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Abstract

This paper uses revealed preference inequalities to provide tight nonparametric bounds on consumer responses to price changes. Price responses are allowed to vary nonparametrically across the income distribution by exploiting micro data on consumer expenditures and incomes over a finite set of discrete relative price changes. This is achieved by combining the theory of revealed preference with the semiparametric estimation of consumer expansion paths (Engel curves). We label these expansion path based bounds as E-bounds. Deviations from revealed preference restrictions are measured by preference perturbations which are shown to usefully characterise taste change.

Key Words: Demand responses, relative prices, revealed preference, semiparametric regression, changing tastes.

JEL Classification: D12, C14, C43.

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1 Introduction

A common situation in applied economics is that we have a set of observations on agents in a fixed environment with particular realised economic variables and we wish to predict their behaviour in the same environment but with new values for the economic variables. For example, we observe demands at particular sets of prices and total expenditures and we wish to predict demands at a new set of prices and total expenditure. With no other structure, the observed behaviour is totally uninformative about the new situation and literally anything that is logically possible is an admissible prediction. One way around this is to use a parametric model and interpolate (or extrapolate). An alternative is adopt a theoretical position on what generates the observed behaviour and to use the theory and the previous observations to make predictions. Usually this leads to bounds on predicted behaviour rather than point predictions. Then the relevant questions become: how plausible is the theory and how tight are the bounds? In this paper we derive bounds on predicted demand behaviour from observations on expansions paths for a finite set of prices and the imposition of the basic (Slutsky or revealed preference) integrability conditions from economic theory. The plausibility of the latter derives from them being, effectively, the observable restrictions from assuming transitivity which is the bedrock of consumer theory in economics. Moreover, the theory implies testable restrictions so it is potentially rejectable. We develop methods to give the tightest possible bounds on demands given observed expansion paths and the basic (nonparametric) theory, if the latter is not rejected by the former. We find that the data and the theory give surprisingly tight bounds if we consider new situations that are within the span of the observed data.

To introduce our methodology, imagine facing a set of individual consumers with a sequence of relative prices and asking them to choose their individual demands, given some overall budget that each can expend. If they behave according to the axioms of revealed preference their vector of demands at each relative price will satisfy certain well known inequalities (see Afriat (1973) and Varian (1982)). If, for any individual, these inequalities are violated then that consumer can be deemed to have failed to behave according to the optimisation rules of revealed preference. This is a very simple and potentially powerful experimental setting for assessing the applicability RP theory. If, as in an experiment, one can choose the budget at which individuals face each price vector then Proposition 1 of Blundell, Browning and Crawford (2003) shows that there is a unique sequence of such budgets, corresponding to the sequence of relative prices, which maximises the chance of finding such a violation. This is the Sequential Maximum Power path. If experimental data are not available then the Blundell, Browning and Crawford study also shows how to use expansion paths (Engel curves) to mimic the experimental choice of this optimal sequence. Thus providing a powerful method of detecting RP violations in observational as well as experimental studies. In this paper we show that these expansion paths, together with revealed preference theory, can also be used to provide tight bounds on demand responses for observational data of the type collected in consumer expenditure surveys.

To construct bounds we extend the analysis introduced in Varian (1983) by considering expansion paths for given relative prices rather than demands at some point. We label these 'expansion path based bounds' as *E-bounds.* The advantages of the *E-bounds* method developed here are that it can describe the complete demand response to a relative price change for any point in the income distribution without recourse to parametric models of consumer behaviour and it gives the tightest possible bounds, given the data and the theory. The measurement of such price responses are at the centre of applied welfare economics, they are a vital ingredient of tax policy reform analysis and is also key to the measurement of market power in modern empirical industrial economics. Robust measurement is therefore a prerequisite of reliable analysis in these fields of applied microeconomics.

In our empirical analysis the relative price variation occurs over time and we consider consumer behaviour as it is recorded in standard repeated consumer expenditure surveys such as the US Consumers Expenditure Survey and the UK Family Expenditure Survey. The later is the source for our empirical analysis. We observe samples of consumers, each of a particular household type, at specific points in time. Assuming consumers are price-takers, we can recover expansion paths by estimating Engel curves at each point in time. We present E-bounds for own and cross price responses using these expansion paths.

Since the expansion paths are estimated, albeit by nonparametric techniques, they are subject to sampling variation. Consequently, violations of the revealed preference conditions may simply reflect estimation error rather than rejections by the individuals in the population under study. We allow for sampling variation in the estimated expansion paths and consider whether perturbations to preferences can be found that allow revealed preference theory to be maintained while lying within standard confidence intervals. For our data we find that preferences are consistent with RP theory over sequences of time periods but rejections do occur. Where such rejections occur the estimated perturbations provide a natural metric against which to measure taste change.

The E-bounds on demand responses we construct are found to be informative. The advantage of adding in more relative price variation is carefully explored, both theoretically and empirically. We show that it is the combination of the new prices *and* the quantity choice implied by the new expansion path that determines whether the new observation is informative. We discuss precisely how such information tightens the bounds. Empirically we show the value of allowing for sampling variation and of introducing perturbations. Bounds on demands are improved and we are also able to detect slow changes in tastes. These bounds on demand responses and the changes in tastes are found to differ across the income distribution.

Freeing-up the variation in relative price responses across the income distribution is one of the key contributions of this research. Historically parametric specifications in the analysis of consumer behavior have been based on the Working-Leser or Piglog form of preferences that underlie the popular Almost Ideal and Translog demand models of Deaton and Muellbauer (1980) and Jorgenson, Lau and Stoker (1982). Even though more recent empirical studies have suggested further nonlinear income terms, (see, for example, Hausman, Newey, Ichimura and Powell (1995), Lewbel (1991), Blundell, Pashardes and Weber (1993), Banks, Blundell and Lewbel (1998)), responses to relative prices at different incomes for these parametric forms remain unnecessarily constrained.

The remainder of the paper is as follows: In section 2 we examine bounds on demand responses and develop a method for generating the best bounds. Section 3 introduces the idea of preference perturbations as a way of imposing the RP inequalities and for characterising changing tastes. In section Section 4 we apply these ideas to describe the demand responses for three broad commodities using the individual household level data in the Family Expenditure Survey. In section 5 we go on to allow for perturbations in preferences and taste change. Section 6 concludes.

2 Expansion Path Bounds on Demands

2.1 Defining E-bounds.

We shall be concerned with predicting demands given particular budgets. To this end, we assume that every agent responds to a given budget $(\mathbf{p}, x)$, where $\mathbf{p}$ is a J-vector of prices and x is total expenditure, with a unique, positive demand J-vector:

Assumption 1. Uniqueness of demands: for each agent there exists a set of demand functions $\mathbf{q}(\mathbf{p}, x) : \mathbb{R}^{J+1}_{++} \rightarrow \mathbb{R}^{J}_{++}$ which satisfy adding-up: $\mathbf{p}'\mathbf{q}(\mathbf{p}, x) = x$ for all prices $\mathbf{p}$ and total outlays x.

For a given price vector $\mathbf{p}_t$ we denote the corresponding J-valued function of x as $\mathbf{q}_t(x)$ (with $q_t^j(x)$ for good j) and refer to this vector of Engel curves as an *expansion path* for the given prices. We shall also have need of the following assumption:

Assumption 2. Weak normality: if $x > x'$ then $q_t^j(x) \geq q_t^j(x')$ for all j and all $\mathbf{p}_t$.

This rules out inferior goods. Adding up and weak normality imply that at least one of the inequalities in this assumption is strict and that expansion paths are continuous.

The question we address is: given a budget $\{\mathbf{p}_0, x_0\}$ and a set of observed prices and expansion paths $\{\mathbf{p}_t, \mathbf{q}_t(x)\}_{t=1,..T}$, what demands are consistent with these observed demands and utility maximisation? Since we are working with a finite set of observed prices, we characterise consistency with utility maximisation in terms of revealed preference axioms. Since we are requiring that demands be single valued (and not correspondences) we work with the Strong Axiom of Revealed Preference (SARP) rather than the more usual Generalised Axiom (GARP).[1] To state SARP we need to define what we mean by revealed preference. If at prices $\mathbf{p}_t$ the agent chooses $\mathbf{q}_t$ and we have $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q}_s$ then we say that $\mathbf{q}_t$ is *directly revealed weakly preferred* to $\mathbf{q}_s$: $\mathbf{q}_t R^0 \mathbf{q}_s$ If we have a chain $\mathbf{q}_t R^0 \mathbf{q}_u$, $\mathbf{q}_u R^0 \mathbf{q}_v$, $...\mathbf{q}_w R^0 \mathbf{q}_s$ then we say that $\mathbf{q}_t$ is *revealed weakly preferred* to $\mathbf{q}_s$: $\mathbf{q}_t R \mathbf{q}_s$. Given this, SARP is defined by the following:

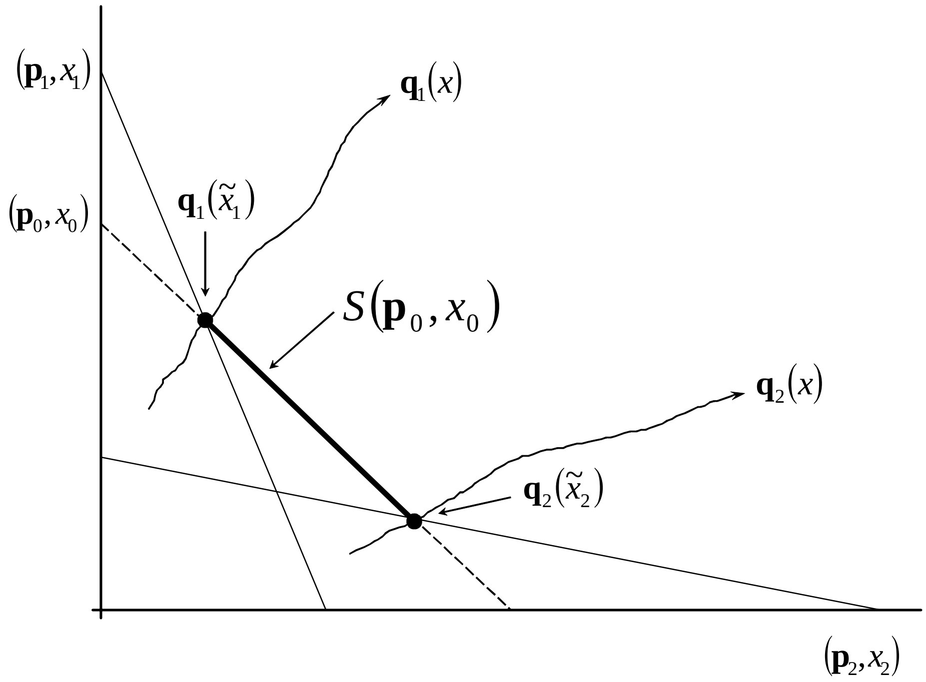
Definition 1 ***SARP:*** $\mathbf{q}_t R \mathbf{q}_s$ *and* $\mathbf{q}_t \neq \mathbf{q}_s$ *implies not* $\mathbf{q}_s R^0 \mathbf{q}_t$ *for all* s, t.

