannot be used to prove $\omega = 2$. In fact, even if we allow a sequence of unstable tensors, we need their size to be unbounded, as the following theorem shows.

▶ **Theorem 16.** *Over $\mathbb{C}$, it is impossible to prove $\omega = 2$ from powers of unstable concise tensors $t$ with bounded $N(t)$ using arbitrary restrictions.*

**Proof.** From Theorem 5 and Theorem 4 it follows that the set of possible values of $\widetilde{SR}(t)$ for tensors in $C^{n\times n\times n}$ is finite. Therefore, the set of possible ratios $\frac{\log N(t)}{\log \widetilde{SR}(t)}$ is also finite. If we let

$$B(n) := \min\left\{\frac{\log N(t)}{\log \widetilde{SR}(t)} \mid t \in \mathbb{C}^{n\times n\times n},\, N(t) \neq \widetilde{SR}(t)\right\},$$

then by Proposition 15 the irreversibility of any unstable concise tensor with $N(t) \leq n$ is bounded from below by $B(n)$, and by Theorem 7 the best bound on $\omega$ we can get is $2B(n) > 2$. ◀

Using the results of Bürgisser et al. [11] for moment polytopes, we can give an explicit bound on irreversibility.

▶ **Theorem 17.** *The irreversibility of an unstable concise tensor in $\mathbb{C}^{n\times n\times n}$ is at least*

$$\left(1 - \frac{3^{-3n}}{18n^4 \ln n}\right)^{-1}.$$

**Proof.** Bürgisser et al. [11] introduced a quantity $\gamma$ called the *weight margin* characterizing a representation of a reductive algebraic group, and Theorems 6.4 and 6.8 from their paper give a lower bound $\gamma \geq \sqrt{3}^{-3n}(3n)^{-1}$ on $\gamma$ for the standard action of $\mathrm{GL}_n \times \mathrm{GL}_n \times \mathrm{GL}_n$ on $\mathbb{C}^{n\times n\times n}$. Denote by $u$ the vector $(\frac{1}{n}, \dots, \frac{1}{n}) \in \mathbb{Q}^n$. By Theorem 3.30 from the same paper, the distance from the point $(u, u, u)$ to the entanglement polytope of a tensor $t$ is at least $\frac{\gamma}{n}$, if the entanglement polytope does not contain this point, that is, $t$ is unstable. Let $(p_1, p_2, p_3)$ be the point of the entanglement polytope closest to $(u, u, u)$.

The entropy function on the set of probability distributions is strongly concave with parameter $\log e$ and the uniform distribution $u$ has maximal entropy $\log n$, so

$$\frac{1}{3}(H(p_1) + H(p_2) + H(p_3)) \leq \log n - \log e\frac{\|(p_1, p_2, p_3) - (u, u, u)\|^2}{2} \leq \log n - \log e\frac{\gamma^2}{2n^2},$$

and $\log \widetilde{SR}(t)$, which is bounded by the maximum of $\frac{1}{3}(H(p_1) + H(p_2) + H(p_3))$, is also at most $\log n - \log e\frac{\gamma^2}{2n^2}$. For the irreversibility, we have a lower bound

$$\frac{\log n}{\log \widetilde{SR}(t)} = \frac{\log n}{\log n - \log e\frac{\gamma^2}{2n^2}} = \left(1 - \frac{\gamma^2}{2n^2 \ln n}\right)^{-1} = \left(1 - \frac{3^{-3n}}{18n^4 \ln n}\right)^{-1}. \qquad \blacktriangleleft$$

This explicit barrier is minuscule and it is likely that it can be improved. Even for $n = 2$, it gives a value less than $1 + 10^{-5}$, which is far from the actual irreversibility value of the only nontrivial unstable $2\times 2\times 2$ tensor $W = e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$, which has $\tilde{R}(W) = 2$ and $\tilde{Q}(W) = \frac{3}{2^{2/3}}$. On the other hand, there is evidence that the difference between minimal irreversibility and 1 is exponential: Bürgisser et al. [11] note that from the results of Kravtsov in combinatorics [21], it follows that $\gamma$ for the trilinear tensor action decreases exponentially in $n$.

At a first glance, these results suggest that one should use semistable tensors to construct algorithms for matrix multiplication. Nevertheless, we need to remember that our reasoning is based on the approximation by Proposition 15, and to actually use a tensor we need to at least bound its asymptotic rank. Known examples of semisimple tensors (except, of course, the diagonal ones) seem to resist this as much as matrix multiplication tensors do. For example, already Coppersmith and Winograd [15] note that it is possible to prove $\omega = 2$ by proving that the tensor $\mathrm{cw}_2$ (which is a semistable $3\times 3\times 3$ tensor) has asymptotic rank 3.

