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The Ramsey property implies no mad families

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We show that if all collections of infinite subsets of \( \mathbb{N} \) have the Ramsey property, then there are no infinite maximal almost disjoint (mad) families. The implication is proved in Zermelo–Fraenkel set theory with only weak choice principles. This gives a positive solution to a long-standing problem that goes back to Mathias [A. R. D. Mathias, Ann. Math. Logic 12, 59–111 (1977)]. The proof exploits an idea which has its natural roots in ergodic theory, topological dynamics, and invariant descriptive set theory: We use that a certain function associated to a purported mad family is invariant under the equivalence relation \( E_{\infty} \) and thus is constant on a “large” set. Furthermore, we announce a number of additional results about mad families relative to more complicated Borel ideals.

Theorem 1. (ZF + DC + R-Unif) If all sets have the Ramsey property, then there are no infinite mad families.

Significance

Certain infinite combinatorial structures in modern mathematics, called mad families, are known to exist only due to indirect, nonconstructive methods arising from a fundamental principle of mathematics, with many paradoxical consequences, called the axiom of choice. This paper shows that if we replace the axiom of choice with a natural assumption of universal combinatorial regularity, a principle known as the Ramsey property for all sets, then no infinite mad families can exist. This solves a problem that has been open in mathematics since the late 1960s.

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We note that *Theorem 1* implies the main results of refs. 5 and 7.

*Theorem 1* may seem all the more surprising given another recent result of Horowitz and Shelah (11), who show that for a variety of measurability notions including the Lebesgue measure, “all sets are measurable” is compatible with the existence of an infinite mad family.

Let us briefly comment on the proof of *Theorem 1* and the difficulties that have to be overcome. For this discussion, suppose \( A \subseteq [N]^\infty \) is an infinite mad family, and assume “all sets are Ramsey” and “Ramsey uniformization” (*Definition 2*).

The first difficulty encountered is that the set of \( x \in [N]^\infty \) which meet exactly 1 element of \( A \) in an infinite set is clearly Ramsey countable when \( A \) is a mad family. The key idea is to overcome this difficulty by associating to each \( z \in [N]^\infty \) a carefully chosen very, very sparse set \( \hat{z} \in [N]^\infty \), which is constructed using a fixed, infinite sequence \((a_n)_{n \in \mathbb{N}}\) chosen from \( A \) (it is here that we use the principle of DC). A basic property of the map \( z \mapsto \hat{z} \) is that it is equivariant under finite differences; that is, if \( z\Delta z' \) is finite, then \( \Delta \hat{z} \) is finite.

Because we assumed that \( A \) is maximal, for each \( z \in [N]^\infty \) there is some \( y, \in A \) such that \( z \cap y \) is infinite, and so R-Unif gives us a function \( f : [N]^\infty \to A \) such that \( f(z) \cap z \) is infinite for \( z \) in a Ramsey positive set. The special way that \( z \mapsto \hat{z} \) will be defined below will ensure that no uniformizing function \( f \) can have the invariance property that \( |z\Delta z'| < \infty \) implies \( f(z) = f(\hat{z}) \). While there is no reason to expect that an abstract application of R-Unif would give us \( f \) with this property, it turns out that with some work we can get dangerously close to having such an invariant \( f \). Indeed, by using the assumption that all sets are Ramsey we can find an infinite set \( W \subseteq N \) such that the restriction \( f(W) \in \mathbb{R}^N \) is continuous, and so the range \( f(W) \) is an analytic set. Using that \( f(W) \) is analytic, we will define a function \( z \mapsto T^z \), where \( T^z \) can be thought of as a tree of approximations to possible, natural uniformization functions. It then turns out that the map \( z \mapsto T^z \) satisfies that if \( |z\Delta z'| < \infty \), then \( T^z = T^{\hat{z}} \). This in turn leads to that \( z \mapsto T^z \) is constant on a Ramsey positive set, which then leads to a contradiction.

