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New factorization relations for nonlinear sigma model amplitudes

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We obtain novel factorization identities for nonlinear sigma model amplitudes using a new integrand in the Cachazo-He-Yuan double-cover prescription. We find that it is possible to write very compact relations using only longitudinal degrees of freedom. We discuss implications for on shell recursion.

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I. INTRODUCTION

Cachazo, He and Yuan (CHY) invented in Ref. [1] a new method for calculating S-matrix elements. This formalism has numerous applications and many interesting connections, see for instance Refs. [2–4]. The CHY construction was formally proven by Dolan and Goddard in Ref. [5]. The main ingredients are the $CP^2$ "-formalism and nonlinear sigma model (NLSM) and provide an alternative formulation that employs a new integrand. Next, we will show how the double-cover formalism naturally factorizes this new CHY formulation in a surprising way.

II. A NEW CHY INTEGRAND

As explained in Ref. [13], the flavor-ordered partial $U(N)$ nonlinear sigma model amplitude in the scattering equation framework is given by the contour integral

$$A_n(\alpha) = \int d\mu_n H_n(\alpha),$$

where $\alpha = (\alpha(1), \ldots, \alpha(n))$ denotes a partial ordering. The integrand is given by

$$H_n(\alpha) = \text{PT}(\alpha) \times (\text{Pf}^A)^2,$$

$$\text{PT}(\alpha) \equiv \frac{1}{z_{\alpha(1)\alpha(2)}z_{\alpha(2)\alpha(3)} \cdots z_{\alpha(n)\alpha(1)}},$$

$$\text{Pf}^A \equiv \prod_{i<j} (-1)^{i+j} \frac{z_{ij}}{z_{ij}} \text{Pf}(A)^{ij}.$$  

Here $\text{PT}(\alpha)$ and $\text{Pf}^A$ are the Parke-Taylor (PT) factor and the reduced Pfaffian of matrix $A$, respectively. The $n \times n$ antisymmetric matrix, $A$, is defined as,

$$A_{ab} \equiv S_{ab} \quad \text{for } a \neq b, \quad \text{and } A_{ab} = 0 \quad \text{for } a = b.$$  

In general, $(A)^{i_1 \cdots i_p}_{j_1 \cdots j_p}$ denotes the reduced matrix obtained by removing the rows, $\{i_1, \ldots, i_p\}$, and columns, $\{j_1, \ldots, j_p\}$,

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from $A$. Note that when the number of external particles $n$ is odd, $\text{Pf'}A = 0$, and $A_n(\alpha)$ vanishes.

Using Eq. (5), we have

$$\text{(Pf'}A)^2 = \frac{(-1)^{i+j+m+p}}{z_{ij}z_{mp}} \text{Pf'}[(A)_{ij}] \times \text{Pf'}[(A)_{mp}].$$  \hspace{1cm} (7)

With the choice $\{i, j\} = \{m, p\}$, this product of Pfaffians becomes a determinant,

$$\text{(Pf'}A)^2 = -PT(m, p) \text{det}(A)_{mp}. \hspace{1cm} (8)$$

We will now discuss the following new matrix identities. On the support of the scattering equations and the massless condition, $\{S_\alpha = 0, k_\beta = 0\}$, we find when $m \neq p \neq q$

$$\text{Pf'}[(A)_{mp}] \times \text{Pf'}[(A)_{pq}] = \text{det}(A)_{pq}. \hspace{1cm} (9)$$

Note that in Eqs. (11) and (12) we have reduced the $A$ matrix with the indices $\{i, j, k\}$ associated with the Faddeev-Popov determinant. This gauge choice will be convenient later. We now have the following equality,

$$A'_n(\alpha) = \int d\mu_n PT(\alpha) \frac{(-1)^{i+j+k}}{z_{ij}z_{jk}} \text{det}(A)_{jk}. \hspace{1cm} (11)$$

