Conservative descent for semi-orthogonal decompositions

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CONSERVATIVE DESCENT FOR SEMI-ORTHOGONAL DECOMPOSITIONS

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Abstract. Motivated by the local flavor of several well-known semi-orthogonal decompositions in algebraic geometry, we introduce a technique called conservative descent, which shows that it is enough to establish these decompositions locally. The decompositions we have in mind are those for projectivized vector bundles and blow-ups, due to Orlov, and root stacks, due to Ishii and Ueda. Our technique simplifies the proofs of these decompositions and establishes them in greater generality for arbitrary algebraic stacks.

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1. Introduction

Semi-orthogonal decompositions (see Definition 5.3) of derived categories are central in the study of non-commutative aspects of algebraic geometry. Such decompositions are well-known for derived categories of projectivized vector bundles, blow-ups and root stacks. More examples are given in a survey by Kuznetsov [Kuz14]. However, the references describing these decompositions usually impose quite restrictive conditions on the geometric objects under consideration (cf. Remark 1.1). For instance, they often only consider smooth and projective varieties over a field. This has been too limited for our purposes, and this article grew out of the need to generalize the semi-orthogonal decompositions mentioned above to the context of algebraic stacks. We use the results of the present article in our recent work [BLS16], where we show that the derived category of a smooth, proper Deligne–Mumford stack is a semi-orthogonal summand of the derived category of a smooth projective variety. We also use them in the follow-up article [BS19], where we generalize a result by Bernardara [Ber09] on semi-orthogonal decompositions of derived

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categories of Brauer–Severi varieties using stacky methods. Furthermore, the results are used in [BGLL17] to define the categorical measure for Deligne–Mumford stacks.

Naively, one would expect that most statements about schemes and varieties generalize via simple descent arguments to algebraic stacks. However, when working with derived categories, it is not clear that such an approach works since derived categories do not satisfy descent in the usual sense. The problem is that the derived pull-back along a faithfully flat morphism is usually not faithful. Indeed, this can already be seen by considering the covering of the projective line by its standard affine charts.

A modern approach to overcome such obstacles is to consider enhancements of derived categories, either by differential graded categories or by $\infty$-categories. However, this comes at the cost of invoking a substantial amount of technical machinery.

Our observation is that there is a technically less demanding way to solve the problem. It is based on the fact that even though the derived pull-back along a faithfully flat morphism is not faithful, it has the weaker property of being conservative. That is, the functor reflects isomorphisms. We formalize a technique, which we call conservative descent, which allows us to descend certain semi-orthogonal decompositions along conservative triangulated functors. Our methods are described entirely in the classical language of triangulated categories.

In the abstract setting of triangulated categories, our conservative descent theorem is formulated as follows.

**Theorem A** (see Theorem 5.16). Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories and let $F: \mathcal{T} \to \mathcal{T}'$ be a conservative triangulated functor. Let $S_1, \ldots, S_n$ and $S'_1, \ldots, S'_n$ be sequences of idempotent comonads (see Definition 4.14) on $\mathcal{T}$ and $\mathcal{T}'$, respectively, and assume that $S_i$ and $S'_i$ are compatible with respect to $F$ for each $i$ in the sense of Definition 4.22. If the sequence

\[(1.1) \quad \text{Im} S'_1, \ldots, \text{Im} S'_n \]

of essential images is semi-orthogonal in $\mathcal{T}'$, then so is the sequence

\[(1.2) \quad \text{Im} S_1, \ldots, \text{Im} S_n \]

in $\mathcal{T}$. Both sequences consist of right admissible subcategories of $\mathcal{T}'$ and $\mathcal{T}$, respectively. Moreover, if $(1.1)$ is a semi-orthogonal decomposition of $\mathcal{T}'$, then $(1.2)$ is a semi-orthogonal decomposition of $\mathcal{T}$.

The important point here is compatibility between the comonads $S_i$ and $S'_i$. In order to formulate a statement which is easy to apply in geometric situations, we introduce the notion of relative Fourier–Mukai transforms (see Definition 3.3). Just as their classical counterparts, these are functors attached to certain geometric data. The comonads in the statement of Theorem A are typically induced by relative Fourier–Mukai transforms. Due to the relative flavor of our definition, these transforms admit a natural notion of flat base change, and the induced comonads automatically satisfy the required compatibility conditions. This allows us to give the following formulation of the conservative descent theorem in the geometric setting.

**Theorem B** (see Theorem 6.1). Let $Z_1, \ldots, Z_n$ and $X$ be algebraic stacks over some base algebraic stack $S$, and assume that $\Phi_i: D_{\text{qc}}(Z_i) \to D_{\text{qc}}(X)$, for $1 \leq i \leq n$, is a Fourier–Mukai transform over $S$ in the sense of Definition 3.3. Let $u: S' \to S$
be a faithfully flat morphism, and denote the base change of the objects above by $Z'_1, \ldots, Z'_n, X'$ and $\Phi'_i : D_{qc}(Z'_i) \to D_{qc}(X')$, respectively. Then for each $i$, the functor $\Phi'$ is fully faithful provided that $\Phi'_i$ is fully faithful.

Assume that all $\Phi'_i$, and therefore also all $\Phi_i$, are fully faithful. If the sequence

\begin{equation}
\text{Im } \Phi'_1, \ldots, \text{Im } \Phi'_n
\end{equation}

of essential images is semi-orthogonal in $D_{qc}(X')$, then so is the sequence

\begin{equation}
\text{Im } \Phi_1, \ldots, \text{Im } \Phi_n
\end{equation}

in $D_{qc}(X)$. Both sequences consist of right admissible subcategories of $D_{qc}(X')$ and $D_{qc}(X)$, respectively. Moreover, if (1.3) is a semi-orthogonal decomposition of $D_{qc}(X')$, then (1.4) is a semi-orthogonal decomposition of $D_{qc}(X)$.

As applications of Theorem B, we obtain generalizations and unified proofs of the semi-orthogonal decompositions for projectivized vector bundles, blow-ups and root stacks. We summarize the results in the following theorem and refer to Section 6 for precise statements.

**Theorem C.** The unbounded derived category $D_{qc}(X)$ of an algebraic stack $X$ admits a naturally defined semi-orthogonal decomposition

(i) if $X$ is a projectivized vector bundle $\mathbb{P}_S(\mathcal{E})$, where $S$ is an algebraic stack and $\mathcal{E}$ is a locally free $\mathcal{O}_S$-module of constant finite rank (Theorem 6.7, Corollary 6.8);

(ii) if $X$ is a blow-up $\tilde{Y}$ of an algebraic stack $Y$ in a regular closed immersion $Z \hookrightarrow Y$ of constant codimension $c$ (Theorem 6.9, Corollary 6.10);

(iii) if $X$ is an $r$-th root stack $\tilde{Y}$ of an algebraic stack $Y$ in an effective Cartier divisor $E \subset Y$ (Theorem 6.11, Corollary 6.12).

Moreover, there are induced semi-orthogonal decompositions of the subcategories $D_{pf}(X)$ of perfect complexes and $D_{pc}^b(X)$ of locally bounded pseudo-coherent complexes as well as of the singularity category $D_{sg}(X)$, which is defined as the Verdier quotient $D_{lb}^b(X)/D_{pf}(X)$. Note that the category $D_{pc}^b(X)$ coincides with the category $D_{coh}^b(X)$ of complexes with bounded coherent cohomology if $X$ is noetherian.

**Remark 1.1.** The semi-orthogonal decompositions mentioned in Theorem C are well known in several special cases. Here we give the relevant references.

(a) Be˘ılinson proves a version of part (i) for $\mathbb{P}^n_k$ where $k$ is a field in [Be˘ı78]. His proof generalizes to arbitrary algebraic stacks $S$.

(b) Orlov proved versions of part (i) and (ii) of Theorem C in [Orl92, Theorem 2.6 and Theorem 4.3]. He works in the context of smooth projective varieties over the field of complex numbers and states the result as a semi-orthogonal decomposition of $D_{coh}^b(X)$. It is not clear to us how to generalize his proof of (ii) to arbitrary algebraic stacks $Y$. A proof of (ii), basically in the same setting as Orlov’s version, also appears in the textbook by Huybrechts [Huy06, Proposition 11.18]. Also this proof seems to be hard to generalize.

(c) Elagin generalizes Orlov’s versions of part (i) and (ii) of Theorem C to stack quotients by linearly reductive group schemes [Ela12, Theorem 10.1, Theorem 10.2]. He uses a cohomological descent argument where he works locally on the source rather than locally on the base as we do.
(d) A version of part (iii) of Theorem C is given by Ishii–Ueda [IU15, Proposition 6.1]. We give a more detailed statement in [BLS16, Theorem 4.7].

Remark 1.2. Thomason provides in [Tho93a] a proof of the Grothendieck–Berthelot–Quillen theorem which describes the algebraic K-theory of a projectivized vector bundle as a product of copies of the algebraic K-theory of the base scheme; in [Tho93b] he describes the algebraic K-theory of a blow-up in a similar way. These two descriptions follow from the semi-orthogonal decompositions of the subcategories of perfect complexes in the instances (i) and (ii) of Theorem C. Without using the terminology, Thomason implicitly establishes these semi-orthogonal decompositions. He works with quasi-compact and quasi-separated schemes. It seems to us that his arguments should generalize to algebraic stacks, which would give a proof of part (i) and (ii) of Theorem C different from the one we give in this article.

1.1. Outline. In Section 2, we summarize some basic facts about the derived category of an algebraic stack. The relative notion of a Fourier–Mukai transform is defined in Section 3. In two following sections, we develop the theory of conservative descent from the abstract point of view of triangulated categories. In Section 4, we review the 2-categorical notions of mates and idempotent comonads. We also show that a flat base change of a relative Fourier–Mukai transform induces a compatibility between the associated idempotent comonads as required later on for deducing Theorem B from Theorem A. In Section 5, we explain that certain vanishing conditions on sequences of idempotent comonads are equivalent to the defining conditions of a semi-orthogonal decomposition, which allows us to prove Theorem A. In Section 6, we deduce the geometric version of the conservative descent theorem, i.e., Theorem B, and use this result to establish the semi-orthogonal decompositions appearing in Theorem C.

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2. Preliminaries

We use the definition of algebraic space and algebraic stack given in the stacks project [SP17, Tag 025Y, Tag 026O]. In particular, we do not assume that algebraic stacks be quasi-separated or have separated diagonals. Algebraic stacks form a 2-category. However, we follow the common practice to suppress 2-categorical details from the language and the notation. For instance, we usually simply write commutative diagram or cartesian diagram instead of 2-commutative diagram or 2-cartesian diagram.

2.1. Derived categories of algebraic stacks. Given an algebraic stack $X$, we consider its derived category $D_{qc}(X)$. There are several approaches to constructing this category in the literature. We briefly recall the one taken in [LMB00]. The $\mathcal{O}_X$-modules in the lisse-étale topos $X_{\text{lis-ét}}$ form a Grothendieck abelian category. We denote its (unbounded) derived category by $D(X_{\text{lis-ét}}, \mathcal{O}_X)$. Now $D_{qc}(X)$ is
defined as the full subcategory of $D(X_{\text{lis-\acute e t}}, \mathcal{O}_X)$ whose objects are complexes with quasi-coherent cohomology sheaves.

Recall that the category $D_{\text{qc}}(X)$ has the structure of a closed symmetric monoidal category, whose operations we denote by

$$- \otimes -, \quad \mathcal{H}om(-,-).$$

Given an arbitrary morphism $f: X \to Y$ of algebraic stacks, we get an induced adjoint pair of functors

$$f^*: D_{\text{qc}}(Y) \to D_{\text{qc}}(X), \quad f_*: D_{\text{qc}}(X) \to D_{\text{qc}}(Y).$$

The precise construction of these functors is somewhat technical. We refer to [HR17, Section 1] for a detailed discussion.

2.2. Concentrated morphisms. It should be emphasized that even though the functors (2.2) exist in the generality stated above, they are not necessarily well behaved without further assumptions. The situation becomes better if $f$ is assumed to be concentrated, as defined by Hall–Rydh [HR17, Definition 2.4]. We recall the definition here.

**Definition 2.1.** An algebraic stack $X$ is concentrated if it is quasi-compact, quasi-separated and has finite cohomological dimension. Recall that an algebraic stack has finite cohomological dimension provided that there exists an integer $d$ such that for any quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we have $H^i(X, \mathcal{F}) = 0$ for every $i > d$. A morphism $f: X \to Y$ is concentrated if for any cartesian diagram as in (3.1) with $Y'$ affine, the stack $X'$ is concentrated.

**Remark 2.2.** The property of a morphism of being concentrated is preserved under arbitrary base change and can be verified after a faithfully flat base change. Moreover, an algebraic stack $X$ is concentrated if and only if it is concentrated over $\text{Spec} \mathbb{Z}$. This follows from [HR17, Lemma 2.5].

**Remark 2.3.** Let $X$ be a quasi-compact and quasi-separated algebraic stack. Assume furthermore that $X$ has finitely presented inertia and affine stabilizers. Then $X$ is concentrated if and only if all stabilizers at points of positive characteristic are linearly reductive. This follows from a slightly more general result by Hall and Rydh [HR15, Theorem C].

**Remark 2.4.** A scheme or an algebraic space is concentrated if and only if it is quasi-compact and quasi-separated. Indeed this is a special case of Remark 2.3. As a consequence, a representable morphism of algebraic stacks is concentrated if and only if it is quasi-compact and quasi-separated.