**Notation and Background Definitions**

In this section we summarize the background needed for the proof. A good general reference for all of the background needed is ref. 12. A comprehensive treatise on modern, infinitary Ramsey theory is ref. 13.

**A. Descriptive Set Theory.** A topological space \( X \) is called Polish if it is separable and admits a complete metric that induces the topology. In this paper we will be working with the Polish space \( 2^N = \{0, 1\}^N \) and \( [N]^\infty \) (with the product topology, taking \( \{0, 1\} \) and \( N \) discrete) and subspaces of these spaces. Recall the following key notion from descriptive set theory:

**Definition 1.** A subset \( A \subseteq X \) of a Polish space \( X \) is analytic if there is a continuous \( f : Y \to X \) from a Polish space \( Y \) to \( X \) such that \( A = \text{ran}(f) \).

Since \([N]^\infty\) maps continuously onto any Polish space, we have that \( A \subseteq X \) is analytic iff there is a continuous \( f : [N]^\infty \to X \) such that \( \text{ran}(f) = A \). We will use this characterization as our definition of the analytic set below.

For the proof of *Theorem 1* we need the following combinatorial description of the topology on \([N]^\infty\). We denote by \( N^n \) the set of all functions \( s : \{1, \ldots, n\} \to N \), and we let \([N]^n = \{0\} \cup \bigcup_{n \in \mathbb{N}} N^n \). (We shall think of \( \emptyset \) as the function with an empty domain, which is why it is included as an element of \([N]^n\).) For \( s, t \in [N]^n \cup [N]^\infty \) we will write \( s \subseteq t \) (“\( t \) extends \( s \)”)

\[ \text{dom}(s) \subseteq \text{dom}(t) \text{ and } (s(i) = t(i) \text{ for all } i \in \text{dom}(s)) \text{ if } s \nsubseteq t \text{ and } t \nsubseteq s. \]

For each \( s \in [N]^n \), let

\[ N_s = \{ f \in [N]^n : (\forall i \in \text{dom}(s)) f(i) = s(i) \}. \]

The family \( \{ N_s : s \in [N]^n \} \) is easily seen to form a basis for the topology on \([N]^n\).

Note that \( N^0 \) is countable, and so \( 2^N \) is a Polish space (isomorphic to \( 2^0 \)) in the product topology, taking \( 0 = \{0, 1\} \) discrete. This view will be important later in the proof of *Theorem 1* where we will describe the properties of a certain continuous function \( f \) defined on \([N]^n \) in terms of a “derived” function \( z \mapsto T^z \) from \([N]^n \) to \((2^N)^{\mathbb{N}}\).

**B. The Ramsey Property.** For any set \( X \) we define

\[ [X]^{\infty} = \{ A \subseteq X : A \text{ is infinite} \}. \]

Recall from the Introduction that a set \( S \subseteq [N]^\infty \) is Ramsey (or has the Ramsey property) if there is \( B \subseteq [N]^\infty \) such that \( [B]^{\infty} \subseteq S \) or \( S \cap [B]^{\infty} = \emptyset \).

To each infinite \( B \subseteq N \) we let \( \hat{B} : \mathbb{N} \to N \) be the unique increasing function with \( B = \text{ran}(\hat{B}) \). For each infinite \( A \subseteq N \) the map \([A]^{\infty} \to [N]^{\mathbb{N}} : B \mapsto \hat{B} \) naturally identifies \([A]^{\infty} \) with a subset of \([N]^{\mathbb{N}} \). The (subspace) topology that \([A]^{\infty} \) inherits under this identification will be called the standard topology on \([A]^{\infty} \).

The crucial use in our proof of the Ramsey property comes from the following well-known fact:

**Proposition 2.** (*ZF + DC*) Suppose all sets have the Ramsey property and let \( \theta : [N]^\infty \to Y \) be a function, where \( Y \) is a separable topological space. Then there is \( A \subseteq [N]^\infty \) such that \( \theta|A|^{\infty} \) is continuous with respect to the standard topology on \([A]^{\infty} \).

