The Ramsey property implies no mad families

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We show that if all collections of infinite subsets of \( \mathbb{N} \) have the Ramsey property, then there are no infinite maximal almost disjoint (mad) families. The implication is proved in Zermelo–Fraenkel set theory with only weak choice principles. This gives a positive solution to a long-standing problem that goes back to Mathias [A. R. D. Mathias, *Ann. Math. Logic* 12, 59–111 (1977)]. The proof exploits an idea which has its natural roots in ergodic theory, topological dynamics, and invariant descriptive set theory: We use that a certain function associated to a purported mad family is invariant under the equivalence relation \( E \), and thus is constant on a “large” set. Furthermore, we announce a number of additional results about mad families relative to more complicated Borel ideals.

**Theorem 1.** \((ZF + DC + R-Unif)\) If all sets have the Ramsey property, then there are no infinite mad families.

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**Significance**

Certain infinite combinatorial structures in modern mathematics, called mad families, are known to exist only due to indirect, nonconstructive methods arising from a fundamental principle of mathematics, with many paradoxical consequences, called the axiom of choice. This paper shows that if we replace the axiom of choice with a natural assumption of universal combinatorial regularity, a principle known as the Ramsey property for all sets, then no infinite mad families can exist. This solves a problem that has been open in mathematics since the late 1960s.

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We note that Theorem 1 implies the main results of refs. 5 and 7.

Theorem 1 may seem all the more surprising given another recent result of Horowitz and Shelah (11), who show that for a variety of measurability notions including the Lebesgue measure, “all sets are measurable” is compatible with the existence of an infinite mad family.

Let us briefly comment on the proof of Theorem 1 and the difficulties that have to be overcome. For this discussion, suppose \( A \subseteq [N]^\infty \) is an infinite mad family, and assume “all sets are Ramsey” and “Ramsey uniformization” (Definition 2).

The first difficulty encountered is the set of \( x \in [N]^\infty \) which meet exactly 1 element of \( A \) in an infinite set is clearly Ramsey conull when \( A \) is a mad family. The key idea is to overcome this difficulty by associating to each \( z \in [N]^\infty \) a carefully chosen very, very sparse set \( \tilde{z} \in [N]^\infty \), which is constructed using a fixed, infinite sequence \( (a_n)_{n \in \mathbb{N}} \) chosen from \( A \) (it is here that we use the principle of DC). A basic property of the map \( z \mapsto \tilde{z} \) is that it is equiuniform under finite differences; that is, if \( z \Delta' z \) is finite, then \( \tilde{z} \Delta \tilde{z} \) is finite.

Because we assumed that \( A \) is maximal, for each \( z \in [N]^\infty \), there is some \( y \in A \) such that \( z \cap y \) is infinite, and so \( \text{R-Unif} \) gives us a function \( f : [N]^\infty \rightarrow A \) such that \( f(z) \cap z \) is infinite for \( z \) in a Ramsey positive set. The special way that \( z \mapsto \tilde{z} \) will be defined below will ensure that no uniformizing function \( f \) can have the invariance property that \( \| z \Delta' z \| < \infty \) implies \( f(z) = f(\tilde{z}) \). While there is no reason to expect that an abstract application of \( \text{R-Unif} \) would give us \( f \) with this property, it turns out that with some work we can get dangerously close to having such an invariant \( f \). Indeed, by using the assumption that all \( A \) we can find an infinite set \( W \subseteq N \) such that the restriction \( f([W]^\infty) \) is continuous, and so the range \( f([W]^\infty) \) is an analytic set. Using that \( [N]^\infty \) is an analytic set. Using that \( \text{R-Unif} \) is analytic, we will define a function \( z \mapsto T^z \), where \( T^z \) can be thought of as a tree of approximations to possible, natural uniformization functions. It then turns out that the map \( z \mapsto T^z \) satisfies that if \( \| z \Delta' z \| < \infty \), then \( T^z = T^\tilde{z} \). This in turn leads to that \( z \mapsto T^z \) is constant on a Ramsey positive set, which then leads to a contradiction.

B. The Ramsey Property. For any set \( X \) we define

\[ [X]^\infty = \{ A \subseteq X : A \text{ is infinite} \}. \]

Recall from the Introduction that a set \( S \subseteq [N]^\infty \) is Ramsey (or has the Ramsey property) if there is \( B \subseteq [N]^\infty \) such that \( B \subseteq S \) or \( S \cap [B]^\infty = \emptyset \).

To each infinite \( B \subseteq \mathbb{N} \) we let \( \hat{B} : \mathbb{N} \rightarrow \mathbb{N} \) be the unique increasing function with \( B = \text{ran} \hat{B} \). For each infinite \( A \subseteq \mathbb{N} \) the map \( [A]^\infty \rightarrow N : \hat{B} \rightarrow \hat{B} \) naturally identifies \( [A]^\infty \) with a subset of \( N \). The (subspace) topology that \( [A]^\infty \) inherits under this identification will be called the standard topology on \( [A]^\infty \).

