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Geometry and physics of pseudodifferential operators on manifolds

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Summary. — A review is made of the basic tools used in mathematics to define a calculus for pseudodifferential operators on Riemannian manifolds endowed with a connection: existence theorem for the function that generalizes the phase; analogue of Taylor's theorem; torsion and curvature terms in the symbolic calculus; the two kinds of derivative acting on smooth sections of the cotangent bundle of the Riemannian manifold; the concept of symbol as an equivalence class. Physical motivations and applications are then outlined, with emphasis on Green functions of quantum field theory and Parker's evaluation of Hawking radiation.

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PACS 02.30.Nw – Fourier analysis.
PACS 02.40.Vh – Global analysis and analysis on manifolds.
PACS 03.70.+k – Theory of quantized fields.

1. – Introduction

In the course of studying partial differential equations on $\mathbb{R}^n$, one discovers one can consider operators whose action is defined, at least formally, by the integral

$$(Pu)(x) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\xi\cdot x} p(x,\xi)\hat{u}(\xi)\mathrm{d}\xi, \tag{1}$$

where $\mathrm{d}\xi$ is the Lebesgue measure on $\mathbb{R}^n$, $p(x,\xi)$ is the *amplitude* of the operator P, $\xi \cdot x = \langle x, \xi \rangle$ is its *phase* function, and

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi\cdot x} u(x)\mathrm{d}x \tag{2}$$

is the Fourier transform of u. These operators are said to be pseudodifferential and form a class large enough to contain the differential operators, the Green operators and the singular integral operators that are used to solve partial differential equations. They also contain, for each elliptic operator, its parametrix, *i.e.* an approximate inverse up to an operator of lower order (see below).

To understand how pseudo-differential operators can be used in solving inhomogeneous partial differential equations, let us begin by considering, for $n \geq 3$ and a smooth function f with compact support, *i.e.* $f \in C_0^\infty(\mathbb{R}^n)$, the inhomogeneous equation $\triangle u = f$, where

$$\triangle \equiv \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

is minus the Laplacian (with our convention, $\triangle$ is defined in the standard way, but the Laplacian has symbol given by $\sum_{k=1}^{n}(\xi_k)^2 = |\xi|^2$). Upon taking the Fourier transform of both sides, one finds

$$-|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi), \tag{3}$$

and hence, for the inverse operator Q of $\triangle$, or fundamental solution, we can write [1, 2]

$$\begin{aligned} u(x) = (Qf)(x) &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} \frac{\hat{f}(\xi)}{|\xi|^2} d\xi \\ &= -\frac{\Gamma\left(\frac{n}{2}-1\right)}{4\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy. \end{aligned} \tag{4}$$

If, instead of the Laplacian on $\mathbb{R}^n$, we deal with a general partial differential operator with constant coefficients which can be written as a polynomial $P = p(D)$, where $D \equiv -i\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, so that

$$p(D)u = f \in C_0^\infty(\mathbb{R}^n), \tag{5}$$

the solution can be formally expressed as

$$u(x) = (Qf)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} q(\xi) \hat{f}(\xi) d\xi, \tag{6}$$

where the amplitude $q(\xi)$ is the inverse of the symbol $p(\xi)$ of $p(D)$, and the integration contour must avoid the zeros of $p(\xi)$. However, if the operator P has variable coefficients on a subset U of $\mathbb{R}^n$, *i.e.*

$$P = p(x, D) = \sum_{\alpha:|\alpha|\leq k} a_\alpha(x) D_x^\alpha, \ a_\alpha \in C^\infty(U), \tag{7}$$

one can no longer solve the equation $Pu = f$ by Fourier transform. One can however *freeze* the coefficients at a point $x_0 \in U$ and consider P as a perturbation of $p(x_0, D)$, which is hence a differential operator with constant coefficients. In this way the amplitude of P reduces to $q(\xi) = \frac{1}{p_0(x_0,\xi)}$, and if we let x_0 vary in U, we obtain the approximate solution operator

$$(Qf)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} q(x,\xi)\hat{f}(\xi)d\xi \tag{8}$$

with amplitude $q(x,\xi) = p(x,\xi)^{-1}$, for $x \in U, f \in C_0^\infty(\mathbb{R}^n)$.

