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The Ramsey property implies no mad families

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We show that if all collections of infinite subsets of $\mathbb{N}$ have the Ramsey property, then there are no infinite maximal almost disjoint (mad) families. The implication is proved in Zermelo–Fraenkel set theory with only weak choice principles. This gives a positive solution to a long-standing problem that goes back to Mathias [A. R. D. Mathias, *Ann. Math. Logic* 12, 59–111 (1977)]. The proof exploits an idea which has its natural roots in ergodic theory, topological dynamics, and invariant descriptive set theory: We use that a certain function associated to a purported mad family is invariant under the equivalence relation E_0 and thus is constant on a "large" set. Furthermore, we announce a number of additional results about mad families relative to more complicated Borel ideals.

Ramsey property | maximal almost disjoint families | invariant descriptive set theory | Borel ideals

In his seminal paper, Mathias (1) established a connection between 3 different ideas in mathematics: the combinatorial set theory of maximal almost disjoint families, infinite-dimensional Ramsey theory, and Cohen's method of forcing. He asked whether the combinatorial statement "all sets have the Ramsey property" implies that there are no infinite maximal almost disjoint (mad) families. In this paper we answer this in the affirmative, working in the theory ZF + DC + R-Unif (*Definition 2*).

Let us recall the key notions: An almost disjoint family (on the natural numbers $\mathbb{N}$) is a family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ such that if $x, y \in \mathcal{A}$, then either $x = y$ or $x \cap y$ is finite. A maximal almost disjoint family ("mad family") is an almost disjoint family which is not a proper subset of an almost disjoint family. Finite mad families are easily seen to exist, e.g., $\{E, O\}$, where E is the set of even numbers and O is the set of odd numbers. The existence of infinite mad families follows easily from Zorn's lemma (equivalently, the axiom of choice).

Given a set X and a natural number $k \in \mathbb{N}$, let $[X]^k$ denote the set of all subsets of X with exactly k elements. The classical infinite Ramsey theorem in combinatorics says that if $S \subseteq [\mathbb{N}]^k$, then there is an infinite set $B \subseteq \mathbb{N}$ such that either $[B]^k \subseteq S$ or $[B]^k \cap S = \emptyset$. Motivated by a question of Erdős and Rado, infinite-dimensional generalizations of this theorem were discovered in the 1960s and 1970s. In this paper, we denote by $[X]^\infty$ the set of countably infinite subsets of X. Moreover, given $S \subseteq [\mathbb{N}]^\infty$, we will say that S has the Ramsey property, or simply is Ramsey, if there is $B \in [\mathbb{N}]^\infty$ such that either $[B]^\infty \subseteq S$ or $[B]^\infty \cap S = \emptyset$. Erdős and Rado showed that the axiom of choice implies that not all sets $S \subseteq [\mathbb{N}]^\infty$ are Ramsey. Later, in refs. 2 and 3, it was shown that Borel and analytic $S \subseteq [\mathbb{N}]^\infty$ are Ramsey, and finally Ellentuck (4) in 1974 characterized the Ramsey property in terms of Baire measurability in the Ellentuck topology on $[\mathbb{N}]^\infty$.

Concurrent with these developments in Ramsey theory, Cohen's introduction of the method of forcing for independence proofs in set theory in the early 1960s set off an explosion of independence results, among the most famous of which is Solovay's model of Zermelo–Fraenkel set theory (ZF) in which only a fragment of the axiom of choice–namely dependent choice (DC)—holds and in which all subsets of $\mathbb{R}$ are Lebesgue and Baire measurable. It was Mathias, in his "Happy families" paper (1) (drafts of which circulated already in the late 1960s), who connected forcing to Erdős and Rado's question and to mad families. Mathias did this by introducing what is now known as Mathias forcing, which he used to show that in Solovay's model all sets $S \subseteq [\mathbb{N}]^\infty$ are Ramsey. Mad families, and their connection to Mathias forcing, play a central role in this proof.

Mathias asked 2 central questions, which his methods did not allow him to answer at the time: 1) Are there infinite mad families in Solovay's model? 2) If all sets $S \subseteq [\mathbb{N}]^\infty$ are Ramsey, does it follow that there are no infinite mad families? A positive answer to question 2 would give a negative answer to question 1.