Note that in Eqs. (11) and (12) we have reduced the $A$ matrix with the indices $\{i, j, k\}$ associated with the Faddeev-Popov determinant. This gauge choice will be convenient later. We now have the following equality,

$$A'_n(\alpha) = A_n(\alpha), \hspace{2cm} (13)$$

when all particles are on shell. When there are off shell particles, the identity is true only if the number of particles is even. When the number of particles is odd and there are off shell particles, one has $A_n(\alpha) = 0$ while $A'_n(\alpha) \neq 0$. Since the $A$ matrix has the corank 2 on the support of the scattering equations and the massless condition, $\{S_\alpha = 0, k_\beta = 0\}$, $A'_{n}(\alpha)$ vanishes trivially. However, when there are off shell particles the amplitude $A'_{n}(\alpha)$ is no longer zero.

These observations will be crucial in obtaining the new factorization relations.

**III. THE DOUBLE-COVER REPRESENTATION**

In the double-cover version of the CHY construction, the $n$-point amplitude is given as a contour integral on the double-covered Riemann sphere with $n$ punctures. The pairs $(\sigma_1, y_1), (\sigma_2, y_2), \ldots, (\sigma_n, y_n)$ provide the new set of doubled variables restricted to the curves

$$0 = C_a \equiv y_a^2 - \sigma_a^2 + \Lambda^2 \quad \text{for } a = 1, \ldots, n.$$  \hspace{1cm} (14)

A translation table has been worked out in detail in Ref. [11]. The double-cover formulation of the NLSM is given by the integral

$$A_n(\alpha) = \int d\mu_n \frac{(-1)^{i+j+k}}{S_r^\alpha} \times T_n(\alpha),$$

$$\text{det}(A)_{ij} = \text{det}(A)_{pq}.$$  \hspace{1cm} (9)

Note that $\text{det}(A)_{pq} = 0$ if $n$ odd. \hspace{1cm} (10)

A proof of these identities will be provided in Ref. [14]. Using the non-anti-symmetric matrix, $(A)_{ij}$, we define the objects (with $i < j < k$)

$$A'_n(\alpha) = \int d\mu_n PT(\alpha) \frac{(-1)^{i+j+k}}{z_{ij}z_{jk}} \text{det}(A)_{jk}.$$  \hspace{1cm} (11)

The $\Gamma$ contour is defined by the $2n - 3$ equations

$$\Lambda = 0 \quad \text{and} \quad C_a = 0, \hspace{1cm} \text{for } d \neq \{i, j, k\} \text{ and } a = 1, \ldots, n.$$  \hspace{1cm} (17)

The integrand is given by

$$T_n(\alpha) = -PT(\alpha) \prod_{a=1}^n \frac{y_{\sigma a}}{y_a} PT(m, p) \text{det}(A)_{mp}.$$  \hspace{1cm} (18)

where $(y_{\sigma a}) = y_a + \sigma_a$. To obtain the kinematic matrix and the Parke-Taylor factors we need to do the following replacements

$$A \rightarrow A^\Lambda, \quad \text{and} \quad PT \rightarrow PT^T \text{ for } z_{ab} \rightarrow T^{-1}_{ab},$$

$$PT \rightarrow PT^T \text{ for } z_{ab} \rightarrow \tau(a, b)^{-1},$$

with $T_{ab} = \frac{1}{(y_{\sigma a} - y_{\sigma b})}$. Analogous to Eq. (11), we can now write down a new form for the integrand

$$T'_n(\alpha) = PT(\alpha) \prod_{a=1}^n \frac{y_{\sigma a}}{y_a} (-1)^{i+j+k} T_{ij} T_{jk} \text{det}(A)_{jk}.$$  \hspace{1cm} (21)
IV. FACTORIZATION

