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Abstract. Recently Yu. Bilu, P. Habegger and L. Kühne proved that no singular modulus can be a unit in the ring of algebraic integers. In this paper we study for which sets $S$ of prime numbers there is no singular modulus that is an $S$-unit. Here we prove that when the set $S$ contains only primes congruent to 1 modulo 3 then no singular modulus can be an $S$-unit. We then give some remarks on the general case and we study the norm factorizations of a special family of singular moduli.

1. Introduction

Elliptic curves with complex multiplication and their $j$-invariants, the so-called singular moduli, have been studied for a long time in number theory because of their many interesting arithmetic properties. For instance, singular moduli are always algebraic integers and they generate ring class fields relative to orders in imaginary quadratic fields ([3]). In 1985, B. Gross and D. Zagier proved, under certain technical assumptions, a formula concerning the prime factorization of the difference of two singular moduli ([5]). This formula has been recently generalized by K. Lauter and B. Viray in [9].

A few years ago David Masser, motivated by some effective results of André-Oort type (see [2], [7]), raised the question whether it is possible that a singular modulus is a unit in the ring of algebraic integers. P. Habegger has shown in [6] that there exists at most a finite number of singular moduli that are algebraic units. However the argument used in the proof is ineffective since it relies on an equidistribution result. In a more recent work of Yu. Bilu, P. Habegger and L. Kühne the answer to this question has been completely settled with the proof of the following

Theorem 1.1 (Theorem 1.1 in [1]). There are no singular moduli that are units in the ring of algebraic integers.

The theorem above opened the way to a number of interesting question. For instance one may ask, inspired by the work of Gross and Zagier, whether there are pairs of singular moduli whose difference is a unit. Work in this direction appears in [13], where Y. Li proves that the difference of two singular moduli of fundamental discriminant is never a unit.

Another possible research path is the following: let $S$ be a finite set of prime numbers. What is the number of singular moduli that are $S$-units? By the theorem above, when

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S = ∅ the answer is zero. However when S ≠ ∅ there can certainly exist singular moduli that are S-units. For instance, when S = {2, 3}, the integers 123, −323 and −963 are three singular moduli that are \(\{2,3\}\)-units (they are the \(j\)-invariants of elliptic curves having complex multiplication by \(\mathbb{Z}[i], \mathbb{Z}\left[\frac{1+i\sqrt{-11}}{2}\right]\) and \(\mathbb{Z}\left[\frac{1+i\sqrt{-19}}{2}\right]\) respectively). We are then looking for some finiteness statement that will in general depend on the choice of the set S. Recently Herrero, Menares and Rivera-Letelier announced a proof of the finiteness of the set of singular moduli that are S-units for any finite set of primes S. However, to the best of our knowledge, their argument is not effective. In this paper we give the first effective results in this direction; in particular our main theorem is the following:

**Theorem 1.2.** Let S be the set of rational primes congruent to 1 modulo 3. If a singular modulus is an S-unit, then it is a unit.

Combining Theorem 1.2 with Theorem 1.1 above we immediately get

**Corollary 1.3.** Let S be the set of rational primes congruent to 1 modulo 3. Then no singular modulus is an S-unit.

Notice that the set of primes considered in Corollary 1.3 is infinite: hence the corollary gives an effective answer to the problem of singular S-units for infinitely many finite sets S of rational primes.

As we will see, the proof of Theorem 1.2 naturally leads to the study of \(j\)-invariants of elliptic curves that have complex multiplication by \(\mathbb{Q}(\sqrt{-3})\). Hence in the final part of this paper some properties of this family of singular moduli are pointed out and studied.

We develop our article as follows. In section 2 we fix the notation and we recall some theorems from the theory of complex multiplication. In section 3 we give the proof of Theorem 1.2. The main ingredient of the proof will be Deuring’s reduction theory for CM elliptic curves. With the same techniques we also prove that for every finite set S of prime numbers there is always a singular modulus that is not an S-unit. In section 4, motivated also by numerical computations, we discuss some properties of the singular \(j\)-invariants relative to orders in \(\mathbb{Q}(\sqrt{-3})\). Here we will make heavy use of the formulas proved in [9]. As an appendix we include a table containing some explicit factorizations for the norms of these \(j\)-invariants. We dedicate a small final section to the study of the case \(j = 1728\), kindly suggested by P. Habegger.

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2. Preliminaries and notation

The main references for this section are [3], [8], [11] and [12]. We begin by recalling the notion of S-units. Let \(K\) be a number field and S a finite set of rational primes. We
say that an element $x \in K$ is an $S$-unit if, for every prime $p \not\in S$ and for every prime $\mathfrak{p}$ of $K$ lying over $p$, the element $x$ is a unit in the ring of integers of $K_\mathfrak{p}$ (here $K_\mathfrak{p}$ denotes the completion of $K$ at the prime $\mathfrak{p}$). In other words $x$ is an $S$-unit if and only if all the primes appearing in the prime factorization of $x\mathcal{O}_K$ lie above primes in $S$. When $x$ is an algebraic integer, this is equivalent to require that all the primes dividing its absolute norm over $\mathbb{Q}$ are in $S$.

The principal object of study of this paper are singular moduli. A singular modulus is the $j$-invariant of an elliptic curve defined over $\mathbb{C}$ with complex multiplication. For a positive integer $D$ congruent to 0 or 3 modulo 4, we say that a singular modulus is of discriminant $-D$ if it is the $j$ invariant of an elliptic curve with complex multiplication by an order of discriminant $-D$. As we already recalled in the introduction, singular moduli are algebraic integers. For a singular modulus of discriminant $-D$ we will denote by $H_D(x) \in \mathbb{Z}[x]$ its minimal polynomial over $\mathbb{Q}$ and we call it the Hilbert class polynomial of discriminant $-D$. It is well known that the roots of the Hilbert class polynomial of discriminant $-D$ are all the singular moduli of discriminant $-D$. In particular $\deg H_D(x) = h_D$, where $h_D$ is the class number of the unique order of discriminant $-D$, and all the singular moduli of discriminant $-D$ have the same absolute norm over the rationals. Indeed, if $N_{L/\mathbb{Q}}(\cdot)$ denotes the usual norm map from a number field $L$ to $\mathbb{Q}$, we see that $|N_{\mathbb{Q}(j)/\mathbb{Q}}(j)| = |H_D(0)|$ for every $j$ of discriminant $-D$. We finally recall that a discriminant $-D$ is called a fundamental discriminant if it is the discriminant of a maximal order in an imaginary quadratic field. If $j$ is a singular modulus relative to a fundamental discriminant $-D$ then $\mathbb{Q}(\sqrt{-D}, j)$ is the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$.