The crucial use in our proof of the Ramsey property comes from the following well-known fact:

Proposition 2. (ZF + DC) Suppose all sets have the Ramsey property and let \( \theta : [N]^\infty \rightarrow Y \) be a function, where \( Y \) is a separable topological space. Then there is \( A \subseteq [N]^\infty \) such that \( \text{ran} \theta[A]^\infty \) is continuous with respect to the standard topology on \( [A]^\infty \).

C. Uniformization. Next we define \( \text{R-Unif} \), the Ramsey uniformization principle.

Definition 2.

1) For Polish spaces \( X, Y, \text{ and } R \subseteq X \times Y \), we say that \( R \) has full projection on \( X \), if for every \( x \in X \) there is \( y \in Y \) such that \((x, y) \in R\).

2) Let \( X, Y \) be Polish spaces, let \( Z \subseteq X \), and let \( R \subseteq X \times Y \) be a set with full projection. A function \( \theta : Z \rightarrow Y \) is said to uniformize \( R \) on \( Z \) if for all \( x \in Z \) we have \((x, \theta(x)) \in R\).

3) The Ramsey uniformization principle, abbreviated \( \text{R-Unif} \), is the following statement: For all \( Y \) Polish and all \( R \subseteq [N]^\infty \times Y \) with full projection, there is an infinite set \( A \subseteq N \) and a function \( \theta : [A]^\infty \rightarrow Y \) which uniformizes \( R \) on \([A]^\infty \).

That the Ramsey uniformization principle holds in Solovay’s model follows by the argument given in ref. 14, section 1.12, p. 46 with Random forcing replaced everywhere by Mathias forcing.

D. Invariance under Finite Changes. The last ingredient for the proof is the notion of “\( E_0 \) invariance.”

Definition 3.

1) \( E_0 \) is the equivalence relation defined on \([N]^\infty \) by

\[ xE_0 y \iff \| x \Delta y \| < \infty. \]

In other words, \( x, y \in [N]^\infty \) are \( E_0 \) equivalent iff they differ only on a finite set.

2) A function \( f : [N]^\infty \rightarrow Y \) is called \( E_0 \) invariant if \( xE_0 y \) implies \( f(x) = f(y) \).
Proof of Theorem 1
We work under the following assumptions: ZF + DC + R-Unif + “all sets have the Ramsey property.”

Let \( A \subseteq [\mathbb{N}]^\omega \) be an infinite almost disjoint family. We will show that \( A \) is not maximal.

Let \( (a_n)_{n \in \mathbb{N}} \) be an injective sequence of elements in \( A \) (here we use that DC implies that all infinite sets are Dedekind infinite). We may assume that \( A = A \setminus \{ (a_n : n \in \mathbb{N}) \} \) is nonempty, since otherwise an easy diagonalization shows that \( A \) is not maximal. Moreover, by possibly replacing \( a_n \) by \( a_n \cup \{v : v \in a_n, \} \) let us assume for simplicity that if \( n \neq m \), then \( a_n \cap a_m = \emptyset \) (each \( a_n \) is now an infinite subset of an element of \( A \)).

Recall that when \( Z \subseteq [\mathbb{N}]^\omega \), then \( Z \cap \mathbb{N} \rightarrow \mathbb{N} \) is the unique increasing function such that ran(\( Z \)) = \( Z \). Using the sequence \( (a_n)_{n \in \mathbb{N}} \) fixed above, define for each \( i \), \( T_i = \{ t m = t m = z \} \) with an infinite subset of \( (\infty \subseteq \infty) \). We will think of \( x = (x \in \mathbb{N}) \) for all \( x \in \mathbb{N} \) and suppose \( x \notin \mathbb{N} \). If \( x \in \mathbb{N} \), then \( x \in \mathbb{N} \).

Now we arrive at a contradiction: If \( x \in \mathbb{N} \) and \( T_i = T_i \), then \( T_i \subseteq T_i \). Since \( T_i \subseteq T_i \), we have \( T_i \subseteq T_i \). Let \( T_i \subseteq T_i \) and \( T_i \subseteq T_i \) with \( \infty \subseteq \infty \) such that for all \( x \in \mathbb{N} \), \( x \in \mathbb{N} \) and \( x \in \mathbb{N} \), we have \( m \in f(x) \cap f(x) \).