Let us start by considering the four-point amplitude, $A_4'(1,2,3,4)$, with the gauge fixing $(ijkr) = (123|4)$. We will denote sums of cyclically-consecutive external momenta (modulo the total number of external momenta) by $P_{i:j} \equiv k_i + k_{i+1} + \cdots + k_{j-1} + k_j$. For expressions involving only two (not necessarily consecutive) momenta, we use the shorthand notation $P_{ij} \equiv k_i + k_j$. We focus on the configuration where the sets of punctures $(\sigma_1, \sigma_2)$ and $(\sigma_3, \sigma_4)$ are respectively on the upper and the lower sheet of the curves

\[
\begin{align*}
(y_1 &= +\sqrt{\sigma_1^2 - \Lambda^2}, \sigma_1), \\
(y_2 &= +\sqrt{\sigma_2^2 - \Lambda^2}, \sigma_2), \\
(y_3 &= -\sqrt{\sigma_3^2 - \Lambda^2}, \sigma_3), \\
(y_4 &= -\sqrt{\sigma_4^2 - \Lambda^2}, \sigma_4).
\end{align*}
\]  

(22)

Expanding all elements in $A_4'(1,2,3,4)$ around $\Lambda = 0$, we obtain (to leading order)

\[
\begin{align*}
PT^*(1,2,3,4) &\bigg|_{1,3,4} = \frac{\Lambda^2}{2} \frac{1}{\sigma_{12}\sigma_{23}\sigma_{P_{34}}\sigma_{P_{34}}} \frac{1}{\sigma_{P_{34}}} \\
\Delta(123)\Delta(123) &\bigg|_{1,3,4} = \frac{\Lambda^2}{2} \frac{1}{\sigma_{12}\sigma_{23}\sigma_{P_{34}}\sigma_{P_{34}}} \frac{1}{\sigma_{P_{34}}} \\
&\quad \times \frac{1}{\sigma_{P_{34}}\sigma_{P_{34}}} \\
&\quad \times (\sigma_{P_{13}}\sigma_{34}\sigma_{P_{12}})^2,
\end{align*}
\]  

(23)

where we have introduced the new fixed punctures $\sigma_{P_{34}} = \sigma_{P_{12}} = 0$. Since we want to arrive at factorization identity for nonlinear sigma model amplitudes (inspired by previous work for Yang-Mills theories, see Ref. [15]) we are now going to introduce polarizations associated with the punctures, $\sigma_{P_{34}} = \sigma_{P_{12}} = 0$, i.e., $e_3^M$ and $e_2^M$. Thus,

\[
s_{14} = 2(k_1 \cdot k_4) = 2(k_1 \times n_\mu \times k_4 \cdot e_4^M)
\]

(25)

employing,

\[
\sum_M e_i^M e_j^M = n_\mu^\mu.
\]

(26)

After separating the labels $\{1,2\}$ and $\{3,4\}$, it is simple to rearrange Eq. (24) as a product of two reduced determinants, therefore

\[
\begin{align*}
\begin{bmatrix}
\frac{1}{\sigma_{12}\sigma_{23}\sigma_{P_{34}}} \frac{1}{\sigma_{P_{34}}} \\
\frac{1}{\sigma_{P_{13}}\sigma_{34}\sigma_{P_{12}}}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\sigma_{P_{34}}} \\
\frac{1}{\sigma_{P_{34}}}
\end{bmatrix}
\begin{bmatrix}
\sqrt{2k_1 \cdot e_1^M} \\
\sqrt{2k_4 \cdot e_4^M}
\end{bmatrix}
\end{align*}
\]