Now let $f: X \to Y$ be a concentrated morphism of algebraic stacks. Then the functor $f_*$ is the restriction of the corresponding derived functor $D(X_{\text{lis-\acute e t}}, \mathcal{O}_X) \to D(Y_{\text{lis-\acute e t}}, \mathcal{O}_Y)$ (see [HR17, Theorem 2.6]). In this situation the functor $f_*$ has a right adjoint ([HR17, Theorem 4.14]), which we denote by

$$f^*: D_{\text{qc}}(Y) \to D_{\text{qc}}(X).$$

2.3. Perfect and pseudo-coherent complexes. The theory of pseudo-coherent and perfect complexes on a ringed topos is worked out in [SGA6, Exposés II] (see also [SP17, Tag 08G5, Tag 08FT] for the definitions). In particular, these definitions
apply to \((X_{\text{lis-ét}}, \mathcal{O}_X)\) when \(X\) is an algebraic stack. This gives us the following inclusions of full triangulated subcategories
\[(2.4) \quad D_{\text{pf}}(X) \subset D_{\text{pc}}^b(X) \subset D_{\text{pc}}(X) \subset D_{\text{qc}}(X),\]
where \(D_{\text{pf}}(X)\) is the subcategory of perfect complexes, \(D_{\text{pc}}(X)\) is the subcategory of pseudo-coherent complexes and \(D_{\text{pc}}^b(X)\) is the subcategory of \(D_{\text{pc}}(X)\) of complexes which locally have bounded cohomology. Furthermore, the singularity category for \(X\) is defined as the Verdier quotient
\[(2.5) \quad D_{\text{sg}}(X) := D_{\text{pc}}^b(X)/D_{\text{pf}}(X).\]

Remark 2.5. The definition of perfect and pseudo-coherent complexes on a general ringed topos is somewhat involved, and it is sometimes convenient to instead use the following characterization.

Let \(X\) be an algebraic stack and let \(\mathcal{F}\) be an object in \(D(X_{\text{lis-ét}}, \mathcal{O}_X)\). Then \(\mathcal{F}\) is perfect (resp. pseudo-coherent) if and only if for any point \(x \in X\) there exists a smooth neighborhood \(U \to X\) of \(x\), where \(U\) is an affine scheme, such that there exists a quasi-isomorphism \(P \to \mathcal{F}|_U\) where \(P\) is a bounded (resp. bounded above) complex of finite locally free \(\mathcal{O}_U\)-modules.

Here the reduction to the local situation is obvious. Assume that \(U = \text{Spec} \, R\). The perfect (resp. pseudo-coherent) objects in \(D(\text{Mod}(R))\) are characterized as the complexes \(M\) admitting a quasi-isomorphism \(P \to M\), where \(P\) is a bounded (resp. bounded above) complex of finitely generated projective \(R\)-modules (see \([SP17, \text{Tag 064U}]\)). Furthermore, we have an equivalence between the category of quasi-coherent modules on the ringed topos \((U_{\text{lis-ét}}, \mathcal{O}_U)\) and the category \(\text{Mod}(R)\) of \(R\)-modules given by taking global sections. This induces an equivalence \(R\Gamma : D_{\text{qc}}(U) \to D(\text{Mod}(R))\) of derived categories. We leave it as an exercise to the reader to verify that this equivalence preserves perfect and pseudo-coherent complexes (cf. \([SP17, \text{Tag 08EB, Tag 08E7, Tag 08HE, Tag 08HG}]\) for the corresponding statement for the small étale topos).

Remark 2.6. The category \(D_{\text{pc}}^b(X)\) is mostly interesting in the case when \(X\) is noetherian, in which case it coincides with the category \(D_{\text{coh}}^b(X)\) of complexes with bounded, coherent cohomology.

Remark 2.7. Note that by default, we do not use any derived decorations for the functors \((2.1), (2.2)\) and \((2.3)\). Also, we use the same notation for the induced functors on the categories \((2.4)\) and \((2.5)\) provided that they exist. The precise meaning of the symbols \(\otimes, \text{Hom}, f^*, f_*, f^!\) should always be clear from the context.

3. Relative Fourier–Mukai transforms

In this section, we discuss Fourier–Mukai transforms in a relative setting. The definition is a straightforward generalization of the usual concept, but involves some technical conditions in order to ensure that such transforms behave well under base change and that the induced functors preserve perfect and locally bounded pseudo-coherent complexes. The required conditions are fairly well understood if we restrict the discussion to quasi-compact and quasi-separated schemes, but the situation for algebraic stacks seems to be more complicated. We do not investigate these conditions systematically in this article. Instead our strategy is to postulate the properties we require in Definition 3.2 and to prove that the types of morphism appearing in our applications satisfy these properties in Proposition 3.9.
Consider a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow{g} & & \downarrow{f} \\
Y' & \xrightarrow{v} & Y
\end{array}
\]

of algebraic stacks, where, for the moment, we do not put any further conditions on the morphisms \(f, g, u\) and \(v\). The obvious isomorphism \(g^*u^* \cong v^*f^*\) together with the adjunction counit \(f^*f_* \to \text{id}\) give a natural transformation

\[
g^*u^* \xrightarrow{\sim} v^*f_* \to v^*.
\]

By adjunction, we get an induced natural transformation

\[
u^*f_* \to g_*v^*
\]

between functors \(\text{D}_{\text{qc}}(X) \to \text{D}_{\text{qc}}(Y')\) which we call the base change transformation for \(f_*\) (along \(u\)). We are interested in situations when the base change transformation is an isomorphism.

**Proposition 3.1** ([HR17, Corollary 4.13]). Let \(f: X \to Y\) be a concentrated morphism of algebraic stacks appearing in a cartesian square (3.1). Then the base change morphism (3.3) for \(f_*\) is an isomorphism provided that \(f\) and \(u\) are tor-independent. In particular, this holds if \(f\) or \(u\) is flat.

Now let \(f: X \to Y\) be a concentrated morphism of algebraic stacks appearing in a cartesian diagram (3.1), and assume that \(u\) is flat. Then we have a natural transformation

\[
g_*v^*f^* \xrightarrow{\sim} u^*f_*f^* \to u^*
\]

constructed from the inverse of the base change transformation (3.3), which is an isomorphism by Proposition 3.1, and the adjunction counit \(f_*f^* \to \text{id}\). By adjunction, we get an induced natural transformation

\[
v^*f^* \to g^*u^*
\]

which we call the base change transformation for \(f^*\) (along \(u\)). Moreover, we have a natural transformation

\[
f_*\left(f^*(\mathcal{O}_Y) \otimes f^*(-)\right) \xrightarrow{\sim} f_*f^*(\mathcal{O}_Y) \otimes (-) \to \text{id},
\]

where the first morphism is induced by the projection formula, which holds since \(f\) is concentrated by [HR17, Corollary 4.12], and the second by the adjunction counit. By adjunction, we get an induced natural transformation

\[
f^*(\mathcal{O}_Y) \otimes f^*(-) \to f^*.
\]

Recall that a ring homomorphism \(A \to B\) is perfect [SP17, Tag 067G] if \(B\) admits a presentation of the form \(B \cong A[x_1, \ldots, x_n]/I\), such that \(B\) is perfect as an \(A[x_1, \ldots, x_n]\)-module. This notion extends, in the usual way, to a property of morphisms of algebraic stacks by demanding that the property be fppf local on both the source and the target (see [SP17, Tag 0685] for the corresponding situation for morphisms of schemes).

**Definition 3.2.** Let \(f: X \to Y\) be a proper, perfect and concentrated morphism of algebraic stacks. We say that \(f\) has property

\[\text{P1}\]

if \(f_*\) preserves perfect complexes.


P2 if \( f_* \) preserves pseudo-coherent complexes;
P3 if (3.7) is an isomorphism and the base change transformation (3.5) for \( f^\times \) along any flat morphism \( u \) is an isomorphism.

Moreover, we say that \( f \) has any of the properties P1–P3 *uniformly* provided that for any cartesian square (3.1) with \( u \) flat, the morphism \( g \) has the corresponding property.

We are now ready to give the definition of a relative Fourier–Mukai transform and to discuss its basic properties.

**Definition 3.3.** Let \( X \) and \( Y \) be algebraic stacks over some base algebraic stack \( S \). A *Fourier–Mukai transform* \( \Phi: X \to Y \) over \( S \) is a diagram

\[
\begin{array}{ccc}
P & \rightarrow & \Phi \\
p \downarrow & & \downarrow q \\
x & \rightarrow & y
\end{array}
\]

of algebraic stacks over \( S \) together with an object \( \Phi \) in \( D_{pf}(K) \) such that

(a) the morphisms \( p \) and \( q \) are proper, perfect and concentrated (see Definition 2.1);
(b) the object \( q^\times \mathcal{O}_Y \) is perfect;
(c) the morphism \( p \) has properties P1 and P2 uniformly in the sense of Definition 3.2;
(d) the morphisms \( q \) has properties P1–P3 uniformly in the sense of Definition 3.2.

*Note:* If we restrict the discussion to quasi-compact and quasi-separated schemes then (c) and (d) are implied by the other conditions by Remark 3.6 below. We do not know if this is true in general.

Given a Fourier–Mukai transform \( \Phi = (K, p, q, \Phi): X \to Y \), we get an induced functor

\[
D_{qc}(X) \to D_{qc}(Y), \quad \mathcal{F} \mapsto q_*(K \otimes p^*(\mathcal{F})).
\]

This functor admits a right adjoint, which is given by

\[
D_{qc}(Y) \to D_{qc}(X), \quad \mathcal{G} \mapsto p_*(K^\vee \otimes q^\times(\mathcal{G})),
\]

where \( K^\vee := \text{Hom}(\mathcal{K}, \mathcal{O}_K) \) denotes the dual of \( K \).

**Remark 3.4.** By abuse of language, we also say that a functor \( \Phi: D_{qc}(X) \to D_{qc}(Y) \) isomorphic to a functor of the form (3.9) is a Fourier–Mukai transform. When doing so, we always fix a corresponding geometric datum \( \Phi = (K, p, q, \Phi) \) which is obvious from the definition of the functor. Note that we use the same symbol \( \Phi \) for the functor and the geometric datum.

The prominent feature of relative Fourier–Mukai transforms is that they admit a natural notion of flat base change. More precisely, if \( \Phi = (K, p, q, \Phi) \) is a Fourier–Mukai transform over \( S \) and \( S' \to S \) is a flat morphism, then we form the base change \( \Phi_{S'} = (K', p', q', \Phi') \), where

\[
\begin{array}{ccc}
K' & \rightarrow & \Phi' \\
p' \downarrow & & \downarrow q' \\
x' & \rightarrow & y'
\end{array}
\]
is the base change of the diagram (3.8) along $S' \to S$, and $\mathcal{K}'$ is the pull-back of $\mathcal{K}$ along the induced morphism $K' \to K$. The conditions on the morphisms $p$ and $q$ assert that $\Phi'$ is again a Fourier–Mukai transform.

**Proposition 3.5.** Let $X, Y$ be algebraic stacks over an algebraic stack $S$, and let $\Phi = (K, p, q, \mathcal{K}) : X \to Y$ be a Fourier–Mukai transform over $S$. Then the functors (3.9) and (3.10) preserve perfect and locally bounded pseudo-coherent complexes.

**Proof.** Derived pull-backs always preserve perfect and pseudo-coherent complexes. Since the morphisms $p$ and $q$ are assumed to be perfect, they have finite tor-dimension. Hence $p^*$ and $q^*$ preserve locally bounded complexes. By assumption, both $p$ and $q$ satisfy $P1$ and $P2$. Hence all the functors $p^*, q^*, p_* q^*$ preserve perfect complexes and locally bounded pseudo-coherent complexes. Since also $q$ satisfies $P3$, we have $q^* \cong q^* (\mathcal{O}_Y) \otimes q^* (-)$. By assumption $q^* (\mathcal{O}_Y)$ is perfect, so also $q^*$ preserves perfect complexes and locally bounded pseudo-coherent complexes. Now the proposition follows from the fact that the functors (3.9) and (3.10) are compositions of the functors $p^*, q^*, p_* q^*$ and tensoring with the perfect complexes $\mathcal{K}$ and $\mathcal{K}^\vee$. □

Note that the restriction of the functor (3.9) to the categories of perfect complexes also admits a left adjoint. Explicitly, this is given by

(3.12) $D_{pf}(Y) \to D_{pf}(X), \; \mathcal{G} \mapsto p_*(\mathcal{K}^\vee \otimes q^*(\mathcal{G})),$

where $p_* : \mathcal{G} \mapsto p_*(\mathcal{G}^\vee)$ denotes the left adjoint of $p^* : D_{pf}(K) \to D_{pf}(X)$ (see e.g. [BLS16, Lemma 4.3]).

We conclude the section by discussing situations when the conditions P1–P3 are satisfied.

**Remark 3.6.** For a moment, let us restrict attention to the category of quasi-compact and quasi-separated schemes. Then the properties in Definition 3.2 are well understood. They are treated thoroughly by Lipman and Neeman in [LN07]. A good overview is also given by Lipman in [Lip09, Section 4.7]. Here we give a brief summary of the results that are relevant to us.

Assume that $f : X \to Y$ is a morphism of quasi-compact and quasi-separated schemes. In particular, such a morphism is concentrated by Remark 2.4. If $f$ is proper and perfect, then it automatically satisfies all properties in Definition 3.2 uniformly. Indeed, since it is proper and pseudo-coherent it is also quasi-proper, i.e., satisfies $P2$ by [Lip09, Corollary 4.3.3.2]. Since it has finite tor-dimension, it is also quasi-perfect, by [LN07, Theorem 1.2], i.e., satisfies $P1$, by [LN07, Proposition 2.1], which also tells us that (3.7) is an isomorphism. Hence it also satisfies $P3$ by [Lip09, Theorem 4.7.4]. In particular, conditions (c) and (d) in Definition 3.3 are redundant in this situation.

**Remark 3.7.** The situation is less explored if we consider arbitrary concentrated morphisms $f : X \to Y$ of algebraic stacks. If $f$ is proper, perfect and representable by schemes, then it satisfies $P1$ and $P2$ uniformly. Indeed, these properties can be verified locally on the target, so this follows from Remark 3.6. If, in addition, $f$ is finite, then also $P3$ holds uniformly by [HR17, Theorem 4.14(4)]. The properties in Definition 3.2 in the context of algebraic stacks are further explored by Neeman in the recent preprint [Nee17].
Remark 3.8. Let us restrict the discussion to the category of smooth and projective schemes over an algebraically closed field. Given objects $X$, $Y$ in this category and an object $K$ in $D_{pf}(X \times Y)$, we obtain the Fourier–Mukai transform $\Phi = (X \times Y, p, q, K): X \to Y$, where $p$ and $q$ are the canonical projections. Indeed, by Remark 3.6, it is enough to verify items (a) and (b) of Definition 3.3. Here (a) is clear (cf. proof of Proposition 3.9 below). Furthermore, (b) follows from Grothendieck–Verdier duality since $q^*\mathcal{O}_Y \cong q_!\mathcal{O}_Y \cong \Sigma^\dim X p^*\omega_X$ (see e.g. [Huy06, Section 3.4]). Usually the term Fourier–Mukai transform, as defined for instance in [Huy06, Definition 5.1], refers to the induced functor $\Phi: D_{pf}(X) \to D_{pf}(Y)$ obtained from such a datum. In particular, our notion is a direct generalization of the standard notion.