This definition does not rule that we might have the same demand for two different price vectors.

The basic idea behind our analysis is shown in figure 1 for a two good, two expansion path example. In this example, the two expansion paths are shown as $\mathbf{q}_1(x)$ and $\mathbf{q}_2(x)$. These intersect the new budget line $\{\mathbf{p}_0, x_0\}$ at $\mathbf{q}_1(\tilde{x}_1)$ and $\mathbf{q}_2(\tilde{x}_2)$ respectively. We term these points *intersection demands*; the two assumptions on demand above ensure that a unique intersection demand exists for any $\{\mathbf{p}_0, x_0\}$ and $\mathbf{q}_t(x)$. By design we have that any point $\mathbf{q}_0$ satisfying $\mathbf{p}_0'\mathbf{q}_0 = x_0$ is weakly directly revealed preferred to all intersection points. We also show the two observed budget lines for the intersection demands (labelled $\{\mathbf{p}_1, x_1\}$ and $\{\mathbf{p}_2, x_2\}$ respectively). As drawn, the two intersection demands satisfy SARP since neither is revealed weakly preferred to the other. The final step is to display the set of points on the new budget line $\{\mathbf{p}_0, x_0\}$ that are consistent with these intersection points and with SARP. This is shown as the interval labelled $S(\mathbf{p}_0, x_0)$; this set includes the intersection demands and, for two goods, it is closed. We term this set the *support set* for $\{\mathbf{p}_0, x_0\}$. Any point on the new budget that is in the support set $S(\mathbf{p}_0, x_0)$ satisfies SARP for the intersection demands and any point outside fails. For example, a point $\mathbf{q}_0$ within the interior of the support set is weakly revealed preferred to the intersection demands, it is distinct from them but the intersection demands are not directly weakly preferred to $\mathbf{q}_0$. Conversely, consider a point $\mathbf{q}_0$ that is not in $S(\mathbf{p}_0, x_0)$. In this case SARP fails immediately since $\mathbf{q}_1(\tilde{x}_1) R^0 \mathbf{q}_0$ (which implies $\mathbf{q}_1(\tilde{x}_1) R \mathbf{q}_0$), $\mathbf{q}_1(\tilde{x}_1) \neq \mathbf{q}_0$ and $\mathbf{q}_0 R^0 \mathbf{q}_1(\tilde{x}_1)$. Finally, the intersection points satisfy SARP and hence are in the support set.

[1] Varian (1982) provides a discussion of the relationship between SARP and GARP; in brief, SARP requires single valued demand curves, whilst GARP allows for set-valued demand correspondences (so that SARP implies GARP).

FIGURE 1. Defining the support set.



Given figure 1 and the definition of intersection demands $\mathbf{q}_t(\tilde{x}_t)$ by $\mathbf{p}_0'\mathbf{q}_t(\tilde{x}_t) = \mathbf{x}_0$, it is straightforward to define the support set algebraically.[2] Given a budget $\{\mathbf{p}_0, x_0\}$ the set of points that are consistent with observed expansion paths $\{\mathbf{p}_t; \mathbf{q}_t(\tilde{x}_t)\}_{t=1,...,T}$ and utility maximisation is given by the *support set*:

$$S(\mathbf{p}_0, x_0) = \left\{ \mathbf{q}_0 : \begin{array}{c} \mathbf{q}_0 \geq \mathbf{0},\ \mathbf{p}_0'\mathbf{q}_0 = \mathbf{x}_0 \\ \{\mathbf{p}_0, \mathbf{p}_t; \mathbf{q}_0, \mathbf{q}_t(\tilde{x}_t)\}_{t=1,...,T} \text{ satisfy SARP} \end{array} \right\}$$

This differs from the support set definition given in Varian (1982) in two major respects. The Varian definition was based on T observed demand bundles whereas the present definition makes use of T expansion paths. Furthermore this support set is defined using expansion paths evaluated at specific budget levels; the intersection demands. We refer to the intervals defined by expansion paths in this way as *E-bounds* - expansion curve based demand bounds. These bounds on demands for the new budget are *best* in the sense that tighter bounds cannot be found without either observing more expansion paths, imposing some additional theoretical structure over and above utility maximisation (such as quasi-homotheticity or separability) or assuming a functional form for preferences. This is set out in the following proposition (the proof is given in the Appendix):

Proposition 1 *If demands are weakly normal* $S(\mathbf{p}_0, x_0) \subseteq S'(\mathbf{p}_0, x_0)$ *where* $S'(\mathbf{p}_0, x_0) =$

[2] In all that follows we assume that the observed prices $\{\mathbf{p}_1, ..., \mathbf{p}_T\}$ are relatively distinct in the sense that $\mathbf{p}_t \neq \lambda \mathbf{p}_s$ for all s, t and any $\lambda > 0$.

$\{\mathbf{q}_0 : \mathbf{p}_0'\mathbf{q}_0 = x_0,\ \mathbf{q}_0 \geq \mathbf{0}$ *and* $\{\mathbf{p}_0, \mathbf{p}_t; \mathbf{q}_0, \mathbf{q}_t(x_t)\}_{t=1,\dots,T}$ *satisfies SARP, and* $x_t \neq \tilde{x}_t$ *for some* $t\}$.

Thus there do not exist alternative bounds (derived from the same data) which are tighter than the E-bounds. The E-bounds therefore make maximal use of the data and the basic nonparametric theory in predicting in a new situation. The properties of the support set are given in the following proposition:

Proposition 2 *(1)* $S(\mathbf{p}_0, x_0)$ *is non-empty if and only if the data set* $\{\mathbf{p}_t,\ \mathbf{q}_t(\tilde{x}_t)_{t=1,\dots T}\}$ *satisfies SARP. (2) If the data set* $\{\mathbf{p}_t, \mathbf{q}_t(\tilde{x}_t)\}_{t=1\dots T}$ *satisfies SARP and* $\mathbf{p}_0 = \mathbf{p}_t$ *for some* t *then* $S(\mathbf{p}_0, x_0)$ *is the singleton* $\{\mathbf{q}_t(\tilde{x}_t)\}$*. (3)* $S(\mathbf{p}_0, x_0)$ *is convex.*

The first statement establishes that there are some predicted demands for $\{\mathbf{p}_0, x_0\}$ if and only if the intersection demands satisfy SARP. The second statement shows that the support set is a single point if the new price vector is one that has been observed. Our decision to consider SARP rather than GARP is largely to give this property; for GARP we would have an interval prediction even for an observed price. The convexity is useful when it comes to solving numerically for E-bounds. Note that, contrary to what figure 1 suggests, with more than two goods the support set is not necessarily closed.

The empirical analysis below requires that we compute E-bounds for given data but the definition of $S(\mathbf{p}_0, x_0)$ is not particularly suited to empirical implementation as it stands. The second set we define gives a set of conditions that allow us to do this in a simple way using linear programming (LP) techniques. If $\{\mathbf{p}_t, \mathbf{q}_t(\tilde{x}_t)\}_{t=1,\dots,T}$ satisfies SARP we define:

$$S^{LP}(\mathbf{p}_0, x_0) = \left\{ \mathbf{q}_0 : \begin{array}{c} \mathbf{q}_0 \geq \mathbf{0},\ \mathbf{p}_0'\mathbf{q}_0 = x_0, \\ \mathbf{p}_t'\mathbf{q}_0 \geq \mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t), t = 1, 2 \dots T \end{array} \right\} \tag{1}$$

The set S^{LP} is closed and convex. We now show that these this set is the same as the support set, except (perhaps) on the boundary of the latter.[3] If we denote the closure of S by $cl(S)$ then we have:

Proposition 3 *(1)* $cl(S(\mathbf{p}_0, x_0)) = S^{LP}(\mathbf{p}_0, x_0)$*. (2)* $S^{LP}(\mathbf{p}_0, x_0) \backslash S(\mathbf{p}_0, x_0) = \{\mathbf{q} \in S^{LP}(\mathbf{p}_0, x_0) : \mathbf{p}_t'\mathbf{q} = \tilde{x}_t$ *and* $\mathbf{q} \neq \mathbf{q}_t(\tilde{x}_t)$ *for some* $t\}$

As we have seen, for two goods $S(\mathbf{p}_0, x_0)$ is closed so that it coincides with $S^{LP}(\mathbf{p}_0, x_0)$ but for more than two goods the set on the right hand side of the second statement is non-empty (so long as $S(\mathbf{p}_0, x_0)$ is non-empty). $S^{LP}(\mathbf{p}_0, x_0)$ gives us a feasible algorithm for displaying E-bounds. We first define intersection demands and test for SARP on the intersection demands. If the intersection demands pass SARP, we can then display bounds for each good. For example, to find the supremum predicted value for good j we maximise q_0^j subject to the constraints in (1). This is a standard linear programming problem.