Using the previous paragraph repeatedly, we can now build sequences

\[
t_0 < u_1 < u_2 \leq \cdots \\
t_0 \leq v_1 \leq v_2 \leq \cdots \\
m_0 < m_1 < \cdots
\]

where \( u_i, v_i \in T \), and for all \( x_0 \in N_{\alpha} \), \( x_1 \in N_{\alpha} \), we have \( m \in f(x_0) \cap f(x_1) \) such that for all \( x_0 \in N_{\alpha} \) and \( x_1 \in N_{\alpha} \), we have \( m \in f(x_0) \cap f(x_1) \).

Subclaim 4. There is a unique \( y^* \in B \) such that for all \( z \in [W_0]^\omega \), we have \( \{ y \cap y^* = \emptyset \} \).

Proof: Since \( y(z) \in B \) for \( z \in [W_0]^\omega \) and \( \{ y \cap y(z) = \emptyset \} \), define \( y(z) \in B \) by definition, for every \( z \in [W_0]^\omega \) there is some \( y^* \in B \) such that \( \{ y \cap y^* = \emptyset \} \).

We must show that there is a unique \( y \in B \) not depending on \( z \) satisfying this.

Suppose not, and let \( x_0, x_1 \in \mathbb{N} \) such that \( f(x_0) \neq f(x_1) \) and for some \( x_0, x_1 \in [W_0]^\omega \) we have \( |y_0 \cap f(x_0)| = \infty \) and \( |y_1 \cap f(x_1)| = \infty \). By continuity of \( f \) we can find \( u_0 \subseteq x_0 \) and \( u_1 \subseteq x_1 \) and \( n_0 \in \mathbb{N} \) such that for all \( x \in N_{\alpha} \) and \( x \in N_{\alpha} \), we have \( n \in f(x) \Delta f(x) \). This contradicts that \( \{ y \cap y^* = \emptyset \} \) is a subset of the almost disjoint family \( A \).

Suppose the claim is false. Then by Ramsey uniformization

\[
\text{arrange that exactly 1 of the following holds:}
\]

1. \( \{ y \cap y^* = \emptyset \} \)
2. \( \{ y \cap y^* = \emptyset \} \)
3. \( \{ y \cap y^* = \emptyset \} \)
4. \( \{ y \cap y^* = \emptyset \} \)

Let \( a_n = a_n \cap \bigcup f(N_\alpha) \).

By our assumptions on \( s^0 \) and \( s^1 \) we have that \( a_n^0 \cap a_n^1 = \emptyset \) for all \( x \in \mathbb{N} \).

Now we arrive at a contradiction: If \( (1^0) \) holds, then since \( a_n^0 \cap a_n^1 = \emptyset \) for all \( n \in W_0^\omega \), we have for all \( n \in W_0^\omega \) and \( n \in W_0^\omega \) such that \( z \cap f(N_{\alpha}) = \emptyset \), contradicting that \( s^0 \in T \). Similarly, if \( (1^1) \) holds, we get for all \( n \in W_0^\omega \) and \( n \in W_0^\omega \) such that \( z \cap f(N_{\alpha}) = \emptyset \), contradicting that \( s^0 \in T \).
Remarks:

1) In the proof above, a crucial point was obtaining $W^*_j \in [W]\infty$ such that $(\forall z \in [W]\infty) \exists \Gamma \in J \exists n \in 2^\omega \forall y \in \Gamma \exists z_0$ or $z \cap a_0 = \emptyset$. Note that an alternative and quick way to obtain such $W^*_j$ is to appeal to Ramsey’s theorem for pairs and take $W^*_j$ to be a homogeneous set for the 2-coloring of unordered pairs $(n, j)$, where assuming $n < j$ we let

$$c(n, j) = \begin{cases} 1 & \text{if } \tilde{a}_n(j) \subseteq a_0, \\ 0 & \text{otherwise.} \end{cases}$$

2) Finally, the following obvious question must be addressed: Can a similar result be obtained without assuming R-Unif? We think it is unlikely and that the assumptions of Theorem 1 are the minimal natural assumptions needed to obtain a positive solution to Mathias’ problem; but we do not know.

Corollaries and Further Results

Corollary 3 (Törnquist (5)). There are no mad families in Solovay’s model.

Proof: By ref. 14, Solovay’s model is a model of ZF + DC. That the Ramsey property holds in this model follows from ref. 1. Finally, the Ramsey uniformization principle holds by our remarks after Definition 2.

We point out that the proof of Theorem 1 above localizes as follows.

Corollary 4. If $\Gamma, \Gamma'$ are reasonable pointclasses such that every relation $R \in \Gamma$ can be uniformized on a Ramsey positive set by a $\Gamma'$-measurable function and all sets in $\Gamma'$ are Ramsey, then there are no infinite mad families in $\Gamma$.