The next proposition summarizes what we need for the applications considered in this article.

Proposition 3.9. Let $f: X \to Y$ be a morphism of algebraic stacks. Then $f$ is perfect, proper, concentrated and satisfies properties $P_1$ and $P_2$ uniformly in the following cases:

(a) $f$ is a projectivized vector bundle;
(b) $f$ is a blow-up in a regular sheaf of ideals;
(c) $f$ is a regular closed immersion;
(d) $f$ is a $\mu_n$-gerbe;
(e) $f$ is a root stack in an effective Cartier divisor.

Moreover, if $f$ is a regular closed immersion then $f^*(\mathcal{O}_Y)$ is perfect and $f$ satisfies $P_3$ uniformly.

Remark 3.10. We follow the definition of regular immersion given in SGA6 [SGA6, Exposé VII, Définition 1.4]. This is what is called a Koszul-regular immersion in the stacks project [SP17, Tag 0638]. A regular sheaf of ideals is simply an ideal sheaf corresponding to a regular closed immersion. The property of being a regular immersion is local on the target for the fpqc topology [SGA6, Exposé VII, Proposition 1.5]. In particular, the definition automatically extends to algebraic stacks.

Proof of Proposition 3.9. The morphism $f$ is clearly proper and of finite presentation in all the cases (a)–(e). It is also perfect by flatness and by being of finite presentation in the cases (a), (d) and (e) and by [SGA6, Exposé VII, Proposition 1.9] in the cases (b) and (c). Moreover, $f$ is concentrated by Remark 2.2, 2.3 and 2.4.

In the cases (a)–(c), $f$ is representable by schemes, so $P_1$ and $P_2$ hold uniformly by Remark 3.7. Assume that we are in one of the cases (d) or (e). The push-forward $f_*$ preserves perfects by [BLS16, Lemma 4.5], so $f$ has property $P_1$. Furthermore, from the proof of the same lemma, we may, after an appropriate base change, assume that the category of quasi-coherent $\mathcal{O}_X$-modules is equivalent to the category of $R$-modules for some not necessarily commutative ring $R$. Hence any pseudo-coherent complex of $\mathcal{O}_X$-modules is quasi-isomorphic to a bounded above complex of finite locally free $\mathcal{O}_X$-modules. Indeed, the proof for this is identical to the proof in the case when $R$ is commutative (cf. [SP17, Tag 068R]). Since $f_*$ is bounded and preserves perfect complexes, this implies that $f$ has property $P_2$. Hence $f$ satisfies property $P_1$ and $P_2$ uniformly, since the property of being a $\mu_n$-gerbe or a root stack in a Cartier divisor is stable under flat base change.
Finally, assume that \( f \) is a regular closed immersion. Then \( f \) is finite so \( P3 \) is satisfied uniformly by Remark 3.7 and \( f^*(\mathcal{O}_Y) \) is perfect by [BLS16, Lemma 4.1].

4. Mates and idempotent comonads

The base change transformations (3.3) and (3.5) are instances of what is called a \textit{mate} in category theory. We recall some aspects of the calculus of mates in 4.1 and use this theory in 4.2 to show that Fourier–Mukai transforms and their right adjoints respect flat base change in a certain sense. In 4.3 we prove \textit{conservative descent for fully faithfulness} and deduce that fully faithfulness of Fourier–Mukai transforms can be tested after a faithfully flat base change.

The essential image of a fully faithful Fourier–Mukai transform is a \textit{right admissible subcategory}, and hence the induced projection functor onto this subcategory is an example of what is called an \textit{idempotent comonad}. If a Fourier–Mukai transform and its flat base change are both fully faithful, their induced idempotent comonads are \textit{compatible}. This result is a consequence of a general categorical result established in 4.7 after the necessary prerequisites on idempotent comonads (and monads) and their compatibilities are explained in 4.4, 4.5 and 4.6.

The general results in this section are formulated for an arbitrary fixed 2-category. The words “object”, “1-morphism”, “2-morphism” refer to this 2-category. We use the symbols \( \rightarrow \) and \( \Rightarrow \) for 1-morphisms and 2-morphisms, respectively. The geometric examples take place in the 2-category of triangulated categories.

4.1. Mates with geometric examples. For the convenience of the reader, we recall some results on mates from [KS74, §2]. We frequently consider the following situation.

\textbf{Setting 4.1.} Let

\[
\begin{array}{ccc}
\text{C}' & \xleftarrow{F} & \text{C} \\
L' & \downarrow & L \\
\text{D}' & \xleftarrow{G} & \text{D}
\end{array}
\]

be a diagram of objects and 1-morphisms and let \((L', R', \eta', \varepsilon')\) and \((L, R, \eta, \varepsilon)\) be adjunctions where \( \eta: \text{id}_{\text{D}} \Rightarrow RL \) and \( \eta' \) are the unit 2-morphisms and \( \varepsilon: LR \Rightarrow \text{id}_{\text{C}} \) and \( \varepsilon' \) are the counit 2-morphisms.

\textbf{Lemma 4.2 ([KS74, Proposition 2.1])}. In Setting 4.1 there is canonical bijection

\[
\{ \text{2-morphisms } \alpha: L'G \Rightarrow FL \} \overset{\sim}{\longrightarrow} \{ \text{2-morphisms } \beta: GR \Rightarrow R'F \}
\]

sending a 2-morphism \( \alpha: L'G \Rightarrow FL \) to the 2-morphism \( \overline{\alpha} \) defined in (4.4) below.

\textbf{Definition 4.3.} We call \( \overline{\alpha} \) the \textit{mate} of \( \alpha \), in the notation of Lemma 4.2.
The 2-morphisms in this lemma are illustrated by the following diagrams.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C' \leftarrow F \downarrow \alpha \rightarrow C \\
\downarrow \beta \rightarrow D' \leftarrow G \\
C' \leftarrow F \downarrow \alpha \rightarrow C \\
\downarrow \beta \rightarrow D' \leftarrow G
\end{array}
\end{array}
\end{array}
\end{array}
\]

Proof. Let \( \alpha : L'G \Rightarrow FL \) be given. The left diagram above can be expanded to the following diagram.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C' \leftarrow F \downarrow \alpha \rightarrow C \\
\downarrow \beta \rightarrow D' \leftarrow G \\
\downarrow \beta \rightarrow D' \leftarrow G \\
C' \leftarrow F \downarrow \alpha \rightarrow C \\
\downarrow \beta \rightarrow D' \leftarrow G
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Define the 2-morphism \( \overline{\alpha} \) as the composition

\[
\alpha = \left( R'F \varepsilon \right) \circ \left( R'GR \right) \circ \left( \eta'GR \right): \quad GR \leftarrow \left( R'F \right) \Rightarrow \left( R'\varepsilon \right)
\]

Similarly, we associate to each 2-morphism \( \beta : GR \Rightarrow R'F \) the following composition of 2-morphisms

\[
\beta = \left( \varepsilon'GR \right) \circ \left( \eta'GR \right): \quad GR \leftarrow \left( \varepsilon'F \right) \Rightarrow \left( \varepsilon \right)
\]

The triangle identities satisfied by the adjunction data show that this map is inverse to the map \( \alpha \mapsto \overline{\alpha} \). \( \square \)

Example 4.4. Given a cartesian diagram (3.1) of algebraic stacks consider the obvious 2-isomorphism \( \alpha : g^*u^* \Rightarrow v^*f^* \). Its mate \( \overline{\alpha} : u^*f_* \Rightarrow g_*v^* \) (with respect to the adjunctions \( (f^*, f_*) \) and \( (g^*, g_*) \)) is the base change transformation (3.3) for \( f_* \).

Assume that \( f \) is concentrated. Then the mate \( \overline{\alpha} \) has an inverse \( \beta := \alpha^{-1} \), and we have adjunctions \( (f_*, f^\times) \) and \( (g_*, g^\times) \). We obtain the mate \( \beta : v^*f^\times \Rightarrow g^\times u^* \), which is the base change transformation (3.5) for \( f^\times \).

Example 4.5. Let \( u : X' \to X \) be a morphism of algebraic stacks and let \( \mathcal{K} \in \mathcal{D}_{qc}(X) \). Consider the obvious 2-isomorphism

\[
\lambda : u^*\mathcal{K} \otimes u^*(-) \cong u^*(\mathcal{K} \otimes -)
\]

between triangulated functors \( \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(X') \). Form its mate

\[
u^*\text{Hom}(\mathcal{K}, -) \Rightarrow \text{Hom}(u^*\mathcal{K}, u^*(-)) : \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(X')
\]

with respect to the adjunctions \( (\mathcal{K} \otimes -, \text{Hom}(\mathcal{K}, -)) \) and \( (u^*\mathcal{K} \otimes -, \text{Hom}(u^*\mathcal{K}, -)) \).

Assume that \( \mathcal{K} \) is perfect. Then it is straightforward to verify that the mate (4.7) is a 2-isomorphism. Indeed, this follows by a standard reduction to the affine case, using the fact that \( \text{Hom}(\mathcal{K}, -) \) coincides with \( R\text{Hom}_{\text{aff}}(\mathcal{K}, -) \) on \( D(X_{\text{aff}}, \mathcal{O}_X) \) when \( \mathcal{K} \) is perfect (see [HR17, Lemma 4.3]). Alternatively, we can use the adjunctions \( (\mathcal{K} \otimes -, \mathcal{K}^\vee \otimes -) \) and \( (u^*\mathcal{K} \otimes -, (u^*\mathcal{K})^\vee \otimes -) \) and obtain as the mate of \( \lambda \) the 2-isomorphism

\[
u : u^*(\mathcal{K}^\vee \otimes -) \cong (u^*\mathcal{K})^\vee \otimes u^*(-).
\]
The following technical lemma on mates is used in the proofs of Proposition 4.12 and Lemma 4.25. The reader may ignore it for now.

**Lemma 4.6.** In Setting 4.1 let \( \alpha : L'G \Rightarrow FL \) be a 2-morphism and let \( \overline{\alpha} : GR \Rightarrow R'F \) be its mate. Then the diagrams

\[
\begin{array}{c}
\alpha R \\
L'GR \\
L'FLR
\end{array}
\]

are commutative.

**Proof.** The upper trapezoids and the left rhombi are obviously commutative. The triangles on the right are commutative by the triangle identities of the adjunctions. The lower trapezoids are commutative by the definition of the bijection in Lemma 4.2. \( \square \)

The following result explains how taking mates is compatible with compositions.

**Lemma 4.7** ([KS74, Proposition 2.2]). Let

\[
\begin{array}{c}
C' \xleftarrow{F} C \\
L' \xrightarrow{\alpha} L \\
D' \xleftarrow{G} D \\
M' \xrightarrow{\beta} M
\end{array}
\]

be diagrams in a 2-category and let \((L', R'), (L, R), (M', P'), (M, P)\) be adjunctions. Then the mate of \((\alpha M) \circ (L' \beta)\) is \((P' \overline{\alpha}) \circ (\overline{\beta} R)\).

In particular, if \(\alpha, \overline{\alpha}, \beta, \overline{\beta}\) are 2-isomorphisms, then \((\alpha M) \circ (L' \beta)\) and its mate are 2-isomorphisms.

**Proof.** See [KS74, Proposition 2.2]. \( \square \)
4.2. Fourier–Mukai transforms and flat base change.

**Setting 4.8.** Let $X$ and $Y$ be algebraic stacks over a base algebraic stack $S$. Let $\Phi = (K, p, q, K): X \to Y$ be a Fourier–Mukai transform over $S$. Let $u: S' \to S$ be a flat morphism of algebraic stacks. The following diagram with cartesian squares is obtained from $X \xleftarrow{p} K \xrightarrow{q} Y$ by base change along $u$.

\[
\begin{array}{ccc}
X' & \xleftarrow{p'} K' & \xrightarrow{q'} Y' \\
\downarrow{f} & \downarrow{h} & \downarrow{g} \\
X & \xleftarrow{P} K & \xrightarrow{Q} Y
\end{array}
\]

Then $\Phi' = (K', p', q', \mathcal{K}' := h^*K): X' \to Y'$ is the base change of $\Phi$ along $u$. We denote the right adjoint functors of the Fourier–Mukai transforms $\Phi: \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(Y)$ and $\Phi': \mathcal{D}_{qc}(X') \to \mathcal{D}_{qc}(Y')$ by $\Psi$ and $\Psi'$, respectively (cf. (3.10)).

In Setting 4.8 we obtain the following diagram where the 2-isomorphisms $\alpha$ and $\lambda$ are the obvious ones, and the 2-isomorphism $\beta$ is the inverse of the base change transformation for $q_*$ along the flat morphism $g$. Note that this base change transformation is indeed a 2-isomorphism by Proposition 3.1 since $q_*$ is concentrated.

\[
\begin{array}{ccc}
\mathcal{D}_{qc}(Y') & \xleftarrow{g^*} & \mathcal{D}_{qc}(Y) \\
\downarrow{q'} & \sim & \downarrow{q_*} \\
\mathcal{D}_{qc}(K') & \xleftarrow{h^*} & \mathcal{D}_{qc}(K) \\
\downarrow{\mathcal{K}' \otimes -} & \sim & \downarrow{\mathcal{K} \otimes -} \\
\mathcal{D}_{qc}(X') & \xleftarrow{f^*} & \mathcal{D}_{qc}(X)
\end{array}
\]

The vertical compositions of the two columns in this diagram are the Fourier–Mukai transforms $\Phi$ and $\Phi'$.

**Proposition 4.9** (Flat base change for Fourier–Mukai transforms). In Setting 4.8 consider the 2-isomorphism

\[
\Phi' (f^*) \cong g^* (\Phi).
\]

obtained from the three 2-isomorphisms in diagram (4.13). Then its mate

\[
f^* (\Psi) \cong \Psi' (g^*)
\]

is a 2-isomorphism as well.