[3] If we had considered GARP rather than SARP then we would have $S = S^{LP}$.

2.2 When is a new observation informative?

We turn now to a consideration of when and how more observations on expansion paths lead to an improvement in our bounds. We consider the situation in which we have T observed prices $\{\mathbf{p}_1, \mathbf{p}_2, ...\mathbf{p}_T\}$. Take a hypothetical budget $\{\mathbf{p}_0, x_0\}$ and suppose that the corresponding intersection demands satisfy SARP; denote the support set by $S^T(\mathbf{p}_0, x_0)$. Now add one more observed price and expansion path, $\{\mathbf{p}_{T+1}, \mathbf{q}_{T+1}(x)\}$, find the corresponding intersection demand $\mathbf{q}_{T+1}(\tilde{x}_{T+1})$ and compute the new support set $S^{T+1}(\mathbf{p}_0, x_0)$. Trivially the support set cannot increase; that is $S^T(\mathbf{p}_0, x_0) \supseteq S^{T+1}(\mathbf{p}_0, x_0)$.[4] For some $\mathbf{p}_{T+1}$ this will be a strict inclusion $(S^T(\mathbf{p}_0, x_0) \supset S^{T+1}(\mathbf{p}_0, x_0))$. We ask when the new observation will lead to a strict shrinkage of the support set. The first result is trivial but is worth formally recording.

Proposition 4 *If* $\mathbf{p}_{T+1} = \mathbf{p}_0 \neq \mathbf{p}_t$ *for* $t = 1, ...T$, $S^T(\mathbf{p}_0, x_0)$ *is non-empty and* $\mathbf{q}_t(\tilde{x}_t) \neq \mathbf{q}_s(\tilde{x}_s)$ *for some* t *and* s *then* $S^T(\mathbf{p}_0, x_0) \supset S^{T+1}(\mathbf{p}_0, x_0)$.

This shows that if the newly observed price just happens to coincide with $\mathbf{p}_0$ then the new support set will be smaller. The proof of this proposition, along with part 2 of proposition 2, establishes that if the intersection points are distinct (which they will almost surely be) then we only point identify a prediction if the new price $\mathbf{p}_0$ is equal to one of the observed prices. More interesting is the case in which $\mathbf{p}_T \neq \mathbf{p}_0$. To present the characterisation for this, we need one more definition:

Definition 2 *The budget plane* $\{\mathbf{p}_{T+1}, \tilde{x}_{T+1}\}$ *intersects with* $S^T(\mathbf{p}_0, x_0)$ *if there exists some* $\mathbf{q}_0 \in S^T(\mathbf{p}_0, x_0)$ *such that* $\mathbf{p}'_{T+1}\mathbf{q}_0 = \tilde{x}_{T+1}$.

We now present necessary and sufficient conditions for strict shrinkage of the support set.

Proposition 5 $S^{T+1}(\mathbf{p}_0, x_0) \subset S^T(\mathbf{p}_0, x_0)$ *iff the new budget plane* $\{\mathbf{p}_{T+1}, \tilde{x}_{T+1}\}$ *intersects with* $S^T(\mathbf{p}_0, x_0)$.

The following three good example serves to illustrate this proposition and to emphasise the point that, if the intersection condition does not hold then a new observation will be uninformative regardless of how close the new price vector is to the hypothetical price vector. Consider the following data for three goods and three periods:

$$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \begin{bmatrix} 0.64 & 0.19 & 0.90 \\ 0.26 & 0.77 & 0.89 \\ 1 & 1 & 1 \end{bmatrix}$$
$$\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} = \begin{bmatrix} 1.895 & 1.768 & 0.399 \\ 1.571 & 1.141 & 1.901 \\ 1.267 & 1.545 & 1.850 \end{bmatrix} \qquad (2)$$

[4]This includes the case in which the original T observations satisfy SARP (so that S^T is non-empty) but the expanded set does not satisfy SARP, in which case S^{T+1} is the empty set.

and take the hypothetical budget given by $(p_0^1, p_0^2, p_0^3) = (0.5, 0.5, 1)$ and $x_0 = 3$.[5] Suppose now that we observe a new price $\mathbf{p}_4$ with an intersection demand:

$$\mathbf{q}_4 = (1, 1, 2)' \tag{3}$$

We ask: what values of $\mathbf{p}_4$ lead to a strict contraction of the support set? With the values given it is easy to show that any:

$$\mathbf{p}_4 = \mathbf{p}_0 - \begin{bmatrix} \varepsilon \\ \varepsilon \\ 0 \end{bmatrix} \tag{4}$$

does not give a strict contraction for any $\varepsilon > 0$. Thus we can take an new price vector that is arbitrarily close to the hypothetical prices but does not lead to an improvement in the bounds. Conversely, any price vector:

$$\mathbf{p}_4 = \mathbf{p}_0 + \begin{bmatrix} 0 \\ \varepsilon \\ 0 \end{bmatrix} \tag{5}$$

gives a strict contraction for any $\varepsilon > 0$, even if ε is large. That is, new prices that are far from the hypothetical prices may give a strict contraction.

3 Changing Tastes and Revealed Preference Violations

The discussion so far has concentrated on the case in which there are no SARP violations at the intersection demands. Now we consider sampling variation in the estimation of the expansion paths. This will allow us to detect significant deviations from revealed preference. Moreover, where rejections occur, our approach also allows us to characterise changing tastes. The starting point is the suggestion by Varian (1985) for testing optimising behaviour in the presence of measurement errors in demands. However, here the measurement error is replaced by estimation error via the stochastic nature of the estimated expansion paths. The significance, or not, of violations will depend therefore on the precision of the estimated Engel curves at the specific income levels corresponding to the intersection points.

The idea is to allow local perturbations to preferences that describe the degree of taste change through a shift in marginal utility. This is achieved by perturbing the intersection demands so that they conform to SARP and then to recover the bounds on demand responses under this restriction. We can then construct a significance test for these perturbations. Since this can be implemented at any point in the income distribution, if rejections occur, we can assess the direction of taste change and how tastes change for rich and poor. Slowly changing

[5] Note that values for the quantities have been rounded and do not exactly satisfy the intersection demand condition $\mathbf{p}_0'\mathbf{q}_t = x_0$.

tastes would be reflected by a systematic evolution of these perturbations. Again these could differ across individuals with different incomes.

Let $\Sigma(p)$ denote the set of all quantity datasets which are SARP-consistent with a given price dataset $\{\mathbf{p}_t\}_{t=1,\ldots,T}$

$$\Sigma(p) = \{\{\mathbf{q}_t\}_{t=1,\ldots,T} : \{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\ldots,T} \text{ satisfies SARP}\}$$

Note firstly that for any prices p, $\Sigma(p)$ is not empty since, for example setting $q_t^j = 1$ for all j,t will satisfy SARP. Secondly $\Sigma(p)$ will generally be non-convex[6].If the intersection demand data violate SARP then

$$\{\widetilde{\mathbf{q}}_t(\widetilde{x}_t)\}_{t=1,\ldots,T} \notin \Sigma(p)$$

Suppose we now define the perturbed intersection demands

$$q_t^{j*} = e_t^j \widetilde{q}_t^j(\widetilde{x}_t) \tag{6}$$

where e_t^j is a (multiplicative) perturbation to demand for the j'th good in the t'th period. The e_t^j can be interpreted as a tilting of the marginal rate of substitution U_j/U_l. For example the marginal conditions between good j and good l are tilted by e_t^j/e_t^l and become

$$\frac{U_j}{U_l}\frac{e_t^j}{e_t^l} = \frac{p_j}{p_l}.$$

In order to estimate the perturbed intersection demands we note that

$$e_t^j - 1 \equiv \frac{q_t^{j*} - \widetilde{q}_t^j(\widetilde{x}_t)}{\widetilde{q}_t^j(\widetilde{x}_t)} \tag{7}$$

and solve for the e_t^j directly in the following nonlinear, constrained minimum distance problem

$$\min_{\{e_t^j\}_{t=1,\ldots,T}^{j=1,\ldots,J}} \{L\left(\{e_t^j\}_{t=1,\ldots,T}^{j=1,\ldots,J}\right) = \sum_{j=1}^{j=J}\sum_{i=1}^{i=J}\sum_{t=1}^{t=T}\left(e_t^j - 1\right)\left(\Omega_t^{-1}\right)^{ij}\left(e_t^i - 1\right)\} \tag{8}$$

subject to

$$\begin{aligned} e_t^j &= q_t^{j*}/\widetilde{q}_t^j(\widetilde{x}_t) \\ \{\mathbf{q}_t^*\}_{t=1,\ldots,T} &\in \Sigma \\ q_t^{j*} &\geq 0 \\ \mathbf{p}_0'\mathbf{q}_t^* &= x_0 \ \forall\ t. \end{aligned}$$

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[6]For example let

$$\{\mathbf{p}_1, \mathbf{p}_2\} = \left\{\begin{bmatrix} 8 \\ 3 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix}\right\}, \{\overline{\mathbf{q}}_1, \overline{\mathbf{q}}_2\} = \left\{\begin{bmatrix} 5 \\ 10 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right\}, \{\widehat{\mathbf{q}}_1, \widehat{\mathbf{q}}_2\} = \left\{\begin{bmatrix} 2 \\ 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right\}$$

then $\{\mathbf{p}_t, \overline{\mathbf{q}}_t\}_{t=1,2}$ and $\{\mathbf{p}_t, \widehat{\mathbf{q}}_t\}_{t=1,2}$ both satisfy SARP and hence $\{\overline{\mathbf{q}}\}_{t=1,2} \in \Sigma(p)$ and $\{\widehat{\mathbf{q}}_t\}_{t=1,2} \in \Sigma(p)$. However, if we set $\mathbf{q}_t = \frac{1}{2}\overline{\mathbf{q}}_t + \left(1 - \frac{1}{2}\right)\widehat{\mathbf{q}}_t$ then $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,2}$ violates SARP and $\{\mathbf{q}_t\}_{t=1,2} \notin \Sigma(p)$.