In the sequel we will need some theorems from reduction theory of elliptic curves with complex multiplication. We recall here the main results that will be used in the article. If $E/L$ is an elliptic curve with complex multiplication defined over a number field $L$, then $E$ has potential good reduction at every prime of $L$ ([12], II. Theorem 6.4). This means that for every prime $\mathfrak{p}$ of $L$ there exists a finite extension $F/L$ such that $E$ has good reduction at every prime of $F$ lying over $\mathfrak{p}$. The reduction type of CM elliptic curves was first described by M. Deuring in [4] and it is nowadays known as "Deuring theory". We recall the following two theorems from that theory, that are respectively Theorem 13.21 in [3] and Theorem 13.12 in [8].

**Theorem 2.1**. Let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be orders in imaginary quadratic fields $K_1$ and $K_2$ respectively, and for $i = 1, 2$ let $\mathfrak{a}_i$ be a proper fractional $\mathfrak{O}_i$-ideal such that $j(\mathfrak{a}_1) \neq j(\mathfrak{a}_2)$. Suppose that $L$ is a number field containing $j(\mathfrak{a}_1)$ and $j(\mathfrak{a}_2)$, and let $\mathfrak{P}$ be a prime of $L$ lying over a rational prime $p$. When $K_1 = K_2$ assume in addition that $p$ divides neither of the conductors of $\mathfrak{O}_1$ and $\mathfrak{O}_2$. If $j(\mathfrak{a}_1) \equiv j(\mathfrak{a}_2) \mod \mathfrak{P}$ then $p$ does not split completely in neither $K_1$ nor $K_2$.

**Theorem 2.2**. Let $E$ be an elliptic curve defined over a number field $L$ and with complex multiplication by an order $\mathfrak{O}$ in an imaginary quadratic field $K$. Let $\mathfrak{P}$ be a prime of $L$ lying over a rational prime $p$ where $E$ has non-degenerate reduction $\tilde{E}$. Then the curve $\tilde{E}$ is a supersingular elliptic curve if and only if $p$ does not split completely in $K$.

These theorems link the splitting of primes in imaginary quadratic fields to the reduction of elliptic curves with complex multiplication and they will be crucial in the proof of our main result.
3. Proof of Theorem 1.2 and remarks on singular S-units

Throughout this section, for a given finite set of rational primes $S$, a singular modulus which is an $S$-unit will be called a singular $S$-unit.

Proof of Theorem 1.2. Fix a set $S$ as in the statement of the theorem. First we show that every singular $S$-unit corresponding to a quadratic field $K \neq \mathbb{Q}(\sqrt{-3})$ is in fact a unit. This follows almost immediately from Theorem 2.1.

Let $j$ be a singular modulus corresponding to a field $K \neq \mathbb{Q}(\sqrt{-3})$, let $L = \mathbb{Q}(j)$ and consider a prime $P$ of $L$ lying over a rational prime $p$ congruent to 1 modulo 3. Notice that $j' = 0$ is the unique singular modulus corresponding to the maximal order of the quadratic field $\mathbb{Q}(\sqrt{-3})$ and that $p$ splits completely in the latter. Then by Theorem 2.1 it follows that the prime $P$ cannot divide $j$ otherwise we would have $j \equiv j' \mod P$, which contradicts the fact that $p$ splits completely in $\mathbb{Q}(\sqrt{-3})$. In particular every singular $S$-unit relative to an imaginary quadratic field of discriminant less than $-3$ is in fact a unit.

We are left to study all the singular $S$-units relative to the field $\mathbb{Q}(\sqrt{-3})$. For these we are going to prove the following

Claim 3.1. Let $j$ be a singular modulus relative to an order $\mathcal{O}_j$ in $\mathbb{Q}(\sqrt{-3})$. Then the primes $2, 3$ and $5$ divide $N_{\mathbb{Q}(j)/\mathbb{Q}}(j)$, where $N_{\mathbb{Q}(j)/\mathbb{Q}}(\cdot)$ denotes the usual norm function on number fields.

It is clear that Theorem 1.2 follows immediately from the claim. Indeed a prime $p$ divides $N_{\mathbb{Q}(j)/\mathbb{Q}}(j)$ if and only if there exists a prime $\mathfrak{p}$ in $\mathbb{Q}(j)$ lying over $p$ such that $\mathfrak{p}$ divides $j$.

We prove Claim 3.1 only for the prime $p = 3$, the other cases being analogous. Fix $j$ as in the statement of the claim and let $E/L$ be an elliptic curve defined over a number field $L$ with complex multiplication by the order $\mathcal{O}_j$. Since $E$ has potential good reduction at every prime of $L$ (Theorem II, 6.4 in [12]) we may assume that $E$ has good reduction at every prime $\mathfrak{p}$ of $L$ lying over 3. Fix such a prime: since 3 ramifies in $\mathbb{Q}(\sqrt{-3})$, by Theorem 2.2 the reduced elliptic curve $\tilde{E} = E \mod \mathfrak{p}$ is supersingular. However by Theorem V, 4.1 (c) in [11] there is only one isomorphism class of supersingular elliptic curves over $\mathbb{F}_3$, one representative being given by

$$E_0 : y^2 = x^3 + x.$$  

Now we have $j(E_0) = 1728 \equiv 0 \mod 3$ and then

$$j \mod \mathfrak{p} = j(\tilde{E}) = j(\tilde{E}) = j(E_0) = 0$$

where $\sim$ denotes the reduction modulo $\mathfrak{p}$. We deduce that $\mathfrak{p}$ divides $j$ and by the discussion above this concludes the proof.

