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# On generalized Stieltjes functions

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**Abstract**

It is shown that a function $f$ is a generalized Stieltjes function of order $\lambda > 0$ if and only if $x^{1-\lambda}(x^{\lambda-1+k}f(x))^{(k)}$ is completely monotonic for all $k \geq 0$, thereby complementing a result due to Sokal. Furthermore, a characterization of those completely monotonic functions $f$ for which $x^{1-\lambda}(x^{\lambda-1+k}f(x))^{(k)}$ is completely monotonic for all $k \leq n$ is obtained in terms of properties of the representing measure of $f$.

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## 1 Introduction

In this paper we investigate a real-variable characterization of generalized Stieltjes functions obtained by Sokal, see [9].

Let $\lambda > 0$ be given. A function $f : (0, \infty) \to \mathbb{R}$ is called a generalized Stieltjes function of order $\lambda$ if

$$f(x) = \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda} + c,$$

where $\mu$ is a positive measure on $[0, \infty)$ making the integral converge for $x > 0$ and $c \geq 0$.

The class of ordinary Stieltjes functions is the class of generalized Stieltjes functions of order 1.

A $C^\infty$-function $f$ on $(0, \infty)$ is completely monotonic if $(-1)^n f^{(n)}(x) \geq 0$ for all $n \geq 0$ and all $x > 0$. Bernstein's theorem characterizes these functions

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as Laplace transforms of positive measures: $f$ is completely monotonic if and only if there exists a positive measure $\mu$ on $[0,\infty)$ such that $t \mapsto e^{-xt}$ is integrable w.r.t. $\mu$ for all $x>0$ and

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

cf. [11, p. 161]. We remark that $f$ is a generalized Stieltjes function of order $\lambda$ if and only if

$$f(x) = \int_0^\infty e^{-xt} t^{\lambda-1} \varphi(t)\, dt + c, \quad x > 0 \tag{1}$$

for some completely monotonic function $\varphi$, and some non-negative number $c$. See [4, Lemma 2.1].

Sokal (see [9]) introduced for $\lambda > 0$ the operators

$$T^\lambda_{n,k}(f)(x) \equiv (-1)^n x^{-(n+\lambda-1)} \left(x^{k+n+\lambda-1} f^{(n)}(x)\right)^{(k)}, \quad n,k \geq 0$$

and obtained the following characterization.

**Theorem 1.1** *The following are equivalent for a $C^\infty$-function $f$ defined on $(0,\infty)$.*

*(a) $f$ is a generalized Stieltjes function of order $\lambda$;*

*(b) $T^\lambda_{n,k}(f)(x) \geq 0$ for all $x > 0$, and $n,k \geq 0$.*

Sokal's characterization is an extension of Widder's characterization of the class of ordinary Stieltjes functions: $f$ is a Stieltjes function if and only if the function $(x^k f(x))^{(k)}$ is completely monotonic for all $k \geq 0$. (See [10].)

In [5, Theorem 1.5] an analogue of Sokal's result where the function $\varphi$ in (1) is absolutely monotonic is obtained. See also [3, Theorem 2] for a result complementing [5, Theorem 1.1].

**Remark 1.2** *Notice that, by Leibniz' rule,*

$$x^{-(n+\lambda-1)} \left(x^{k+n+\lambda-1} f^{(n)}(x)\right)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} \frac{\Gamma(n+k+\lambda)}{\Gamma(n+j+\lambda)} x^j f^{(n+j)}(x).$$

In this paper we first show that condition (b) in Sokal's theorem above can be replaced by the condition that

$$c^\lambda_k(f)(x) \equiv x^{1-\lambda}(x^{\lambda-1+k} f(x))^{(k)}$$

is completely monotonic for all $k$. There is a simple relation between $T^\lambda_{n,k}(f)$ and $c^\lambda_k(f)$:

**Proposition 1.3** *The relation*

$$T_{n,k}^{\lambda}(f)(x) = (-1)^n c_k^{\lambda}(f)^{(n)}(x)$$

*holds for any* $n, k \geq 0$ *and* $x > 0$.