This finds the nearest set of nonnegative intersection demands (defined by the distance function (8)) which are consistent with SARP. The weights $\left(\Omega_t^{-1}\right)^{i,j}$ are the i, j'th elements of the inverse of the perturbations in (7). If the intersection demands satisfy SARP then the objective function will be minimised at zero. The support set can then be defined using the perturbed intersection demands.

The distance function evaluated at optimal perturbation values is a test statistic for the null hypothesis of the revealed preference conditions. We construct a bootstrap confidence interval for this statistic.

4 Empirical Analysis

4.1 Data

In this analysis we take three broad consumption goods: food, other nondurables, and services and examine the E-bounds on demand responses. For this we draw on 25 years of British Family Expenditure Surveys from 1975 to 1999. In many contexts these three consumption goods represent an important grouping as the price responsiveness of food relative to services and to other non-durables is of particular interest. For example, the price responsiveness at different income levels is a key parameter in the indirect tax debate. Although food is largely free of value added tax (VAT) in the UK, the discussions over the harmonisation of indirect tax rates across Europe and the implications of a flat expenditure tax raised uniformly across all consumption items requires a good understanding of food demand responses across the income distribution. It is also important in general discussions of cost of living changes across the income distribution. Relative food prices saw some abrupt rises as the tariff structure and food import quotas were changed in Europe early in the period under study. To study further disaggregations of goods with any precision some form of separability has to be assumed. Although separability restrictions in revealed preference are of interest they are beyond the scope of this study and here we keep to this leading three good example.

The Family Expenditure Survey is a repeated cross-section survey consisting of around 7,000 households in each year. From these data we draw a relatively homogeneous sub-sample of couples with children who own a car. This gives us between 1,421 and 1,906 observations per year and 40,731 observations over the entire period. We use total spending on non-durables to define our total expenditure variable. Figure 2 shows the mean budget shares for these goods over the period[7]. As can be seen, the mean budget share for food exhibits a large fall whereas services are rising steadily over our data period.

Annual price indices for these commodities are taken from the annual Retail Prices Index.

[7] Precise details of the categories are available from the authors.

Nondurables are treated as the numeraire good and prices are normalised so that the price of non-durables is always one and so that the mean of each of the prices over the period 1975 to 1999 is also one for each good. Figure 3 shows the price data for the three commodity groups over the period under analysis. We see a steadily rising price for services relative to food and non-durables.

FIGURE 2. Mean budget shares, 1975 to 1999

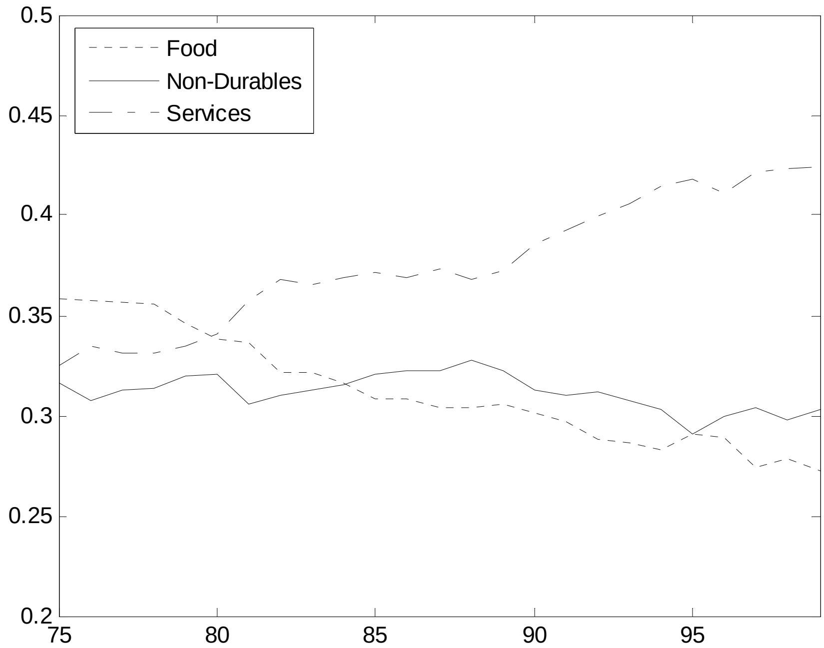
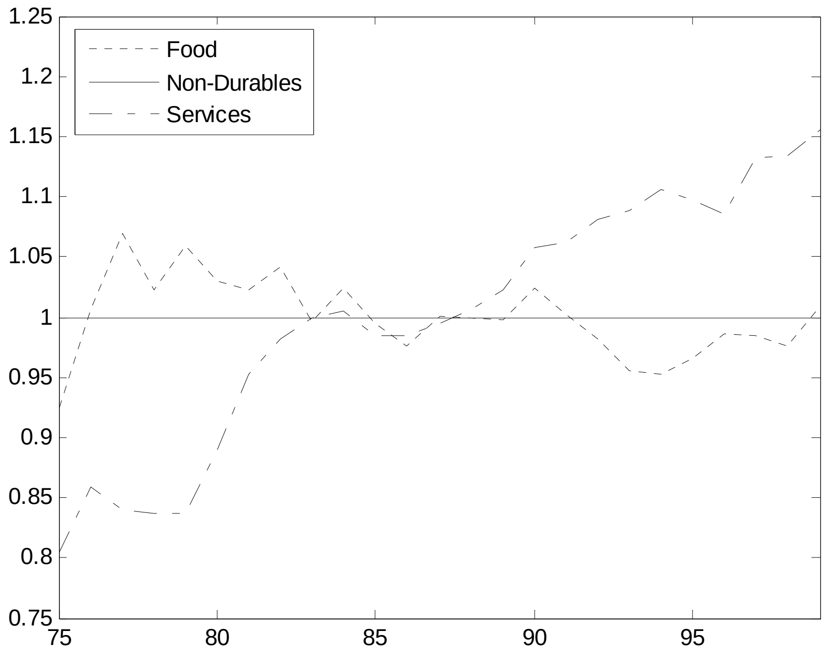


FIGURE 3. Price indices, 1975 to 1999



4.2 Estimating Expansion Paths

Consumers observed in the same time period and location are assumed to face the same relative prices. Under this assumption, Engel curves for each location and period correspond to expansion paths for each price regime. Blundell and Duncan (1998) have shown the attraction of nonparametric Engel curves when trying to capture the shape of income effects on consumer behaviour across a wide range of the income distribution. As in Blundell, Browning and Crawford (2003) we adopt a shape invariant specification for pooling over different demographic types of households. This semiparametric specification for Engel curves turns out to be a parsimonious, yet accurate, description of behaviour. We also account for the endogeneity of total expenditure using the control function approach (see Blundell and Powell (2003)).[8]

Let $\mathbf{d}^i$ represent a $(D \times 1)$ vector of household composition variables relating to household i. Our specification takes the form

$$w_j^i = g_j \left(\ln x_i - \phi(\mathbf{d}_i' \boldsymbol{\alpha}) \right) + \mathbf{d}_i' \boldsymbol{\gamma}_j + \varepsilon_j^i \tag{9}$$

where w_j^i is the expenditure share for household i on good j. To account for the endogeneity of $\ln x$ we specify

$$\ln x_i = \mathbf{z}_i' \pi + v_i \tag{10}$$

where $\mathbf{z}$ are a set of variables which include the demographic variables $\mathbf{d}_i$ and earned income as an excluded instrument. The control function approach assumes:

$$E(\varepsilon_j^i | \ln x_i, \mathbf{d}_i, v_i) = 0 \tag{11}$$

so that semiparametric regression using an augmented equation (9) that includes v_i will produce consistent estimates of g_j, $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ (see Newey, Powell and Vella (1999)).

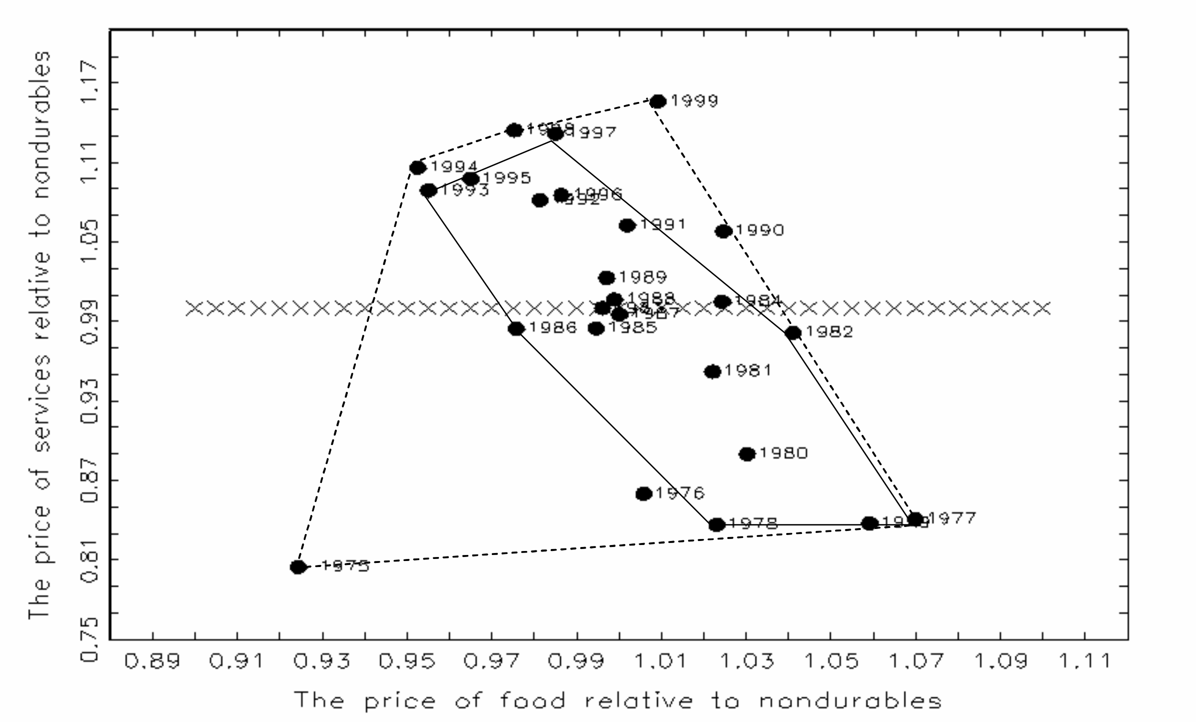
4.3 Empirical E-Bounds on Demand Responses