**Proof.** The mate of $\alpha$ is a 2-isomorphism since $f$ is flat and $p$ is concentrated (Proposition 3.1 and Example 4.4). The mate of $\lambda$ is a 2-isomorphism by Example 4.5 since $\mathcal{K}$ is perfect. The mate of $\beta$ is a 2-isomorphism since it is the base change
transformation for $q^*$ (see Example 4.4) which is a 2-isomorphism since $q$ satisfies property $P3$ from Definition 3.2. Hence Lemma 4.7 yields the statement. □

4.3. Conservative descent for fully faithfulness.

**Definition 4.10.** A functor $G: \mathcal{D} \to \mathcal{D}'$ between categories is called conservative if it reflects isomorphisms: if $d: D \to D'$ is any morphism in $\mathcal{D}$ such that $G(d)$ is an isomorphism, then $d$ is an isomorphism.

**Example 4.11.** A triangulated functor is conservative if and only if it reflects zero objects. For example, if $g: Y' \to Y$ is a faithfully flat morphism of algebraic stacks, then $g^*: D\text{qc}(Y) \to D\text{qc}(Y')$ is conservative.

**Proposition 4.12** (Conservative descent for fully faithfulness). Assume that we are in Setting 4.1 in the 2-category of categories (resp. triangulated categories). Let $\alpha: L'G \Rightarrow FL$ be a 2-isomorphism and assume that its mate $\alpha: GR \Rightarrow R'F$ is a 2-isomorphism as well.

(a) If the functor $G$ is conservative and the functor $L'$ is fully faithful, then $L$ is fully faithful.

(b) If $F$ is conservative and $R'$ is fully faithful, then $R$ is fully faithful.

In particular, we get conservative descent for equivalences: If both $F$ and $G$ are conservative and $(L', R')$ is an equivalence, then so is $(L, R)$.

**Proof.** (a) Observe that the commutative diagram (4.10) in Lemma 4.6 yields the equality $(\alpha L) \circ (G\eta) = (R'\alpha) \circ (\eta'G)$ of 2-morphisms, so that $G\eta$ is a 2-isomorphism if and only if $\eta'G$ is a 2-isomorphism. Now assume that $L'$ is full and faithful. This is equivalent to the condition that $\eta'$ is a 2-isomorphism. Hence $\eta'G$ and $G\eta$ are 2-isomorphisms. Since $G$ is conservative, this means that $\eta$ is a 2-isomorphism, i.e., $L$ is fully faithful.

(b) This follows similarly from the equality $(F\varepsilon) \circ (\alpha R) = (\varepsilon'F) \circ (L'\tau)$ obtained from the commutative diagram (4.9) in Lemma 4.6. □

**Proposition 4.13** (Faithfully flat descent for fully faithfulness of Fourier–Mukai transforms). In Setting 4.8 assume that the flat base change morphism $u: S' \to S$ is surjective, i.e., faithfully flat. Then the Fourier–Mukai transform $\Phi$ is fully faithful if its base change $\Phi'$ is fully faithful.

**Proof.** Thanks to Proposition 4.9 and Example 4.11, Proposition 4.12.(a) applies. □

4.4. Comonads and idempotent comonads. We continue to work in our fixed 2-category. Standard references for monads and comonads are [ML98, Section VI], [Bor94, Section 4] and [Str72].

**Definition 4.14.** A comonad is a quadruple $(C, S, \varepsilon, \delta)$ where $C$ is an object, $S: C \to C$ is a 1-morphism, and counit $\varepsilon: S \Rightarrow \text{id}_C$ and comultiplication $\delta: S \Rightarrow S^2$ are 2-morphisms such that the two diagrams

\[
\begin{array}{ccc}
S & \xrightarrow{\delta} & S^2 \\
\downarrow \varepsilon & & \downarrow \text{id} \\
S^2 & \xrightarrow{\text{id}} & S \\
\end{array}
\]

\[
\begin{array}{ccc}
S & \xrightarrow{\delta} & S^2 \\
\downarrow & & \downarrow S\delta \\
S^2 & \xrightarrow{\delta S} & S^3 \\
\end{array}
\]
are commutative.

A comonad \((C,S,\varepsilon,\delta)\) is idempotent if its comultiplication \(\delta\) is a 2-isomorphism (see Remark 4.19 below for a shorter equivalent definition).

By abuse of notation we often say that \(S\) is a comonad on \(C\) or just that \(S\) or \((S,\varepsilon,\delta)\) is a comonad.

**Example 4.15.** Let \((L: D \rightleftarrows C: R, \eta, \varepsilon)\) be an adjunction where \(\eta: \text{id}_D \Rightarrow RL\) is the unit and \(\varepsilon: LR \Rightarrow \text{id}_C\) is the counit. Then we obtain an associated comonad
\[(S := LR, \varepsilon, \delta := L\eta R)\]
on \(C\). It is idempotent if the unit \(\eta\) is a 2-isomorphism.

If we work in the 2-category of (triangulated) categories, then the functor \(L\) is fully faithful if and only if \(\eta\) is a 2-isomorphism. In this case, the essential images of \(L\) and \(S = LR\) coincide, i.e., \(\text{Im} L = \text{Im} S\), because \(L\eta: L \Rightarrow LRL = SL\) is a 2-isomorphism. In particular, any fully faithful left adjoint functor \(L\) (which is part of an adjunction as above) gives rise to an idempotent comonad with the same essential image.

**Example 4.16.** Let \(\Phi = (K,p,q,K): X \to Y\) be a relative Fourier–Mukai transform. Assume that the induced functor \(\Phi: D_{qc}(X) \to D_{qc}(Y)\) is fully faithful, and denote its right adjoint by \(\Psi\). Then
\[(4.17) \quad S\Phi := \Phi \Psi\]
is an idempotent comonad on \(D_{qc}(Y)\) whose essential image is the essential image of \(\Phi\). This is a special case of Example 4.15.

**Lemma 4.17.** A comonad \((S,\varepsilon,\delta)\) is idempotent if and only if \(\varepsilon S\) (resp. \(S\varepsilon\)) is a 2-isomorphism. If these conditions are satisfied then \(\varepsilon S = S\varepsilon = \delta^{-1}\) and \(\delta S = S\delta\).

*Proof.* This follows from the defining commutative diagrams in (4.16). \(\square\)

**Lemma 4.18.** Let \(S: C \to C\) be a 1-morphism and \(\varepsilon: S \Rightarrow \text{id}_C\) a 2-morphism. If \(S\varepsilon\) and \(\varepsilon S\) are equal 2-isomorphisms, then there is a unique \(\delta: S \Rightarrow S^2\), namely \(\delta = (S\varepsilon)^{-1} = (\varepsilon S)^{-1}\), such that \((C,S,\varepsilon,\delta)\) is a comonad. This comonad is idempotent.

*Proof.* The left diagram in (4.16) shows that we have to put \(\delta = (S\varepsilon)^{-1} = (\varepsilon S)^{-1}\). But then \(S\delta\) and \(\delta S\) are invertible and \((S\delta)^{-1} = S\delta^{-1} = S\varepsilon S = \delta^{-1} S = (\delta S)^{-1}\). This implies that the right diagram in (4.16) is commutative. \(\square\)

**Remark 4.19** (Alternative definition of an idempotent comonad). Giving an idempotent comonad \((C,S,\varepsilon,\delta)\) is the same thing as giving a triple \((C,S,\varepsilon)\) where \(S: C \to C\) is a 1-morphism and \(\varepsilon: S \Rightarrow \text{id}_C\) is a 2-morphism such that \(S\varepsilon = \varepsilon S\) is a 2-isomorphism. This is obvious from Lemmas 4.17 and 4.18. Hence it suffices to write \((C,S,\varepsilon)\) when referring to an idempotent comonad. Again, we often just write \((S,\varepsilon)\) or \(S\) for an idempotent comonad.

**Remark 4.20** (Idempotent comonads versus colocalizations). An endofunctor \(S: C \to C\) of a (triangulated) category is a colocalization functor in the sense of [Kra10, 2.4, 2.8] if there exists a 2-morphism \(\varepsilon: S \Rightarrow \text{id}_C\) such that \((S,\varepsilon)\) is an idempotent comonad (use Remark 4.19); if \(\varepsilon': S \Rightarrow \text{id}_C\) is another 2-morphism turning \(S\) into an idempotent comonad, there is a unique 2-automorphism \(\mu: S \cong S\) such that \(\varepsilon = \varepsilon' \circ \mu\) (see [Kra10, Remark 2.5.5.(1)] for the corresponding statement for colocalizations). In particular, the difference between the definition of an idempotent
comonad and a colocalization functor consists in making \( \varepsilon \) part of the datum or not.

**Remark 4.21.** If \((S, \varepsilon)\) is an idempotent comonad on a (triangulated) category \( \mathcal{C} \), then the inclusion functor \( \text{Im} \, S \to \mathcal{C} \) admits a right adjoint, e.g., the functor \( \mathcal{C} \to \text{Im} \, S \) induced by \( S \).

### 4.5. Compatibilities of idempotent comonads.

**Definition 4.22.** Let \((\mathcal{C}, S, \varepsilon)\) and \((\mathcal{C}', S', \varepsilon')\) be idempotent comonads and let \( F: \mathcal{C} \to \mathcal{C}' \) be a 1-morphism. An \( F \)-compatibility from \( S \) to \( S' \), written \( \sigma: S \xrightarrow{F} S' \), is a 2-isomorphism \( \sigma: F \circ S \Rightarrow S' \circ F \) such that the diagram

\[
\begin{array}{ccc}
FS & \xrightarrow{\sigma} & S'F \\
\downarrow \varepsilon & & \downarrow \varepsilon' \\
F & \equiv & F
\end{array}
\]

commutes. If there exists at least one \( F \)-compatibility from \( S \) to \( S' \), we say that \( S \) and \( S' \) are compatible with respect to \( F \).

**Remark 4.23.** Our notion of an \( F \)-compatibility is closely related to the notion of a morphism of comonads as defined in [Str72]: a morphism \((F, \sigma): (\mathcal{C}, S, \varepsilon, \delta) \to (\mathcal{C}', S', \varepsilon', \delta')\) of comonads consists of a 1-morphism \( F: \mathcal{C} \to \mathcal{C}' \) and a 2-morphism \( \sigma: FS \Rightarrow S'F \) such that two obvious diagrams are commutative.

If we assume that \( S \) and \( S' \) are idempotent comonads and fix a 1-morphism \( F: \mathcal{C} \to \mathcal{C}' \), then the set of \( F \)-compatibilities coincides precisely with the set of 2-isomorphisms \( \sigma: FS \Rightarrow S'F \) such that \((F, \sigma)\) is a morphism of comonads. The proof of this result is not difficult and uses Lemma 4.18. We omit the details since we do not use arbitrary morphisms of comonads in the following.

### 4.6. Monads and compatibilities.

All results we have proven for (idempotent) comonads have analogs for (idempotent) monads. We quickly review what we need.

A **monad** is a quadruple \((\mathcal{D}, T, \eta, \mu)\) where \( \mathcal{D} \) is an object, \( T: \mathcal{D} \to \mathcal{D} \) is a 1-morphism, and unit \( \eta: \text{id}_\mathcal{D} \Rightarrow T \) and multiplication \( \mu: T^2 \Rightarrow T \) are 2-isomorphisms making the obvious two diagrams commutative; the shape of these two diagrams is obtained from the two diagrams in (4.16) by reversing the direction of all 2-morphisms. Such a monad is idempotent if its multiplication \( \mu \) is a 2-isomorphism.

An idempotent monad is equivalently given by a triple \((\mathcal{D}, T, \eta)\) where \( T\eta \) and \( \eta T \) are equal and invertible (by the monadic version of Remark 4.19).

**Example 4.24** (cf. Example 4.15). Let \((L: \mathcal{D} \rightleftarrows \mathcal{C}: R, \eta, \varepsilon)\) be an adjunction where \( \eta: \text{id}_\mathcal{D} \Rightarrow RL \) is the unit and \( \varepsilon: LR \Rightarrow \text{id}_\mathcal{C} \) is the counit. Then we obtain an associated monad

\[
(T := RL, \eta, \mu := R\varepsilon L)
\]

on \( \mathcal{C} \). It is idempotent if the counit \( \varepsilon \) is a 2-isomorphism.

Given idempotent monads \((\mathcal{D}, T, \eta)\) and \((\mathcal{D}', T', \eta')\) and a 1-morphism \( G: \mathcal{D} \to \mathcal{D}' \), a **\( G \)-compatibility** from \( T \) to \( T' \), written \( \tau: T \xrightarrow{G} T' \), is a 2-isomorphism \( \tau: GT \Rightarrow \)
such that the diagram
\[
\begin{array}{ccc}
GT & \xrightarrow{\tau} & T'G \\
\downarrow{G\eta} & & \downarrow{G\eta'} \\
G & = & G
\end{array}
\]
is commutative. The direction of \(\tau\) is chosen with regard to our later applications.

4.7. **Compatibilities from mates.**

**Lemma 4.25.** In Setting 4.1 assume that the units \(\eta\) and \(\eta'\) of both adjunctions \((L, R, \eta, \varepsilon)\) and \((L', R', \eta', \varepsilon')\) are 2-isomorphisms. Assume that there is a 2-isomorphism \(\alpha: L'G \Rightarrow FL\) whose mate \(\alpha\) is a 2-isomorphism. Then
\[
(L'R'F\varepsilon) \circ (\varepsilon'FLR)^{-1}: (S = LR, \varepsilon) \xrightarrow{\sim} (S' = L'R', \varepsilon')
\]
is an \(F\)-compatibility between the associated idempotent comonads.