**C. Uniformization.** Next we define R-Unif, the Ramsey uniformization principle.

**Definition 2.**

1. **Proposition 2.** (*ZF + DC*) Suppose all sets have the Ramsey property and let \( \theta : [N]^\infty \to Y \) be a function, where \( Y = \mathbb{R} \) is a separable topological space. Then there is \( A \subseteq [N]^\infty \) such that \( \theta|A|^{\infty} \) is continuous with respect to the standard topology on \([A]^{\infty} \).

That the Ramsey uniformization principle holds in Solovay’s model follows by the argument given in ref. 14, section 1.2, p. 46 with Random forcing replaced everywhere by Mathias forcing.

**D. Invariance under Finite Changes.** The last ingredient for the proof is the notion of “\( E_0 \) invariance.”

**Definition 3.**

1. \( E_0 \) is the equivalence relation defined on \([N]^\infty \) by

\[ xE_0y \iff |x \Delta y| < \infty. \]

In other words, \( x, y \in [N]^\infty \) are \( E_0 \) equivalent iff they differ only on a finite set.

2. A function \( f : [N]^\infty \to Y \) is called \( E_0 \) invariant if \( xE_0y \implies f(x)E_0f(y) \).
Proof of Theorem 1
We work under the following assumptions: ZF + DC + R-Unif + “all sets have the Ramsey property.”
Let $A \subseteq [\mathbb{N}]^\infty$ be an infinite almost disjoint family. We will show that $A$ is not maximal.

Let $(a_n)_{n \in \mathbb{N}}$ be an injective sequence of elements in $A$ (here we use that DC implies that all infinite sets are Dedekind-infinite). We may assume that $A' = A \setminus \{(a_n : n \in \mathbb{N})\}$ is nonempty, since otherwise an easy diagonalization shows that $A$ is not maximal. Moreover, by possibly replacing $a_n$ by $a_n \setminus \bigcup_{i < n} a_i$, let us assume for simplicity that if $n \neq m$, then $a_n \cap a_m = \emptyset$ (each $a_n$ is now an infinite subset of an element of $A$).

Recall that when $z \in [\mathbb{N}]^\infty$, then $\tilde{z} : \mathbb{N} \to \mathbb{N}$ is the unique increasing function such that $\text{ran}(\tilde{z}) = z$. Using the sequence $(a_n)_{n \in \mathbb{N}}$ fixed above, define for each $z \in [\mathbb{N}]^\infty$,

$$\hat{z} = \{a_n(\tilde{z}(n + 1)) : n \in \mathbb{N}\}.$$ 

Note that $|\hat{z} \cap a_n| \leq 1$ for all $n$, so proving the following claim will prove the theorem:

**Main Claim.** There is $z \in [\mathbb{N}]^\infty$ such that for all $y \in A'$, $z \cap y < \infty$.

Suppose the claim is false. Then by Ramsey uniformization there is $W \subseteq [\mathbb{N}]^{<\infty}$ and $\vartheta : [W]^\infty \to A'$ such that

$$|\vartheta(z) \cap z| = \infty$$

for all $z \in [W]^\infty$. By Proposition 2 we may assume (after possibly replacing $W$ with an infinite subset of $W$) that $\vartheta([W]^\infty)$ is continuous. Then $B = \vartheta([W]^\infty)$ is an analytic subset of $A'$, and we can fix a continuous function $f : N \to [\mathbb{N}]^\infty$ such that $\text{ran}(f) = B$.

For $z \in [\mathbb{N}]^\infty$, let

$$T'^{z} = \{x \in N^\infty : \exists z \in N \cap f(x) = \infty\}.$$

By identifying $P(N^{<\infty})$ with $2^{N^{<\infty}}$, we will think of $z \mapsto T'^{z}$ as a map $[\mathbb{N}]^\infty \to 2^{N^{<\infty}}$. The reader can easily verify that $T'^{z}$ is a tree in the sense of ref. 12 and that $\emptyset \in T'^{z}$ for all $z \in [W]^\infty$.