There was only modest progress on these questions until very recently, when suddenly the research in mad families and forcing experienced a renaissance. Question 1 was solved in 2014 in ref. 5, and shortly after, Horowitz and Shelah showed in ref. 6 that a model of ZF in which there are no mad families can be achieved without using an inaccessible cardinal, which is otherwise a crucial ingredient in the construction of Solovay's model. Neeman and Norwood in ref. 7 and independently, Bakke Haga in joint work with the present authors in ref. 8 proved a number of further results, among them that $V = L(\mathbb{R})$ + AD implies there are no mad families. Horowitz and Shelah also solved a number of related questions that had been formulated over the years; in particular, they showed the existence of a Borel "med" family in ref. 9; see also ref. 10 for a simpler proof.

We denote by R-Unif the principle of uniformization on Ramsey positive sets (*Definition 2*). R-Unif is a weak choice principle, which is weak enough that it holds in Solovay's model, and this can be seen quite easily. In this paper we give the following positive solution to Mathias' question.

Theorem 1. (*ZF + DC + R-Unif*) *If all sets have the Ramsey property, then there are no infinite mad families.*

Significance

Certain infinite combinatorial structures in modern mathematics, called mad families, are known to exist only due to indirect, nonconstructive methods arising from a fundamental principle of mathematics, with many paradoxical consequences, called the axiom of choice. This paper shows that if we replace the axiom of choice with a natural assumption of universal combinatorial regularity, a principle known as the Ramsey property for all sets, then no infinite mad families can exist. This solves a problem that has been open in mathematics since the late 1960s.

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MATHEMATICS

We note that *Theorem 1* implies the main results of refs. 5 and 7.

Theorem 1 may seem all the more surprising given another recent result of Horowitz and Shelah (11), who show that for a variety of measurability notions including the Lebesgue measure, "all sets are measurable" is compatible with the existence of an infinite mad family.

Let us briefly comment on the proof of *Theorem 1* and the difficulties that have to be overcome. For this discussion, suppose $\mathcal{A} \subseteq [\mathbb{N}]^\infty$ is an infinite mad family, and assume "all sets are Ramsey" and "Ramsey uniformization" (*Definition 2*).

The first difficulty encountered is that the set of $x \in [\mathbb{N}]^\infty$ which meet exactly 1 element of $\mathcal{A}$ in an infinite set is clearly Ramsey conull when $\mathcal{A}$ is a mad family. The key idea is to overcome this difficulty by associating to each $z \in [\mathbb{N}]^\infty$ a carefully chosen very, very sparse set $\tilde{z} \in [\mathbb{N}]^\infty$, which is constructed using a fixed, infinite sequence $(a_n)_{n\in\mathbb{N}}$ chosen from $\mathcal{A}$ (it is here that we use the principle of DC). A basic property of the map $z \mapsto \tilde{z}$ is that it is equivariant under finite differences; that is, if $z \triangle z'$ is finite, then $\tilde{z} \triangle \tilde{z}'$ is finite.

Because we assumed that $\mathcal{A}$ is maximal, for each $z \in [\mathbb{N}]^\infty$ there is some $y_z \in \mathcal{A}$ such that $\tilde{z} \cap y$ is infinite, and so R-Unif gives us a function $f : [\mathbb{N}]^\infty \to \mathcal{A}$ such that $f(z) \cap \tilde{z}$ is infinite for z in a Ramsey positive set. The special way that $z \mapsto \tilde{z}$ will be defined below will ensure that no uniformizing function f can have the invariance property that $|z \triangle z'| < \infty$ implies $f(z) = f(z')$. While there is no reason to expect that an abstract application of R-Unif would give us f with this property, it turns out that with some work we can get dangerously close to having such an invariant f. Indeed, by using the assumption that all sets are Ramsey we can find an infinite set $W \subseteq \mathbb{N}$ such that the restriction $f \restriction [W]^\infty$ is continuous, and so the range $f([W]^\infty)$ is an analytic set. Using that $f([W]^\infty)$ is analytic, we will define a function $z \mapsto T^z$, where T^z can be thought of as a tree of approximations to possible, natural uniformization functions. It then turns out that the map $z \mapsto T^z$ satisfies that if $|z \triangle z'| < \infty$, then $T^z = T^{z'}$. This in turn leads to that $z \mapsto T^z$ is constant on a Ramsey positive set, which then leads to a contradiction.