**Corollary 1.4** *The following are equivalent for a function* $f \in C^{\infty}((0,\infty))$.

*(i)* $f$ *is a generalized Stieltjes function of order* $\lambda$*;*

*(ii)* $c_k^{\lambda}(f)$ *is completely monotonic for all* $k \geq 0$.

*(iii)* $T_{n,k}^{\lambda}(f) \geq 0$ *for all* $n \geq 0$ *and all* $k \geq 0$.

In [4] the generalized Stieltjes functions corresponding to measures having moments of all orders were characterized in terms of properties of remainders in asymptotic expansions. (A measure $\mu$ has moments of all orders if any polynomial is integrable w.r.t. $\mu$.) In view of the results in the present paper we notice the following corollary. The proof follows by combining Corollary 1.4 with [4, Theorem 3.2] and [4, Lemma 3.1].

**Corollary 1.5** *The following are equivalent for a function* $f : (0,\infty) \to \mathbb{R}$.

*(i)* $f$ *is a generalized Stieltjes function corresponding to a measure* $\mu$ *having moments of all orders;*

*(ii)* $c_k^{\lambda}(f)$ *is completely monotonic for all* $k \geq 0$ *and the function* $x^{\lambda-1}f(x)$ *admits for any* $n$ *an asymptotic expansion*

$$x^{\lambda-1}f(x) = \sum_{k=0}^{n-1} \frac{\alpha_k}{x^{k+1}} + r_n(x),$$

*in which* $x^n r_n(x) \to 0$ *as* $x \to \infty$.

*In the affirmative case,* $\alpha_k = (-1)^k(\lambda)_k s_k(\mu)/k!$ *where* $s_k(\mu)$ *is the* $k$*'th moment of* $\mu$*, and* $r_n$ *has the representation*

$$r_n(x) = \frac{(-1)^n x^{\lambda-1}}{\Gamma(\lambda)} \int_0^{\infty} e^{-xt} t^{\lambda-1} \xi_n(t)\, dt,$$

*where* $\xi_n$ *belongs to* $C^{\infty}([0,\infty))$*, and satisfies* $\xi_n^{(j)}(0) = 0$ *for* $j \leq n-1$ *and* $0 \leq \xi_n^{(n)}(t) \leq s_n(\mu)$ *for* $t \geq 0$. *Furthermore,*

$$c_n^{\lambda}(f)(x) = x^{1-\lambda}(x^n r_n(x))^{(n)} = c_n^{\lambda}\left(\mathcal{L}\left(\frac{t^{\lambda-1}(-1)^n \xi_n(t)}{\Gamma(\lambda)}\right)\right)(x).$$

Our aim is also to characterize, for any given positive integer $N$, those functions $f$ for which $c_0^\lambda(f), \ldots, c_N^\lambda(f)$ are completely monotonic. In the case where $\lambda = 1$ this has been carried out in [6], but the case of general $\lambda$ requires, as we shall see, additional insight.

We thus introduce the classes $\mathcal{C}_N^\lambda$ as

$$\mathcal{C}_N^\lambda = \{f \in C^\infty((0,\infty)) \,|\, c_k^\lambda(f) \text{ is completely monotonic for } \; k = 0, \ldots, N\}.$$

We shall use some distribution theory so we briefly describe our notation. The action of a distribution $u$ on a test function $\varphi$ (an infinitely often differentiable function of compact support in $(0,\infty)$) is denoted by $\langle u, \varphi \rangle$. The distribution $\partial u$ is defined via $\langle \partial u, \varphi \rangle = -\langle u, \varphi' \rangle$. A standard reference to distribution theory is [7].

Our results can be formulated as follows.