**Proof.** Since \(\eta\) and \(\eta'\) are 2-isomorphisms, the associated comonads \(S\) and \(S'\) are idempotent by Example 4.15. Consider the commutative diagram (4.9) of Lemma 4.6. Invertibility of \(\alpha, \alpha\) and \(\eta'G\) shows that the two 2-morphisms \(\varepsilon'FLR\) and \(L'R'F\varepsilon\) are invertible. Hence \((L'R'F\varepsilon) \circ (\varepsilon'FLR)^{-1}\) is a 2-isomorphism whose composition with \(\varepsilon'F\) is \(F\varepsilon\). This just means that it is an \(F\)-compatibility from the idempotent comonad \(S\) to the idempotent comonad \(S'\). \(\square\)

**Proposition 4.26.** In Setting 4.8 assume that the Fourier–Mukai transform \(\Phi\) and its base change \(\Phi'\) are fully faithful. Then the associated idempotent comonads \(S_{\Phi}\) on \(\text{D}_{qc}(Y)\) and \(S_{\Phi'}\) on \(\text{D}_{qc}(Y')\) (see Example 4.16) are compatible with respect to \(g^* : \text{D}_{qc}(Y) \rightarrow \text{D}_{qc}(Y')\).

**Proof.** This is a consequence of Proposition 4.9 and Lemma 4.25. \(\square\)

5. **Semi-orthogonal decompositions and descent**

The main goal of this section is the conservative descent Theorem 5.16.

5.1. **Semi-orthogonal decompositions.** General results on admissible subcategories and semi-orthogonal decompositions of triangulated categories can be found in [BK89] and [LS16, Appendix A]. We recall the basic definitions here.

**Definition 5.1.** Let \(\mathcal{T}\) be a triangulated category. A right (resp. left) admissible subcategory of \(\mathcal{T}\) is a strictly full triangulated subcategory \(\mathcal{U}\) of \(\mathcal{T}\) such that the inclusion functor \(\mathcal{U} \rightarrow \mathcal{T}\) admits a right (resp. left) adjoint. An admissible subcategory is a subcategory that is both left and right admissible.

We remind the reader that an adjoint functor of a triangulated functor is triangulated (in a canonical way if the adjunction is fixed), see e.g. [SP17, Tag 0A8D].

**Example 5.2.** Let \((S, \varepsilon)\) be an idempotent comonad on a triangulated category \(\mathcal{T}\). Then \(\text{Im} S\) is a right admissible subcategory of \(\mathcal{T}\), by Remark 4.21. Any right admissible subcategory is of this form (by Example 4.15). Similarly, the essential image of an idempotent monad is a left admissible subcategory, and any left admissible subcategory is of this form.
Definition 5.3. Let $\mathcal{T}$ be a triangulated category. A sequence $\mathcal{T}_1, \ldots, \mathcal{T}_n$ of strictly full triangulated subcategories of $\mathcal{T}$ is called semi-orthogonal if $\mathcal{T}(A_i, A_j) = 0$ for all objects $A_i \in \mathcal{T}_i$ and $A_j \in \mathcal{T}_j$ whenever $i > j$. It is called full (in $\mathcal{T}$) if $\mathcal{T}$ coincides with the smallest strictly full triangulated subcategory of $\mathcal{T}$ that contains all the categories $\mathcal{T}_i$. A semi-orthogonal decomposition of $\mathcal{T}$ is a full semi-orthogonal sequence $\mathcal{T}_1, \ldots, \mathcal{T}_n$ and is denoted as $\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle$.

5.2. Complementary idempotent comonads and monads. In the rest of this section, we always work in the 2-category of triangulated categories. In particular, if we say that $(\mathcal{T}, S, \varepsilon, \delta)$ is a comonad then $\mathcal{T}$ is a triangulated category, $S$ is a triangulated functor and $\varepsilon$ and $\delta$ are morphisms of triangulated functors.

Definition 5.4. Let $\mathcal{T}$ be a triangulated category. We say that an idempotent comonad $(S, \varepsilon)$ and an idempotent monad $(T, \eta)$ on $\mathcal{T}$ are complementary if for any object $A \in \mathcal{T}$ there is a morphism $\partial_A$ such that

$$SA \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} TA \xrightarrow{\partial_A} \Sigma SA$$

is a triangle. We also just say that $T$ is complementary to $S$ or that $S$ is complementary to $T$.

Lemma 5.5. If an idempotent comonad $S$ and an idempotent monad $T$ on a triangulated category $\mathcal{T}$ are complementary, then

$$\text{(Im } S)^\perp = \text{Im } T = \ker S,$$

$$\text{Im } T = \text{Im } S = \ker T.$$

In particular, Im $S$ and Im $T$ are closed under direct summands in $\mathcal{T}$. Moreover, the morphism $\partial_A$ in triangle (5.1) is unique, and mapping $A$ to the triangle (5.1) extends uniquely to a functor from $\mathcal{T}$ to the category of triangles in $\mathcal{T}$.

Proof. We claim that $\mathcal{T}(\text{Im } S, \text{Im } T) = 0$. Let $f: SA \to TB$ be a morphism where $A$ and $B$ are objects of $\mathcal{T}$. Consider the following commutative diagram

$$
\begin{array}{ccc}
S^2 A & \xrightarrow{\varepsilon_{SA}} & SA & \xrightarrow{\eta_{SA}}& TSA \\
\downarrow{f} & & \downarrow{Tf} & & \downarrow{Tf} \\
TB & \xrightarrow{\eta_{TB}} & TTB.
\end{array}
$$

Its upper row can be completed to a triangle, and $\varepsilon_{SA}$ and $\eta_{TB}$ are isomorphisms. Hence $TSA = 0$ and $f = 0$. This proves the claim, and the four equalities in the lemma follow immediately. Uniqueness of $\partial_A$ follows from [BBD82, Corollaire 1.1.10]. Functoriality of the triangle follows from [BBD82, Corollaire 1.1.9].

Example 5.6. Given an idempotent comonad $S$ and a complementary idempotent monad $T$ on a triangulated category $\mathcal{T}$, there is semi-orthogonal decomposition $\mathcal{T} = \langle \text{Im } T, \text{Im } S \rangle$. Any semi-orthogonal decomposition with two components arises in this way.

Remark 5.7 (Existence of complementary comonads and monads). Any idempotent comonad (resp. monad) on a triangulated category $\mathcal{T}$ admits a complementary idempotent monad (resp. comonad). This follows from well-known arguments, e.g., see [Kra10, Proposition 4.12.1] using Remark 4.20 and the corresponding statement.
for idempotent monads. Given an idempotent comonad \((S, \varepsilon)\), the idea is to complete for each object \(A \in \mathcal{T}\) the morphism \(\varepsilon_A\) to a triangle (5.1), and to show that this extends to morphisms and in fact defines an idempotent monad \((T, \eta)\) in the 2-category of triangulated categories.

**Remark 5.8** (Uniqueness of complementary comonads and monads). If an idempotent comonad \((S, \varepsilon)\) on \(\mathcal{T}\) admits two complementary idempotent monads \((T, \eta)\) and \((T', \eta')\), then there is a unique 2-isomorphism \(\nu: T \Rightarrow T'\) such \(\nu \circ \eta = \eta'\). This follows from [BBD82, Corollaire 1.1.9] or Proposition 5.13 below for \(\mathcal{T} = \mathcal{T}'\) and \(F = \text{id}_{\mathcal{T}}\). The dual statement is also true.

**Remark 5.9.** Given an idempotent comonad \((S, \varepsilon)\) on \(\mathcal{T}\) we often use the notation \((S^\perp, \bar{\eta})\) for a complementary monad. Such a complementary monad always exists (Remark 5.7) and is as unique as possible (Remark 5.8). The notation \(S^\perp\) is motivated by the equality \((\text{Im } S)^\perp = \text{Im}(S^\perp)\), see Lemma 5.5.

### 5.3. Semi-orthogonal decompositions and idempotent comonads

Given a sequence of idempotent comonads, we reformulate the defining conditions that their essential images form a semi-orthogonal decomposition in terms of vanishing conditions on certain compositions of these comonads and their complementary monads. This reformulation is a key ingredient in the proof of the conservative descent Theorem 5.16 below.

**Proposition 5.10.** Let \(S_1, S_2, \ldots, S_n\) be a sequence of idempotent comonads on a triangulated category \(\mathcal{T}\). Then the following statements are equivalent:

(a) The sequence \(\text{Im } S_1, \ldots, \text{Im } S_n\) of essential images is semi-orthogonal.

(b) The composition \(S_i S_j\) vanishes for all \(i > j\).

**Proof.** It is certainly enough to prove the statement in case \(n = 2\). So we need to prove that \(\mathcal{T}(\text{Im } S_2, \text{Im } S_1) = 0\) if and only if \(S_2 S_1 = 0\).

The condition \(\mathcal{T}(\text{Im } S_2, \text{Im } S_1) = 0\) is equivalent to \(\text{Im } S_1 \subset (\text{Im } S_2)^\perp\). Lemma 5.5 shows \((\text{Im } S_2)^\perp = \text{ker } S_2\) since \(S_2\) has a complementary monad (see Remark 5.7). Finally, the condition \(S_2 S_1 = 0\) is equivalent to \(\text{Im } S_1 \subset \text{ker } S_2\). □

**Proposition 5.11.** Let \(S_1, S_2, \ldots, S_n\) be a sequence of idempotent comonads on a triangulated category \(\mathcal{T}\), with complementary idempotent monads \(S_1^\perp, \ldots, S_n^\perp\). Assume that the sequence \(\text{Im } S_1, \ldots, \text{Im } S_n\) is semi-orthogonal. Then the following statements are equivalent:

(a) The sequence \(\text{Im } S_1, \ldots, \text{Im } S_n\) is full, i.e., it forms a semi-orthogonal decomposition of \(\mathcal{T}\).

(b) The composition \(S_1^\perp S_2^\perp \cdots S_n^\perp\) vanishes on \(\mathcal{T}\).

**Proof.** Let \(\mathcal{T}'\) denote the triangulated hull of \(\text{Im } S_1, \ldots, \text{Im } S_n\) in \(\mathcal{T}\).

Assume that \(S_1^\perp S_2^\perp \cdots S_n^\perp\) vanishes. We want to show that any object \(A\) in \(\mathcal{T}\) lies in \(\mathcal{T}'\). Diagram (5.7) below illustrates the following proof in case \(n = 3\). By assumption, we have \(S_1 S_2 \cdots S_n A = 0\), and this object trivially lies in \(\mathcal{T}'\). By induction over \(i \in \{1, \ldots, n\}\) assume that \(S_1^\perp \cdots S_i^\perp A\) lies in \(\mathcal{T}'\) and consider the triangle

\[
S_i S_{i+1}^\perp \cdots S_n^\perp A \xrightarrow{\varepsilon_{i+1}} S_i^\perp S_{i+1}^\perp \cdots S_n^\perp A \xrightarrow{\eta_i} S_i^\perp S_{i+1}^\perp \cdots S_n^\perp A \rightarrow .
\]

Since the first object lies in \(\text{Im } S_i \subset \mathcal{T}'\), we deduce that also \(S_{i+1}^\perp \cdots S_n^\perp A\) lies in \(\mathcal{T}'\). So eventually we get \(A \in \mathcal{T}'\). This shows that \(\mathcal{T}' = \mathcal{T}\), i.e., the sequence
Im \( S_1, \ldots, \) Im \( S_n \) is full. Note that semi-orthogonality was in fact not used for this implication.

Conversely, assume that \( \mathcal{T} = \mathcal{T}' \). We need to show \( S_1^\perp \cdots \cdots S_n^\perp A = 0 \) for any object \( A \in \mathcal{T} \). It is enough to show that \( \mathcal{T}(B, S_1^\perp \cdots \cdots S_n^\perp A) = 0 \) for any \( j \in \{1, \ldots, n\} \) and any \( B \in \text{Im} \( S_j \) \). Fix such \( j \) and \( B \).

We prove that \( \mathcal{T}(B, S_1^\perp \cdots \cdots S_n^\perp A) = 0 \) by descending induction over \( i \in \{1, \ldots, j\} \).

By induction, this proves what we need.

\[ (5.6) \quad 0 = \mathcal{T}(B, S_{i+1}^\perp \cdots \cdots S_n^\perp A) \sim \mathcal{T}(B, S_i^\perp S_{i+1}^\perp \cdots \cdots S_n^\perp A). \]

Example 5.12. The following diagram illustrates the first argument of the above proof in case \( n = 3 \). It shows that any object \( A \) can be written as an iterated extension of an object of Im \( S_1 \) by an object of Im \( S_2 \) by an object of Im \( S_3 \).

(5.7) \[ \begin{array}{c}
A \xrightarrow{\eta_3} S_3^\perp A \\
\downarrow \varepsilon_3 \\
S_3 A
\end{array} \quad \begin{array}{c}
S_2^\perp A \xrightarrow{\eta_2} S_2^\perp S_3^\perp A \\
\downarrow \varepsilon_2 \\
S_2 S_3^\perp A
\end{array} \quad \begin{array}{c}
S_1^\perp S_2^\perp S_3^\perp A \\
\downarrow \varepsilon_1 \sim \\
0
\end{array} \quad \begin{array}{c}
S_1 S_2 S_3^\perp A \\
\downarrow \varepsilon_1 \\
0
\end{array} \]

5.4. Compatibilities and complementary comonads and monads.

Proposition 5.13. Let \((S, \varepsilon)\) and \((S', \varepsilon')\) be idempotent comonads on triangulated categories \( \mathcal{T} \) and \( \mathcal{T}' \), and let \((S^\perp, \eta)\) and \((S'^\perp, \eta')\) be complementary idempotent monads. Let \( F: \mathcal{T} \to \mathcal{T}' \) be a triangulated functor. Given any \( F \)-compatibility

\[ (5.8) \quad \sigma: S \xrightarrow{F} S' \]

of comonads there is a unique \( F \)-compatibility

\[ (5.9) \quad \sigma^\perp: S^\perp \xrightarrow{F} S'^\perp \]

of monads such that

\[ (5.10) \quad \begin{array}{c}
FS \xrightarrow{F \varepsilon} F \xrightarrow{F \eta} FS^\perp \xrightarrow{\Sigma F} S' F \xrightarrow{\varepsilon' F} F \xrightarrow{\eta' F} S'^\perp F \xrightarrow{\Sigma} \Sigma F S
\end{array} \]

is a functorial isomorphism of triangles, i.e. plugging in any object \( A \in \mathcal{T} \) yields an isomorphism of triangles in \( \mathcal{T}' \), and these morphisms are compatible with morphisms in \( \mathcal{T} \).