When $p = 2$ or $p = 5$ the argument can be repeated; in these cases the only supersingular elliptic curves over $\mathbb{F}_p$ are given by

$$E_1 : y^2 + y = x^3$$

when $p = 2$ and by

$$E_2 : y^2 = x^3 + 1$$

when $p = 5$. In both cases the $j$-invariant is zero. $\square$
In Table 1 we have collected the factorizations of $N_{\mathbb{Q}(j)/\mathbb{Q}}(j)$ for all the singular moduli $j$ of discriminant $-3f^2$ with $f = 1, \ldots, 50$. The prime factorizations in the table agree with the statement made in Claim 3.1.

Note that the result in Corollary 1.3 is a special effective case of Theorem 1.1 for $S$-units. As we already mentioned in the introduction, Herrero, Menares and Rivera-Letelier announced that, for every finite set of primes $S$, there are at most finitely many singular $S$-units. The details are not accessible so far. We then give here a much weaker statement than the one mentioned above, which however provides evidence for the stronger claim. The main ingredient of the proof, besides Theorem 1.1, is Deuring Theory.

**Proposition 3.2.** Let $S = \{p_1, \ldots, p_r\}$ be a finite set of rational primes. Then there exists a singular modulus that is not an $S$-unit.

**Remark 3.3.** When $S$ does not contain 2 the above result is clear. Indeed consider any fundamental discriminant $-D \equiv 0 \mod 4$: then 2 ramifies inside $\mathbb{Q}(\sqrt{-D})$ and by Theorem 2.2 it is a prime of supersingular reduction for any elliptic curve with complex multiplication by the ring of integers in $\mathbb{Q}(\sqrt{-D})$. Since 0 is the only supersingular invariant modulo 2, we deduce that the singular moduli of discriminant $D$ cannot be $S$-units. Hence we can assume in the proof that $2 \in S$.

**Proof.** We know by Theorem 1.1 that no singular modulus is a unit. In particular there is always a rational prime that divides the norm of any singular modulus. If $j$ is a singular modulus of discriminant $-D$ and $p$ divides $N_{\mathbb{Q}(j)/\mathbb{Q}}(j)$, then by Theorem 2.1 the prime $p$ cannot split completely in $\mathbb{Q}(\sqrt{-D})$. The idea for the proof of the proposition is then to find a (fundamental) discriminant $-D$ such that all the primes in $S$ split completely in $\mathbb{Q}(\sqrt{-D})$. In this way the set of primes dividing the norm of any singular modulus of discriminant $-D$ (this set is nonempty by the above discussion) has trivial intersection with $S$.

Let $q$ be a prime number such that
- $q \equiv -1 \mod p_i$ for every $p_i \in S$.
- $q \equiv -1 \mod 8$.

We know that such a prime exists by Dirichlet’s theorem on primes in arithmetic progression and the Chinese reminder theorem. We claim that in $\mathbb{Q}(\sqrt{-q})$ all the primes of $S$ split completely. First of all notice that $\text{disc } \mathbb{Q}(\sqrt{-q}) = -q$ because clearly $q$ is squarefree and $-q \equiv 1 \mod 4$ by assumption. To prove that every prime in $S$ splits in this field we want to compute the Kronecker symbols $\left(\frac{-q}{p_i}\right)$. We have three cases:

- If $p_i = 2$ for some $i$ then $\left(\frac{-q}{p_i}\right) = 1$ because $-q \equiv 1 \mod 8$.
- If $p_i \equiv 1 \mod 4$ then
  \[\left(\frac{-q}{p_i}\right) = \left(\frac{-1}{p_i}\right) \left(\frac{q}{p_i}\right) = (-1)^{\frac{p_i-1}{2}} \cdot \left(\frac{q}{p_i}\right) = \left(\frac{-1}{p_i}\right) = 1\]
  where we used the fact that $-1$ is a square modulo $p_i$.
- If $p_i \equiv 3 \mod 4$ then
  \[\left(\frac{-q}{p_i}\right) = \left(\frac{-1}{p_i}\right) \left(\frac{q}{p_i}\right) = (-1)^{\frac{p_i-1}{2}} \cdot \left(\frac{q}{p_i}\right) = (-1) \cdot \left(\frac{-1}{p_i}\right) = 1\]
  where we used the fact that $-1$ is not a square modulo $p_i$. 

Since all the Kronecker symbols above are equal to 1, we deduce that all the primes in $S$ split completely in $\mathbb{Q}(\sqrt{-\eta})$. This proves the proposition. \hfill \Box

4. On singular moduli of discriminant $d = -3f^2$

We have seen that the proof of Theorem 1.2 naturally leads to the study of singular moduli relative to orders $\mathcal{O}_D \subseteq \mathbb{Q}(\sqrt{-3})$ i.e. of singular moduli of discriminant $-3f^2$, $f \in \mathbb{N}_{>0}$ being the conductor of the corresponding order. We have collected some factorizations for the absolute value of the norm of these singular moduli in Table 1. By looking at the table one immediately notices a main difference between the singular moduli in this family and all other $j$-invariants. Indeed we know (and we showed in the proof of Theorem 1.2) that for a singular modulus $j$ relative to an order $\mathcal{O} \not\subseteq \mathbb{Q}(\sqrt{-3})$ the primes dividing $N_{\mathcal{O}_U}/\mathbb{Q}(j)$ cannot be congruent to 1 modulo 3. The situation is different for singular moduli of discriminant $-3f^2$. For instance we have:

\[
\begin{align*}
|N_{\mathcal{O}_U}/\mathbb{Q}(j_{21})| &= 2^{10} \cdot 3^9 \cdot 5^6 \cdot 7 \cdot 17^3 \\
|N_{\mathcal{O}_U}/\mathbb{Q}(j_{39})| &= 2^{36} \cdot 3^{21} \cdot 5^{12} \cdot 11^6 \cdot 13 \cdot 23^3 \\
|N_{\mathcal{O}_U}/\mathbb{Q}(j_{57})| &= 2^{10} \cdot 3^9 \cdot 5^6 \cdot 11^6 \cdot 19 \cdot 29^6 \cdot 41^3 \cdot 53^3 
\end{align*}
\]

where $j_D$ is any singular modulus of discriminant $-D$. We see that 7, 13 and 19 are all primes congruent to 1 modulo 3 and they appear in the above factorizations. A closer inspection at the table reveals more: whenever the conductor $f = p^n$ is an odd prime power, the prime $p$ divides the norm of the corresponding $j$ invariants with order exactly 1. For instance for $f = 3, 9, 27, 81$ we have:

\[
\begin{align*}
f = 3, \quad D &= -3 \cdot 3^2 \quad |N_{\mathcal{O}_U}/\mathbb{Q}(j)| = 2^{15} \cdot 3 \cdot 5^3 \\
f = 9, \quad D &= -3 \cdot 3^4 \quad |N_{\mathcal{O}_U}/\mathbb{Q}(j)| = 2^{45} \cdot 3^3 \cdot 5^9 \cdot 11^3 \cdot 23^3 \\
f = 27, \quad D &= -3 \cdot 3^6 \quad |N_{\mathcal{O}_U}/\mathbb{Q}(j)| = 2^{144} \cdot 3^9 \cdot 5^{27} \cdot 11^{15} \cdot 17^9 \cdot 23^3 \cdot 29^6 \cdot 53^6 \\
f = 81, \quad D &= -3 \cdot 3^8 \quad |N_{\mathcal{O}_U}/\mathbb{Q}(j)| = 2^{432} \cdot 3^9 \cdot 5^{81} \cdot 11^{30} \cdot 17^{27} \cdot 23^6 \cdot 29^{12} \cdot 41^6 \cdot 47^9 \cdot 53^{12} \cdot 59^6 \cdot 71^6 \cdot 131^6 \cdot 167^6 \cdot 179^3 \cdot 191^6 \cdot 227^3 \cdot 239^3
\end{align*}
\]

while for $f = 5, 25, 125$ we have

\[
\begin{align*}
f = 5, \quad D &= -3 \cdot 5^2 \quad |N_{\mathcal{O}_U}/\mathbb{Q}(j)| = 2^{30} \cdot 3^6 \cdot 5 \cdot 11^3 \\
f = 25, \quad D &= -3 \cdot 5^4 \quad |N_{\mathcal{O}_U}/\mathbb{Q}(j)| = 2^{156} \cdot 3^{48} \cdot 5 \cdot 11^9 \cdot 17^6 \cdot 23^6 \cdot 47^6 \cdot 59^3 \cdot 71^3 \\
f = 125, \quad D &= -3 \cdot 5^6 \quad |N_{\mathcal{O}_U}/\mathbb{Q}(j)| = 2^{810} \cdot 3^{150} \cdot 5 \cdot 11^{54} \cdot 17^{48} \cdot 23^{24} \cdot 29^{30} \cdot 41^{18} \cdot 53^{12} \cdot 59^{12} \cdot 71^{18} \cdot 83^{12} \cdot 89^{12} \cdot 107^6 \cdot 113^6 \cdot 131^6 \cdot 167^6 \cdot 179^9 \cdot 227^6 \cdot 251^6 \cdot 263^6 \cdot 311^3 \cdot 347^6 \cdot 359^3
\end{align*}
\]

We want to remark that the results displayed above are peculiar of non-maximal orders and can never be spotted for singular moduli of fundamental discriminants. Indeed we have the following

**Proposition 4.1.** Let $j$ be a singular modulus of fundamental discriminant $-D$ and let $p$ be either 2, 3 or 5. If $p$ divides $N_{\mathcal{O}_U}/\mathbb{Q}(j)$ then $p^2$ also divides $N_{\mathcal{O}_U}/\mathbb{Q}(j)$. 

Proof. One easily verifies the statement (for instance using SAGE [10]) for all the fundamental discriminants $-D$ of class number $h_D \leq 2$. Hence we may suppose that $h_D > 2$.

Since $p$ divides $N_{\mathbb{Q}(\sqrt{-D})/\mathbb{Q}(j)}$ there must exist a prime ideal $p$ of $\mathbb{Q}(j)$ lying over $p$ such that $p | j$. In particular $j \equiv 0 \mod p$ and since 0 is a singular modulus we can apply Theorem 2.1 to deduce that $p$ does not split in $\mathbb{Q}(\sqrt{-D})$. Let now $E/L$ be an elliptic curve with complex multiplication whose $j$-invariant is a singular modulus of discriminant $-D$. Here we can assume that $L$ is a number field where $E$ has good reduction at all primes lying over $p$. Fix such a prime $\mathfrak{p}$: since $p$ does not split in $\mathbb{Q}(\sqrt{-D})$, the reduction $\tilde{E} = E \mod \mathfrak{p}$ is a supersingular elliptic curve by Theorem 2.2. But now we know ([11], Theorem V, 4.1) that the only supersingular invariant modulo $\mathfrak{p}$ is 0. We deduce that all the singular moduli of discriminant $-D$ must reduce to 0 modulo $p$. In other words we have

$$H_D(x) \equiv x^{h_D} \mod p$$

where $H_D(x)$ is the Hilbert class polynomial of discriminant $-D$ and $h_D$ is the class number of the unique order of discriminant $-D$.

Suppose now by contradiction that $p^2$ does not divide $N_{\mathbb{Q}(\sqrt{-D})/\mathbb{Q}(j)}$. Then, by what we showed above, the Hilbert class polynomial $H_D(x)$ must be Eisenstein at $p$. Since $H_D(x)$ is the minimal polynomial of $j$, we deduce that $p$ has to be totally ramified in $\mathbb{Q}(j)$. Now we look at the field $\mathbb{Q}(\sqrt{-D}, j)$: since $[\mathbb{Q}(\sqrt{-D}, j) : \mathbb{Q}(\sqrt{-D})] = [\mathbb{Q}(j) : \mathbb{Q}] = h_D \geq 3$ there exist a prime of $\mathbb{Q}(\sqrt{-D})$ lying over $p$ that ramifies in $\mathbb{Q}(\sqrt{-D}, j)$. This contradicts the fact that $\mathbb{Q}(\sqrt{-D}, j)$ is the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$. Hence $p^2$ divides $N_{\mathbb{Q}(\sqrt{-D})/\mathbb{Q}(j)}$ and the proposition is proved. □

Remark 4.2. The same argument works if we consider discriminants of orders whose conductor is not divided by 2, 3 and 5. Indeed in this case the associated ring class field is unramified at these primes.