Notation and Background Definitions

In this section we summarize the background needed for the proof. A good general reference for all of the background needed is ref. 12. A comprehensive treatise on modern, infinitary Ramsey theory is ref. 13.

A. Descriptive Set Theory. A topological space X is called Polish if it is separable and admits a complete metric that induces the topology. In this paper we will be working with the Polish space $2^\mathbb{N} = \{0,1\}^\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ (with the product topology, taking $\{0,1\}$ and $\mathbb{N}$ discrete) and subspaces of these spaces. Recall the following key notion from descriptive set theory:

Definition 1. *A subset $A \subseteq X$ of a Polish space X is analytic if there is a continuous $f : Y \to X$ from a Polish space Y to X such that $A = \operatorname{ran}(f)$.*

Since $\mathbb{N}^\mathbb{N}$ maps continuously onto any Polish space, we have that $A \subseteq X$ is analytic iff there is a continuous $f : \mathbb{N}^\mathbb{N} \to X$ such that $\operatorname{ran}(f) = A$. We will use this characterization as our definition of the analytic set below.

For the proof of *Theorem 1* we need the following combinatorial description of the topology on $\mathbb{N}^\mathbb{N}$. We denote by $\mathbb{N}^n$ the set of all functions $s : \{1, \dots, n\} \to \mathbb{N}$, and we let $\mathbb{N}^{<\mathbb{N}} = \{\emptyset\} \cup \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$. (We shall think of $\emptyset$ as the function with an empty domain, which is why it is included as an element of $\mathbb{N}^{<\mathbb{N}}$.) For $s, t \in \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N}$ we will write $s \subseteq t$ ("t extends s") if $\operatorname{dom}(s) \subseteq \operatorname{dom}(t)$ and $s(i) = t(i)$ for all $i \in \operatorname{dom}(s)$; we will write $s \perp t$ ("s and t are incompatible") if $s \not\subseteq t$ and $t \not\subseteq s$.

For each $s \in \mathbb{N}^{<\mathbb{N}}$, let

$$N_s = \{f \in \mathbb{N}^\mathbb{N} : (\forall i \in \operatorname{dom}(s))\, f(i) = s(i)\}.$$

The family $\{N_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ is easily seen to form a basis for the topology on $\mathbb{N}^\mathbb{N}$.

Note that $\mathbb{N}^{<\mathbb{N}}$ is countable, and so $2^{\mathbb{N}^{<\mathbb{N}}}$ is a Polish space (isomorphic to $2^\mathbb{N}$) in the product topology, taking $2 = \{0,1\}$ discrete. This view will be important later in the proof of *Theorem 1* where we will describe the properties of a certain continuous function f defined on $\mathbb{N}^\mathbb{N}$ in terms of a "derived" function $z \mapsto T^z$ from $\mathbb{N}^\mathbb{N}$ to $2^{\mathbb{N}^{<\mathbb{N}}}$.

B. The Ramsey Property. For any set X we define

$$[X]^\infty = \{A \subseteq X : A \text{ is infinite}\}.$$

Recall from the Introduction that a set $S \subseteq [\mathbb{N}]^\infty$ is Ramsey (or has the Ramsey property) if there is $B \in [\mathbb{N}]^\infty$ such that $[B]^\infty \subseteq S$ or $S \cap [B]^\infty = \emptyset$.

To each infinite $B \subseteq \mathbb{N}$ we let $\hat{B} : \mathbb{N} \to \mathbb{N}$ be the unique increasing function with $B = \operatorname{ran}(\hat{B})$. For each infinite $A \subseteq \mathbb{N}$ the map $[A]^\infty \to \mathbb{N}^\mathbb{N} : B \mapsto \hat{B}$ naturally identifies $[A]^\infty$ with a subset of $\mathbb{N}^\mathbb{N}$. The (subspace) topology that $[A]^\infty$ inherits under this identification will be called the standard topology on $[A]^\infty$.