Remark 5.14. The map \( \sigma \mapsto \sigma^\perp \) defines in fact a bijection between the set of \( F \)-compatibilities from \( S \) to \( S' \) and the set of \( F \)-compatibilities from \( S^\perp \) to \( S'^\perp \).
Proof. Let $\sigma$ be an $F$-compatibility as above. For any $A \in \mathcal{T}$ complete the partial morphism
\[
\begin{array}{ccccccc}
FSA & \xrightarrow{F \varepsilon_A} & FA & \xrightarrow{F \eta_A} & FS^\perp A & \xrightarrow{} & \Sigma FSA \\
\sigma_A \downarrow \sim & & & & \tilde{\sigma}_A \downarrow \sim & & \Sigma \sigma_A \downarrow \sim \\
S'FA & \xrightarrow{\varepsilon'_F A} & FA & \xrightarrow{\eta'_F A} & S'^\perp FA & \xrightarrow{} & \Sigma S'FA
\end{array}
\]
of triangles by the dotted arrow to an (iso)morphism of triangles. This dotted arrow is already uniquely specified by the requirement that the square in the middle is commutative, by [BBD82, Corollaire 1.1.10], since the object $S'^\perp FA$ lies in $\text{Im}(S'^\perp) = (\text{Im} S')^\perp$ (by Lemma 5.5), and the objects $FSA$ and $\Sigma FSA$ lie in $\text{Im} S'$ since $\sigma_A$ is an isomorphism. Similarly, the morphisms $(\sigma_A, \text{id}_{F A}, \tilde{\sigma}_A)$ of triangles, for $A \in \mathcal{T}$, are easily seen to be compatible with morphisms $A \to A'$ in $\mathcal{T}$. In particular, $\sigma^\perp : FS^\perp \xrightarrow{\sim} S'^\perp F$ is a 2-isomorphism between functors, and it is easy to see that it is compatible with suspensions, i.e., it is a 2-isomorphism between triangulated functors. This proves the proposition. \qed

Example 5.15. The conclusion of Proposition 4.26 can be extended: the complementary idempotent monads $S^\perp_\Phi$ and $S^\perp_\Phi'$ are also $g^*$-compatible, by Proposition 5.13.

5.5. Conservative descent.

Theorem 5.16 (Conservative descent). Let $F : \mathcal{T} \to \mathcal{T}'$ be a conservative triangulated functor. Let $S_1, \ldots, S_n$ and $S'_1, \ldots, S'_n$ be sequences of idempotent comonads on $\mathcal{T}$ and $\mathcal{T}'$, respectively, and assume that $S_i$ and $S'_i$ are compatible with respect to $F$ for each $i$. If the sequence
\[
\text{Im } S'_1, \ldots, \text{Im } S'_n
\]
of essential images is semi-orthogonal in $\mathcal{T}'$, then so is the sequence
\[
\text{Im } S_1, \ldots, \text{Im } S_n
\]
in $\mathcal{T}$. Both sequences consist of right admissible subcategories of $\mathcal{T}'$ and $\mathcal{T}$, respectively. Moreover, if (5.12) is a semi-orthogonal decomposition of $\mathcal{T}'$, then (5.13) is a semi-orthogonal decomposition of $\mathcal{T}$.

Proof. By Example 5.2, both sequences consist of right admissible subcategories of $\mathcal{T}'$ and $\mathcal{T}$, respectively.

Choose $F$-compatibilities $S_i \xrightarrow{E} S'_i$. They are given by 2-isomorphisms $\sigma_i : FS_i \xrightarrow{\sim} S'_i F$. We obtain 2-isomorphisms
\[
FS_i S_j \xrightarrow{\sigma_i S_j \sim} S'_i FS_j \xrightarrow{S'_i \sigma_j \sim} S'_i S'_j F.
\]
We use Proposition 5.10 twice. Semi-orthogonality of $\text{Im } S'_1, \ldots, \text{Im } S'_n$ gives $S'_i S'_j = 0$ for $i \neq j$ and hence $FS_i S_j = 0$. Since $F$ is conservative we deduce $S_i S_j = 0$ (Example 4.11), so $\text{Im } S_1, \ldots, \text{Im } S_n$ is semi-orthogonal.

Proposition 5.10 provides $F$-compatibilities $S_i^\perp \xrightarrow{E} S'_i^\perp$ given by 2-isomorphisms $\sigma^\perp_i : FS_i^\perp \xrightarrow{\sim} S'_i^\perp F$. We obtain 2-isomorphisms
\[
FS_1^\perp S_2^\perp \ldots S_n^\perp \xrightarrow{\sigma^\perp_1 S^\perp_2 \ldots S^\perp_n \sim} S'_1^\perp FS_2^\perp \ldots S_n^\perp \xrightarrow{\sim} \ldots \xrightarrow{\sim} S'_1^\perp S'_2^\perp \ldots S'_n^\perp F.
\]
Assume that (5.12) is a semi-orthogonal decomposition of \( \mathcal{T} \). Then the expression on the right vanishes by Proposition 5.11. Hence the expression on the left vanishes and we get \( S_1^1 \cdots S_n^1 = 0 \) because \( F \) is conservative. Proposition 5.11 then shows that (5.13) is a semi-orthogonal decomposition of \( \mathcal{T} \) since we already know that it is a semi-orthogonal sequence. □

5.6. **Induced semi-orthogonal decompositions.** Let \( \mathcal{T}' \) be a strictly full triangulated subcategory of a triangulated category \( \mathcal{T} \). Let \((S, \varepsilon)\) be an idempotent comonad on \( \mathcal{T} \) such that \( S \) restricts to a functor \( S' : \mathcal{T}' \to \mathcal{T}' \). Then we obtain by restriction an idempotent comonad \((S', \varepsilon')\) on \( \mathcal{T}' \). Moreover, the universal property of the Verdier quotient \( Q : \mathcal{T} \to \mathcal{T}' \) shows that there is a unique triangulated functor \( S'' : \mathcal{T}' \to \mathcal{T}' \) satisfying \( S''Q = QS \). The 2-morphism \( \varepsilon \) descends similarly and we obtain an idempotent comonad \((S'', \varepsilon'')\) on \( \mathcal{T}' \).

**Lemma 5.17.** Let \( S \) be an idempotent comonad on a triangulated category \( \mathcal{T} \). Let \( \mathcal{T}' \) be a strictly full triangulated subcategory of \( \mathcal{T} \) such that \( S \) preserves \( \mathcal{T}' \). Denote the induced idempotent comonads on \( \mathcal{T}' \) and \( \mathcal{T}/\mathcal{T}' \) by \( S' \) and \( S'' \), respectively. Let \( S^\perp \) be an idempotent monad which is complementary to \( S \). Then \( S^\perp \) restricts to an idempotent monad \((S^\perp)'\) which is complementary to \( S' \) and induces an idempotent monad \((S^\perp)''\) which is complementary to \( S'' \), i.e., \((S^\perp)' = S^\perp_1 \) and \((S^\perp)'' = (S'')^\perp_1 = (S''^\perp)^1_1 \).

**Proof.** For any object \( A \) of \( \mathcal{T} \) we have the triangle \( SA \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} (S^\perp)'A \xrightarrow{\partial} \Sigma S'A' \)

in \( \mathcal{T}' \) for \( A' \in \mathcal{T}' \) and

\[
S''A'' \xrightarrow{\varepsilon_A''} A'' \xrightarrow{\eta_A''} (S^\perp)''A'' \xrightarrow{\partial} \Sigma S''A''
\]

in \( \mathcal{T}/\mathcal{T}' \) for \( A'' \in \mathcal{T}/\mathcal{T}' \). From this, the lemma follows. □

**Proposition 5.18.** Let \( S_1, \ldots, S_n \) be a sequence of idempotent comonads on a triangulated category \( \mathcal{T} \). Let \( \mathcal{T}' \subset \mathcal{T} \) be a strictly full triangulated subcategory such that all \( S_i \) restrict to \( \mathcal{T}' \). Denote the restricted idempotent comonads on \( \mathcal{T}' \) by \( S'_i \) and the induced idempotent comonads on \( \mathcal{T}/\mathcal{T}' \) by \( S''_i \). If the sequence

\[
\text{Im } S_1, \ldots, \text{Im } S_n
\]

of essential images is semi-orthogonal in \( \mathcal{T} \), then so are the sequences

\[
\text{Im } S'_1, \ldots, \text{Im } S'_n
\]

and

\[
\text{Im } S''_1, \ldots, \text{Im } S''_n
\]

in \( \mathcal{T}' \) and \( \mathcal{T}/\mathcal{T}' \), respectively. Moreover, if (5.18) is a semi-orthogonal decomposition of \( \mathcal{T} \), then (5.19) and (5.20) are semi-orthogonal decompositions of \( \mathcal{T}' \) and \( \mathcal{T}/\mathcal{T}' \), respectively.

**Proof.** The equality \( S_iS_j = 0 \) implies the equalities \( S'_iS'_j = 0 \) and \( S''_iS''_j = 0 \), for all \( i, j \). Lemma 5.17 shows that \((S^\perp)' = (S'_1)^\perp \) and \((S^\perp)'' = (S''_1)^\perp \). Therefore the equality \( S_1^\perp S_2^\perp \cdots S_n^\perp = 0 \) implies the equalities \((S'_1)^\perp(S'_2)^\perp \cdots (S'_n)^\perp = 0 \) and \((S''_1)^\perp(S''_2)^\perp \cdots (S''_n)^\perp = 0 \). Now use Propositions 5.10 and 5.11. □
Corollary 5.19. Let $L_i: \mathcal{T}_i \to \mathcal{T}$ be fully faithful triangulated functors admitting right adjoints $R_i$, for $i = 1, \ldots, n$. Let $\mathcal{T}' \subset \mathcal{T}$ and $\mathcal{T}'_i \subset \mathcal{T}_i$ be strictly full triangulated subcategories such that all $L_i$ and all $R_i$ restrict to functors $L'_i: \mathcal{T}'_i \to \mathcal{T}'$ and $R'_i: \mathcal{T}' \to \mathcal{T}'_i$, respectively. Then the $L'_i$ are fully faithful triangulated functors and descend to fully faithful triangulated functors $L''_i: \mathcal{T}'/\mathcal{T}'_i \to \mathcal{T}/\mathcal{T}'$, and all functors $L'_i$ and $L''_i$ admit right adjoints. Moreover, if the sequence

\begin{equation}
\text{Im } L_1, \ldots, \text{Im } L_n
\end{equation}

is semi-orthogonal in $\mathcal{T}$, then so are the sequences

\begin{equation}
\text{Im } L'_1, \ldots, \text{Im } L'_n,
\end{equation}

\begin{equation}
\text{Im } L''_1, \ldots, \text{Im } L''_n
\end{equation}

in $\mathcal{T}'$ and $\mathcal{T}''$, respectively. Furthermore, if (5.21) is a semi-orthogonal decomposition of $\mathcal{T}$, then (5.22) and (5.23) are semi-orthogonal decompositions of $\mathcal{T}'$ and $\mathcal{T}/\mathcal{T}'$, respectively.

Proof. Clearly, $L'_i$ is fully faithful triangulated and has $R'_i$ as a right adjoint. The associated idempotent comonad $S'_i = L'_iR'_i$ satisfies $\text{Im } L'_i = \text{Im } S'_i$ by Example 4.15. The universal property of the Verdier quotient shows that $L_i$ descends to a triangulated functor $L''_i: \mathcal{T}'/\mathcal{T}'_i \to \mathcal{T}/\mathcal{T}'$. More precisely, the adjunction $(L'_i, R'_i, \eta_i, \varepsilon_i)$ descends to an adjunction $(L''_i, R''_i, \eta''_i, \varepsilon''_i)$, so $L''_i$ has a right adjoint. Since $L_i$ is full and faithful, $\eta_i: \text{id} \cong R_iL_i$ is an isomorphism, and hence so is $\eta''_i$. This shows that $L'_i$ is full and faithful. As above, the associated idempotent comonad $S''_i = L''_iR''_i$ satisfies $\text{Im } L''_i = \text{Im } S''_i$.

Clearly, the idempotent comonad $S_i = L_iR_i$ satisfies $\text{Im } L_i = \text{Im } S_i$ and restricts to $S'_i$ and induces $S''_i$. Hence Proposition 5.18 proves what we need. \qed

6. Applications

In this section, we combine the formalism of Fourier–Mukai transforms developed in Section 3 with the abstract version of the conservative descent theorem (Theorem 5.16) from the previous section. This gives a geometric version of the conservative descent theorem (Theorem 6.1) which is easy to apply in practice. We illustrate the usefulness of this theorem by giving new proofs of the existence of semi-orthogonal decompositions associated to projectivized vector bundles, blow-ups and root stacks.

We start by reformulating our main theorem in a geometric context.

Theorem 6.1 (Conservative descent). Let $Z_1, \ldots, Z_n$ and $X$ be algebraic stacks over some base algebraic stack $S$, and assume that $\Phi_i: \text{D}_{\text{qc}}(Z_i) \to \text{D}_{\text{qc}}(X)$, for $1 \leq i \leq n$, are Fourier–Mukai transforms over $S$. Let $u: S' \to S$ be a faithfully flat morphism, and denote the base change of the objects above by $Z'_1, \ldots, Z'_n$, $X'$ and $\Phi'_i: \text{D}_{\text{qc}}(Z'_i) \to \text{D}_{\text{qc}}(X')$, respectively. Then for each $i$, the functor $\Phi_i$ is fully faithful provided that $\Phi'_i$ is fully faithful.

Assume that all $\Phi'_i$, and therefore also all $\Phi_i$, are fully faithful. If the sequence

\begin{equation}
\text{Im } \Phi'_1, \ldots, \text{Im } \Phi'_n
\end{equation}

of essential images is semi-orthogonal in $\text{D}_{\text{qc}}(X')$, then so is the sequence

\begin{equation}
\text{Im } \Phi_1, \ldots, \text{Im } \Phi_n
\end{equation}
in $\text{D}_{\text{qc}}(X)$. Both sequences consist of right admissible subcategories of $\text{D}_{\text{qc}}(X)$ and $\text{D}_{\text{qc}}(X)$, respectively. Moreover, if (6.1) is a semi-orthogonal decomposition of $\text{D}_{\text{qc}}(X)$, then (6.2) is a semi-orthogonal decomposition of $\text{D}_{\text{qc}}(X)$.