However, to solve the inhomogeneous equation $Pu = f$, we do not strictly need the full inverse operator or fundamental solution, but, as we said before, it is enough to know a parametrix, *i.e.* a quasi-inverse modulo a regularizing operator. One can provide a first example of parametrix by reverting to the study of constant coefficient operators. A parametrix Q can then be constructed by choosing its amplitude [1] as

$$q(\xi) = \frac{\chi(\xi)}{p(\xi)}, \tag{9}$$

where $\chi(\xi)$ is a suitably chosen $C^\infty(\mathbb{R}^n)$ function which is identically zero in a disk about the origin and identically 1 for large ξ. In this way the integral formula

$$(Qf)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} q(\xi)\hat{f}(\xi)d\xi, \tag{10}$$

is not affected by the convergence problems that would be met if the amplitude were taken to be just $p(\xi)^{-1}$. If Q is the integral operator in (10) it is no longer true that $PQ = QP = I$, but we have [1]

$$PQf = f + Rf, \ f \in C^\infty(U), \tag{11}$$

$$(Rf)(x) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} r(x-\xi)f(\xi)d\xi, \tag{12}$$

where the Fourier transform of r is $\chi - 1$. Thus r is a smooth function and R turns out to be a smoothing operator. Such a class of smoothing operators is fully under control, and hence a parametrix Q serves just as well as a full fundamental solution, for which R vanishes identically.

In the following we will see how to extend the theory of pseudodifferential operators on $\mathbb{R}^n$ to more general pseudodifferential operators defined on compact manifolds. In particular, we will outline some basic symbolic calculus for such operators. In the last two sections of this paper, some applications to physics will be shown.

2. – Pseudodifferential operators on manifolds

First, note that the material in the appendix can be re-expressed by saying that a pseudodifferential operator A acting on functions in $C_0^\infty(\mathbb{R}^n)$ has a symbol given by

$$\sigma_A(x,\xi) = e^{-i\xi\cdot x} A e^{i\xi\cdot x} \iff \sigma_A(x_0,\xi) = \left[A e^{i\xi\cdot(x-x_0)} \right]\Big|_{x=x_0}, \tag{13}$$

and hence formal application of Taylor's formula yields

$$\mathrm{e}^{-\mathrm{i}\xi\cdot x}Af(x)\mathrm{e}^{\mathrm{i}\xi\cdot x}=\sum_{k_1\ldots k_n}\frac{\mathrm{i}^{-(k_1+\ldots+k_n)}}{k_1!\ldots k_n!}\left(\frac{\partial^{k_1}}{\partial\xi_1^{k_1}}\cdots\frac{\partial^{k_n}\sigma_A}{\partial\xi_n^{k_n}}\right)\left(\frac{\partial^{k_1}}{\partial x_1^{k_1}}\cdots\frac{\partial^{k_n}f}{\partial x_n^{k_n}}\right), \tag{14}$$

the sum being taken over all values of the multi-index $k=(k_1,\ldots,k_n)$. The symbol of the product AB of two pseudodifferential operators A and B is then defined by

$$\begin{aligned}\sigma_{AB}(x,\xi)&=\mathrm{e}^{-\mathrm{i}\xi\cdot x}AB\mathrm{e}^{\mathrm{i}\xi\cdot x}=\mathrm{e}^{-\mathrm{i}\xi\cdot x}A\mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{e}^{-\mathrm{i}\xi\cdot x}B\mathrm{e}^{\mathrm{i}\xi\cdot x}\\&=\mathrm{e}^{-\mathrm{i}\xi\cdot x}A\mathrm{e}^{\mathrm{i}\xi\cdot x}\sigma_B\\&=\sum_{k_1\ldots k_n}\frac{\mathrm{i}^{-(k_1+\ldots+k_n)}}{k_1!\ldots k_n!}\left(\frac{\partial^{k_1}}{\partial\xi_1^{k_1}}\cdots\frac{\partial^{k_n}\sigma_A}{\partial\xi_n^{k_n}}\right)\left(\frac{\partial^{k_1}}{\partial x_1^{k_1}}\cdots\frac{\partial^{k_n}\sigma_B}{\partial x_n^{k_n}}\right).\end{aligned} \tag{15}$$

Note here the crucial role played by the function

$$l(x_0,\xi,x)=\xi\cdot(x-x_0)=\sum_{l=1}^{n}\xi_l(x^l-x_0^l) \tag{16}$$

which, for each x_0, is linear in x and ξ. Its derivative with respect to x is ξ, while its derivative with respect to ξ is $x-x_0$.