## 5 Instability of block tensors

In this section we will prove a bound on the slice rank for block tensors which implies a better barrier for block tensors with subtight support, including structure tensors of algebras with radical. We do this by extending some of the techniques for dealing with slice rank introduced in [8, 13] to block tensors.

On a qualitative level, we study the stability of block tensors with fixed support. It may so happen that all such block tensors are unstable.

▶ **Definition 18.** *A set $S \subset I_1 \times I_2 \times I_3$ is an* unstable support in block format $(n_1, n_2, n_3)$ *if all block tensors $t \in V_1 \otimes V_2 \otimes V_3$ with index sets $I_1, I_2, I_3$ having support $S$ and block format $(n_1, n_2, n_3)$ are unstable under the action of* $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3)$.

Sometimes the instability of support can be certified by a 1-parameter subgroup in SL which acts as scalar multiplication on each block $V_{1,i_1} \otimes V_{2,i_2} \otimes V_{3,i_3}$. This can be alternatively expressed in combinatorial terms as follows.

▶ **Definition 19.** *A set* $S \subset I_1 \times I_2 \times I_3$ *is a* combinatorially unstable support in block format $(n_1, n_2, n_3)$ *if there exist* exponents $u_k \colon I_k \to \mathbb{R}$ *such that* $\sum_{i \in I_k} n_k(i) u_k(i) = 0$ *for each* $k$ *and* $u_1(i_1) + u_2(i_2) + u_3(i_3) > 0$ *for each* $(i_1, i_2, i_3) \in S$.

▶ **Proposition 20.** *Every combinatorially unstable support is unstable.*

**Proof.** Note that the notion of combinatorial instability does not change if we restrict the exponents to be rationals or integers. If $u_k$ are integer exponents, consider the 1-parameter subgroup $(g_1(\varepsilon), g_2(\varepsilon), g_3(\varepsilon))$ where the operators $g_k(\varepsilon)$ on $V_k$ are defined on the direct summands $V_{k,i}$ by $g_k(\varepsilon)v = \varepsilon^{u_k(i)} v$.

The condition $\sum_{i \in I_k} n_k(i) u_k(i) = 0$ ensures that $g_k(\varepsilon) \in \mathrm{SL}(V_k)$ for all $\varepsilon \neq 0$. For every block tensors $t$ with support $S$, we have

$$(g_1(\varepsilon), g_2(\varepsilon), g_3(\varepsilon))t = \sum_{(i_1,i_2,i_3) \in S} \varepsilon^{u_1(i_1)+u_2(i_2)+u_3(i_3)} t_{i_1 i_2 i_3} \to 0 \text{ as } \varepsilon \to 0,$$

so $t$ is unstable. ◀

▶ **Theorem 21.** *Suppose* $S \subset I_1 \times I_2 \times I_3$ *is a s-subtight set with numbering* $(a_1, a_2, a_3)$. *Let* $\bar{a}_k = \sum_{i \in I_k} f_k(i) a_k(i)$. *If* $\bar{a}_1 + \bar{a}_2 + \bar{a}_3 > s$, *then* $S$ *is combinatorially unstable for any block format with relative block format* $(f_1, f_2, f_3)$.

**Proof.** Combinatorial instability is certified by exponents $u_k(i) = \bar{a}_k - a_k(i)$. Indeed,

$$\sum_{i \in I_k} n_k(i) u_k(i) = N_k \sum_{i \in I_k} f_k(i) u_k(i) = N_k \left( (\sum_{i \in I_k} f_k(i)) \bar{a}_k - \sum_{i \in I_k} f_k(i) a_k(i) \right) = 0,$$

and for $(i_1, i_2, i_3) \in S$

$$u_1(i_1) + u_2(i_2) + u_3(i_3) = \bar{a}_1 + \bar{a}_2 + \bar{a}_3 - (a_1(i_1) + a_2(i_2) + a_3(i_3)) \geq \bar{a}_1 + \bar{a}_2 + \bar{a}_3 - s > 0.$$ ◀

A quantitative measure of the instability of a support can be given by an uniform upper bound on the slice rank of tensors with this support. Such bounds can be given using the techniques of [8, 1].

▶ **Theorem 22.** *Suppose* $S \subset I_1 \times I_2 \times I_3$ *is a s-subtight set with numbering* $(a_1, a_2, a_3)$. *As in Theorem 21, let* $\bar{a}_k = \sum_{i \in I_k} f_k(i) a_k(i)$.