The crucial use in our proof of the Ramsey property comes from the following well-known fact:

Proposition 2. (*ZF + DC*) *Suppose all sets have the Ramsey property and let $\vartheta : [\mathbb{N}]^\mathbb{N} \to Y$ be a function, where Y is a separable topological space. Then there is $A \in [\mathbb{N}]^\infty$ such that $\vartheta \restriction [A]^\infty$ is continuous with respect to the standard topology on $[A]^\infty$.*

C. Uniformization. Next we define R-Unif, the Ramsey uniformization principle.

Definition 2.

1) *For Polish spaces X, Y, and $R \subseteq X \times Y$, we say that R has full projection on X, if for every $x \in X$ there is $y \in Y$ such that $(x,y) \in R$.*
2) *Let X, Y be Polish spaces, let $Z \subseteq X$, and let $R \subseteq X \times Y$ be a set with full projection. A function $\vartheta : Z \to Y$ is said to uniformize R on Z if for all $x \in Z$ we have $(x, \vartheta(x)) \in R$.*
3) *The Ramsey uniformization principle, abbreviated R-Unif, is the following statement: For all Y Polish and all $R \subseteq [\mathbb{N}]^\infty \times Y$ with full projection, there is an infinite set $A \subseteq \mathbb{N}$ and a function $\vartheta : [A]^\infty \to Y$ which uniformizes R on $[A]^\infty$.*

That the Ramsey uniformization principle holds in Solovay's model follows by the argument given in ref. 14, section 1.12, p. 46 with Random forcing replaced everywhere by Mathias forcing.

D. Invariance under Finite Changes. The last ingredient for the proof is the notion of "E_0 invariance."

Definition 3.

1) *E_0 is the equivalence relation defined on $[\mathbb{N}]^\infty$ by*

$$xE_0y \iff |x \triangle y| < \infty.$$

In other words, $x, y \in [\mathbb{N}]^\infty$ are E_0 equivalent iff they differ only on a finite set.

2) *A function $f : [\mathbb{N}]^\infty \to Y$ is called E_0 invariant if xE_0y implies $f(x)E_0f(y)$.*

Proof of Theorem 1

We work under the following assumptions: ZF + DC + R-Unif + "all sets have the Ramsey property."

Let $\mathcal{A} \subseteq [\mathbb{N}]^\infty$ be an infinite almost disjoint family. We will show that $\mathcal{A}$ is not maximal.

Let $(a_n)_{n\in\mathbb{N}}$ be an injective sequence of elements in $\mathcal{A}$ (here we use that DC implies that all infinite sets are Dedekind infinite). We may assume that $\mathcal{A}' = \mathcal{A} \setminus \{a_n : n \in \mathbb{N}\}$ is nonempty, since otherwise an easy diagonalization shows that $\mathcal{A}$ is not maximal. Moreover, by possibly replacing a_n by $a_n \setminus (\bigcup_{i<n} a_i)$, let us assume for simplicity that if $n \neq m$, then $a_n \cap a_m = \emptyset$ (each a_n is now an infinite subset of an element of $\mathcal{A}$).

Recall that when $z \in [\mathbb{N}]^\infty$, then $\hat{z} : \mathbb{N} \to \mathbb{N}$ is the unique increasing function such that $\mathrm{ran}(\hat{z}) = z$. Using the sequence $(a_n)_{n\in\mathbb{N}}$ fixed above, define for each $z \in [\mathbb{N}]^\infty$,

$$\tilde{z} = \{\widehat{a_{\hat{z}(n)}}(\hat{z}(n+1)) : n \in \mathbb{N}\}.$$

Note that $|\tilde{z} \cap a_n| \leq 1$ for all n, so proving the following claim will prove the theorem:

Main Claim. There is $z \in [\mathbb{N}]^\infty$ such that for all $y \in \mathcal{A}'$, $|\tilde{z} \cap y| < \infty$.