When the conductor $f$ is a power of 2 the regularity in the factorizations appears a bit different from the previous cases. For instance for $f = 2, 4, 8, 16, 32$ we have

\[
\begin{align*}
&f = 2, \quad D = -3 \cdot 2^2 \quad |N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}(j)}| = 2^4 \cdot 3^3 \cdot 5^3 \\
&f = 4, \quad D = -3 \cdot 2^4 \quad |N_{\mathbb{Q}(\sqrt{4})/\mathbb{Q}(j)}| = 2^4 \cdot 3^5 \cdot 5^6 \cdot 11^3 \\
&f = 8, \quad D = -3 \cdot 2^8 \quad |N_{\mathbb{Q}(\sqrt{8})/\mathbb{Q}(j)}| = 2^4 \cdot 3^{12} \cdot 5^{12} \cdot 11^6 \cdot 17^6 \cdot 23^3 \\
&f = 16, \quad D = -3 \cdot 2^{16} \quad |N_{\mathbb{Q}(\sqrt{16})/\mathbb{Q}(j)}| = 2^4 \cdot 3^{42} \cdot 5^{24} \cdot 11^6 \cdot 17^6 \cdot 23^3 \cdot 29^6 \cdot 41^6 \cdot 47^3 \\
&f = 32, \quad D = -3 \cdot 2^{32} \quad |N_{\mathbb{Q}(\sqrt{32})/\mathbb{Q}(j)}| = 2^4 \cdot 3^{48} \cdot 5^{48} \cdot 11^{24} \cdot 17^{16} \cdot 23^{12} \cdot 29^{12} \cdot 47^9 \cdot 53^6 \cdot 59^6 \cdot 71^3 \cdot 83^6 \cdot 89^6
\end{align*}
\]

All the data above suggest the following

Conjecture 4.3. For a prime number $p$ let $v_p : \mathbb{Q}^* \to \mathbb{Z}$ be the usual $p$-adic valuation and for a discriminant $D = -3f^2$ let $j_D$ be any singular modulus relative to that discriminant. Then:

- if $f = p^n$ with $p$ odd prime, $v_p(N_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}(j_D)}) = 1$.
- if $f = 2^n$, $v_2(N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}(j_D)}) = 4$.

We are able to prove part of conjecture 4.3 in the following
THEOREM 4.4. Let \( j \) be a singular modulus of discriminant \( d = -3f^2 \), i.e. a singular modulus relative to an order \( \mathcal{O}_j \subseteq \mathbb{Q}(\sqrt{-3}) \) of conductor \( f \). Assume that \( f = p^n \) is a perfect prime power with \( n \) a positive even natural number.

- If \( p \neq 3 \) is odd then \( p \) divides exactly \( N_{\mathbb{Q}(j)/\mathbb{Q}}(j) \).
- If \( p = 2 \) then \( 2^4 \) divides exactly \( N_{\mathbb{Q}(j)/\mathbb{Q}}(j) \).

**Proof.** The proof of Theorem 4.4 will rely on the formulas proved by K. Lauter and B. Viray in [9]. Following the same notation of their paper, set for \( n \) positive even

\[ d_1 = -3, \quad f_1 = 1, \quad d_2 = -3p^{2n}, \quad f_2 = p^n \]

so that \( j_2 = j \) is a singular modulus of discriminant \( d_2 \) and \( j_1 = 0 \) is the only singular modulus of discriminant \( d_1 \). Then

\[
\prod_{j_1, j_2 \text{ disc } j_1 = d_1} (j_1 - j_2) = \pm N_{\mathbb{Q}(j)/\mathbb{Q}}(j)
\]

since all the singular moduli of the same discriminant are conjugated. If now \( w_i \) denotes the number of units in the order \( \mathcal{O}_{d_i} \) for \( i = 1, 2 \), then by our assumptions we have \( w_1 = 6 \) and \( w_2 = 2 \). By Theorem 1.1 in [9] we get

\[ |N_{\mathbb{Q}(j)/\mathbb{Q}}(j)|^{2/3} = \prod_{x^2 \leq 9p^{2n}} F \left( \frac{9p^{2n} - x^2}{4} \right) \]

where \( F \) is a function that takes non-negative integers of the form \( \frac{9p^{2n} - x^2}{4} \) to possibly fractional prime powers. Identity (1) shows that in order to understand the factorization of \( N_{\mathbb{Q}(j)/\mathbb{Q}}(j) \) one should study the function \( F \left( \frac{9p^{2n} - x^2}{4} \right) \) for different values of \( x \). We begin by studying the case \( x = \pm 3p \), i.e. the factorization of \( F(0) \). We denote by \( v_p(\cdot) \) the usual \( p \)-adic valuation. Then by the final part of Theorem 1.5 in [9], since \( f_1 = 1 \) and \( d_2 = d_1p^{2n} \) we have

\[ v_p(F(0)) = \frac{2}{6} \# \text{ Pic}(\mathcal{O}_{d_1}) = \frac{1}{3} \]

because \( 2[1 + \sqrt{-3}] \) is a principal ideal domain. Combining this with equation (1) gives

\[ |N_{\mathbb{Q}(j)/\mathbb{Q}}(j)|^{2/3} = p^{2/3} \prod_{x^2 \leq 9p^{2n}} F \left( \frac{9p^{2n} - x^2}{4} \right) \]

In what follows we will distinguish between the cases \( p \) odd and \( p = 2 \). In the first case we will have to prove that none of the factors appearing in the product on the right-hand side of equation (2) is a power of \( p \). In the second case we shall prove that there are exactly two factors in the same product that are equal to 2.