Note that in particular, this gives an additional proof of Mathias’ classical result that there are no analytic infinite mad families: Apply the corollary taking the pointclass of analytic subsets of $[\omega]^\omega$ for $\Gamma$, and the $\sigma$ algebra generated by the analytic sets for $\Gamma'$, and use the Jankov-von Neumann uniformization theorem (ref. 12, theorem 18.1).

Our first corollary also allows us to draw consequences regarding the axiom of projective determinacy (in short, PD).

Corollary 5 (7, 8). PD implies there are no projective infinite mad families.

Proof: The hypotheses of the previous theorem hold with $\Gamma' = \Gamma$ equal to the class of projective sets: PD implies that this pointclass has the uniformization property, and by ref. 15, all projective sets are completely Ramsey under PD.

Another consequence of our proof is that Mathias forcing destroys mad families from the ground model:

Theorem 6. In the Mathias extension, there is no infinite mad family which is definable by a $\sum_1$ formula in the language of set theory with parameters in the ground model. In particular, no infinite almost disjoint family from the ground model is maximal in the Mathias extension.

Proof: Suppose $x$ is Mathias over $V$ and that in $V[x], A = \{ z \in [\omega]^{<\omega} \mid \Psi(z) \}$ is an infinite almost disjoint family, where $\Psi(z)$ is $\Sigma_1$ (with parameter in $V$). We show that $\tilde{x}$, where $\tilde{x}$ is defined in $V[x]$ as in the proof of Theorem 1, is almost disjoint from every $z \in A$. Otherwise, we can choose a Mathias condition $(s, A)$ with $s \subseteq x \subseteq A$ and a name $\dot{y}$ such that $p \Vdash \dot{y} \cap \tilde{x}_c = \emptyset$ and $\Psi(y)$ (where $\tilde{x}_c$ is a name for the Mathias real). By a well-known property of Mathias forcing (so-called continuous reading of names) we can assume that there is a continuous function $\vartheta : [\omega]^{<\omega} \rightarrow [\omega]^{<\omega}$ with code in $V$ such that $p$ forces that $\dot{y} = \vartheta(\tilde{x}_c)$. It is easy to see that $\vartheta(\tilde{y}) \in A$ for any $y \in [\omega]^{<\omega}$ (here we use the definability of $A$). But then $\vartheta(\tilde{y})$ would be an analytic almost disjoint family such that any element of $\{\dot{y} : y \in [\omega]^{<\omega}\}$ has infinite intersection with some element of ran($\vartheta$), which is impossible by the proof of Theorem 1.

Surprisingly, the connection between the Ramsey property in $[\omega]^{<\omega}$ and mad families relative to the ideal of finite sets extends to much more complicated Borel ideals. We construct a family of ideals using the familiar Fibuni sum: Given, for each $n \in \mathbb{N}$, an ideal $\mathcal{J}_n$ on a countable set $S_n$ we obtain an ideal $\mathcal{J}$ on $S = \bigcup_n S_n$ as follows:

$$\mathcal{J} = \bigoplus_{n} \mathcal{J}_n \equiv \{ X \subseteq S \mid (\forall n) X \cap S_n \in \mathcal{J}_n \}$$

where $(\forall n) m$ means “for all but finitely many $n$.” The Fibuni sum $\bigoplus_{n} \mathcal{J}_n$ (where $\mathcal{J}_n$ denotes the ideal of finite sets on $N$) is also known as FIN $\times$ FIN or FIN$^2$; iterating Fibuni sums into the transfinite we obtain FIN$^\beta$, $\alpha < \omega_1$. This family of ideals of lies cofinally in the Borel hierarchy in terms of complexity.

The notion of the mad family can be extended to arbitrary ideals on a countable set: If $\mathcal{J}$ is such an ideal, a $\mathcal{J}$-almost disjoint family is a subfamily $\mathcal{A}_0$ of $\mathcal{P}(S) \setminus \mathcal{J}$ such that for any 2 distinct $\mathcal{A}, \mathcal{A}' \in \mathcal{A}_0$, $\mathcal{A} \cap \mathcal{A}' \in \mathcal{J}$. A $\mathcal{J}$-mad family is of course a $\mathcal{J}$-almost disjoint family which is maximal under $\subseteq$ among such families.

In ref. 16 we show the following:

Theorem 7. (ZF + DC + R-Unif) Let $\alpha < \omega_1$. If all sets have the Ramsey property, then there are no infinite FIN$^\alpha$-mad families.

As for classical mad families, we immediately obtain corollaries regarding the axiom of projective determinacy and Solovay’s model. The first corollary was already shown in ref. 8 using forcing over inner models.

Corollary 8. (ZF + PD) For each $\alpha < \omega_1$ there are no infinite projective FIN$^\alpha$-mad families.

Corollary 9. For each $\alpha < \omega_1$ there are no infinite FIN$^\alpha$-mad families in Solovay’s model.

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