Suppose the claim is false. Then by Ramsey uniformization there is $W \in [\mathbb{N}]^{<\infty}$ and $\vartheta : [W]^\infty \to \mathcal{A}'$ such that

$$|\vartheta(z) \cap \tilde{z}| = \infty$$

for all $z \in [W]^\infty$. By *Proposition 2* we may assume (after possibly replacing W with an infinite subset of W) that $\vartheta \restriction [W]^\infty$ is continuous. Then $\mathcal{B} = \vartheta([W]^\infty)$ is an analytic subset of $\mathcal{A}'$, and we fix a continuous function $f : \mathbb{N}^\mathbb{N} \to [\mathbb{N}]^\infty$ such that $\mathrm{ran}(f) = \mathcal{B}$.

For $z \in [\mathbb{N}]^\infty$, let

$$T^z = \{s \in \mathbb{N}^{<\mathbb{N}} : (\exists x \in N_s)\ |\tilde{z} \cap f(x)| = \infty\}.$$

By identifying $\mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ with $2^{\mathbb{N}^{<\mathbb{N}}}$, we will think of $z \mapsto T^z$ as a map $[\mathbb{N}]^\infty \to 2^{\mathbb{N}^{<\mathbb{N}}}$. The reader can easily verify that T^z is a tree in the sense of ref. 12 and that $\emptyset \in T^z$ for all $z \in [W]^\infty$.

Subclaim 1. The function $z \mapsto T^z$ is E_0 invariant.

Proof: Suppose $|z' \triangle z| < \infty$. Then we can find k_0, k_0' such that $\hat{z}(k_0 + i) = \hat{z}'(k_0' + i)$ for all $i \in \mathbb{N}$. Then

$$\widehat{a_{\hat{z}(k_0+i)}}(\hat{z}(k_0+i+1)) = \widehat{a_{\hat{z}'(k_0'+i)}}(\hat{z}'(k_0'+i+1))$$

for all $i \in \mathbb{N}$. It follows that $\tilde{z} E_0 \tilde{z}'$, but then $|\tilde{z} \cap f(x)| = \infty$ if and only if $|\tilde{z}' \cap f(x)| = \infty$, so $T^z = T^{z'}$. Subclaim 1.$\dashv$

Subclaim 2. There is $W_0 \in [W]^\infty$ such that $z \mapsto T^z$ is constant on $[W_0]^\infty$.

Proof: By *Proposition 2* there is $W_0 \in [W]^\infty$ such that $z \mapsto T^z$ is continuous on $[W_0]^\infty$. Since $z \mapsto T^z$ is E_0 invariant, it follows that $z \mapsto T^z$ is constant on $[W_0]^\infty$. Subclaim 2.$\dashv$

From now on we fix $W_0 \in [W]^\infty$ and $\tilde{T} \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $T^z = \tilde{T}$ for all $z \in [W_0]^\infty$. The next claim echoes the claim in ref. 5, top of p. 65.

Subclaim 3. Suppose there are $t^0, t^1 \in \tilde{T}$ and $n_0 \in \mathbb{N}$ such that for all $y_0 \in f(N_{t^0})$ and $y_1 \in f(N_{t^1})$ we have $n_0 \in f(y_0) \triangle f(y_1)$. Then there are $s^0, s^1 \in \tilde{T}$ and $k \in \mathbb{N}$ such that $s^0 \supseteq t^0$, $s^1 \supseteq t^1$, and for all $y_0 \in f(N_{s_0})$ and $y_1 \in f(N_{s_1})$ we have $y_0 \cap y_1 \subseteq \{1, \ldots, k\}$.

Proof: Suppose no $s^0 \supseteq t^0$ and $s^1 \supseteq t^1$, with $s^0, s^1 \in \tilde{T}$, satisfies the claim. Then for every $m \in \mathbb{N}$, $t^0 \subseteq u \in \tilde{T}$ and $t^1 \subseteq v \in \tilde{T}$ we can find $m' > m$, $u \subseteq u' \in \tilde{T}$ and $v \subseteq v' \in \tilde{T}$ such that for some $x_0 \in N_{u'}$ and $x_1 \in N_{v'}$ we have $m' \in f(x_0) \cap f(x_1)$. By the continuity of f we can then find $u' \subseteq u'' \in \tilde{T}$ and $v' \subseteq v'' \in \tilde{T}$ such that for all $x_0 \in N_{u''}$ and $x_1 \in N_{v''}$ we have $m' \in f(x_0) \cap f(x_1)$.