**Case 1:** \( p \neq 3 \) odd. We are now supposing that \( x \neq \pm 3p \), i.e. that \( m = \frac{9p^{2n} - x^2}{4} > 0 \). By the final part of Theorem 1.1 in [9] we can have \( v_p(F(m)) \neq 0 \) only if \( p \) divides \( m \). Hence we only have to study the values of \( F(m) \) with \( p \mid m \). By definition of \( m \) this implies that \( p \) divides \( x \) and we can then write \( x = p^rk, \ r \leq n \) (here we use the fact that \( p \) is odd), \( k \) coprime with \( p \). Hence \( m \) can be factored as

\[ m = \frac{9p^{2n} - k^2p^{2r}}{4} = p^{2r}A, \quad A = \frac{9p^{2(n-r)} - k^2}{4} \]
Notice that $p$ does not divide $A$.
By Theorem 1.5 in \[9\] (which we can apply since $f_1 = 1$) we have that

\[ v_p(F(m)) = \rho(m) \mu \left( \frac{m}{p^{l+n}} \right) \]

where $\rho(\cdot)$ and $\mu(\cdot)$ are two functions defined for every integer $m, N$ as follows:

\[
\rho(m) = \begin{cases} 
0 & \text{if } (-3, -m)_3 = -1 \\
2 & \text{if } 3 \nmid m \\
4 & \text{otherwise}
\end{cases}
\]

\[
\mu(N) = \# \left\{ a \subseteq \mathbb{Z} \left[ \frac{1 + \sqrt{-3}}{2} \right] \text{ ideals : } N(a) = N \right\}.
\]

In particular the values of these two functions depend on the prime factorization of $m$.
By Theorem 7.12 in \[9\] we have that the right-hand side is zero if either $m$ is not an integer or $p \nmid d_1$ and $v_p \left( \frac{m}{p^{l+n}} \right) \equiv 1 \mod 2$. But now $p \neq 3$ by assumption and we have

\[ v_p \left( \frac{m}{p^{l+n}} \right) = v_p \left( p^{2^{r-1-n} A} \right) = 2r - 1 - n \equiv 1 \mod 2 \]

since by hypothesis $n$ is even. Hence the right-hand side of equation (3) is 0 for every $m \neq 0$ and this concludes the proof in this case.

**Case 2:** $p = 2$.
As in the previous case, we have that $v_2(F(m))$ can be nonzero only if 2 divides $m$, and this leads us to consider integers $m$ of the form

\[ m = \frac{2^ng - 2^{2r}k^2}{4} > 0 \]

where $k$ is either 0 or coprime with 2. First we study what happens when $k = 0$. In this case we have $m = 2^{2n-2g}$ and as above

\[ v_2(F(m)) = \rho(m) \mu \left( \frac{m}{2^{l+n}} \right) \]

where the quantity on the right-hand side is zero when $v_2 \left( \frac{m}{2^{l+n}} \right) \equiv 1 \mod 2$. But we see that $\frac{m}{2^{l+n}} = 2^{n-3g}$ and since $n$ is even by assumption, we deduce that $v_2(F(m)) = 0$ in this case. Hence we may assume $k \neq 0$ and coprime with 2. Notice that in order for $m$ to be strictly positive as in (4) two possibilities may occur:

(i) We may have $r \leq n$. In this case we can write

\[ m = \frac{2^{2r}(2^{2(n-r)}g - k^2)}{4} = 2^{2r-2}(2^{2(n-r)}g - k^2). \]

As in the previous case we have

\[ v_2(F(m)) = \rho(m) \mu \left( \frac{m}{2^{l+n}} \right) \]

and we need to study

\[ \frac{m}{2^{l+n}} = 2^{2r-n-3}(2^{2(n-r)}g - k^2). \]
Notice now that, since $k$ is coprime with 2, the quantity inside the parenthesis cannot be divided by 2 unless $n = r$ and $k = \pm 1$ or $k = \pm 2$. Suppose first that $n \neq r$; then
\[ v_2 \left( \frac{m}{2^{1+r}} \right) = 2r - n - 3 \equiv 1 \mod 2 \]
since $n$ is even by assumption. Using Theorem 7.12 in [9] we deduce that $v_2(F(m)) = 0$ in this case.
Suppose now that $n = r$ and $k = 1$: under these hypotheses we have $m = 2^{2n+1}$ and
\[ v_2 \left( \frac{m}{2^{1+r}} \right) = v_2(2^n) = n \equiv 0 \mod 2 \]
by our assumptions on $n$. This means that in this case $v_2(F(m))$ could be nonzero. To compute the value of this valuation we have to use the full strength of Theorem 7.12 in [9]: using the same notation of that theorem we have
\[ v_2(F(m)) = \varepsilon_2(2^n) \prod_{p \mid 2^n, p \neq 2} (*) \]
where we see that the product on the right is empty, hence equal to 1, and by definition of $\varepsilon_2(\cdot)$ we have $\varepsilon_2(2^n) = 1$. Hence when $k = \pm 2$, we have $v_2(F(m)) = 1$. Using the same strategy for the case $n = r$ and $k = \pm 2$ one could see that in this case we have $v_2(F(m)) = 0$.

(ii) The second possibility occurs when $r = n + 1$ and $k = \pm 1$. In this case $m = 2^{2n-25}$ and similar arguments as the ones shown above allow to conclude that $v_2(F(m)) = 0$ in this case.

To sum up, when $p = 2$ the only integers $m$ of the form $m = \frac{9p^{2n} - 2}{4}$ for which $F(m)$ is a power of 2 are $m = 0$ ($x = \pm 2^n 3$) and $m = 2^{2n+1}$ ($x = \pm 2^n$), in which cases we obtain
\[ F(0) = 2^{1/3}, \quad F(2^{2n+1}) = 2. \]
Combining these results with equation (1) we get
\[ |N_{Q(j)/Q}(j)|^{2/3} = 2^{2/3} \cdot 2^2 \cdot A \]
where $A$ is an integer coprime with 2. This concludes the proof.

The main problem encountered in trying to generalize Theorem 4.4 to the cases $n$ odd or $p = 3$ is that, in these cases, the use of Lauter-Viray formulas requires some knowledge on the prime factorization of integers of the form $\frac{9p^{2n-25} - k^2}{4}$ and, in particular, on some congruence conditions modulo 3 satisfied by these primes.

5. The case $j - 1728$

In this final section we prove a result analogous to Theorem 1.2 for the difference $j - 1728$, with $j$ is a singular modulus different from 1728. In this case we have the following result.

**Theorem 5.1.** Let $S$ be the set of primes congruent to 1 modulo 4 and let $j$ be a singular modulus. If the difference $j - 1728$ is an $S$-unit, then it is a unit.
ON SINGULAR MODULI THAT ARE $S$-UNITS

Proof. The argument is analogous to the one given in the proof of Theorem 1.2, so we will omit the details.