**Theorem 1.6** *Let $\lambda > 0$ be given, and let $N \geq 1$. The following properties of a function $f : (0,\infty) \to \mathbb{R}$ are equivalent.*

*(a)* $f \in \mathcal{C}_N^\lambda$;

*(b)* $f$ *can be represented as*

$$f(x) = c + \int_0^\infty e^{-xs} s^{\lambda-1} \, d\mu(s),$$

*where $c \geq 0$, and $\mu$ is a positive measure on $(0,\infty)$ for which $\mu_k \equiv (-1)^k s^k \partial^k \mu$, (in distributional sense) is a positive measure such that*

$$\int_0^\infty e^{-xs} s^{\lambda-1} \, d\mu_k(s) < \infty, \quad k = 0, \ldots, N.$$

*In the affirmative case,*

$$c_k^\lambda(f)(x) = x^{1-\lambda} \left( x^{\lambda-1+k} f(x) \right)^{(k)} = \int_0^\infty e^{-xs} s^{\lambda-1} \, d\mu_k(s) + (\lambda)_k c$$

*for $k = 0, \ldots, N$.*

We notice the following corollary characterizing those non-negative functions $f$ for which $c_1^\lambda(f)$ is completely monotonic. The proof follows from Propostion 2.3 and Lemma 3.5.

**Corollary 1.7** *Let $f$ be a non-negative $C^\infty$-function defined on $(0,\infty)$. Then $x^{1-\lambda}\left(x^\lambda f(x)\right)' = \lambda f(x) + xf'(x)$ is completely monotonic if and only if*

$$f(x) = \alpha + \frac{\beta}{x^\lambda} + \int_0^\infty e^{-xs} s^{\lambda-1} \int_s^\infty \frac{d\mu(t)}{t^\lambda}\, ds,$$

*for some non-negative numbers $\alpha$ and $\beta$ and some positive measure $\mu$ on $(0,\infty)$ making the integral convergent.*

**Remark 1.8** *It is easy to see that $e^{-xs}s^{\lambda-1}\int_s^\infty d\mu(t)/t^\lambda$ is integrable on $(0,\infty)$ if and only if $s^{\lambda-1}\int_s^\infty d\mu(t)/t^\lambda$ is integrable at $0$, and that this is the case if and only if $\int_0^1 d\mu(t) < \infty$ and $\int_1^\infty d\mu(t)/t^\lambda < \infty$.*

Corollary 1.7 can be reformulated as follows. Let $g$ be a non-negative $C^\infty$-function on $(0,\infty)$. Then $x^{1-\lambda}g'(x)$ is completely monotonic if and only if

$$g(x) = \alpha x^\lambda + \beta + \int_0^\infty \int_0^{xt} e^{-u}u^{\lambda-1}\, du \frac{d\mu(t)}{t^\lambda}. \tag{2}$$

Formulated in this way the corollary is related to the class of Bernstein functions. A Bernstein function is by definition a non-negative function $g$ on $(0,\infty)$ for which $g'$ is completely monotonic. These functions admit an integral representation (see [8, Theorem 3.2] or [2]), which we for the reader's convenience state here: $g$ is a Bernstein function if and only if

$$g(x) = \alpha x + \beta + \int_0^\infty (1-e^{-xt})\, d\nu(t),$$

where $\alpha$ and $\beta$ are non-negative numbers, and $\nu$, called the Lévy measure, is a positive measure on $(0,\infty)$ satisfying $\int_0^1 t d\nu(t) < \infty$ and $\int_1^\infty d\nu(t) < \infty$.

When $\lambda = 1$, we have

$$\int_0^{xt} e^{-u}u^{\lambda-1}\, du = 1 - e^{-xt},$$

and (2) reduces to the integral representation of a Bernstein function with the corresponding Lévy measure being $d\mu(t)/t$. Corollary 1.7 contains a characterization of what could be called "generalized Bernstein functions of order $\lambda$".

## 2 Proofs

*Proof of Proposition 1.3:* The key to the proof is the following relation

$$T_{n,k}^{\lambda}(f)(x) = (-1)^n \left( \sum_{j=0}^{k} (\lambda - 1)_{k-j} \binom{k}{j} \left( x^j f(x) \right)^{(j)} \right)^{(n)}, \tag{3}$$

which we verify now. A standard application of Leibniz' formula yields

$$\begin{aligned}
\left( x^j f(x) \right)^{(j+n)} &= \sum_{l=0}^{n+j} \binom{n+j}{l} \left( x^j \right)^{(l)} f^{(n+j-l)}(x) \\
&= \sum_{l=0}^{j} \binom{n+j}{l} \frac{j!}{(j-l)!} x^{j-l} f^{(n+j-l)}(x) \\
&= \sum_{m=0}^{j} \binom{n+j}{j-m} \frac{j!}{m!} x^m f^{(n+m)}(x).
\end{aligned}$$