Using the previous paragraph repeatedly, we can now build sequences

$$t^0 \subseteq u_1 \subseteq u_2 \subseteq \cdots$$
$$t^1 \subseteq v_1 \subseteq v_2 \subseteq \cdots$$
$$m_0 < m_1 < \cdots$$

where $u_i, v_i \in \tilde{T}$, and for all $x_0 \in N_{u_i}$ and $x_1 \in N_{v_i}$ we have $m_i \in f(x_0) \cap f(x_1)$ when $i > 0$. Let $u_\infty, v_\infty \in \mathbb{N}^\mathbb{N}$ be such that $u_i \subseteq u_\infty$ and $v_i \subseteq v_\infty$ for all $i \in \mathbb{N}$. Then $|f(u_\infty) \cap f(v_\infty)| = \infty$ since $m_i \in f(u_\infty) \cap f(v_\infty)$ for all $i > 0$, but $f(u_\infty) \neq f(v_\infty)$ since $n_0 \in f(u_\infty) \triangle f(v_\infty)$. This contradicts that $\mathrm{ran}(f)$ is a subset of the almost disjoint family $\mathcal{A}$. Subclaim 3.$\dashv$

Subclaim 4. There is a unique $y^* \in \mathcal{B}$ such that for all $z \in [W_0]^\infty$ we have $|\tilde{z} \cap y^*| = \infty$.

Proof: Since $\vartheta(z) \in \mathcal{B}$ for $z \in [W_0]^\infty$ and $|\tilde{z} \cap \vartheta(z)| = \infty$ by definition, for every $z \in [W_0]^\infty$ there is some $y^* \in \mathcal{B}$ such that $|\tilde{z} \cap y| = \infty$. We must show that there is a unique $y \in \mathcal{B}$ not depending on z satisfying this.

Suppose not, and let $x_0, x_1 \in \mathbb{N}^\mathbb{N}$ such that $f(x_0) \neq f(x_1)$ and for some $z_0, z_1 \in [W_0]^\infty$ we have $|\tilde{z}_0 \cap f(x_0)| = \infty$ and $|\tilde{z}_1 \cap f(x_1)| = \infty$. By continuity of f we can find $t_0 \subseteq x_0$ and $t_1 \subseteq x_1$ and $n_0 \in \mathbb{N}$ such that for all $x \in N_{t^0}$ and $x' \in N_{t^1}$ we have $n_0 \in f(x) \triangle f(x')$.

Note that $t^0, t^1 \in \tilde{T}$ since $t^i \in T^{z_i}$ and $T^{z_i} = \tilde{T}$ for $i \in \{0, 1\}$. By the previous subclaim we can find $t^0 \subseteq s^0 \in \tilde{T}$ and $t^1 \subseteq s^1 \in \tilde{T}$ and $k \in \mathbb{N}$ such that for all $y_0 \in f(N_{s^0})$ and $y_1 \in f(N_{s^1})$ we have $y_0 \cap y_1 \subseteq \{1, \ldots, k\}$.

Let

$$a_n^i = a_n \cap \bigcup f(N_{s^i}).$$

By our assumptions on s^0 and s^1 we have that $a_n^0 \cap a_n^1 = \emptyset$ for n sufficiently large. By possibly removing a finite initial segment from W_0, we may assume that $a_n^0 \cap a_n^1 = \emptyset$ for all $n \in W_0$.

Below, for $A \subseteq \mathbb{N}$, we let $A/n = \{i \in A : i > n\}$. Clearly, for each $n \in W_0$ at least 1 of the following holds:

0) $(\exists^\infty j \in W_0)\ \widehat{a_n}(j) \in a_n^0$;
1) $(\exists^\infty j \in W_0)\ \widehat{a_n}(j) \notin a_n^0$.

By refining W_0 to $W_0' \in [W_0]^\infty$ we can then arrange that for each $n \in W_0'$ exactly 1 of the following holds:

0′) $(\forall j \in W_0'/n)\ \widehat{a_n}(j) \in a_n^0$;
1′) $(\forall j \in W_0'/n)\ \widehat{a_n}(j) \notin a_n^0$.