If the singular modulus $j$ is not relative to an order in $\mathbb{Q}(i)$, then an argument identical to the one given in the proof of Theorem 1.2 allows to conclude. For singular moduli relative to orders in $\mathbb{Q}(i)$ we have the following

Claim 5.2. Let $j$ be a singular modulus relative to an order $\mathfrak{O}_j$ in $\mathbb{Q}(i)$. Then the primes 2, 3 and 7 divide $N_{\mathbb{Q}(j)/\mathbb{Q}}(j - 1728)$.

It is clear that from Claim 5.2 the theorem follows. We prove the claim for $p = 7$, the other cases being analogous.

Fix $j$ as in the statement of the claim and let $E/L$ be an elliptic curve defined over a number field $L$ with complex multiplication by the order $\mathfrak{O}_j$ and assume that $E$ has good reduction at every prime $\mathfrak{P}$ of $L$ lying over 7. Fix such a prime: since 7 is inert in $\mathbb{Q}(i)$, by Theorem 2.2 the reduced elliptic curve $\tilde{E} = E \mod \mathfrak{P}$ is supersingular. However there is only one isomorphism class of supersingular elliptic curves over $\mathbb{F}_7$, one representative being given by

$$E_0 : y^2 = x^3 + x$$

with $j$-invariant $j(E_0) = 1728$. Then

$$j \mod \mathfrak{P} = j(\tilde{E}) = j(E) = j(E_0) = 1728$$

where $\sim$ denotes the reduction modulo $\mathfrak{P}$. We deduce that $\mathfrak{P}$ divides $j - 1728$ and this proves the claim.

□

Notice that we cannot deduce from Theorem 5.1 a result similar to Corollary 1.3 because at present it is not known whether the difference $j - 1728$, for all singular moduli $j$, can be a unit in the ring of algebraic integers. The same techniques used in [1] are very likely to be applicable also in this case, and this will be object of future investigation.

6. Appendix: some numerical computations.

In this appendix we collect in a table some numerical computations, obtained using SAGE ([10]), concerning the norm factorizations for singular moduli of discriminant $-3f^2$. In the first column of the table we list the conductors $f$ of different orders of complex multiplication inside $\mathbb{Q}(\sqrt{-3})$; in the second column we compute, up to a sign, the norm factorizations of the corresponding singular moduli (since singular moduli relative to the same order form a Galois orbit in $\mathbb{Q}$, they all have the same norm). The factorizations are obtained simply by factoring the constant term in the Hilbert class polynomial of discriminant $-3f^2$. 

Table 1. Norm factorizations of singular moduli of discriminant $-3f^2$ for $f \in \{1, \ldots, 50\}$