Hence, the right hand side of (3) equals

$$\begin{aligned}
&(-1)^n \sum_{j=0}^{k} (\lambda - 1)_{k-j} \binom{k}{j} \left( x^j f(x) \right)^{(j+n)} \\
&= (-1)^n \sum_{j=0}^{k} (\lambda - 1)_{k-j} \binom{k}{j} \sum_{m=0}^{j} \binom{n+j}{j-m} \frac{j!}{m!} x^m f^{(n+m)}(x) \\
&= (-1)^n \sum_{m=0}^{k} \left\{ \sum_{j=m}^{k} (\lambda - 1)_{k-j} \binom{k}{j} \binom{n+j}{j-m} \frac{j!}{m!} \right\} x^m f^{(n+m)}(x).
\end{aligned}$$

The expression in the brackets can be written in another form. Indeed

$$\sum_{j=m}^{k} (\lambda - 1)_{k-j} \binom{k}{j} \binom{n+j}{j-m} \frac{j!}{m!} = \binom{k}{m} \frac{\Gamma(n+k+\lambda)}{\Gamma(n+m+\lambda)},$$

by a corollary to the Chu-Vandermonde identity (see [1, p. 70]). This gives us

$$\begin{aligned}
&(-1)^n \left( \sum_{j=0}^{k} (\lambda - 1)_{k-j} \binom{k}{j} \left( x^j f(x) \right)^{(j)} \right)^{(n)} \\
&= (-1)^n \sum_{m=0}^{k} \binom{k}{m} \frac{\Gamma(n+k+\lambda)}{\Gamma(n+m+\lambda)} x^m f^{(n+m)}(x) = T_{n,k}^{\lambda}(f)(x).
\end{aligned}$$

For $n = 0$ the identity reads

$$\sum_{j=0}^{k} (\lambda-1)_{k-j} \binom{k}{j} \left(x^j f(x)\right)^{(j)} = T^{\lambda}_{0,k}(f)(x) = c^{\lambda}_k(f)(x),$$

and the proposition is proved. □

To prove Theorem 1.6 we need a few preliminary results.

**Lemma 2.1** *For $f \in C^\infty((0,\infty))$ we have*

$$c^{\lambda}_{k+1}(f)(x) = (\lambda+k)c^{\lambda}_k(f)(x) + xc^{\lambda}_k(f)'(x).$$

*Proof.* This follows by computation:

$$\begin{aligned} c^{\lambda}_{k+1}(f)(x) &= x^{1-\lambda}(xx^{\lambda-1+k} f(x))^{(k+1)} \\ &= x^{2-\lambda}(x^{\lambda-1+k} f(x))^{(k+1)} + (k+1)x^{1-\lambda}(x^{\lambda-1+k} f(x))^{(k)} \\ &= x^{2-\lambda}(x^{\lambda-1} c^{\lambda}_k(f)(x))' + (k+1)c^{\lambda}_k(f)(x) \\ &= xc^{\lambda}_k(f)'(x) + (\lambda+k)c^{\lambda}_k(f)(x). \end{aligned}$$

□

**Proposition 2.2** *Suppose that $f \in \mathcal{C}^{\lambda}_N$, and let for $k = 0, \ldots, N$*

$$c^{\lambda}_k(f)(x) = \int_0^\infty e^{-xs}\, d\mu_k(s) + b_k,$$

*where $\mu_k$ is a positive measure on $(0,\infty)$ and $b_k \geq 0$. Then, in the distributional sense,*

$$(-1)^k s^k \partial^k (s^{1-\lambda}\mu_0) = s^{1-\lambda}\mu_k.$$

*Proof.* From Lemma 2.1 it follows that (for $k \leq N-1$)

$$\begin{aligned} \int_0^\infty e^{-xs}\, d\mu_{k+1}(s) + b_{k+1} = (\lambda+k) \int_0^\infty e^{-xs}\, d\mu_k(s) + (\lambda+k)b_k \\ - x\int_0^\infty se^{-xs}\, d\mu_k(s). \end{aligned}$$