By refining W_0' 1 more time to $W_0'' \in [W_0']^\infty$ we can then arrange that exactly 1 of the following holds:

0″) $(\forall n \in W_0'')(\forall j \in W_0''/n)\ \widehat{a_n}(j) \in a_n^0$;
1″) $(\forall n \in W_0'')(\forall j \in W_0''/n)\ \widehat{a_n}(j) \notin a_n^0$.

Now we arrive at a contradiction: If (0″) holds, then since $a_n^0 \cap a_n^1 = \emptyset$ for all $n \in W_0''$ we have for all $z \in [W_0'']^\infty$ that $\tilde{z} \cap \bigcup f(N_{s^1}) = \emptyset$, contradicting that $s^1 \in \tilde{T} = T^z$. Similarly, if (1″) holds, we get for all $z \in [W_0'']^\infty$ that $\tilde{z} \cap \bigcup f(N_{s^0}) = \emptyset$, contradicting that $s^0 \in \tilde{T}$. Subclaim 4.$\dashv$

To finish the proof of the *Main Claim*, let $y^* \in \mathcal{B}$ be as in the previous claim. Since $y^* \in \mathcal{A}'$, we have that $a_n \cap y^*$ is finite for all $n \in \mathbb{N}$. Let $z \in [W_0]^\infty$ be such that

$$\widehat{a_{\hat{z}(n)}}(\hat{z}(n+1)) > \max(a_{\hat{z}(n)} \cap y^*)$$

for all $n \in \mathbb{N}$. Then $\tilde{z} \cap y^* = \emptyset$, contradicting *Subclaim 4*. This contradiction establishes the *Main Claim*, and as noted above, the *Main Claim* easily implies that $\mathcal{A}$ is not maximal, which is what we needed to prove. □

Remarks:

1) In the proof above, a crucial point was obtaining $W_0'' \in [W_0]^\infty$ such that $(\forall z \in [W_0'']^\infty)\, \tilde{z} \subseteq a_n^0$ or $\tilde{z} \cap a_n^0 = \emptyset$. Note that an alternative and quick way to obtain such W_0'' is to appeal to Ramsey's theorem for pairs and take W_0'' to be a homogeneous set for the 2-coloring of unordered pairs $\{n, j\}$, where assuming $n < j$ we let

$$c(n,j) = \begin{cases} 1 & \text{if } \widehat{a_n}(j) \in a_n^0, \\ 0 & \text{otherwise.} \end{cases}$$

2) Finally, the following obvious question must be addressed: Can a similar result be obtained without assuming R-Unif? We think it is unlikely and that the assumptions of *Theorem 1* are the minimal natural assumptions needed to obtain a positive solution to Mathias' problem; but we do not know.

Corollaries and Further Results

Corollary 3 (Törnquist (5)). *There are no mad families in Solovay's model.*

Proof: By ref. 14, Solovay's model is a model of ZF + DC. That the Ramsey property holds in this model follows from ref. 1. Finally, the Ramsey uniformization principle holds by our remarks after *Definition 2*. □

We point out that the proof of *Theorem 1* above localizes as follows.

Corollary 4. *If* Γ, Γ' *are reasonable pointclasses such that every relation* $R \in \Gamma$ *can be uniformized on a Ramsey positive set by a* Γ'*-measurable function and all sets in* Γ' *are Ramsey, then there are no infinite mad families in* Γ.

Note that in particular, this gives an additional proof of Mathias' classical result that there are no analytic infinite mad families: Apply the corollary taking the pointclass of analytic subsets of $[\mathbb{N}]^\infty$ for Γ, and the σ algebra generated by the analytic sets for Γ', and use the Jankov–von Neumann uniformization theorem (ref. 12, theorem 18.1).

Our first corollary also allows us to draw consequences regarding the axiom of projective determinacy (in short, PD).