Proof. We may verify that the Fourier–Mukai transforms $\Phi_i$ are fully faithful after a faithfully flat base change by Proposition 4.13. Now assume that all our Fourier–Mukai transforms are fully faithful. By Example 4.16, we get sequences of idempotent comonads $S_i$ and $S'_i$ with the same essential images as $\Phi_i$ and $\Phi'_i$, respectively. Furthermore, by Proposition 4.26, we get $g^*$-compatibilities $S_i \Rightarrow S'_i$, where $g: X' \to X$ is the morphism induced by the base change. By Example 4.11, the functor $g^*$ is conservative. Hence we are in a situation where we can apply Theorem 5.16, with $g^*: \text{D}_{\text{qc}}(X) \to \text{D}_{\text{qc}}(X')$ playing the role of $F: \mathcal{T} \to \mathcal{T}'$ in the statement of the theorem. This proves the rest of the theorem. □

In situations where Theorem 6.1 applies, we also get semi-orthogonal decompositions of the categories of perfect complexes, locally bounded pseudo-coherent complexes and of the singularity category. Keep the notation from the statement of Theorem 6.1. Recall from Proposition 3.5 that the functors $\Phi_i$ restrict to functors

$$\Phi_i^{\text{pf}}: \text{D}_{\text{pf}}(Z_i) \to \text{D}_{\text{pf}}(X), \quad \Phi_i^{\text{pc}}: \text{D}_{\text{pc}}(Z_i) \to \text{D}_{\text{pc}}(X),$$

(6.3)

between categories of perfect complexes and locally bounded pseudo-coherent complexes, respectively. We also get induced functors

$$\Phi_i^{\text{sg}}: \text{D}_{\text{sg}}(Z_i) \to \text{D}_{\text{sg}}(X),$$

(6.4)

between the singularity categories. Note that the functors (6.3) and (6.4), again by Proposition 3.5, have right adjoints. Furthermore, as already discussed in Section 3, the functors $\Phi_i^{\text{pf}}$ also admit left adjoints given by (3.12). In particular, we get the following theorem as a direct application of Corollary 5.19.

**Theorem 6.2.** Let $Z_1, \ldots, Z_n$ and $X$ be algebraic stacks over some base algebraic stack $S$, and assume that $\Phi_i: \text{D}_{\text{qc}}(Z_i) \to \text{D}_{\text{qc}}(X)$, for $1 \leq i \leq n$, are Fourier–Mukai transforms over $S$. Consider the induced functors (6.3) and (6.4).

Assume that each $\Phi_i$ is fully faithful and that

$$\text{D}_{\text{qc}}(X) = \langle \text{Im} \Phi_1, \ldots, \text{Im} \Phi_n \rangle$$

(6.5)

is a semi-orthogonal decomposition. Then

$$\text{D}_{\text{pf}}(X) = \langle \text{Im} \Phi_1^{\text{pf}}, \ldots, \text{Im} \Phi_n^{\text{pf}} \rangle$$

(6.6)

is a semi-orthogonal decomposition into admissible subcategories, and

$$\text{D}_{\text{pc}}(X) = \langle \text{Im} \Phi_1^{\text{pc}}, \ldots, \text{Im} \Phi_n^{\text{pc}} \rangle,$$

(6.7)

$$\text{D}_{\text{sg}}(X) = \langle \text{Im} \Phi_1^{\text{sg}}, \ldots, \text{Im} \Phi_n^{\text{sg}} \rangle$$

(6.8)

are semi-orthogonal decompositions into right admissible subcategories.

6.1. **Some auxiliary results.** Before turning to the actual applications, we state some auxiliary results. They will help us to determine whether a given semi-orthogonal sequence is full (Lemma 6.4 and Lemma 6.5) and to show that pull-back functors are fully faithful (Lemma 6.6).

We start by recalling the definition of a generator for a triangulated category.

**Definition 6.3.** Let $\mathcal{T}$ be a triangulated category. An object $G$ in $\mathcal{T}$ is a generator for $\mathcal{T}$ if for any object $F$ we have $F = 0$ if and only if $\mathcal{T}(\Sigma^m G, F) = 0$ for all $m \in \mathbb{Z}$. 
Lemma 6.4. Let $\mathcal{T}$ be a triangulated category with a generator $G$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_n$ be a semi-orthogonal sequence of right admissible subcategories of $\mathcal{T}$, and let $\mathcal{T}'$ denote its triangulated hull in $\mathcal{T}$. Then $\mathcal{T} = \mathcal{T}'$ in and only if $G \in \mathcal{T}'$.

Proof. Trivially, $\mathcal{T} = \mathcal{T}'$ implies $G \in \mathcal{T}'$. Note that $\mathcal{T}'$ is right admissible in $\mathcal{T}$ and that $\mathcal{T} = \langle (\mathcal{T}')^\perp, \mathcal{T}' \rangle$ is a semi-orthogonal decomposition (see e.g. [LS16, Lemma A.9 and A.11]). Assume that $G$ lies in $\mathcal{T}'$. Then the same holds for all shifts of $G$, so $(\mathcal{T}')^\perp = 0$ since $G$ is a generator. Hence $\mathcal{T} = \mathcal{T}'$ as desired. \hfill $\square$

For quasi-projective schemes, one can explicitly construct generators from ample line bundles.

Lemma 6.5. Let $\mathcal{L}$ be an ample line bundle on a scheme $X$ (see [SP17, Tag 01PS]). Then there exists an integer $n_0$ such that the vector bundle $\mathcal{L}^a \oplus \mathcal{L}^{a+1} \oplus \cdots \oplus \mathcal{L}^{a+n_0}$ is a generator for $D_{qc}(X)$ for each integer $a$. Moreover, if there are sections $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ such that the open subschemes $X_{s_i}$ are affine and cover $X$, then $n_0$ can be taken to be $n$.

Proof. This is a well-known fact. The proof is almost identical to the proof of [SP17, Tag 0A9V]. \hfill $\square$

We remind the reader of the following useful criterion for fully faithfulness.

Lemma 6.6. If $f : X \to Y$ is a concentrated morphism of algebraic stacks, then $f^* : D_{qc}(Y) \to D_{qc}(X)$ is fully faithful if and only if the evaluation $\eta_{\mathcal{O}_Y} : \mathcal{O}_Y \to f_* f^* \mathcal{O}_Y$ of the adjunction unit $\eta$ at the structure sheaf is an isomorphism.

Proof. It is enough to show that the adjunction unit $\eta : \text{id} \to f_* f^*$ is an isomorphism if $\eta_{\mathcal{O}_Y}$ is an isomorphism. This is a direct consequence of the projection formula [HR17, Corollary 4.12]. Indeed, for any $\mathcal{F} \in D_{qc}(Y)$, the adjunction morphism $\eta_{\mathcal{F}} : \mathcal{F} \to f_* f^* \mathcal{F}$ is in the obvious way identified with the upper horizontal arrow in the commutative diagram

$$
\begin{array}{ccc}
\mathcal{F} \otimes \mathcal{O}_Y & \xrightarrow{\eta_{\mathcal{F}} \otimes \mathcal{O}_Y} & f_* f^* (\mathcal{F} \otimes \mathcal{O}_Y) \\
\text{id} \otimes \eta_{\mathcal{O}_Y} & \sim & f_*(f^* \mathcal{F} \otimes f^* \mathcal{O}_Y)
\end{array}
$$

where the lower horizontal isomorphism is the projection formula and the right vertical arrow is the obvious isomorphism. Checking that this diagram is commutative essentially boils down to the definition of the morphism in the projection formula. \hfill $\square$

6.2. Projectivized vector bundles. As our first application of conservative descent, we generalize the semi-orthogonal decomposition associated to a projectivized vector bundle, first established by Beilinson [Be78] in the case of a projective space over a field and by Orlov [Orl92, Lemma 2.5, Theorem 2.6] in the relative setting.

Theorem 6.7. Let $S$ be an algebraic stack and let $\pi : \mathbb{P}(\mathcal{E}) \to S$ be the projectivization of a finite locally free $\mathcal{O}_S$-module $\mathcal{E}$ of rank $n + 1$. Then the functors

$$
\Phi_i : D_{qc}(S) \to D_{qc}(\mathbb{P}(\mathcal{E})), \quad \mathcal{F} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(i) \otimes \pi^*(\mathcal{F}),
$$

where...
are fully faithful Fourier–Mukai transforms. For any integer $a$, we have a semi-orthogonal decomposition
\begin{equation}
D_{qc}(\mathcal{P}(\mathcal{E})) = \langle \text{Im } \Phi_a, \ldots, \text{Im } \Phi_{a+n} \rangle
\end{equation}
into right admissible subcategories.

**Proof.** By Proposition 3.9, each $\Phi_i$ is a Fourier–Mukai transform over $S$ with $K = \mathcal{P}(\mathcal{E})$, $\mathcal{K} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(i)$, $p = \pi$, $q = \text{id}$ in the notation from Definition 3.3. Moreover, base change along any morphism is compatible with taking the projectivization of a vector bundle (cf. [SP17, Tag 0103]). Hence the theorem can be verified after a faithfully flat base change, by the geometric conservative descent Theorem 6.1. This reduces the problem to the situation where $S = \text{Spec } R$ is an affine scheme and $\mathcal{E} = \mathcal{O}_{S}^{n+1}$ is a trivial vector bundle.

First we verify that $\Phi_i$ is fully faithful. Since twisting with a line bundle induces an equivalence of categories, it is enough to show that $\pi^*$ is fully faithful. By Lemma 6.6, it is enough to show that the canonical morphism $\mathcal{O}_S \to \pi_* \pi^* \mathcal{O}_S$ is an isomorphism. Since $S$ is affine, this follows from the well known fact that $H^q(\mathbb{P}_R^n, \mathcal{O}_{\mathbb{P}_R^n})$ is equal to $R$ for $q = 0$ and vanishes otherwise (see e.g. [SP17, Tag 01XT]).

Next we show that the sequence of categories in (6.10) is semi-orthogonal. Let $\mathcal{F}$ and $\mathcal{G}$ be objects of $D_{qc}(S)$.
\begin{equation}
\text{Hom}(\Phi_i(\mathcal{F}), \Phi_j(\mathcal{G})) \cong \text{Hom}(\mathcal{F}, \pi_* (\mathcal{O}_{\mathbb{P}(\mathcal{E})}(j-i) \otimes \pi^* \mathcal{G}))
\cong \text{Hom}(\mathcal{F}, \pi_* (\mathcal{O}_{\mathbb{P}(\mathcal{E})}(j-i)) \otimes \mathcal{G}),
\end{equation}
where the first bijection follows from the adjunction and twisting with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-i)$, and the second bijection follows from the projection formula. But the sheaf cohomology of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(j-i)$ vanishes whenever $-n \leq j-i < 0$, by [SP17, Tag 01XT], so $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(j-i)) = 0$. Hence (6.11) vanishes for $-n \leq j-i < 0$, so the sequence in (6.10) is indeed semi-orthogonal.

Finally, we show that the sequence is full. By Lemma 6.5, the object $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(a) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(a+n)$ is a generator for $D_{qc}(\mathbb{P}(\mathcal{E}))$, so by Lemma 6.4 it is enough to verify that this object is in the triangulated hull of the semi-orthogonal sequence $\text{Im } \Phi_a, \ldots, \text{Im } \Phi_{a+n}$ of right admissible subcategories. But this is obvious since $\Phi_i(\mathcal{O}_S) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(i)$.

By Theorem 6.2, we immediately get the following corollary of Theorem 6.7.

**Corollary 6.8.** *Keep the notation from Theorem 6.7. Similarly as in the statement of Theorem 6.2, we let $\Phi_i^{pf}$, $\Phi_i^{pc}$ and $\Phi_i^{sg}$ denote the induced functors between derived categories of perfect complexes, derived categories of locally bounded pseudo-coherent complexes, and singularity categories, respectively.*

*Then we have a semi-orthogonal decomposition
\begin{equation}
D_{pf}(\mathbb{P}(\mathcal{E})) = \langle \text{Im } \Phi_{a}^{pf}, \ldots, \text{Im } \Phi_{a+n}^{pf} \rangle
\end{equation}
into admissible subcategories and semi-orthogonal decompositions
\begin{equation}
D_{pc}(\mathbb{P}(\mathcal{E})) = \langle \text{Im } \Phi_{a}^{pc}, \ldots, \text{Im } \Phi_{a+n}^{pc} \rangle
\end{equation}
\begin{equation}
D_{sg}(\mathbb{P}(\mathcal{E})) = \langle \text{Im } \Phi_{a}^{sg}, \ldots, \text{Im } \Phi_{a+n}^{sg} \rangle
\end{equation}
into right admissible subcategories.*
6.3. **Blow-ups.** In this subsection, we describe the semi-orthogonal decomposition associated to the blow-up of an algebraic stack in a regular sheaf of ideals. This semi-orthogonal decomposition was first described by Orlov [Orl92, Theorem 4.3] in the less general setting of a blow-up of a smooth variety in a smooth subvariety.

The standard reference for blow-ups in the generality we work in is [SGA6, Exposé VII]. Recall from Remark 3.10 that we follow the conventions about regular immersions from this source. Let \( \iota: Z \to X \) be a regular closed immersion of algebraic stacks. By blowing up \( X \) in \( Z \), we obtain a cartesian diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\pi} & \tilde{X} \\
\downarrow{\rho} & & \downarrow{\pi} \\
Z & \xrightarrow{\iota} & X
\end{array}
\]

where \( E \) is the exceptional divisor of the blow-up. The morphism \( \pi \) is projective with twisting sheaf \( \mathcal{O}_{\tilde{X}}(1) \cong \mathcal{O}_{\tilde{X}}(-E) \). Recall that the conormal bundle \( N_{Z/X} \) of the regular immersion is locally free and that \( \rho: E \to Z \) is isomorphic to the projectivization of \( N_{Z/X} \). If the rank of \( N_{Z/X} \) is constant \( c \), then we say that \( Z \) has constant codimension \( c \) in \( X \).