Letting $x \to \infty$ yields $b_{k+1} = (\lambda+k)b_k$ so that

$$\begin{aligned} \frac{1}{x}\int_0^\infty e^{-xs}\, d\mu_{k+1}(s) = (\lambda+k)\frac{1}{x}\int_0^\infty e^{-xs}\, d\mu_k(s) \\ - \int_0^\infty se^{-xs}\, d\mu_k(s). \end{aligned}$$

By the uniqueness of the Laplace transform we obtain

$$s\mu_k = ((\lambda + k)\mu_k - \mu_{k+1}) * m.$$

(Here, $m$ denotes Lebesgue measure on $(0, \infty)$.) We get by differentiation (as distributions) that

$$s\partial\mu_k = (\lambda + k - 1)\mu_k - \mu_{k+1}.$$

We shall obtain the assertion in the proposition by induction, using this recursive relation: for $k = 0$ the assertion is valid. Before verifying the induction step notice that

$$s\partial(s^k\partial^k(s^{1-\lambda}\mu_0)) = ks^k\partial^k(s^{1-\lambda}\mu_0) + s^{k+1}\partial^{k+1}(s^{1-\lambda}\mu_0).$$

Suppose now that the assertion holds for $k$. Then

$$\begin{aligned}
s^{k+1}\partial^{k+1}(s^{1-\lambda}\mu_0) &= s\partial(s^k\partial^k(s^{1-\lambda}\mu_0)) - ks^k\partial^k(s^{1-\lambda}\mu_0) \\
&= s\partial((-1)^k s^{1-\lambda}\mu_k) - k(-1)^k s^{1-\lambda}\mu_k \\
&= (-1)^k\{s(1-\lambda)s^{-\lambda}\mu_k + s^{1-\lambda}s\partial\mu_k - ks^{1-\lambda}\mu_k\} \\
&= (-1)^k s^{1-\lambda}\{(1-\lambda)\mu_k + (\lambda + k - 1)\mu_k - \mu_{k+1} - k\mu_k\} \\
&= (-1)^{k+1}s^{1-\lambda}\mu_{k+1}.
\end{aligned}$$

The assertion holds also for $k + 1$, and the assertion follows. □

*Proof that (a) implies (b) in Theorem 1.6.* If $f \in \mathcal{C}_N^\lambda$ then the function $c_k^\lambda(f)$ is completely monotonic for $0 \le k \le N$. In particular

$$f(x) = c_0^\lambda(f)(x) = \int_0^\infty e^{-xs}\, d\mu_0(s) + b_0 = \int_0^\infty e^{-xs}s^{\lambda-1}\, d(s^{1-\lambda}\mu_0)(s) + b_0.$$

Let $\mu = s^{1-\lambda}\mu_0$ and notice that by Proposition 2.2 $(-1)^k s^k\partial^k\mu\,(= s^{1-\lambda}\mu_k)$ is a positive measure with the property that

$$\int_0^\infty e^{-xs}s^{\lambda-1}\, d((-1)^k s^k\partial^k\mu)(s) < \infty.$$

Thus (b) follows. □

The next result is a special case of (b) implies (a) in Theorem 1.6. We state and prove it separately in order to describe the method, which will be alluded to in the following proof.