On going from $\mathbb{R}^n$ to compact manifolds, we look for a real-valued function

$$\begin{aligned}l:T^*M\times M&\to\mathbb{R}\\(v,x)&\mapsto l(v,x),\end{aligned}$$

which generalizes the function (16). Linearity in ξ becomes linearity in v on each fiber of the cotangent bundle of M, but linearity in x has no obvious counterpart. However, if there exists a connection ∇ on T^*M, linearity at x_0 can be defined by stating that, for all integer $k\geq 2$, the symmetrized k-th covariant derivative vanishes at x_0. The desired linear function is then a real-valued function $l\in C^\infty(T^*M\times M)$ such that the image $l(v,x)$ is, for fixed x, linear in each fiber of T^*M, and such that, for each $v\in T^*M$,

$$\partial^k l(v,x)|_{x=\pi(v)}=\begin{cases}v, & \text{if } k=1,\\ 0, & \text{otherwise.}\end{cases} \tag{17}$$

With this notation, ∂^k is the symmetrized k-th covariant derivative with respect to x, while π is the projection map $\pi:T^*M\to M$.

Once a connection ∇ is assigned, a definition of symbol of a pseudodifferential operator A on $C^\infty(M)$ is provided by [3]

$$\sigma_A(v)=\left[A\mathrm{e}^{\mathrm{i}l(v,x)}\right]\Big|_{x=\pi(v)}, \tag{18}$$

which is a generalization of formula (13). However, the function $l(v,x)$, whose existence will be proved in the next section, is not uniquely determined by the linearity conditions above. Therefore, different functions l would lead to different symbol maps. On the other hand, it can be proved [3] that the difference between any two symbol maps corresponding to different choices of functions l belongs to a certain class of functions, therefore the symbol of a pseudodifferential operator will be actually defined as an element of a quotient space suitably defined.

3. – Existence theorem for the function l

Following our main source [3], we are now going to prove that there exists a function $l \in C^\infty(T^*M \times M)$ with the properties listed above. Indeed, we have the following result.

Proposition 3.1. [3] *There exists a function $l \in C^\infty(T^*M \times M)$ such that $l(\cdot, x)$ is, for each $x \in M$, linear on the fibers of T^*M and such that, for each cotangent vector $v \in T^*M$, eq. (17) holds.*

Proof. The desired function l is first constructed locally. If U is a coordinate neighbourhood in M with local coordinates x^i and an m-th order covariant tensor $\tau_{i_1 \dots i_m}$, its covariant derivative is given by

$$\tau_{i_1 \dots i_m ; i} = \frac{\partial}{\partial x^i} \tau_{i_1 \dots i_m} - \sum_\nu \Gamma^j_{\ i i_\nu} \, \tau_{i_1 \dots i_{\nu-1} j i_{\nu+1} \dots i_m}, \tag{19}$$

where $\Gamma^j_{\ ik}$ is the standard notation for Christoffel symbols. Thus by induction, for any scalar function f, one has

$$f_{; i_1 \dots ; i_k} = \frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}} + \sum_{j : |j| < k} \gamma_{i_1 \dots i_k j} \frac{\partial^j f}{\partial x^j}, \tag{20}$$

where j are multiindices and γ's are polynomials in the derivatives of Christoffel symbols. This implies that, for $k > 1$, condition (17) is equivalent to each term

$$\frac{\partial^k l}{\partial x^{i_1} \dots \partial x^{i_k}},$$

being equal, at $\pi(v)$, to some linear combination of lower-order derivatives. To sum up, starting with the requirements

$$l(v, \pi(v)) = 0, \qquad \left. \frac{\partial l(v,x)}{\partial x^i} \right|_{x = \pi(v)} = v\left(\frac{\partial}{\partial x^i}\right), \tag{21}$$

specifies what $\left.\frac{\partial^k l}{\partial x^k}\right|_{x=\pi(v)}$ must be, in order eq. (17) to hold. Borel's theorem ensures that there exists a C^∞ function having partial derivatives arbitrarily prescribed, and the proof shows that l may be chosen to be both linear and C^∞ in v [3].

Having established that, for each coordinate neighbourhood U_i in M, there exists an $l_i \in C^\infty(T^*U_i \times U_i)$ with the desired properties, we can take finitely many U_i covering M with a partition of unity given by functions $\varphi_i \in C_0^\infty(U_i)$, and yet other smooth functions with compact support $\psi_i \in C_0^\infty(U_i)$ equal to 1 on a neighbourhood of the support of φ_i. The function

$$l(v, x) \equiv \sum_i \varphi_i(\pi(v)) l_i(v, x) \psi_i(x) \tag{22}$$

is then globally defined and satisfies all requirements. □

4. – Analogue of Taylor's theorem

For a smooth function f on M, $f^{(k)}(x_0)$ is replaced by $\nabla^k f(x_0)$. To obtain the analogue of $x - x_0$ note that, since for fixed x_0 and $x \in M$ the function $l(v, x)$ is linear for $v \in T^*_{x_0}$, we may think of it as an element of the tangent space at x_0. We can instead regard $l(\cdot, x)$ as a vector field on M, so that