The substantial relative price variation seen in figure 3 can be also be seen in the dated points in figure 4. The dotted figure shows the convex hull of observed relative prices (the other features shown in the figure will be explained below). The relative prices show a dramatic change in the mid to late-1970's. To map out the E-bounds we consider variations in relative prices around a central $\mathbf{p}_0' = [1, 1, 1]'$. In particular, we choose a sequence in which the price of food varies around this central value by $\pm 10\%$ in 41 steps of half of one percent. That is we vary the $\mathbf{p}_0$ vector from $\mathbf{p}_0 = [0.9, 1, 1]'$ to $\mathbf{p}_0 = [1.1, 1, 1]'$. The line of crosses in figure 4 shows

[8] This is analysed in Blundell, Chen and Kristensen (2003) and compared to a the fully nonparametric instrument variables (NPIV) case. It is found to account quite well for the endogeneity of total expenditure in comparison to a full NPIV approach.

this particular sequence of the $\mathbf{p}_0$ vector as we vary the price of food, holding other prices constant. Note that this passes through a dense part of the relative price distribution where we might expect to be able to produce quite informative bounds of likely demand responses. The path also starts and finishes in areas of very sparse price information where, without extrapolation, we would not expect to have much to say about likely demand responses.

FIGURE 4. Scatter plot of the relative price data: 1975 to 1999



Our first step is to estimate the E-Bounds on demand responses to own price changes for food at some base level expenditure. This base level is set to be the median of the sample distribution of total expenditure (= £96.68). To construct the E-bounds at each new price vector we first check the revealed preference conditions for the data $\{\mathbf{p}_t, \mathbf{q}_t(\tilde{x}_t)\}$. For the first bounds we present, we only include a subset of observations that do not violate RP, ignoring sampling variation in the expansion paths used to construct $\mathbf{q}_t(\tilde{x}_t)$. Based on this sample we construct support sets for each of the 41 grid points for the food prices.

In figures 5a to 5c we present the resulting E-bounds for own and cross price responses using the reduced set of observations. As can be seen from a comparison of Figures 4 and 5, the bounds on the demand curve are particularly tight when the $\mathbf{p}_0$ vector is in the dense part of the observed price data. Outside the convex hull of the data the E-Bounds widen and we cannot rule out extreme responses (such as households not buying food if the price rises by more than 5%). These figures show the power of E-Bounds. Through the use of revealed preference inequalities and without appealing to parametric models or extrapolation we have been able to construct tight bounds on own and cross price elasticities.

FIGURE 5A. Own price demand bounds for Food

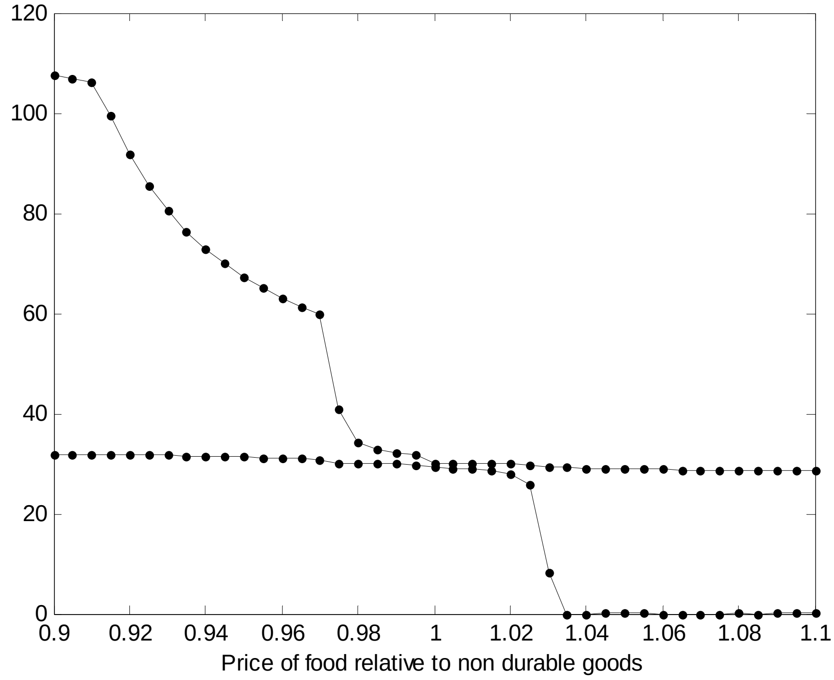


FIGURE 5B. Cross-price demand bounds for Non-durable

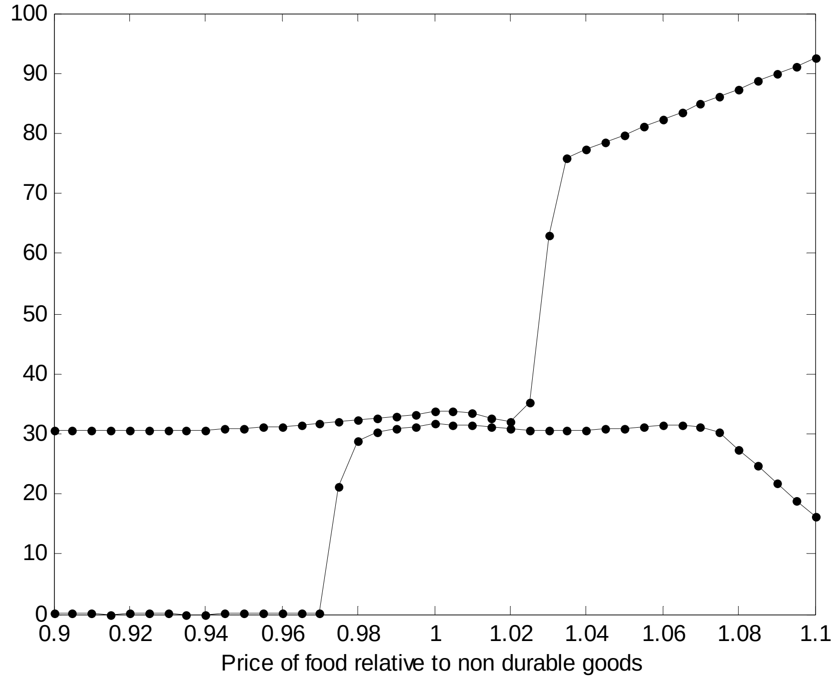
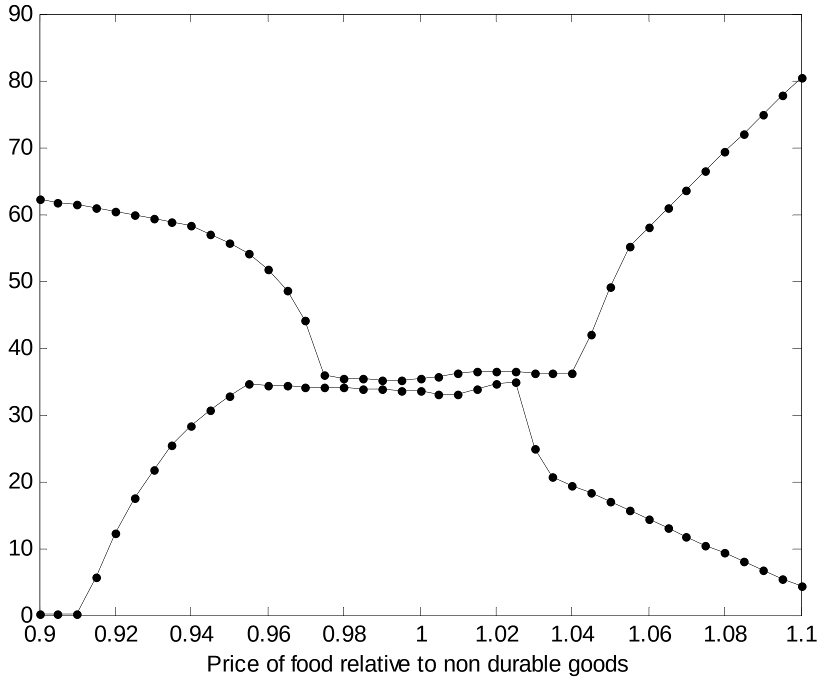