**Proposition 2.3** *Let $f$ have the representation*

$$f(x) = c + \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu(s),$$

*where $c \geq 0$ and $\mu$ is a positive measure on $(0, \infty)$. If $-s\partial\mu(s)$ is a positive measure then*

$$c_1^\lambda(f)(x) = \lambda c + \int_0^\infty e^{-xs} s^{\lambda-1}\, d(-s\partial\mu(s))$$

*is completely monotonic.*

*Proof.* Let $n \geq 1$ and take $\varphi_n \in C^\infty((0,\infty))$ such that $0 \leq \varphi_n(t) \leq 1$,

$$\varphi_n(t) = \begin{cases} 0, & t < 1/(2n) \\ 1, & 1/n \leq t \leq n \\ 0, & t > 2n \end{cases},$$

$|\varphi_n'(t)| \leq \text{Const} \cdot n$ for $t \in (1/(2n), 1/n)$, and $|\varphi_n'(t)| \leq \text{Const}$ for $t \in (n, 2n)$. By definition of the derivative in distributional sense we have

$$\begin{aligned}
&\int_0^\infty e^{-xs} s^{\lambda-1} \varphi_n(s)\, d(-s\partial\mu(s)) \\
&= \langle -s\partial\mu(s), e^{-xs} s^{\lambda-1}\varphi_n(s)\rangle \\
&= \langle \mu, (e^{-xs} s^{\lambda}\varphi_n(s))'\rangle \\
&= -x\int_0^\infty e^{-xs} s^{\lambda}\varphi_n(s)\, d\mu(s) + \int_0^\infty e^{-xs}\lambda s^{\lambda-1}\varphi_n(s)\, d\mu(s) \\
&\quad + \int_0^\infty e^{-xs} s^{\lambda}\varphi_n'(s)\, d\mu(s).
\end{aligned}$$

Using dominated convergence it follows that the sum of the first and second term on the right hand side tends to

$$-x\int_0^\infty e^{-xs} s^{\lambda}\, d\mu(s) + \lambda\int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu(s).$$

The third term tends to zero, again due to dominated convergence and the estimate (using $|s\varphi'(s)| \leq \text{Const}$ for $s \leq 1/n$)

$$\begin{aligned}
&\int_0^\infty \left| e^{-xs} s^{\lambda}\varphi_n'(s)\right|\, d\mu(s) \leq \\
&\text{Const}\left(\int_{1/2n}^{1/n} e^{-xs} s^{\lambda-1}\, d\mu(s) + \int_n^{2n} e^{-xs} s^{\lambda}\, d\mu(s)\right).
\end{aligned}$$

Hence, letting $n$ tend to infinity, we obtain that

$$\begin{aligned} c_1^\lambda(f)(x) &= \lambda f(x) + x f'(x) \\ &= \lambda c + \lambda \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu(s) - x \int_0^\infty e^{-xs} s^{\lambda}\, d\mu(s) \\ &= \lambda c + \int_0^\infty e^{-xs} s^{\lambda-1}\, d(-s\partial\mu(s)). \end{aligned}$$

Thus $c_1^\lambda(f)$ is completely monotonic, and $e^{-xs}s^{\lambda-1}$ is integrable w.r.t. the measure $-s\partial\mu(s)$. □

*Proof that (b) implies (a) in Theorem 1.6.* We suppose that $f$ has the representation

$$f(x) = c + \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu(s),$$

with $c \geq 0$, and $\mu_k \equiv (-1)^k s^k \partial^k \mu$ being a positive measure for $k = 0, \ldots, N$. It is easy to verify that $\mu_{k+1} = k\mu_k - s\partial\mu_k$ for $k = 0, \ldots, N-1$. Proposition 2.3 yields that $c_1^\lambda(f)$ is completely monotonic and has the representation

$$c_1^\lambda(f)(x) = \lambda c + \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu_1(s).$$

Let us now assume that $c_k^\lambda(f)$ is completely monotonic and has the representation

$$c_k^\lambda(f)(x) = b_k + \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu_k(s).$$

Then

$$\begin{aligned} c_{k+1}^\lambda(f)(x) &= (\lambda + k) c_k^\lambda(f)(x) + x c_k^\lambda(f)'(x) \\ &= (\lambda + k) \int_0^\infty e^{-xs} s^{\lambda-1}\, d\mu_k(s) - x \int_0^\infty e^{-xs} s^{\lambda}\, d\mu_k(s) \\ &\quad + (\lambda + k) b_k. \end{aligned}$$