$$l(\cdot, x)^k = l(\cdot, x) \otimes \cdots \otimes l(\cdot, x) \tag{23}$$

is a symmetric k-th order contravariant tensor field, and hence

$$\nabla^k f(x_0) \cdot l(x_0, x)^k$$

is defined and, by virtue of symmetry of $l(x_0, x)^k$, it coincides with $\partial^k f(x_0) \cdot l(x_0, x)^k$. A basic theorem [3] holds according to which, for each point $x_0 \in M$ and each integer N, one has

$$\partial^k f(x_0) = \partial^k \sum_{n=0}^{N} \frac{1}{n!} \nabla^n f(x_0) \cdot l(x_0, x)^n \Big|_{x=x_0}, \tag{24}$$

for $k \leq N$.

5. – Torsion and curvature terms in the symbolic calculus

In the symbolic calculus, one encounters frequently the unsymmetrized covariant derivatives

$$\nabla^k l(v) \equiv \nabla^k l(v, x)\big|_{x=\pi(v)}, \tag{25}$$

which turn out to be polynomials in the torsion tensor $T^p_{\ ij}$ and curvature tensor $R^p_{\ ijk}$. Indeed, by virtue of the Ricci identity, the difference of second covariant derivatives of an m-th order covariant tensor $\tau_{i_1 \dots i_m}$ is given by

$$\tau_{i_1 \dots i_m; j; k} - \tau_{i_1 \dots i_m; k; j} = \sum_\nu \tau_{i_1 \dots i_{\nu-1} p i_{\nu+1} \dots i_m} \ R^p_{\ i_\nu jk} - \tau_{i_1 \dots i_m; p} T^p_{\ jk}. \tag{26}$$

Thus, for any permutation α of $1, \dots, k$, the difference

$$f_{;i_1 \dots ; i_k} - f_{;i_{\alpha(1)}; \dots ; i_{\alpha(k)}}$$

is a sum of terms each of which is a product of $(k-1)$st or lower-order covariant derivatives of f and covariant derivatives of T and R followed by contraction. In particular, if $f = l$, one has at $x = \pi(v)$

$$\sum_{\alpha} l_{;i_{\alpha(1)}\cdots;i_{\alpha(k)}} = 0. \tag{27}$$

In particular, one finds, using the Einstein summation convention,

$$(\nabla^2 l)_{ij} = l_{;ij} = \frac{1}{2} v_p T^p_{\ ij}, \tag{28}$$

and, in the case of vanishing torsion,

$$(\nabla^3 l)_{ijk} = l_{;ijk} = \frac{1}{3} v_p \left(R^p_{\ ijk} + R^p_{\ jik} \right). \tag{29}$$

6. – The derivatives D^k and ∇^k

Given a function $\sigma \in C^\infty(T^*M)$, one defines $D^k\sigma$ to be the k-th derivative of σ in the direction of fibers of T^*M. Thus, for $\pi(v) = x_0$, think of σ as a function on the cotangent space at x_0, and take its k-th derivative $D^k\sigma$ evaluated at v. This is a k-linear function on T^*x_0 and may be identified with an element of the tensor product $\otimes_k T_{x_0}$. This means that $D^k\sigma$ is a contravariant k-tensor, and it is the analogue of $\frac{\partial^k \sigma}{\partial \xi^k}$, but of course k is an integer in D^k and a multi-index in $\frac{\partial^k}{\partial \xi^k}$.

The covariant derivatives ∇^k act on $C^\infty(M)$, and to define their action on $\sigma \in C^\infty(T^*M)$ we set [3]

$$\nabla^k \sigma(v) \equiv \nabla^k \sigma \left(\mathrm{d}_x l(v, x) \right) \big|_{x=\pi(v)}. \tag{30}$$

Although the function $l(v, \cdot)$ is not unique as we said before, all its derivatives are determined at $\pi(v)$, and hence (30) defines $\nabla^k\sigma$ unambiguously as a covariant k-tensor. The mixed derivatives $\nabla^k D^j \sigma$ may also occur and are defined by

$$\nabla^k D^j \sigma(v) \equiv \nabla_x^k D_v^j \ \sigma(\mathrm{d}_x l(v, x))\big|_{x=\pi(v)}. \tag{31}$$