FIGURE 5C. Cross-price demand bounds for Services



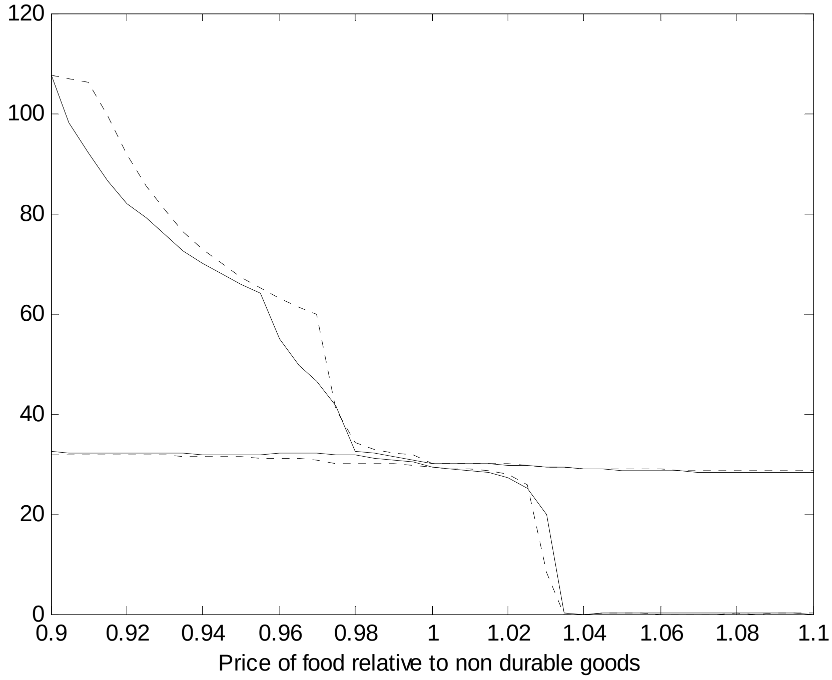
5 Revealed Preference Restrictions and Taste Change

5.1 Constrained E-Bounds

By perturbing preferences we can impose the RP restrictions across all the intersection demands used to construct the E-Bounds. Here we allow for perturbations that minimise the distance function (8) developed in section 3. At each $\{\mathbf{p}_0, x_0\}$ this imposes the revealed preference conditions using (8) and weighting the minimum distance procedure by the pointwise variance covariance matrix of the estimated expansion paths (evaluated at the intersection demands values).

The resulting demand curve bounds are illustrated in figure 6 along with, for comparison, the bounds recovered by dropping SARP rejections in figure 5a (the dashed lines). As can be seen, there is an improvement/narrowing of the bounds when all of the observations are used and constrained to be revealed preference consistent, compared to the case in which some data points are just dropped. Nevertheless, the improvement is quite small in the central part of the demand curve where the existing bounds were already fairly tight. Note also that there is no reason for the new bounds to lie everywhere inside the old bounds. The perturbed intersection demands can lead to the bounds widening at some relative price points. The general pattern of the bounds are similar however, with typically wider bounds the further the new price vector is from the most dense part of the observed price distribution.

Figure 6. Constrained E-Bounds for Food



5.2 Preference Perturbations and Taste Change

In figure 7 we present a graphical display of the perturbation terms e_t^j. Each sub-panel records the perturbation to each good for each observation. Since there are 25 intersection demands (one on each annual expansion path) there are 25 adjustments. Also shown are the pointwise 95% bootstrapped confidence intervals at 1975, 1977, 1979 and 1981. These show the significance of the deviation from RP conditions in the early period. That is there does not appear to be a stable set of preferences that represent the consumer choices in this data for the whole period.

Examining the pattern of the perturbations we see that rather than being random, as might be expected if the violations were the result of classical measurement error or truly random behaviour, they appear to follow a reasonably systematic pattern. Slowly changing tastes would be reflected by a systematic evolution of these perturbations. The perturbation to food for example is positive (and significant) to begin with requiring an upward adjustment of around 15% to begin with and then indicate a systematic shift in preferences away from food in the early period of the data. The adjustment gets progressively smaller until by 1980 almost no adjustment at all is needed (the multiplicative perturbation is close to one). The adjustments to non-durables goods are also trending, exhibiting a steady increase over time. For all goods the adjustments is greatest (and significant) for the earlier (pre 1980) observations.

Figure 7. RP Perturbations

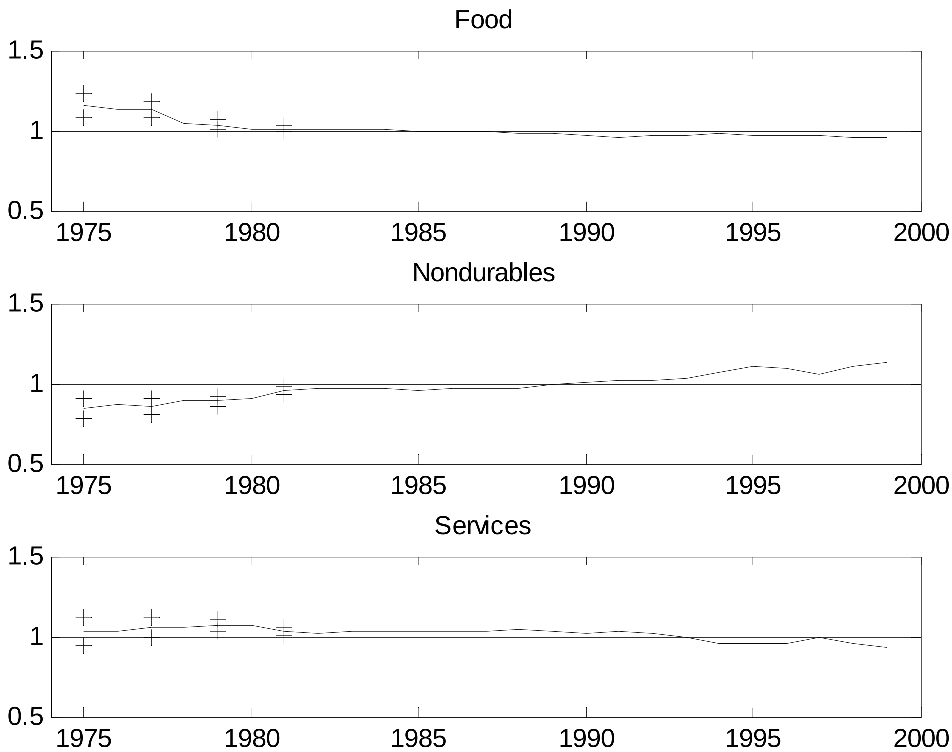
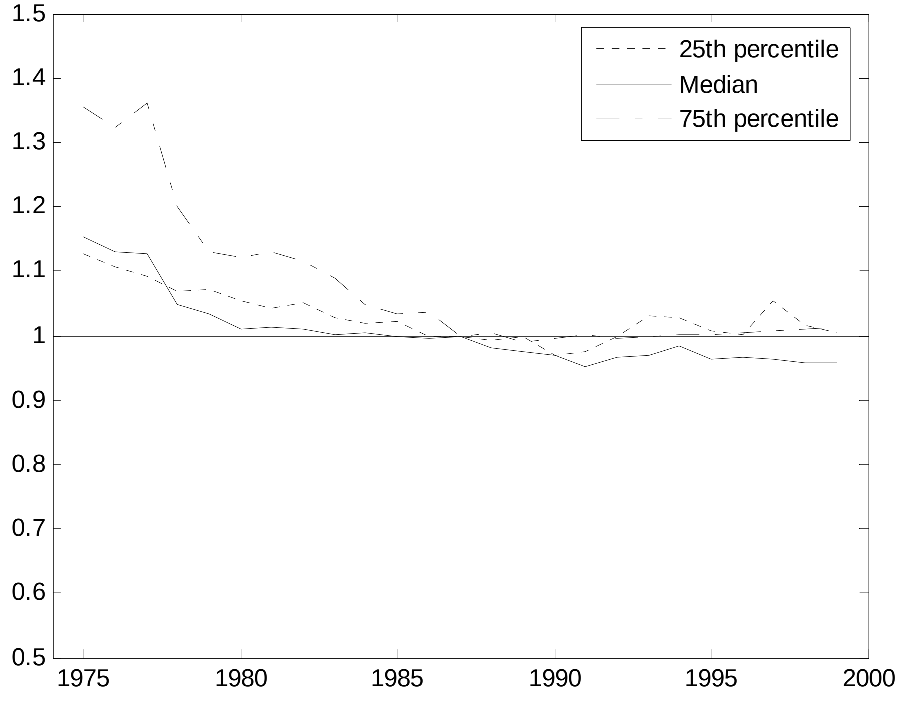


Figure 8. The Perturbations to Food Demands by Budget

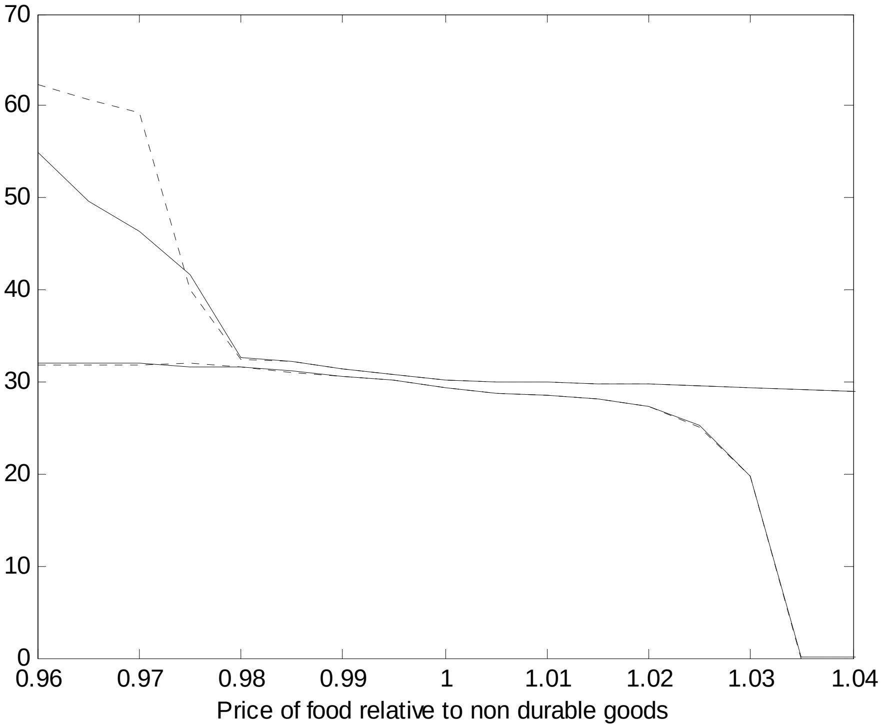


Interestingly this broad pattern of perturbations is maintained across different percentiles in the income (total budget) distribution. Figure 8 shows the relative perturbations for food at the same new price vector for three different budgets: the 25th and 75th percentiles of the empirical distribution as well as the median (already illustrated in the top panel of figure 7). In all cases there is a drift of relative preferences away from food in the early period. More so for those on higher incomes. This could also be reflecting a relative increase in the quality of other goods over this period.