**Theorem 6.9.** Let \( X \) be an algebraic stack and \( \iota: Z \to X \) a regular closed immersion of constant codimension \( c \geq 0 \). Consider the blow-up diagram (6.15), and define the functors

\[
\begin{align*}
\Phi_i &: D_{qc}(Z) \to D_{qc}(\tilde{X}), & \mathcal{F} &\mapsto \mathcal{O}_{\tilde{X}}(-iE) \otimes \kappa^* \rho^*(\mathcal{F}), & i < 0, \\
\Phi_0 &: D_{qc}(X) \to D_{qc}(\tilde{X}), & \mathcal{F} &\mapsto \pi^*(\mathcal{F}).
\end{align*}
\]

Then each functor \( \Phi_i \) is a Fourier–Mukai transform over \( X \). Moreover, for each \( i \in \{-c + 1, \ldots, 0\} \), the functor \( \Phi_i \) is fully faithful and we have a semi-orthogonal decomposition

\[
D_{qc}(\tilde{X}) \cong \langle \text{Im } \Phi_{-c+1}, \ldots, \text{Im } \Phi_0 \rangle
\]

into right admissible subcategories.

**Proof.** By Proposition 3.9 and the projection formula, each \( \Phi_i \) is a Fourier–Mukai transform over \( X \). Indeed, for \( i < 0 \), we let \( K = E, K = \mathcal{O}_{E(i)} \cong \kappa^* \mathcal{O}_{\tilde{X}}(i), p = \rho, q = \kappa \) in the notation from Definition 3.3. Similarly, for \( i = 0 \), we let \( K = \tilde{X}, K = \mathcal{O}_{\tilde{X}}, p = \pi, q = \text{id} \). Hence by conservative descent, as stated in Theorem 6.1, and the fact that base change along any flat morphism preserves blow-ups (cf. [SP17, Tag 0805]), the theorem can be verified after a faithfully flat base change. In particular, we may assume that \( X = \text{Spec } R \) is an affine scheme and that the ideal defining \( Z \) is generated by a regular sequence of length \( c \). The theorem is trivial when \( c \leq 1 \), so let us assume that \( c > 1 \).

First we prove that \( \Phi_i \) is fully faithful for each \( i \). For \( i = 0 \) it is enough to verify that the canonical morphism \( \mathcal{O}_X \to \pi^* \pi^* \mathcal{O}_X \) is an isomorphism, by Lemma 6.6. This follows from the fact that \( H^r(\bar{X}, \mathcal{O}_{\bar{X}}) \) vanishes for \( r > 0 \) and is isomorphic to \( R \) for \( r = 0 \), as is shown in [SGA6, Exposé VII, Lemme 3.5]. Assume that \( i < 0 \) and let \( \mathcal{E} \) and \( \mathcal{F} \) be objects of \( D_{qc}(Z) \). The map \( \text{Hom}(\mathcal{E}, \mathcal{F}) \to \text{Hom}(\Phi_i(\mathcal{E}), \Phi_i(\mathcal{F})) \)
factors as
\[
\text{(6.17)} \quad \text{Hom}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \text{Hom}(\rho^*\mathcal{E}, \rho^*\mathcal{F}) \xrightarrow{\sim} \text{Hom}(\kappa^*\kappa_*\rho^*\mathcal{E}, \rho^*\mathcal{F}) \\
\xrightarrow{\sim} \text{Hom}(\kappa_*\rho^*\mathcal{E}, \kappa_*\rho^*\mathcal{F}) \xrightarrow{\sim} \text{Hom}(\Phi_i(\mathcal{E}), \Phi_i(\mathcal{F})),
\]
where the first map is bijective by Theorem 6.7, and the last two maps are the obvious bijections. The map $\varepsilon$ is obtained from the evaluation $\kappa^*\kappa_*\rho^*\mathcal{E} \to \rho^*\mathcal{E}$ of the adjunction counit at $\rho^*\mathcal{E}$. It suffices to prove that $\varepsilon$ is an isomorphism. Since $E$ is an effective Cartier divisor, the adjunction counit evaluated at an arbitrary object $\mathcal{M}$ of $\mathcal{D}_{\text{qc}}(E)$ fits into a triangle
\[
\text{(6.18)} \quad \Sigma\mathcal{M}(1) \to \kappa^*\kappa_*\mathcal{M} \to \mathcal{M} \to \Sigma^2\mathcal{M}(1)
\]
in $\mathcal{D}_{\text{qc}}(E)$ by [BLS16, Lemma 4.2] or [Tho93b, Porisme 3.5], where $\mathcal{M}(1)$ denotes the usual Serre twist $\mathcal{M} \otimes O_X(1)$.

Consider the particular case $\mathcal{M} = \rho^*\mathcal{E}$. Note that $E$ is the projectivization of the conormal bundle of the inclusion $Z \subset X$, which has rank $c$ by assumption. By the semi-orthogonal decomposition in Theorem 6.7, the functor $\text{Hom}(-, \rho^*\mathcal{F})$ vanishes on the first and the last object in (6.18). Hence $\varepsilon$ is an isomorphism as required.

Next we verify that the sequence of categories in (6.16) is semi-orthogonal. First assume that $-c + 1 \leq j < i < 0$ and let $\mathcal{E}$ and $\mathcal{F}$ be objects in $\mathcal{D}_{\text{qc}}(Z)$. Using the adjunction isomorphism and twisting with $O_X(-i)$, we get
\[
\text{Hom}(\Phi_i(\mathcal{E}), \Phi_j(\mathcal{F})) \cong \text{Hom}(\kappa^*\kappa_*\rho^*\mathcal{E}, (\rho^*\mathcal{F})(j-i)).
\]
Similarly as above, we apply the functor $\text{Hom}(-, (\rho^*\mathcal{F})(j-i))$ to the triangle (6.18) with $\mathcal{M} = \rho^*\mathcal{E}$. By the semi-orthogonal decomposition in Theorem 6.7 it vanishes on the first and third object because $-c + 1 < j - i < 0$. This implies that both sides of (6.19) also vanish, as desired. Now assume instead that $\mathcal{E}$ is an object of $\mathcal{D}_{\text{qc}}(X)$, that $\mathcal{F}$ is an object of $\mathcal{D}_{\text{qc}}(Z)$, and that $-c + 1 \leq i < 0$. Then
\[
\text{Hom}(\Phi_0(\mathcal{E}), \Phi_i(\mathcal{F})) \cong \text{Hom}(\kappa^*\pi^*\mathcal{E}, (\rho^*\mathcal{F})(i)) \\
\cong \text{Hom}(\rho^*\tau^*\mathcal{E}, (\rho^*\mathcal{F})(i))
\]
vanishes, again by the semi-orthogonal decomposition in Theorem 6.7. Hence the sequence in (6.16) is indeed semi-orthogonal.

Finally, we prove that our sequence is full. Let $\mathcal{T}$ denote the triangulated hull in $\mathcal{D}_{\text{qc}}(\tilde{X})$ of the subcategories $\text{Im} \Phi_{-c+1}, \ldots, \text{Im} \Phi_0$. It clearly contains $O_{\tilde{X}}$ and $O_E(i)$ for $-c < i < 0$. By twisting the exact sequence
\[
\text{(6.21)} \quad 0 \to O_{\tilde{X}}(1) \to O_{\tilde{X}} \to O_E \to 0
\]
with $O_{\tilde{X}}(i)$ for $-c < i < 0$, we see that $\mathcal{T}$ also contains $O_{\tilde{X}}(i)$ for $-c < i < 0$. In particular, the category $\mathcal{T}$ contains $\mathcal{G} = O_{\tilde{X}}(-c+1) \oplus \cdots \oplus O_{\tilde{X}}(0)$. Since the ideal defining $Z$ is generated by $c$ elements, the blow-up $\tilde{X}$ embeds into $\mathbb{P}^{c-1}$. Hence the bundle $\mathcal{G}$ generates $\mathcal{D}_{\text{qc}}(\tilde{X})$ by Lemma 6.5. Therefore $\mathcal{T} = \mathcal{D}_{\text{qc}}(\tilde{X})$ by Lemma 6.4. This concludes the proof. \hfill \Box

By Theorem 6.2, we immediately get the following corollary of Theorem 6.9.

**Corollary 6.10.** Keep the notation from Theorem 6.9. Similarly as in the statement of Theorem 6.2, we let $\Phi^{\text{fl}}_i$, $\Phi^{\text{qc}}_i$ and $\Phi^{\text{sg}}_i$ denote the induced functors between the derived categories of perfect complexes, derived categories of locally bounded pseudo-coherent complexes, and singularity categories, respectively.
Then we have a semi-orthogonal decomposition
\[(6.22) \quad D_{pf}(\tilde{X}) = \langle \text{Im } \Phi_{c+1}^{pf}, \ldots, \text{Im } \Phi_{0}^{pf} \rangle\]
into admissible subcategories and semi-orthogonal decompositions
\[(6.23) \quad D_{pc}^{lb}(\tilde{X}) = \langle \text{Im } \Phi_{c+1}^{pc}, \ldots, \text{Im } \Phi_{0}^{pc} \rangle,\]
\[(6.24) \quad D_{sg}(\tilde{X}) = \langle \text{Im } \Phi_{c+1}^{sg}, \ldots, \text{Im } \Phi_{0}^{sg} \rangle\]
into right admissible subcategories.

6.4. **Root stacks.** The *root construction* is a construction which formally adjoins a root of some order \(r > 0\) to an effective Cartier divisor on a scheme or an algebraic stack. This is a purely stacky construction, which yields a genuine algebraic stack in all but the trivial cases where we take the first root of a divisor. The root construction was originally described by Cadman in [Cad07]. We refer to [BLS16, Section 3] for a summary of its most important properties.

A root stack is a birational modification which has many similarities with a blow-up. In particular, its derived category admits a semi-orthogonal decomposition, as first noted by Ishii–Ueda [IU15, Proposition 6.1]. We gave a proof in a more general setting in [BLS16, Theorem 4.7]. In the proof we left out the details of the reduction to the local setting with a reference to this article. We restate the theorem here as Theorem 6.11 and fill in the missing part of the proof.

Let \(X\) be an algebraic stack and let \(\iota : D \hookrightarrow X\) be an effective Cartier divisor. For a given integer \(r > 0\), the root construction gives a diagram
\[(6.25) \quad \begin{array}{ccc}
E & \xrightarrow{\kappa} & \tilde{X} \\
\rho \downarrow & & \downarrow \pi \\
D & \xrightarrow{\iota} & X.
\end{array}\]
Note that this diagram fails to be cartesian whenever \(r > 1\). Rather the pull-back of \(D\) along \(\pi\) can be identified with \(rE\). This is the motivation for the term *root construction*.

**Theorem 6.11.** Let \(X\) be an algebraic stack and \(\iota : D \hookrightarrow X\) an effective Cartier divisor. Fix an integer \(r > 0\) and consider the root diagram (6.25) associated to the \(r\)-th root construction. Define the functors
\[
\Phi_i : D_{qc}(D) \rightarrow D_{qc}(\tilde{X}), \quad F \mapsto \mathcal{O}_{\tilde{X}}(-iE) \otimes \kappa^* \rho^*(F), \quad -r < i < 0,
\]
\[
\Phi_0 : D_{qc}(X) \rightarrow D_{qc}(\tilde{X}), \quad F \mapsto \pi^*(F).
\]
Then each functor \(\Phi_i\) is a Fourier–Mukai transform over \(X\). Moreover, for each \(i \in \{-r+1, \ldots, 0\}\), the functor \(\Phi_i\) is fully faithful and we have a semi-orthogonal decomposition
\[(6.26) \quad D_{qc}(\tilde{X}) = \langle \text{Im } \Phi_{-r+1}, \ldots, \text{Im } \Phi_0 \rangle\]
into right admissible subcategories.

**Proof.** By Proposition 3.9 and the projection formula, each \(\Phi_i\) is a Fourier–Mukai transform over \(X\). Indeed, for \(i \in \{-r+1, \ldots, -1\}\), we let \(K = E, \mathcal{K} = \kappa^* \mathcal{O}_{\tilde{X}}(-iE), \mathcal{P} = \rho^*, \mathcal{Q} = \kappa^*\) in the notation from Definition 3.3. For \(i = 0\), we let \(K = \tilde{X}, \mathcal{K} = \mathcal{O}_{\tilde{X}}, \mathcal{P} = \pi^*, \mathcal{Q} = \text{id}\). Hence by conservative descent, as stated in Theorem 6.1, the theorem can be verified after a faithfully flat base change. In particular, we may assume
that $X = \text{Spec } R$ is an affine scheme and that the ideal defining $D$ is generated by a single regular element.

The rest of the proof, which is very similar to the proof of Theorem 6.9, is written down in full detail in [BLS16, Theorem 4.7]. The semi-orthogonal decomposition in the statement of the cited theorem refers to the category $D_{\text{pf}}(\tilde{X})$, but the proof applies equally well to $D_{\text{qc}}(\tilde{X})$. Note that there is an obvious typographical error in the formulation of the cited theorem; two occurrences of $D(\tilde{X})$ should be replaced by $D_{\text{pf}}(\tilde{X})$. □

By Theorem 6.2, we immediately get the following corollary of Theorem 6.11.

**Corollary 6.12.** Keep the notation from Theorem 6.11. Similarly as in the statement of Theorem 6.2, we let $\Phi_{i}^{\text{pf}}$, $\Phi_{i}^{\text{pc}}$ and $\Phi_{i}^{\text{sg}}$ denote the induced functors between the derived categories of perfect complexes, derived categories of locally bounded pseudo-coherent complexes, and singularity categories, respectively.

Then we have a semi-orthogonal decomposition

$$D_{\text{pf}}(\tilde{X}) = \langle \text{Im } \Phi_{r+1}^{\text{pf}}, \ldots, \text{Im } \Phi_{0}^{\text{pf}} \rangle$$

(6.27)

into admissible subcategories and semi-orthogonal decompositions

$$D_{\text{lb}}^{\text{pc}}(\tilde{X}) = \langle \text{Im } \Phi_{r+1}^{\text{pc}}, \ldots, \text{Im } \Phi_{0}^{\text{pc}} \rangle,$$

(6.28)

$$D_{\text{sg}}(\tilde{X}) = \langle \text{Im } \Phi_{r+1}^{\text{sg}}, \ldots, \text{Im } \Phi_{0}^{\text{sg}} \rangle$$

(6.29)

into right admissible subcategories.

**References**


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