This is a contravariant (respectively, covariant) j-tensor (respectively, k-tensor), or tensor of type (j, k). For example, given a Riemannian manifold (M, g), if σ is the squared norm of v, *i.e.*

$$\sigma(v) = |v|^2 = g^{ij} v_i v_j, \tag{32}$$

one has

$$(D\sigma)^i = 2g^{ij} v_j, \ (D^2\sigma)^{ij} = 2g^{ij}. \tag{33}$$

Moreover, since

$$\sigma(\mathrm{d}l) = g^{ij} l_i l_j, \tag{34}$$

and both l_i and g^{ij} have vanishing covariant derivatives at $\pi(v)$, one finds

$$\nabla\sigma = 0, \tag{35}$$

while, by virtue of (29),

$$(\nabla^2\sigma)_{kl} = \frac{2}{3} v_p v_q \left(R^{pq}{}_{kl} + R^{p\ \ q}_{\ k\ \ l}\right), \tag{36}$$

and

$$(\nabla^2 D\sigma)^p{}_{kl} = \frac{4}{3} v_q \left(R^{pq}{}_{kl} + R^{p\ \ q}_{\ k\ \ l}\right). \tag{37}$$

Note also that $\sigma(v) = |v|^2$ is the symbol of the Laplacian on a Riemannian manifold, if the connection with respect to which the symbol is taken is the Levi-Civita connection.

7. – The symbol as an equivalence class

Let $S^\omega_\rho(\mathbb{R}^n \times \mathbb{R}^m)$, for $1/2 < \rho \leq 1$, be the space of functions $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ such that, for all multi-indices j and k,

$$\frac{\partial^j}{\partial x^j}\frac{\partial^k}{\partial \xi^k}\sigma(x,\xi) = O\left((1+|\xi|)^{\omega-\rho|k|+(1-\rho)|j|}\right), \tag{38}$$

uniformly on compact x-sets.

On going from $\mathbb{R}^n$ to a (compact) manifold M, the spaces $S^\omega_\rho(M\times\mathbb{R}^m)$ and $S^\omega_\rho(T^*M)$ consist of functions satisfying (38) in terms of local coordinates. If it is sufficiently clear what the underlying space is, one writes simply S^ω_ρ, and one sets

$$S^\infty_\rho \equiv \bigcup_{\omega\in\mathbb{R}} S^\omega_\rho, \qquad S^{-\infty} \equiv \bigcap_{\omega\in\mathbb{R}} S^\omega_\rho, \tag{39}$$

where $S^{-\infty}$ is independent of ρ.

Consider now a manifold M endowed with a connection. We denote by $L^\omega_\rho(M)$ the space of operators on $C^\infty(M)$ which locally are pseudodifferential operators with symbols (as defined in (18)) in S^ω_ρ. Given a linear function $l(v,x)$ as defined in sect. **3**, let $\psi \in C^\infty(M\times M)$ such that it is 1 on a neighbourhood of the diagonal and such that $\mathrm{d}_x l(v,x) \neq 0$ for $\psi(x_0,x)\neq 0$ and $0 \neq v \in T^*_{x_0}$

Then, for any operator $A \in L^\omega_\rho$, define

$$\sigma_A(v) = \left[A\psi(\pi(v),x)\mathrm{e}^{\mathrm{i}l(v,x)}\right]\Big|_{x=\pi(v)}. \tag{40}$$

It can be proved [3] that such a function belongs to S^ω_ρ, and that different choices of functions ψ and l lead to the same function σ_A modulo on element of $S^{-\infty}$. Therefore, the symbol σ_A of the operator A is defined as the corresponding equivalence class in $S^\omega_\rho/S^{-\infty}$.

8. – Symbols in quantum field theory

On the side of physical applications, let us here reconsider the photon propagator in the Euclidean version of quantum electrodynamics. In modern language, the functional integral tells us that the photon propagator is obtained by first evaluating the gauge-field operator $P_{\mu\nu}$, $\mu, \nu = 0, \ldots, 3$, resulting from the particular choice of gauge-averaging functional, then taking its symbol $\sigma(P_{\mu\nu})$ and inverting such a symbol to find $\sigma^{-1}(P_{\mu\nu}) = \Sigma^{\mu\nu}$ for which $\sigma\Sigma = \Sigma\sigma = I$. The photon propagator reads eventually [4]

$$\triangle^{\mu\nu}(x,y) = (2\pi)^{-4} \int_{\zeta} \mathrm{d}^4k \; \Sigma^{\mu\nu} \mathrm{e}^{\mathrm{i}k\cdot(x-y)} \tag{41}$$

for some contour ζ. The gauge-field Lagrangian turns out to be

$$\mathcal{L} = \partial^{\mu}\rho_{\mu} + \frac{1}{2} A^{\mu} P_{\mu\nu} A^{\nu}, \tag{42}$$

where

$$\rho_{\mu} = \frac{1}{2} A_{\nu} \left(\partial_{\mu} A^{\nu} - \partial^{\nu} A_{\mu}\right) + \frac{1}{2\alpha} A_{\mu} \partial^{\nu} A_{\nu}, \tag{43}$$

and

$$P_{\mu\nu} = -g_{\mu\nu}\Box + \left(1 - \frac{1}{\alpha}\right) \partial_{\mu}\partial_{\nu}. \tag{44}$$