5.3 Improving E-Bounds

Given the information on the pattern and significance of the perturbations we might expect that, taken together, the period from 1980 would not strongly reject the RP conditions. To assess this we construct a general test for the RP restrictions over this period using the minimised distance function (8). We compute a bootstrap CV for this statistic and find that this corresponds to a *p-value* of 0.16, confirming that the post 1980 period does not reject the RP conditions (see the Appendix for details). Given the consistency with RP, we can reasonably use the post 1980 data to recompute the bounds on the demand curve. This is presented in figure 9. Using fewer observations does widen the bounds as expected, particularly over the price range 0.96 to 0.98 however elsewhere the bounds are very close showing that restricting the data to a shorter period of revealed preference consistent consecutive demand observations does not cause the bounds to deteriorate greatly over the $\pm 2\frac{1}{2}\%$ range.

FIGURE 9. Demand Bounds for Food: 80 to 99 and 75 to 99

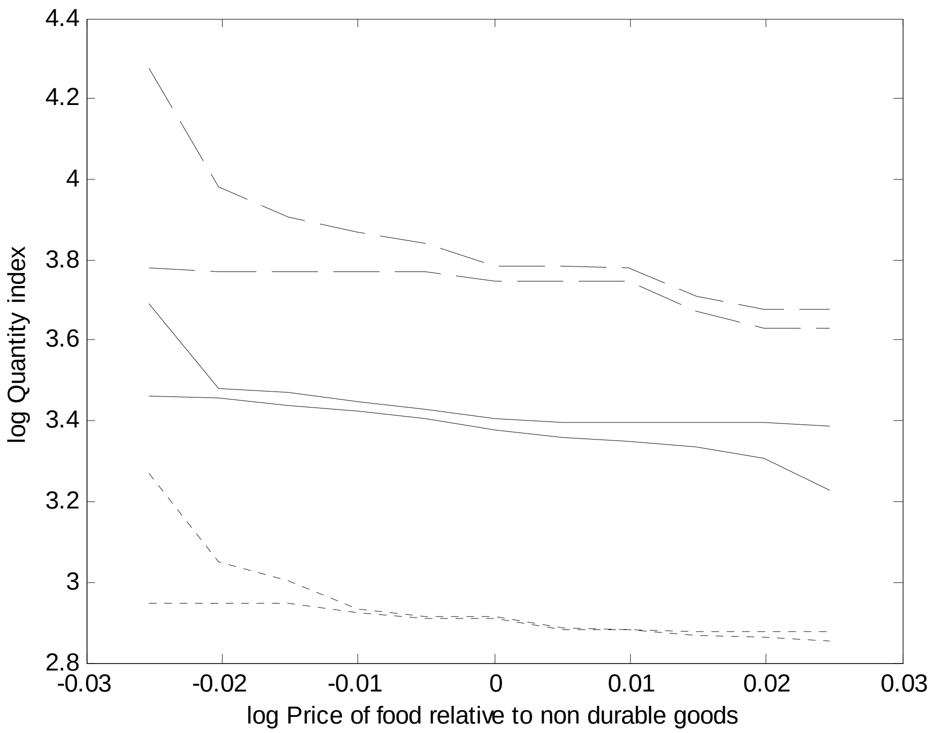


5.4 Price Responses Across the Income Distribution

The demand bounds on price responses that we have estimated so far have been constructed at the median income (expenditure). But we might expect demand responses to vary with income levels. Figure 10 shows how the demand bounds vary according to the total budget. Three sets of bounds are calculated corresponding to the 25th, 50th and 75th percentiles of the x_0 distribution (the solid lines for the median are identical to the dashed lines in the preceding figure over this range).

We note that the bounds are wider as the total budget increases; this is a result of the budget expansion paths "spreading out" as the budgets increase. At low expenditure levels the bounds are very tight indeed. It will be clear from this figure that there is not a single elasticity that summarises price response behaviour. Indeed the elasticity appears to be highly variable both along each demand curve and also across income levels. However, we can give an indication of the range of price responses. For example, if we consider a 1% drop in the relative price of food from the baseline 1985 price vector, then for the 25 percentile group the corresponding elasticity is -0.51 at the midpoint of the bounds. At the median income the response elasticity is -1.01 and at the 75 percentile it is -1.52. Indeed, at median income the elasticity measured this way is generally greater than unity and even higher at the 75 percentile. However the range is also large with a range of 0.0 to -1.02 at the 25 percentile and 0.0 to -3.05 at the 75 percentile.

FIGURE 10. Demand Bounds for Food By Budget Percentile (log-log)



Elasticities in excess of unity for food may be unusual but it should be noted that we have included food consumed outside the home in our broad definition of food. Moreover, the range of price responsiveness highlights the local nature of our nonparametric analysis. The price responsiveness are local to both income and relative prices. Unlike in the Stone-Geary model, for example, there is no reason why price elasticities should not be increasing or decreasing with income. For some broad aggregates like food a price elasticity which is increasing with income would seem sensible while for more disaggregated food items - rice and potatoes, for example - the reverse could equally well be true.

6 Summary and Conclusions

The aim of this paper has been to bound demands at a new set of relative prices and total expenditure using revealed preference inequalities alone. We have shown how to derive bounds on predicted demand behaviour from observations on expansions paths for a finite set of prices and the imposition of the basic (Slutsky or revealed preference) integrability conditions from economic theory. We find that these *E-bounds* give surprisingly tight bounds especially if we consider new situations that are within the span of the observed data. Our approach allows allow price responses to vary nonparametrically across the income distribution by exploiting micro data on consumer expenditures and incomes over a finite set of discrete relative price changes. We have introduced the concept of preference perturbations, local to each income percentile, which characterise the degree of congruence with RP conditions and provide a useful metric for describing taste change.

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Appendix: Proofs of Propositions

Proof of Proposition 1.
Let $S'(\mathbf{p}_0, x_0)$ denote the support set

$$S'(\mathbf{p}_0, x_0) = \left\{ \mathbf{q}_0 : \begin{array}{c} \mathbf{p}_0'\mathbf{q}_0 = x_0, \ \mathbf{q}_0 \geq \mathbf{0} \text{ and} \\ \{\mathbf{p}_0, \mathbf{p}_t; \mathbf{q}_0, \mathbf{q}_t(x)\}_{t=1,\ldots,T} \text{ satisfies SARP} \\ \text{and } x_t \neq \tilde{x}_t \text{ for some } t \end{array} \right\}$$

where the $\mathbf{q}_t(x)$ data are demands on expansion paths at arbitrary budget levels. Suppose that there exists some demand vector $\mathbf{q}_0 \geq \mathbf{0}$ and $\mathbf{p}_0'\mathbf{q}_0 = x_0$ such that $\mathbf{q}_0 \in S(\mathbf{p}_0, x_0)$ but $\mathbf{q}_0 \notin S'(\mathbf{p}_0, x_0)$. Then by definition of $S'(\mathbf{p}_0, x_0)$ it must be the case that $\{\mathbf{p}_0, \mathbf{p}_t; \mathbf{q}_0, \mathbf{q}_t(x)\}_{t=1,\ldots,T}$ contains a violation of SARP. That is there is some element of $\{\mathbf{q}_t(x)\}_{t=1,\ldots,T}$ (call it $\mathbf{q}_t(x)$) such that either $\mathbf{q}_t(x) R \mathbf{q}_0$ and $\mathbf{q}_0 R^0 \mathbf{q}_t(x)$ or $\mathbf{q}_0 R \mathbf{q}_t(x)$ and $\mathbf{q}_t(x) R^0 \mathbf{q}_0$. Consider the first case where $\mathbf{q}_0 R^0 \mathbf{q}_t(x)$. If demands are weakly normal then the corresponding intersection demand $\mathbf{q}_t(\tilde{x}_t)$ used to define $S(\mathbf{p}_0, x_0)$ must be such that $\mathbf{q}_t(\tilde{x}_t) R^0 \mathbf{q}_t(x)$. But $\mathbf{q}_t(x) R \mathbf{q}_0$ and hence $\mathbf{q}_t(x) R \mathbf{q}_t(\tilde{x}_t)$ and there is a contradiction of SARP. Now consider the second case where $\mathbf{q}_t(x) R^0 \mathbf{q}_0$. Since $\mathbf{q}_0 \in S(\mathbf{p}_0, x_0)$ we know that by definition $\mathbf{p}_t'\mathbf{q}_0 \geq \mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t)$ and hence $\mathbf{q}_t(x) R^0 \mathbf{q}_t(\tilde{x}_t)$. Therefore we have another contradiction of SARP. Hence $\mathbf{q}_0 \notin S'(\mathbf{p}_0, x_0) \Rightarrow \mathbf{q}_0 \notin S(\mathbf{p}_0, x_0)$.■