**Subclaim 1.** The function $z \mapsto T'^{z}$ is $E_0$ invariant.

**Proof:** Suppose $|z' \Delta z| < \infty$. Then we can find $k_0, k_0'$ such that $\tilde{z}(k_0 + i) = \tilde{z}'(k_0' + i)$ for all $i \in \mathbb{N}$. Then

$$a^{\tilde{z}(k_0 + i)} = a^{\tilde{z}'(k_0' + i)} = a^{\tilde{z}'(k_0' + i)} = a^{\tilde{z}(k_0 + i + 1)}$$

for all $i \in \mathbb{N}$. It follows that $\tilde{z} = \tilde{z}'$, but then $|z \cap z'| = \infty$ if and only if $|z' \cap f(x)| = \infty$, so $T'^{z} = T'^{z'}$.

By refining $W_0$ to $W_0^\prime \subseteq [\mathbb{N}]^\infty$ we can then arrange that for each $n \in W_0$ exactly 1 of the following holds:

0) $(\exists j \in \mathbb{N})\ a^n_0(j) \in a^n_1$
1) $\emptyset \in T'^{z(\tilde{n}(n + 1))} = \max(a^n_0 \cap y^n)$.

For refining $W_0$ we can then arrange that exactly 1 of the following holds:

0) $(\forall y \in [\mathbb{N}]^\infty)\ a^n_0(j) \in a^n_1$
1) $(\forall y \in [\mathbb{N}]^\infty)\ a^n_0(j) \notin a^n_1$.$\square$

Proof: Since $\vartheta(z) \in B$ for $z \in [W_0]^\infty$ and $\tilde{z} \cap \vartheta(z) = \infty$ by definition, for every $z \in [\mathbb{N}]^\infty$ there is some $y \in [\mathbb{N}]^\infty$ such that $z \cap y = \infty$. We must show that there is a unique $y \in B$ not depending on $z$ satisfying this.

Suppose not, and let $x_0, x_1 \in \mathbb{N}^{<\infty}$ such that $f(x_0) \neq f(x_1)$ and for some $y_0, y_1 \in [\mathbb{N}]^\infty$ we have $\vartheta_0 \cap f(x_0) = \infty$ and $\vartheta_1 \cap f(x_1) = \infty$. By continuity of $f$ we can find $u \subseteq x_0$ and $v \subseteq x_1$ and $n_0 \in N$ such that for all $x_0 \in N_0$ and $x_1 \in N_1$, we have $n_0 \in f(x_0)$ and $\Delta f(x_0)$. This contradicts that $\text{ran}(f)$ is a subset of the almost disjoint family $A$.

**Subclaim 4.** There is a unique $y^* \in B$ such that for all $z \in [W_0]^\infty$ we have $z \cap y^* = \infty$.

**Proof:** Since $\vartheta(z) \in B$ for $z \in [W_0]^\infty$ and $\tilde{z} \cap \vartheta(z) = \infty$ by definition, for every $z \in [\mathbb{N}]^\infty$ there is some $y \in [\mathbb{N}]^\infty$ such that $z \cap y = \infty$. We must show that there is a unique $y \in B$ not depending on $z$ satisfying this.

Suppose not, and let $x_0, x_1 \in \mathbb{N}^{<\infty}$ such that $f(x_0) \neq f(x_1)$ and for some $y_0, y_1 \in [\mathbb{N}]^\infty$ we have $\vartheta_0 \cap f(x_0) = \infty$ and $\vartheta_1 \cap f(x_1) = \infty$. By continuity of $f$ we can find $u \subseteq x_0$ and $v \subseteq x_1$ and $n_0 \in N$ such that for all $x_0 \in N_0$ and $x_1 \in N_1$, we have $n_0 \in f(x_0)$ and $\Delta f(x_0)$. This contradicts that $\text{ran}(f)$ is a subset of the almost disjoint family $A$.

Let $a_n = a_n \cap \bigcup f(x_i)$.