Note that here we denoted by ∂_{μ} the standard partial derivative with respect to the μ component, that is $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$. Also, $\alpha \in \mathbb{R} \setminus \{0\}$, $g_{\mu\nu} = \mathrm{diag}(1,1,1,1)$ and $\Box = g^{\mu\nu}\partial_{\mu}\partial_{\nu}$. Of course, the term ρ_{μ} only contributes to a total divergence and hence does not affect the photon propagator, while the parameter α can be set equal to 1 (Feynman choice) so that calculations are simplified. Thus, we can eventually obtain the gauge-field operator

$$P_{\mu\nu}(\alpha = 1) = -g_{\mu\nu}\Box. \tag{45}$$

Its symbol, which results from Fourier analysis of the $\Box$ operator, reads as

$$\sigma(P_{\mu\nu}(\alpha = 1)) = k^2 g_{\mu\nu}, \tag{46}$$

and hence the Euclidean photon propagator is (cf. eq. (4))

$$\triangle_E^{\mu\nu}(x,y) = (2\pi)^{-4} \int_{\Gamma} \mathrm{d}^4k \; \frac{g^{\mu\nu}}{k^2} \mathrm{e}^{\mathrm{i}k\cdot(x-y)}, \tag{47}$$

where the points x and y refer to the indices μ and ν, respectively. Note that integration along the real axis for k_0, k_1, k_2, k_3 avoids poles of the integrand, which are located at the complex points for which $k^2 = \sum_{\mu=0}^{3} (k_{\mu})^2 = 0$.

Strictly speaking, the gauge parameter α in (43) and (44) is the bare value α_B of α before renormalization, and we should express the bare symbol of the gauge-field operator in QED in the form

$$\sigma(P_{\mu\nu}) = k^2 g_{\mu\nu} + \left(\frac{1}{\alpha_B} - 1\right) k_\mu k_\nu = \sigma_{\mu\nu}(k). \tag{48}$$

Its inverse $\Sigma^{\mu\nu}$ is a combination of $g^{\mu\nu}$ and $k^\mu k^\nu$ with coefficients $\mathcal{A}$ and $\mathcal{B}$, respectively, determined from the condition

$$\sigma_{\mu\nu}\Sigma^{\nu\lambda} = \delta_\mu^{\ \lambda}, \tag{49}$$

which implies

$$\mathcal{A} = \frac{1}{k^2},\ \mathcal{B} = \frac{(\alpha_B - 1)}{k^4}. \tag{50}$$

At this stage, the bare photon propagator takes the form

$$\triangle^{\mu\nu}(x,y) = \int_\Gamma \frac{\mathrm{d}^4 k}{(2\pi)^4}\left[\frac{g^{\mu\nu}}{k^2} + \frac{(\alpha_B - 1)k^\mu k^\nu}{k^4}\right]\mathrm{e}^{\mathrm{i}k\cdot(x-y)}. \tag{51}$$

9. – Hawking radiation

Following ref. [5] we now consider a completely different setting, *i.e.* the spacetime of a collapsing star, for which, outside the horizon, the wave operator appears as it would in a Schwarzschild spacetime for a suitable choice of coordinates. Let r be such a radial coordinate with associated hypersurface Σ having equation $r = 0$, and let t be Killing time, so that $k = \frac{\partial}{\partial t}$ is the timelike Killing vector field outside the Killing horizon B_k. The wave operator reads as

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{1}{r^2}\left(1 - \frac{2m}{r}\right)\frac{\partial}{\partial r}\left(r^2\left(1 - \frac{2m}{r}\right)\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\left(1 - \frac{2m}{r}\right)L, \tag{52}$$

where the operator L is independent of t or r and takes the same form as in Minkowski spacetime. The radial part of the wave operator is

$$\Box_R = \frac{\partial^2}{\partial t^2} - \frac{1}{r^2}\left(1 - \frac{2m}{r}\right)\frac{\partial}{\partial r}\left(r^2\left(1 - \frac{2m}{r}\right)\frac{\partial}{\partial r}\right). \tag{53}$$