Proof of Proposition 2.
(1) $S(\mathbf{p}_0, x_0)$ *is non-empty if and only if the data set* $\{\mathbf{p}_t, \mathbf{q}_t(\tilde{x}_t)\}_{t=1,\ldots T}$ *satisfies SARP.*
If $\{\mathbf{p}_t, \mathbf{q}_t(\tilde{x}_t)\}_{t=1,\ldots T}$ fail SARP than so does $\{\mathbf{p}_0, \mathbf{p}_t; \mathbf{q}_0, \mathbf{q}_t(\tilde{x}_t)\}_{t=1,\ldots,T}$ for any $\{\mathbf{p}_0; \mathbf{q}_0\}$ so that the support set is empty. Conversely, if $\{\mathbf{p}_t, \mathbf{q}_t(\tilde{x}_t)\}_{t=1,\ldots T}$ pass SARP then these points satisfy the conditions for inclusion in $S(\mathbf{p}_0, x_0)$ which is thus non-empty.
(2) $S(\mathbf{p}_0, x_0)$ *is the singleton* $\mathbf{q}_t(\tilde{x}_t)$ if $\mathbf{p}_0 = \mathbf{p}_t$ *and the data set* $\{\mathbf{p}_t, \mathbf{q}_t(\tilde{x}_t)\}_{t=1,\ldots T}$ *satisfies SARP.*
Let $\mathbf{p}_0 = \mathbf{p}_t$ and suppose there is a $\mathbf{q}_0 \in S(\mathbf{p}_0, x_0)$ with $\mathbf{q}_0 \neq \mathbf{q}_t(\tilde{x}_t)$. We have $\mathbf{p}_0'\mathbf{q}_0 = x_0$. By construction $\mathbf{q}_t(\tilde{x}_t) R^0 \mathbf{q}_0$ which implies $\mathbf{q}_t(\tilde{x}_t) R \mathbf{q}_0$. Since $\mathbf{q}_0$ satisfies SARP and $\mathbf{q}_0 \neq \mathbf{q}_t(\tilde{x}_t)$ we have *not* $(\mathbf{q}_0 R^0 \mathbf{q}_t(\tilde{x}_t))$ which is equivalent to $\mathbf{p}_0'\mathbf{q}_0 < \mathbf{p}_0'\mathbf{q}_t(\tilde{x}_t) = \mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t)$. Since both sides of this strict inequality are equal to x_0 this gives a contradiction.
(3) $S(\mathbf{p}_0, x_0)$ *is convex.*
Let the support set contain $\check{\mathbf{q}}_0$ and $\tilde{\mathbf{q}}_0$. The convex combination $\lambda\check{\mathbf{q}}_0 + (1-\lambda)\tilde{\mathbf{q}}_0$ for $\lambda \in [0,1]$ satisfies the non-negativity constraint and $\mathbf{p}_0'(\lambda\check{\mathbf{q}}_0 + (1-\lambda)\tilde{\mathbf{q}}_0) = \lambda\mathbf{x}_0 + (1-\lambda)\mathbf{x}_0 = \mathbf{x}_0$. Finally, we have $\mathbf{p}_t'\check{\mathbf{q}}_0 \geq \mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t)$ and $\mathbf{p}_t'\tilde{\mathbf{q}}_0 \geq \mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t)$ so that $\mathbf{p}_t'(\lambda\check{\mathbf{q}}_0 + (1-\lambda)\tilde{\mathbf{q}}_0) \geq \mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t)$.■

Proof of Proposition 3.
If $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,2\ldots T}$ fails SARP then both sets are empty and the proposition holds trivially. In the following we shall assume that $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,2\ldots T}$ passes SARP. We shall first show $S^{LP} \supseteq S$, then part 2 of the proposition and then $cl(S) \supseteq S^{LP}$.
$S^{LP}(\mathbf{p}_0, x_0) \supseteq S(\mathbf{p}_0, x_0)$.
Take any $\mathbf{q}_0 \in S(\mathbf{p}_0, x_0)$. We have $\mathbf{q}_0 \geq \mathbf{0}$ and $\mathbf{p}_0'\mathbf{q}_0 = x_0$ and $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,2\ldots T}$ satisfies SARP. Thus we only need to check the last condition in S^{LP}. Since $\mathbf{p}_0'\mathbf{q}_0 = x_0 = \mathbf{p}_0'\mathbf{q}_t$ we have $\mathbf{q}_0 R^0 \mathbf{q}_t$ which implies $\mathbf{q}_0 R \mathbf{q}_t$. The definition of SARP then gives $\mathbf{p}_t'\mathbf{q}_t < \mathbf{p}_t'\mathbf{q}_0$ which is the condition in the definition of $S^{LP}(\mathbf{p}_0, x_0)$.

For part 2 of the proposition we have:

$$S^{LP} \setminus S = \left\{ \begin{array}{c} \mathbf{q}_0 : \mathbf{q}_0 \geq 0, \mathbf{p}_0'\mathbf{q}_0 = x_0, \\ \mathbf{p}_t'\mathbf{q}_0 \geq \mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t), t = 1, 2, ...T \\ \{\mathbf{p}_0, \mathbf{p}_t, \mathbf{q}_0, \mathbf{q}_t(\tilde{x}_t)\}_{t=1,...T} \text{ fails SARP} \end{array} \right\}$$

If $\mathbf{q}_0 = \mathbf{q}_t(\tilde{x}_t)$ $\mathbf{q}_0 \in S$ so that we only need to consider $\mathbf{q}_0 \neq \mathbf{q}_t(\tilde{x}_t)$ for all t. This and the failure of SARP implies either:
(A) $\mathbf{q}_t(\tilde{x}_t) R\mathbf{q}_0$ and $\mathbf{p}_0'\mathbf{q}_0 \geq \mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t)$ for some t. The first statement requires that there is some s such that $\mathbf{q}_s(\tilde{x}_s) R^0\mathbf{q}_0$ which implies $\mathbf{p}_s'\mathbf{q}_s(\tilde{x}_s) \geq \mathbf{p}_s'\mathbf{q}_0$. Combining this with the condition $\mathbf{p}_s'\mathbf{q}_0 \geq \mathbf{p}_s'\mathbf{q}_s(\tilde{x}_s)$ gives $\mathbf{p}_s'\mathbf{q}_0 = \mathbf{p}_s'\mathbf{q}_s(\tilde{x}_s)$ as in the statement in the proposition.
or:
(B) $\mathbf{q}_0 R\mathbf{q}_t(\tilde{x}_t)$ and $\mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t) \geq \mathbf{p}_0'\mathbf{q}_0$. In this case the latter statement and $\mathbf{p}_t'\mathbf{q}_0 \geq \mathbf{p}_t'\mathbf{q}_t(\tilde{x}_t)$ give the statement in the proposition.
$cl(S) \supseteq S^{LP}$.
We have just shown that it is only boundary of S^{LP} that are not in S. Thus the closure of S contains S^{LP}.■

Proof of Proposition 4.
Since $\mathbf{p}_{T+1} = \mathbf{p}_0$ we have that S^{T+1} is a singleton (by part 2 of proposition 2). Since S^T is convex and there are two distinct intersection points in S^T, there are a continuum of points in S^T. Hence S^Tstrictly includes S^{T+1}.■

Proof of Proposition 5.
1) We first show that intersection of the budget plane $\{\mathbf{p}_{T+1}, x_{T+1}\}$ with $S^T(\mathbf{p}_0, x_0)$ implies that $S^{T+1}(\mathbf{p}_0, x_0) \subset S^T(\mathbf{p}_0, x_0)$. The definition of intersection between the new budget plane $\{\mathbf{p}_{T+1}, x_{T+1}\}$ and $S^T(\mathbf{p}_0, x_0)$ implies that $\mathbf{q}_{T+1}R^0\mathbf{q}_0$. Since $\mathbf{q}_0 \in S^T(\mathbf{p}_0, x_0)$ the definition of an intersection demand implies $\mathbf{q}_0 R^0\mathbf{q}_{T+1}$. This gives a violations of SARP in the dataset $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=0,...,T+1}$. Therefore $\mathbf{q}_0 \notin S^{T+1}(\mathbf{p}_0, x_0)$ and hence $S^{T+1}(\mathbf{p}_0, x_0) \subset S^T(\mathbf{p}_0, x_0)$.

2) We now show that $S^{T+1}(\mathbf{p}_0, x_0) \subset S^T(\mathbf{p}_0, x_0)$ implies intersection of the budget plane $\{\mathbf{p}_{T+1}, x_{T+1}\}$ with $S^T(\mathbf{p}_0, x_0)$. Suppose $S^{T+1}(\mathbf{p}_0, x_0) \subset S^T(\mathbf{p}_0, x_0)$. This implies that there exists at least one $\mathbf{q}_0 \in S^T(\mathbf{p}_0, x_0)$ such that $\mathbf{q}_0 \notin S^{T+1}(\mathbf{p}_0, x_0)$. In the following $\overline{R^0}$ denotes "not R^0". Since $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=0,...,T}$ satisfies SARP, and since $\mathbf{q}_0 R^0 \{\mathbf{q}_t\}_{t=1,...,T}$ by the definition of intersection demands, this implies that $\{\mathbf{q}_t\}_{t=1,...,T} \overline{R^0}\mathbf{q}_0$. Since $\mathbf{q}_0 \notin S^{T+1}(\mathbf{p}_0, x_0)$ the dataset $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=0,...,T+1}$ violates SARP. Given $\{\mathbf{q}_t\}_{t=1,...,T} \overline{R^0}\mathbf{q}_0$ this violation must result from $\mathbf{q}_{T+1}R^0\mathbf{q}_0 \Rightarrow x_{T+1} \geq \mathbf{p}_{T+1}'\mathbf{q}_0$. Hence $\mathbf{q}_0$ must lie in the intersection of the convex set $S^T(\mathbf{p}_0, x_0)$ and the closed half-space $\mathbf{p}_{T+1}'\mathbf{q}_0 \leq x_{T+1}$. If there exists some $\mathbf{q}_0 \in S^T(\mathbf{p}_0, x_0)$ such that $\mathbf{p}_{T+1}'\mathbf{q}_0 < x_{T+1}$ then there must also exist some $\mathbf{q}_0 \in S^T(\mathbf{p}_0, x_0)$ such that $\mathbf{p}_{T+1}'\mathbf{q}_0 = x_{T+1}$ and therefore the new budget plane $\{\mathbf{p}_{T+1}, x_{T+1}\}$ intersects with $S^T(\mathbf{p}_0, x_0)$.■