By our assumptions on $s^0$ and $s^1$ we have that $a^n_0 \cap a^n_* = \emptyset$ for $n$ sufficiently large. By possibly removing a finite initial segment from $W_0$, we may assume that $a^n_0 \cap a^n_* = \emptyset$ for all $n \in W_0$.

Below, for $A \subseteq N$, we let $A/n = \{i : i \in A : i > n\}$. Clearly, for each $n \in W_0$ at least 1 of the following holds:

0) $(\exists j \in W_0)\ a^n_0(j) \in a^n_1$
1) $(\exists j \in W_0)\ a^n_0(j) \notin a^n_1$.

Now we arrive at a contradiction: If $(0')$ holds, then since $a^n_0 \cap a^n_* = \emptyset$ for all $n \in W_0$, we have for all $z \in [W_0]^\infty$ that $\tilde{z} \cap \bigcup f(N_1) = \emptyset$, contradicting that $s^1 \in T = T'$. Similarly, if $(1')$ holds, we get for all $z \in [W_0]^\infty$ that $\tilde{z} \cap \bigcup f(N_0) = \emptyset$, contradicting that $s^0 \in T$.

To finish the proof of the Main Claim, let $y^* \in B$ be as in the previous claim. Since $y^* \in A'$, we have that $a_n \cap y^*$ is finite for all $n \in N$. Let $z \in [W_0]^\infty$ be such that

$$a^n_0(\tilde{z}(n + 1)) = \max(a^n_0 \cap y^*)$$

for all $n \in N$. Then $\tilde{z} \cap y^* = \emptyset$, contradicting Subclaim 4. This contradiction establishes the Main Claim, and as noted above, the Main Claim easily implies that $A$ is not maximal, which is what we needed to prove.
Remarks:
1) In the proof above, a crucial point was obtaining \( W_\alpha^\gamma \in [W_0]^{\infty} \)
such that \( (\forall z \in [W_\alpha^\gamma]^{\infty}) \exists y \in \mathcal{A}_0 \) or \( z \cap a^\alpha = \emptyset \). Note that an alternative and quick way to obtain such \( W_\alpha^\gamma \) is to appeal to Ramsey’s theorem for pairs and take \( W_\alpha^\gamma \) to be a homogeneous set for the 2-coloring of unordered pairs \( \{n,j\} \), where assuming \( n < j \) we let

\[
e(n,j) = \begin{cases} 1 & \text{if } \mathcal{A}_0(n,j) \in a_0^n, \\ 0 & \text{otherwise.} \end{cases}
\]

2) Finally, the following obvious question must be addressed: Can a similar result be obtained without assuming R-Unif? We think it is unlikely and that the assumptions of Theorem 1 are the minimal natural assumptions needed to obtain a positive solution to Mathias’s problem; but we do not know.

Corollaries and Further Results

Corollary 3 (Törnquist (5)). There are no mad families in Solovay’s model.

Proof: By ref. 14, Solovay’s model is a model of ZF + DC. That the Ramsey property holds in this model follows from ref. 1. Finally, the Ramsey uniformization principle holds by our remarks after Definition 2.

We point out that the proof of Theorem 1 above localizes as follows.

Corollary 4. If \( \Gamma \), \( \Gamma' \) are reasonable pointclasses such that every relation \( R \in \Gamma \) can be uniformized on a Ramsey positive set by a \( \Gamma' \)-measurable function and all sets in \( \Gamma' \) are Ramsey, then there are no infinite mad families in \( \Gamma \).

Note that in particular, this gives an additional proof of Mathias’ classical result that there are no analytic infinite mad families: Apply the corollary taking the pointclass of analytic subsets of \( [\mathbb{N}]^\infty \) for \( \Gamma' \), and the \( \sigma \) algebra generated by the analytic sets for \( \Gamma' \), and use the Jankov-von Neumann uniformization theorem (ref. 12, theorem 18.1).

Our first corollary also allows us to draw consequences regarding the axiom of projective determinacy (in short, PD).

Corollary 5 (7, 8). PD implies there are no projective infinite mad families.