The new matrices in Eq. (27) can be obtained from the $A$ matrix by replacing the off shell momenta, $P_{34}$ and $P_{12}$, by their corresponding off shell polarization vectors,

\[
det(A_{\frac{12}{34}}) \to \det \left[ \frac{\sqrt{2k_1 \cdot e_1^M}}{\sigma_{P_{34}}} \right] \quad \text{for } P_{34} \to \frac{1}{\sqrt{2}} e_3^M, \quad \text{(28)}
\]

\[
det(A_{\frac{P_{12}}{34}}) \to \det \left[ \frac{\sqrt{2k_4 \cdot e_4^M}}{\sigma_{P_{12}}} \right] \quad \text{for } P_{12} \to \frac{1}{\sqrt{2}} e_2^M, \quad \text{(29)}
\]

where the $A$ matrix in Eq. (28) is the $3 \times 3$ matrix related with the punctures $(\sigma_1, \sigma_2, \sigma_{P_{34}})$, while the matrix in Eq. (29) corresponds to the punctures $(\sigma_{P_{12}}, \sigma_3, \sigma_4)$.

Using the measure, $dA^\Lambda = \frac{1}{2} dA^\Lambda$, we now perform the $A$ integral and the amplitude becomes

\[
A_4'(1,2,3,4) \bigg|_{1,3,4} = \frac{1}{2} \sum_M A_3' (1,2,3,4) \times A_3' (P_{12},3,4) \to \frac{s_{14}}{2}.
\]

(30)

where the notation, $P_{12}^M$, means one must make the replacement, $P_{12} \to \frac{1}{\sqrt{2}} e_2^M$, and use Eq. (26). The overall factor $1/2$ cancels out after summing over mirrored configurations, i.e., $A_4'(1,2,3,4) \bigg|_{1,3,4} + A_4'(1,2,3,4) \bigg|_{1,2} = s_{14}$. Following the integration rules in Ref. [15], we also have the contribution (up to summing over mirrored configurations)

\[
A_4'(1,2,3,4) \bigg|_{1,3} = \frac{1}{2} \sum_M A_3' (P_{12}^M,3,4) \times A_3' (P_{41},2,3) \to \frac{s_{14}}{2}.
\]

(31)
Thus, the final result is
\[
A'_4(1,2,3,4) = \sum_M A'_4(1,2, P'^M_{34};1,2) \times A_3^{(P_{12};3)}(P'^M_{12};3,4) \quad s_{12}
\]
\[
+ A_3^{(P_{12};3)}(1, P'^M_{23};4) \times A'_3(P'^M_{23};2,3) \quad s_{14}
\]
\[ = -s_{13}. \quad (32)\]

The four-point amplitude is factorized in terms of three-point functions. The general three-point functions where some or all particles can be off shell, are
\[
A'_4(P_a, P_b, P_c) = s_{P_a P_b} = -(P_a^2 - P_b^2 + P_c^2),
\]
\[
A_3^{(P_{12};3)}(P_a, P_b, P_c) = s_{P_a P_b} s_{P_b P_c} = (P_a^2 - P_b^2 + P_c^2).
\]

Since the nonlinear sigma model is a scalar theory it is an interesting proposition to consider longitudinal degrees of freedom only
\[
\det L = \sum_L \sum_{i,j} \frac{k_i^0 k_j^0}{k_i \cdot k_j}. \quad (35)
\]

Doing so we arrive at the equation
\[
A'_4(1,2,3,4) = 2 \sum_{L} \left[ (-1)^3 A'_4(1,2, P'^M_{34};1,2) \times A_3^{(P_{12};3)}(P'^M_{12};3,4) \quad s_{12}
\]
\[+ (-1)^3 A_3^{(P_{12};3)}(1, P'^M_{23};4) \times A'_3(P'^M_{23};2,3) \quad s_{14}\]
\[= -s_{13}. \quad (36)\]

Surprisingly, it is possible to generalize this equation to higher point amplitudes. Here the overall sign of each contribution depends of the number of points of the subamplitudes. In Ref. [14], we will give more details on this phenomenon.