*The following bounds on the asymptotic slice rank of every block tensor* $t \in \mathbb{C}^{N \times N \times N}$ *with support* $S$ *and relative block format* $(f_1, f_2, f_3)$ *hold:*

$$\widetilde{SR}(t) \leq N \frac{m^{-s/3}}{\sqrt[3]{\prod_{k=1}^{3} \sum_{i_k \in I_k} f_k(i_k) m^{a_k(i_k)}}}$$

*where* $m$ *is the solution of the equation*

$$\sum_{k=1}^{3} \frac{\sum_{i_k \in I_k} f_k(i_k) m^{a_k(i_k)} a_k(i_k)}{\sum_{i_k \in I_k} f_k(i_k) m^{a_k(i_k)}} = s, \tag{1}$$

*and*

$$\widetilde{SR}(t) \leq N \exp\left( -\frac{(\bar{a}_1 + \bar{a}_2 + \bar{a}_3 - s)^2}{6 \sum_{k=1}^{3} \sum_{i_k \in I_k} a_k^2} \right).$$

**Proof.** To bound the asymptotic slice rank, we use the second bound of Corollary 12 with $\theta(1) = \theta(2) = \theta(3) = \frac{1}{3}$. To obtain the bound, we need to solve the following optimization problem:

$$\log N - \frac{1}{3}\sum_{k=1}^{3}(D(p_k \| f_k)) \to \max$$

where $p_1, p_2, p_3$ are the marginals of $p \in \mathcal{P}(S)$.

This is a convex optimization problem, but the polytope of all marginals of probability distributions on $S$ is unwieldy. Instead we optimize with relaxed requirements on $p_1$, $p_2$, $p_3$. Note that for any $p \in \mathcal{P}(S)$ we have

$$\sum_{i=(i_1,i_2,i_3)\in S} p(i)(a_1(i_1) + a_2(i_2) + a_3(i_3)) \leq s$$

On the other hand, if the marginals of $p$ are $p_1, p_2, p_3$, then this expression can be rewritten as

$$\sum_{i_1\in I_1} p_1(i_1)a_1(i_1) + \sum_{i_2\in I_2} p_2(i_2)a_2(i_2) + \sum_{i_3\in I_3} p_3(i_3)a_3(i_3).$$

Consider the relaxed optimization problem

$$\log N - \frac{1}{3}\sum_{k=1}^{3}(D(p_k \| f_k)) \to \max$$

where $\sum_{i_k\in I_k} p_k(i_k) = 1$,

$$\sum_{i_1\in I_1} p_1(i_1)a_1(i_1) + \sum_{i_2\in I_2} p_2(i_2)a_2(i_2) + \sum_{i_3\in I_3} p_3(i_3)a_3(i_3) \leq s.$$

This is still a convex optimization problem. If $\mu$ is the Lagrange multiplier for the inequality restriction and $\lambda_k$ are multipliers for the three equality restrictions, then the KKT conditions for this problem are

$$\begin{aligned}
\frac{1}{3}(-\ln p_k(i_k) + \ln f_k(i_k) - 1) - \mu a_k(i_k) - \lambda_k = 0,\\
\mu \geq 0,\\
\sum_{i_k\in I_k} p_k(i_k) = 1,\\
\sum_{i_1\in I_1} p_1(i_1)a_1(i_1) + \sum_{i_2\in I_2} p_2(i_2)a_2(i_2) + \sum_{i_3\in I_3} p_3(i_3)a_3(i_3) \leq s,\\
\mu(\sum_{i_1\in I_1} p_1(i_1)a_1(i_1) + \sum_{i_2\in I_2} p_2(i_2)a_2(i_2) + \sum_{i_3\in I_3} p_3(i_3)a_3(i_3) - s) = 0.
\end{aligned}$$

The solution is obtained on the boundary

$$\sum_{i_1\in I_1} p_1(i_1)a_1(i_1) + \sum_{i_2\in I_2} p_2(i_2)a_2(i_2) + \sum_{i_3\in I_3} p_3(i_3)a_3(i_3) = s,$$

because if $\mu = 0$, then the only probability distributions $p_k$ satisfying the stationary condition are $p_k = f_k$, but this is incompatible with the inequality $\bar{a}_1 + \bar{a}_2 + \bar{a}_3 > s$. Denoting

$l_k = \exp(-3\lambda_k - 1)$ and $m = \exp(-3\mu)$, we have $p_k(i_k) = f_k(i_k) l_k m^{a_k(i_k)}$. From the feasibility conditions we obtain $l_k = (\sum_{i_k \in I_k} f_k(i_k) m^{a_k(i_k)})^{-1}$ and the required restriction (1) on $m$.