Corollary 5 (7, 8). *PD implies there are no projective infinite mad families.*

Proof: The hypotheses of the previous theorem hold with $\Gamma' = \Gamma$ equal to the class of projective sets: PD implies that this pointclass has the uniformization property, and by ref. 15, all projective sets are completely Ramsey under PD. □

Another consequence of our proof is that Mathias forcing destroys mad families from the ground model:

Theorem 6. *In the Mathias extension, there is no infinite mad family which is definable by a* Σ_1 *formula in the language of set theory with parameters in the ground model. In particular, no infinite almost disjoint family from the ground model is maximal in the Mathias extension.*

Proof: Suppose x is Mathias over V and that in $V[x]$, $\mathcal{A} = \{z \in [\mathbb{N}]^\infty : \Psi(z)\}$ is an infinite *almost disjoint* family, where $\Psi(z)$ is Σ_1 (with parameter in V). We show that $\tilde{x}$, where $\tilde{x}$ is defined in $V[x]$ as in the proof of *Theorem 1*, is almost disjoint from every $z \in \mathcal{A}$. Otherwise, we can choose a Mathias condition (s, A) with $s \subseteq x \subseteq A$ and a name $\dot{y}$ such that $p \Vdash \dot{y} \cap \tilde{\dot{x}}_G$ is infinite and $\Psi(\dot{y})$ (where $\dot{x}_G$ is a name for the Mathias real). By a well-known property of Mathias forcing (so-called continuous reading of names) we can assume that there is a continuous function $\vartheta\colon [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ with code in V such that p forces that $\dot{y} = \vartheta(\dot{x}_G)$. It is easy to see that $\vartheta(y) \in \mathcal{A}$ for any $y \in [x]^\infty$ (here we use the definability of $\mathcal{A}$). But then $\operatorname{ran}(\vartheta)$ would be an analytic almost disjoint family such that any element of $\{\tilde{y} : y \in [x]^\infty\}$ has infinite intersection with some element of $\operatorname{ran}(\vartheta)$, which is impossible by the proof of Theorem 1. □

Surprisingly, the connection between the Ramsey property in $[\mathbb{N}]^\infty$ and mad families relative to the ideal of finite sets extends to much more complicated Borel ideals. We construct a family of ideals using the familiar Fubini sum: Given, for each $n \in \mathbb{N}$, an ideal $\mathcal{J}_n$ on a countable set S_n we obtain an ideal $\mathcal{J}$ on $S = \bigsqcup_n S_n$ as follows:

$$\mathcal{J} = \bigoplus_n \mathcal{J}_n = \{X \subseteq S : (\forall^\infty n)\, X \cap S_n \in \mathcal{J}_n\}$$

where $(\forall^\infty n)$ means "for all but finitely many n." The Fubini sum $\bigoplus_n \mathrm{FIN}$ (where FIN denotes the ideal of finite sets on $\mathbb{N}$) is also known as $\mathrm{FIN} \times \mathrm{FIN}$ or FIN^2; iterating Fubini sums into the transfinite we obtain FIN^α, $\alpha < \omega_1$. This family of ideals of lies cofinally in the Borel hierarchy in terms of complexity.

The notion of the mad family can be extended to arbitrary ideals on a countable set: If $\mathcal{J}$ is such an ideal, a $\mathcal{J}$-almost disjoint family is a subfamily $\mathcal{A}$ of $\mathcal{P}(S) \setminus \mathcal{J}$ such that for any 2 distinct $A, A' \in \mathcal{A}$, $A \cap A' \in \mathcal{J}$. A $\mathcal{J}$-mad family is of course a $\mathcal{J}$-almost disjoint family which is maximal under $\subseteq$ among such families.

In ref. 16 we show the following:

Theorem 7. (*ZF + DC + R-Unif*) *Let* $\alpha < \omega_1$. *If all sets have the Ramsey property, then there are no infinite* FIN^α*-mad families.*

As for classical mad families, we immediately obtain corollaries regarding the axiom of projective determinacy and Solovay"s model. The first corollary was already shown in ref. 8 using forcing over inner models.

Corollary 8. (*ZF + PD*) *For each* $\alpha < \omega_1$ *there are no infinite projective* FIN^α*-mad families.*

Corollary 9. *For each* $\alpha < \omega_1$ *there are no infinite* FIN^α*-mad families in Solovay's model.*

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MATHEMATICS