By letting γ_d (here d is for detector, since a calculation along γ_d yields the spectrum as measured by an observer whose worldline is γ_d) be an integral curve of $\frac{\partial}{\partial t}$ and defining $\tau \equiv \kappa t$, the radial part of the wave operator can be decomposed in the form

$$\Box_R = (\kappa\dot{\gamma}_d)^2 - \triangle_R = (\kappa\dot{\gamma}_d)^2 - \frac{1}{r^2}\left(1 - \frac{2m}{r}\right)\frac{\partial}{\partial r}\left(r^2\left(1 - \frac{2m}{r}\right)\frac{\partial}{\partial r}\right), \tag{54}$$

where the dot denotes $\frac{\partial}{\partial \tau}$.

Since the operator $\sqrt{\triangle_R}$ is on firm ground, the operator $\Box_R$ can be decomposed as

$$\Box_R = \left(\kappa\dot{\gamma}_d - \sqrt{\triangle_R}\right)\left(\kappa\dot{\gamma}_d + \sqrt{\triangle_R}\right).$$

In the following, we will investigate the spectral properties of the operator $\kappa\dot{\gamma}_d - \sqrt{\triangle_R}$, corresponding to the outgoing part of the radiation. Let us start by considering the first-order pseudodifferential operator (our conventions for numerical factors follow here our ref. [5])

$$P \equiv \sqrt{\triangle_R} + \mathrm{i}\frac{\kappa}{2\pi}\xi, \tag{55}$$

obtained by taking the Fourier transform with respect to the variable τ.

In order to study the spectrum of $\sqrt{\triangle_R}$, we consider the eigenvalue equation $\triangle_R u + \lambda^2 u = 0$, where u depends only on r. Upon denoting by a prime the differentiation with respect to r, this reads as

$$\left(r^2\left(1-\frac{2m}{r}\right)u'\right)' + \lambda^2 r^2\left(1-\frac{2m}{r}\right)^{-1} u = 0. \tag{56}$$

This ordinary differential equation implies that $u(2m) = 0$ while $u'(2m)$ is finite. At large r, eq. (56) reduces to

$$(r^2 u')' + \lambda^2 r^2 u = 0. \tag{57}$$

The general solution of eq. (57) that is bounded at infinity is [5]

$$u(r) = a\frac{\sin(\lambda r)}{r} + b\frac{\cos(\lambda r)}{r}, \tag{58}$$

where the parameters a and b are constant. From the condition $u(2m) = 0$ one finds

$$-\frac{b}{a} = \tan(2m\lambda), \tag{59}$$

and [5]

$$\lambda_n = \frac{\theta}{2m} + n\frac{\pi}{2m},\ n \in Z \tag{60}$$

is the large-r limit of the spectrum of $\triangle_R$, having set $\theta \equiv \arctan(-b/a)$. Since the action of the wave operator on smooth functions should be smooth on the hypersurface Σ, one also requires boundedness of u as $r \to 0^+$. This implies in turn that $b = \theta = 0$ and λ_n reduces to n upon rescaling the radial variable r.

The spectral ζ-function for the operator P defined in (55) is therefore expressed, in the large-r limit, by the asymptotic expansion

$$\widetilde{\zeta}_P(s) = \sum_{n=1}^{\infty}\left(n + \mathrm{i}\frac{\kappa}{2\pi}\xi\right)^{-s}, \tag{61}$$

where the tilde is used to denote removal of the degeneracy of the vanishing eigenvalue. By relabelling the lower limit of summation, we can re-express this spectral ζ-function in terms of the Hurwitz ζ-function, *i.e.* [5]

$$\widetilde{\zeta}_P(s) \sim \sum_{l=0}^{\infty} \left(l + 1 + \mathrm{i}\frac{\kappa}{2\pi}\xi \right)^{-s} = \zeta_H \left(s, 1 + \mathrm{i}\frac{\kappa}{2\pi}\xi \right). \tag{62}$$

If ω is the variable dual to ξ in the framework of Fourier transform, one can write that the Fourier transform with respect to ξ of $\widetilde{\zeta}_P(s)$ is approximated by [6]

$$\mathcal{F}(\widetilde{\zeta}_P(s)) \sim \frac{\left(\frac{2\pi}{\kappa}\right)^s \omega^{s-1}}{\Gamma(s)(\mathrm{e}^{2\pi\omega/\kappa} - 1)}. \tag{63}$$

If one takes the limit as $s \to 1$ and recalls, from ref. [7], that $\kappa = \frac{1}{4m}$ in the spacetime of a collapsing star, one finds

$$\mathcal{F}(\widetilde{\zeta}_P(s)) \sim \frac{8\pi m}{(\mathrm{e}^{8\pi m\omega} - 1)}. \tag{64}$$

The spectral density ρ is now given by

$$\rho = \kappa \mathcal{F}(\widetilde{\zeta}_P(s)) \sim \frac{2\pi}{(\mathrm{e}^{8\pi m\omega} - 1)}, \tag{65}$$

which is the famous result of Hawking in ref. [8].