| $f$  | $|N_{Q(j)}/Q(j)|$                           |
|------|---------------------------------------------|
| 1    | 0                                           |
| 2    | $2^4 \cdot 3^3 \cdot 5^3$                  |
| 3    | $2^{15} \cdot 3 \cdot 5^3$                 |
| 4    | $2^4 \cdot 3^9 \cdot 5^6 \cdot 11^3$      |
| 5    | $2^{30} \cdot 3^6 \cdot 5 \cdot 11^3$     |
| 6    | $2^{12} \cdot 3^3 \cdot 5^9 \cdot 11^6 \cdot 17^3$ |
| 7    | $2^{30} \cdot 3^9 \cdot 5^6 \cdot 7 \cdot 17^3$ |
| 8    | $2^4 \cdot 3^{12} \cdot 5^{12} \cdot 11^6 \cdot 17^6 \cdot 23^3$ |
| 9    | $2^{45} \cdot 3 \cdot 5^9 \cdot 11^3 \cdot 23^3$ |
| 10   | $2^{24} \cdot 3^{30} \cdot 5^3 \cdot 11^6 \cdot 17^6 \cdot 23^6 \cdot 29^3$ |
| 11   | $2^{63} \cdot 3^{12} \cdot 5^{12} \cdot 11 \cdot 17^3 \cdot 29^3$ |
| 12   | $2^{12} \cdot 3^6 \cdot 5^{18} \cdot 11^6 \cdot 17^6 \cdot 23^6 \cdot 29^6$ |
| 13   | $2^{66} \cdot 3^{21} \cdot 5^{12} \cdot 11^6 \cdot 13 \cdot 23^3$ |
| 14   | $2^{24} \cdot 3^{18} \cdot 5^{18} \cdot 11^{12} \cdot 17^3 \cdot 23^6 \cdot 29^6 \cdot 41^3$ |
| 15   | $2^{96} \cdot 3^6 \cdot 5^{3} \cdot 11^6 \cdot 17^6 \cdot 29^3 \cdot 41^3$ |
| 16   | $2^4 \cdot 3^{42} \cdot 5^{24} \cdot 11^6 \cdot 17^6 \cdot 23^3 \cdot 29^6 \cdot 41^6 \cdot 47^3$ |
| 17   | $2^{96} \cdot 3^{18} \cdot 5^{18} \cdot 11^6 \cdot 17 \cdot 23^6 \cdot 47^3$ |
| 18   | $2^{46} \cdot 3^3 \cdot 5^{27} \cdot 11^{12} \cdot 17^9 \cdot 29^6 \cdot 29^3 \cdot 41^6 \cdot 47^6 \cdot 53^3$ |
| 19   | $2^{83} \cdot 3^{27} \cdot 5^{18} \cdot 11^6 \cdot 19 \cdot 29^6 \cdot 41^3 \cdot 53^3$ |
| 20   | $2^{24} \cdot 3^{30} \cdot 5^6 \cdot 11^{15} \cdot 17^{12} \cdot 23^6 \cdot 29^6 \cdot 41^6 \cdot 47^6 \cdot 53^6 \cdot 59^3$ |
| 21   | $2^{96} \cdot 3^6 \cdot 5^{18} \cdot 11^{12} \cdot 17^3 \cdot 47^3 \cdot 59^3$ |
| 22   | $2^{48} \cdot 3^{60} \cdot 5^{36} \cdot 17^9 \cdot 23^6 \cdot 29^6 \cdot 41^3 \cdot 47^6 \cdot 53^6 \cdot 59^6$ |
| 23   | $2^{126} \cdot 3^{24} \cdot 5^{24} \cdot 11^9 \cdot 17^9 \cdot 23 \cdot 41^6 \cdot 53^3$ |
| 24   | $2^{12} \cdot 3^{12} \cdot 5^{36} \cdot 11^{12} \cdot 17^6 \cdot 23^{12} \cdot 29^6 \cdot 41^6 \cdot 47^3 \cdot 53^6 \cdot 59^6 \cdot 71^3$ |
| 25   | $2^{156} \cdot 3^{48} \cdot 5 \cdot 11^9 \cdot 17^6 \cdot 23^6 \cdot 47^6 \cdot 59^4 \cdot 71^5$ |
| 26   | $2^{48} \cdot 3^{36} \cdot 5^{36} \cdot 11^{12} \cdot 17^{12} \cdot 23^6 \cdot 29^6 \cdot 41^6 \cdot 47^6 \cdot 53^3 \cdot 59^6 \cdot 71^6$ |
| 27   | $2^{144} \cdot 3 \cdot 5^{27} \cdot 11^{15} \cdot 17^9 \cdot 23^3 \cdot 29^6 \cdot 53^6$ |
| 28   | $2^{24} \cdot 3^{66} \cdot 5^{36} \cdot 11^{12} \cdot 17^6 \cdot 23^6 \cdot 41^6 \cdot 47^6 \cdot 53^6 \cdot 59^3 \cdot 71^6 \cdot 83^3$ |
| 29   | $2^{150} \cdot 3^{30} \cdot 5^{30} \cdot 11^{12} \cdot 17^{12} \cdot 23^3 \cdot 29 \cdot 59^6 \cdot 71^3 \cdot 83^3$ |
| \( f \) | \( |N_{Q(f)}/Q(j)| \) |
|---|---|
| 30 | \( 2^{72}, 3^{18}, 5^{9}, 11^{24}, 17^{12}, 23^{12}, 29^{9}, 41^{9}, 47^{6}, 53^{6}, 59^{6}, 71^{6}, 83^{6}, 89^{3} \) |
| 31 | \( 2^{156}, 3^{51}, 5^{30}, 11^{15}, 17^{9}, 23^{3}, 29^{3}, 31^{41}, 89^{3} \) |
| 32 | \( 2^{4}, 3^{48}, 5^{48}, 11^{24}, 17^{6}, 23^{12}, 29^{12}, 47^{9}, 53^{6}, 59^{6}, 71^{3}, 83^{6}, 89^{6} \) |
| 33 | \( 2^{189}, 3^{12}, 5^{36}, 17^{9}, 23^{12}, 29^{3}, 47^{6}, 71^{6}, 83^{3} \) |
| 34 | \( 2^{72}, 3^{96}, 5^{54}, 11^{24}, 23^{6}, 29^{6}, 41^{6}, 53^{9}, 59^{6}, 71^{6}, 83^{6}, 89^{6}, 101^{3} \) |
| 35 | \( 2^{192}, 3^{36}, 5^{6}, 11^{12}, 17^{6}, 23^{12}, 29^{6}, 41^{3}, 53^{6}, 89^{6}, 101^{3} \) |
| 36 | \( 2^{36}, 3^{6}, 5^{54}, 11^{21}, 17^{18}, 23^{6}, 29^{12}, 41^{6}, 47^{6}, 59^{9}, 71^{6}, 83^{3}, 89^{6}, 101^{6}, 107^{3} \) |
| 37 | \( 2^{186}, 3^{63}, 5^{36}, 11^{15}, 17^{6}, 23^{6}, 37, 47^{3}, 59^{6}, 83^{6}, 89^{3}, 101^{3} \) |
| 38 | \( 2^{72}, 3^{54}, 5^{54}, 11^{24}, 17^{18}, 23^{12}, 29^{6}, 41^{6}, 47^{6}, 53^{6}, 71^{6}, 83^{6}, 89^{3}, 101^{6}, 107^{6}, 113^{3} \) |
| 39 | \( 2^{186}, 3^{12}, 5^{36}, 11^{12}, 17^{12}, 23^{6}, 29^{3}, 41^{6}, 53^{9}, 89^{6}, 101^{3}, 113^{3} \) |
| 41 | \( 2^{228}, 3^{42}, 5^{42}, 11^{12}, 17^{6}, 23^{9}, 29^{12}, 41^{6}, 47^{6}, 59^{3}, 71^{6}, 107^{3} \) |
| 42 | \( 2^{72}, 3^{18}, 5^{57}, 11^{12}, 17^{15}, 23^{12}, 29^{12}, 41^{3}, 47^{6}, 53^{6}, 59^{6}, 83^{6}, 89^{6}, 101^{3}, 107^{6}, 113^{6} \) |
| 43 | \( 2^{222}, 3^{75}, 5^{45}, 11^{12}, 17^{12}, 23^{6}, 29^{3}, 43, 53^{6}, 101^{6}, 113^{3} \) |
| 44 | \( 2^{48}, 3^{72}, 5^{78}, 11^{13}, 17^{18}, 23^{18}, 29^{6}, 41^{18}, 53^{6}, 59^{6}, 71^{6}, 83^{9}, 89^{6}, 101^{6}, 107^{3}, 113^{6}, 131^{3} \) |
| 45 | \( 2^{204}, 3^{6}, 5^{9}, 11^{21}, 17^{18}, 23^{12}, 29^{9}, 41^{3}, 59^{6}, 71^{3}, 83^{6}, 107^{6}, 131^{3} \) |
| 46 | \( 2^{96}, 3^{126}, 5^{78}, 11^{30}, 17^{15}, 29^{6}, 41^{6}, 47^{12}, 59^{6}, 71^{6}, 89^{9}, 101^{6}, 107^{6}, 113^{3}, 131^{6}, 137^{3} \) |
| 47 | \( 2^{258}, 3^{48}, 5^{51}, 11^{24}, 17^{12}, 23^{9}, 29^{9}, 41^{3}, 47, 89^{6}, 113^{6}, 137^{3} \) |
| 49 | \( 2^{222}, 3^{90}, 5^{42}, 7, 11^{18}, 17^{12}, 23^{6}, 29^{6}, 47^{3}, 71^{6}, 83^{3}, 131^{3} \) |
References