Multiplying each component of the stationary condition by its variable $p_k(i_k)$ and summing the obtained equalities we get

$$\begin{aligned}-\frac{1}{3}\sum_{k=1}^{3} D(p_k\|f_k) &= \frac{1}{3}\sum_{k=1}^{3}\sum_{i_k\in I_k}(-p_k(i_k)\log p_k(i_k) + p_k(i_k)\log f_k(i_k)) \\ &= \log e\left(1+\lambda_1+\lambda_2+\lambda_3+\mu\sum_{k=1}^{3}\sum_{i_k\in I_k} p_k(i_k)a_k(i_k)\right) \\ &= \log e(1+\lambda_1+\lambda_2+\lambda_3+\mu s) = \frac{1}{3}\log\frac{1}{l_1 l_2 l_3 m^s},\end{aligned}$$

from which we get the required bound on $\widetilde{SR}(t)$.

The second bound follows from the estimate of the distance between triples of distributions $p = (p_1, p_2, p_3)$ and $f = (f_1, f_2, f_3)$ in the Euclidean metric. Consider these triples and the triple $a = (a_1, a_2, a_3)$ as vectors in the Euclidean space of dimension $|I_1| + |I_2| + |I_3|$. We have $\langle p, a\rangle = s$ and $\langle f, a\rangle = s + z$ for $z = \bar{a}_1 + \bar{a}_2 + \bar{a}_3 - s$. Therefore, $\|p - f\| \geq \frac{z}{\|a\|}$. Since on the space of probability distributions the functions $D(p_k\|f_k)$ are strongly convex with parameter $\log e$ and minimum 0 attained at $p_k = f_k$, we have

$$\frac{1}{3}(D(p_1\|f_1) + D(p_2\|f_2) + D(p_3\|f_3)) \geq \log e\frac{z^2}{6\|a\|^2},$$

which gives the required upper bound on the asymptotic slice rank. ◀

The first bound of Theorem 22 is better, but requires the solution of an algebraic equation possibly of high degree. The second (coarser) bound is similar to the upper bound from [8] in terms of instability.

▶ **Corollary 23.** *Suppose $A$ is a non-semisimple associative algebra. Then its structure tensor is unstable and irreversible. Moreover, the irreversibility of the structure tensor is at least*

$$\left(1-\left(\frac{(\sum_{i=0}^{d-1} ir_i)^2}{18N^2\ln N\sum_{i=0}^{d-1} r_i^2}\right)\right)^{-1}$$

*where $N = \dim A$, $d$ is minimal such that $(\operatorname{rad} A)^d = 0$ and $r_i = \dim(\operatorname{rad} A)^i/(\operatorname{rad} A)^{i+1}$.*

**Proof.** We apply the previous theorems to the block structure with subtight support explained in Example 14. Since $A$ is non-semisimple, it has a nontrivial radical ($r_1 > 0$) and

$$\bar{a}_1 + \bar{a}_2 + \bar{a}_3 - s = \frac{\sum_{i=0}^{d-1} ir_i}{N} > 0,$$

so the structure tensor is unstable. We have $\widetilde{SR}(t_A) \leq N\exp\left(-\frac{(\sum_{i=0}^{d-1} ir_i)^2}{18N^2\sum_{i=0}^{d-1} r_i^2}\right)$.

Recall that the structure tensor of an algebra is always concise. Thus, from Proposition 15 it follows that it is irreversible and

$$\mathbf{i}(t_A) \geq \frac{\log N}{\log\widetilde{SR}(A)} \geq \left(1-\left(\frac{(\sum_{i=0}^{d-1} ir_i)^2}{18N^2\ln N\sum_{i=0}^{d-1} r_i^2}\right)\right)^{-1}. \quad ◀$$

▶ **Corollary 24.** *It is impossible to prove $\omega = 2$ using powers of structure tensors of non-semisimple algebras with bounded dimension.*

▶ **Corollary 25.** *Denote by $X_N$ the set of all binding tensors $t \in \mathbb{C}^{N\times N\times N}$ with $\underline{R}(t) = N$ and $R(t) > N$. It is impossible to prove $\omega = 2$ using powers of binding tensors from $X_N$ with bounded $N$.*

**Proof.** By Theorem 1 every tensor from $X_N$ is equivalent to a structure tensor of some commutative algebra. The only semisimple commutative algebra of dimension $N$ is $\mathbb{C}^N$. It has rank $N$, so tensors equivalent to its structure tensor and equivalent tensors are not contained in $X_N$. Thus, the previous corollary applies. ◀

We conjecture that not only binding tensors, but all concise tensors of minimal border rank (except tensors equivalent to $\langle N\rangle$) are unstable.

▶ Remark 26. The results of Section 4 use properties of entanglement polytopes, which are defined only over $\mathbb{C}$. But the results on block tensors and associative algebras are combinatorial and hold over arbitrary field. The result on tensors of minimal border rank applies over algebraically closed fields.

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