V. NEW RELATIONS

As will be shown in great detail elsewhere [14], using the double-cover prescription for a partial nonlinear sigma model amplitude one is led to the following general formula where an \( n \)-point amplitude is factorized into a product of two (single-cover) lower-point amplitudes:
\[
A'_n(1,2,3,4,\ldots,n)
\]
\[= \sum_{i=4,M} \frac{A'_n(P_{i+1};i+1,\ldots,n)A_3^{(P_{i+1};3)}(P'^M_{i+1};2,3,\ldots,i)}{s_{i+1:2}}\]
\[+ \frac{A'_3(P'^M_{i+1};2,3) \times A_3^{(P_{i+1};3)}(1, P'^M_{i+1};3,4,\ldots,n)}{s_{i+1:2}} \quad (37)\]

Here \( n \) is an even integer and we have used Eq. (26). The above expression is valid using the Möbius and scale-invariance gauge choice \((ijk|r) = (123|4)\). From the decomposition obtained by the double-cover method in Eq. (37), we are able to write down a new factorization relation, where only longitudinal degrees of freedom contribute,
\[
A'_n(1,2,3,4,\ldots,n)
\]
\[= 2 \sum_{i=4,L} \frac{A'_n(P';i+1;3,4,\ldots,n)A_3^{(P_{i+1};3)}(P'^{L}_{i+1};2,3,\ldots,i)}{s_{i+1:2}}\]
\[+ \frac{A'_3(P'^{L}_{i+1};2,3) \times A_3^{(P_{i+1};3)}(1, P'^{L}_{i+1};3,4,\ldots,n)}{s_{i+1:2}} \quad (38)\]

where Eq. (35) was used. We checked this formula up to ten points.

Since the above factorization relation includes only longitudinal contributions, we can rewrite it in a more elegant form, involving only the \( A'_q \) amplitudes. Using the definitions given in Eqs. (11)–(12) and under the gauge fixing \((ijk)\), with \( i < j < k \), we have the following two identities [14]
\[
A_q^{(ij)}(\ldots P_i, \ldots) = P^2_i A_q^{(ij)}(\ldots P_i, \ldots), \quad q = 2m + 1
\]
\[
A_q^{(ij)}(\ldots P_i, \ldots) = -P^2_i A_q^{(ij)}(\ldots P_i, \ldots), \quad q = 2m. \quad (39)\]

where \( P^2_i \neq 0 \). In addition, \( A_q^{(ij)} \) satisfies the useful identities
\[
A_q^{(ij)}(1,\ldots,i,P_i,\ldots,k,q) = A_q^{(jk)}(1,\ldots,i,P_j,\ldots,k,q),
\]
\[
A_q^{(ij)}(1,\ldots,i,P_j,\ldots,k,q) = A_q^{(ij)}(2,\ldots,i,P_j,\ldots,k,q),
\]
\[\ldots = A_q^{(jk)}(\ldots P_j,\ldots,k,q,q), \ldots, (40)\]

Applying the identities Eqs. (39)–(40), it is straightforward to obtain
\[
A'_n(1,2,3,4,\ldots,n)
\]
\[= \sum_{i=4}^{n} \frac{A'_n(P_{i+1};i+1,\ldots,n)A_3^{(P_{i+1};3)}(P'^{L}_{i+1};2,3,\ldots,i)}{s_{i+1:2}}\]
\[+ \frac{A'_3(P'^{L}_{i+1};2,3) \times A_3^{(P_{i+1};3)}(1, P'^{L}_{i+1};3,4,\ldots,n)}{s_{i+1:2}} \quad (41)\]
where the factorization formula has been written in terms of the generalized amplitude $A'_n$. Other gauge choices will naturally with lead to alternative factorization formulas.