10. – Concluding remarks

Pseudodifferential [9] and Fourier-Maslov integral operators [10] play a key role in the modern theory of elliptic and hyperbolic equations on manifolds, respectively, and the physical applications form an equally rich family, ranging from the Cauchy problem of classical field theory [11] to the Green functions of quantum field theory and black-hole physics, as we have shown.

Here we would like to add that, in the sixties, DeWitt discovered that the advanced and retarded Green functions of the wave operator on metric perturbations in the de Donder gauge make it possible to define classical Poisson brackets on the space of functionals that are invariant under the action of the full diffeomorphism group of spacetime. He therefore tried to exploit this property to define invariant commutators for the quantized gravitational field [12], but the operator counterpart of the classical Poisson brackets turned out to be a hard task. On the other hand, we know from sect. **1** that, rather than inverting exactly a partial differential operator, it is more convenient to build a parametrix. This makes it possible to solve inhomogeneous equations with the desired accuracy. Interestingly, it remains to be seen whether such a construction might be exploited in canonical quantum gravity, provided one understands what is the counterpart of classical smoothing operators in the quantization procedure.

* * *

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Appendix A.

Symbol map for partial differential operators; the space of pseudodifferential operators on $\mathbb{R}^n$

The definitions of sect. **1** are best suited to deal with the analysis of inhomogeneous partial differential equations $Pu = f$ and use a nomenclature very close to the one appropriate for Fourier-Maslov integral operators for hyperbolic equations. However, we should also recall the basic properties summarized below [13].

A linear partial differential operator P of order d on $\mathbb{R}^n$ is a polynomial expression

$$P(x, D) = \sum_{\alpha:|\alpha|\leq d} a_\alpha(x) D_x^\alpha, \; a_\alpha \in C^\infty(\mathbb{R}^n), \tag{A.1}$$

where, for the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, the modulus $|\alpha|$ and the derivative operator D_x^α are defined by

$$|\alpha| \equiv \alpha_1 + \cdots + \alpha_n, \; D_x^\alpha \equiv (-\mathrm{i})^{|\alpha|} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}. \tag{A.2}$$

The symbol of P is then defined by

$$\sigma(P) = \sigma(x, \xi) \equiv \sum_{\alpha:|\alpha|\leq d} a_\alpha(x) \xi^\alpha, \tag{A.3}$$

and is a polynomial of order d in the dual variable ξ, where (x, ξ) defines a point of the cotangent bundle of $\mathbb{R}^n$. The *leading symbol* is the highest order part of $\sigma(x, \xi)$, *i.e.*

$$\sigma_L(P) = \sigma_d(x, \xi) \equiv \sum_{\alpha:|\alpha|=d} a_\alpha(x) \xi^\alpha, \tag{A.4}$$

and the action of P can be re-expressed in integral form as

$$Pf(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\xi\cdot x} \sigma(x, \xi) \hat{f}(\xi) \mathrm{d}\xi. \tag{A.5}$$

In general, one can consider the set S^d of all symbols $\sigma(x, \xi)$ such that

i) σ is smooth in (x, ξ) with compact x support.

ii) For all multi-indices (α, β), there exist constants $C_{\alpha,\beta}$ for which

$$\left| D_x^\alpha D_\xi^\beta \sigma(x,\xi) \right| \leq C_{\alpha,\beta}(1+|\xi|)^{d-|\beta|}. \tag{A.6}$$

For $\sigma \in S^d$, one defines the associated operator ($\mathcal{S}$ being the Schwartz space of smooth complex-valued functions with fast decrease) $P : \mathcal{S} \to C_0^\infty(\mathbb{R}^n)$ as in (A.5), *i.e.*

$$Pf(x) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x}\sigma(x,\xi)\hat{f}(\xi)d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)}\sigma(x,\xi)f(y)dy\, d\xi. \tag{A.7}$$

The space ψ^d of such operators is the set of pseudo-differential operators of order d [14].

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