Proof: The hypotheses of the previous theorem hold with \( \Gamma' = \Gamma \) equal to the class of projective sets: PD implies that this pointclass has the uniformization property, and by ref. 15, all projective sets are completely Ramsey under PD.

Another consequence of our proof is that Mathias forcing destroys mad families from the ground model:

Theorem 6. In the Mathias extension, there is no infinite mad family which is definable by a \( \Sigma_1^1 \) formula in the language of set theory with parameters in the ground model. In particular, no infinite almost disjoint family from the ground model is maximal in the Mathias extension.

Proof: Suppose \( z \) is Mathias over \( V \) and that in \( V[x] \), \( A = \{ z \in [\mathbb{N}]^\infty : \mathcal{P}(z) \} \) is an infinite almost disjoint family, where \( \mathcal{P}(z) = \Sigma_1^1 \) (with parameter in \( V \)). We show that \( x \), where \( x \) is defined in \( V[x] \) as in the proof of Theorem 1, is almost disjoint from every \( z \in A \). Otherwise, we can choose a Mathias condition \( \langle s, \bar{A} \rangle \) with \( s \subseteq x \subset A \) and a name \( y \) such that \( p \forces y \cap \dot{x}_c \) is infinite and \( \mathcal{P}(y) \) (where \( \dot{x}_c \) is a name for the Mathias real). By a well-known property of Mathias forcing (so-called continuous reading of names) we can assume that there is a continuous function \( \vartheta : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty \) with code in \( V \) such that \( p \forces \vartheta(y) = \vartheta(y) \). It is easy to see that \( \vartheta(y) \in A \) for any \( y \in [x]^\infty \) (here we use the definability of \( A \)). But then \( ran(\vartheta) \) would be an analytic almost disjoint family such that any element of \( \{ y : y \in [x]^\infty \} \) has infinite intersection with some element of \( ran(\vartheta) \), which is impossible by the proof of Theorem 1.

Surprisingly, the connection between the Ramsey property in \( [\mathbb{N}]^\infty \) and mad families relative to the ideal of finite sets extends to much more complicated Borel ideals. We construct a family of ideals using the familiar Fubini sum: Given, for each \( n \in \mathbb{N} \), an ideal \( J_n \) on a countable set \( S_n \) we obtain an ideal \( J \) on \( S = \bigcup_n S_n \) as follows:

\[
J = \bigoplus_n J_n = \{ X \subseteq S : \forall n (X \cap S_n \in J_n) \}
\]

where \( (\forall n) \) means “for all but finitely many \( n \).” The Fubini sum \( \bigoplus \) FIN (where FIN denotes the ideal of finite sets on \( \mathbb{N} \)) is also known as FIN × FIN or FIN2; iterating Fubini sums into the transfinite we obtain FIN\(^{\alpha} \), \( \alpha < \omega_1 \). This family of ideals of lies cofinally in the Borel hierarchy in terms of complexity.

The notion of the mad family can be extended to arbitrary ideals on a countable set: If \( J \) is such an ideal, a \( J \)-almost disjoint family is a subfamily \( A \) of \( \mathcal{P}(S) \setminus J \) such that for any 2 distinct \( A, A' \in A \), \( A \cap A' \notin J \). A \( J \)-mad family is of course a \( J \)-almost disjoint family which is maximal under \( \subseteq \) among such families.

In ref. 16 we show the following:

Theorem 7. (ZF + DC + R-Unif) Let \( \alpha < \omega_1 \). If all sets have the Ramsey property, then there are no infinite FIN\(^{\alpha} \)-mad families.

As for classical mad families, we immediately obtain corollaries regarding the axiom of projective determinacy and Solovay’s model. The first corollary was already shown in ref. 8 using forcing over inner models.

Corollary 8. (ZF + PD) For each \( \alpha < \omega_1 \) there are no infinite projective FIN\(^{\alpha} \)-mad families.

Corollary 9. For each \( \alpha < \omega_1 \) there are no infinite projective FIN\(^{\alpha} \)-mad families in Solovay’s model.

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