**A. BCFW recursion**

It is interesting to analyze the new factorization identities in comparison with expressions originating from the Britto-Cachazo-Feng-Witten (BCFW) recursion \[16\]. We introduce the momentum deformation

$$k'^n(z) = k^n - zq^n, k'^n(z) = k^n - zq^n, z \in \mathbb{C}, \quad (42)$$

where $q^n$ satisfies $k_2 \cdot q = k_3 \cdot q = q \cdot q = 0$. Deformed momenta are conserved and on shell: $k_1 + k_2(z) + k_3(z) + k_4 + \cdots + k_n = 0$ and $k'^2(z) = k^2(z) = 0$. We consider the general amplitude, $A_n(1, \ldots, n)$, where $n$ is an even integer. From Eq. (41) using Cauchy’s theorem we have

$$A_n(1, 2, \ldots, n)$$

$$= - \sum_{i=1}^{n/2} \text{Res}_{z=a_i} A'_{n-2i+4}(1, 2, P_{3:2i-1}, 2i, \ldots, n)$$

$$\times A'_{2i-2}(P_{3:2i}, 3, \ldots, 2i-1)$$

$$\frac{zP_{2i-2}(z)}{P_{2i+2}(z)}$$

$$- \text{Res}_{z=\infty} \frac{A'_n(1, 2, \ldots, n)(z)}{z}. \quad (43)$$

Only the even amplitudes, namely $A'_{2q}$, contribute to the physical residues. This is simple to understand as we have the identity, $A_{2q}(1, \ldots, 2q) = A'_{2q}(1, \ldots, 2q)$, so only subamplitudes with an even number of particles produce physical factorization channels. On the other hand, when the number of particles is odd, the off shell ($P^2_i \neq 0$) amplitude, $A'_{2q+1}(...)P_i(...)$, is proportional to $P^2_i$, since it must vanish when all particles are on shell. So, the poles, $P^2_{2i+1}, i = 3, \ldots, \frac{n}{2} + 1$ and $P_{2m}$, are all spurious and the subamplitudes with an odd number of particles only contribute the boundary term at $z = \infty$.

Finally, it is important to remark that after evaluating the residues, $P^2_{2i+1}(z) = 0$, in Eq. (43), one obtains extra nonphysical contributions, which cancel out combining with terms associated with the residue at $z = \infty$. Therefore, the effective boundary contribution is just given by the subamplitudes with an odd number of particles

$$\text{Res}_{z=\infty} \left[ A'_n(1, 2, \ldots, n)(z) \right]$$

$$= \partial_z \sum_{i=1}^{n/2} A'_{n-2i+4}(1, 2, P_{3:2i-1}, 2i, \ldots, n)$$

$$\times A'_{2i-2}(P_{3:2i}, 3, \ldots, 2i-1)$$

$$\frac{zP_{2i-2}(z)}{P_{2i+2}(z)}$$

$$+ A'_n(P_{4:1}, 2, 3) \times A'_{2m+1}(P_{23}, 4, \ldots, n, 1) \quad (44)$$

**VI. CONCLUSIONS**

We have proposed a new CHY integrand for the $U(N)$ nonlinear sigma model. For this new integrand, the kinematic matrix, $(A)_{ij}$, is no longer antisymmetric. We have found two new factorization identities, Eqs. (37) and (38). We have written the second factorization formula in an elegant way, which only involves the generalized amplitude, $A'_n$. This formula turns out to be surprisingly compact (we have checked agreement of the soft limit of this formula with Ref. [17]).

This has implications for the BCFW recursion since the two new factorization formulas can be split among even and odd subamplitudes, e.g., $A'_{2q} \times A'_{2m}$ and $A'_{2q+1} \times A'_{2m+1}$ respectively. Using this we are able to give a physical meaning to the odd subamplitudes as boundary contributions under such recursions.

Work in progress \[14\] is going to present a new recurrence relation and investigate its connection to Berends-Giele \[18\–22\] currents and Bern-Carrasco-Johansson (BCJ) numerators \[23–25\]. Similar relations for other effective field theories \[13,17,22\] are expected and will be another focus.

Despite similarities between the three-point amplitudes with the Feynman vertices obtained in Ref. [26], the construction presented here is different. For example, the numerators found in Eq. (32) are not reproduced by the Feynman rules found in Ref. [26]. Understanding the relationship between the formalisms would be interesting.

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