Now, taking $\varphi_n$ as before it follows that

$$\begin{aligned}
&\int_0^\infty e^{-xs}s^{\lambda-1}\varphi_n(s)\,d\mu_{k+1}(s)\\
&=\int_0^\infty e^{-xs}s^{\lambda-1}\varphi_n(s)\,d(k\mu_k-s\partial\mu_k)(s)\\
&=\langle k\mu_k-s\partial\mu_k(s),e^{-xs}s^{\lambda-1}\varphi_n(s)\rangle\\
&=\langle k\mu_k,e^{-xs}s^{\lambda-1}\varphi_n(s)\rangle+\langle\mu_k,(e^{-xs}s^{\lambda}\varphi_n(s))'\rangle\\
&=-x\int_0^\infty e^{-xs}s^{\lambda}\varphi_n(s)\,d\mu_k(s)+(k+\lambda)\int_0^\infty e^{-xs}s^{\lambda-1}\varphi_n(s)\,d\mu_k(s)\\
&\quad+\int_0^\infty e^{-xs}s^{\lambda}\varphi_n'(s)\,d\mu_k(s).
\end{aligned}$$

As before, letting $n$ tend to infinity, and applying dominated convergence we get that

$$c_{k+1}^{\lambda}(f)(x)=\int_0^\infty e^{-xs}s^{\lambda-1}\,d\mu_{k+1}(s)+(\lambda+k)b_k$$

is completely monotonic.

In the affirmative case we infer that $b_{k+1}=(\lambda+k)b_k,\ \ldots,\ b_1=\lambda c$ yielding $b_k=(\lambda)_k c$. □

# 3 Additional results and comments

Suppose that $c_k^{\lambda}(f)$ is completely monotonic for some $k\geq 1$. What can be said about the functions $c_0^{\lambda}(f),\ldots,c_{k-1}^{\lambda}(f)$? Are they also completely monotonic? The answer is given in Proposition 3.1.

**Proposition 3.1** *Let $k\geq 1$, $f\in C^\infty((0,\infty))$ and suppose that the functions $c_0^{\lambda}(f),\ldots,c_{k-1}^{\lambda}(f)$ are non-negative. If $c_k^{\lambda}(f)$ is completely monotonic then $c_j^{\lambda}(f)$ is also completely monotonic for $j\leq k-1$.*

The proof of this proposition requires some preliminary results. Define

$$\gamma_k^{\lambda}(f)(x)\equiv x^{-\lambda}(x^{\lambda-1+k}f(x))^{(k-1)},\quad k\geq 1.$$

Notice that $\gamma_k^{\lambda}(f)=c_{k-1}^{\lambda+1}(f)$.

**Lemma 3.2** *For $f\in C^{(k+1)}((0,\infty))$ we have*

$$\gamma_k^{\lambda}(f)(x)=c_{k-1}^{\lambda}(f)(x)+(k-1)\gamma_{k-1}^{\lambda}(f)(x).$$

*Proof:* This follows by a direct computation. □

The next lemma is an immediate consequence of Lemma 3.2.

**Lemma 3.3** *If* $c_j^\lambda(f)(x) \geq 0$ *for all* $j = 0, \ldots, k$ *then* $\gamma_j^\lambda(f)(x) \geq 0$ *for all* $j = 1, \ldots, k+1$

**Lemma 3.4** *Let* $k \geq 1$ *be given and assume that* $c_j^\lambda(f)(x) \geq 0$ *for all* $j = 0, \ldots, k$*. Then:*

*(i)* $(x^{\lambda-1+k} f(x))^{(j)} \geq 0$ *for all* $j = 0, \ldots, k$*;*

*(ii)* $\lim_{x\to 0}(x^{\lambda-1+k} f(x))^{(j)} = 0$ *for all* $j = 0, \ldots, k-2$*;*

*(iii)* $\lim_{x\to 0}(x^{\lambda-1+k} f(x))^{(k-1)} \in [0, \infty)$*;*

*Proof.* We use induction in $k$. For $k = 1$ (i) is clearly satisfied, (ii) needs not be checked, and (iii) follows by noticing that $x^\lambda f(x)$ is non-negative and increasing. For $k = 2$, $(x^{\lambda+1} f(x))'' = x^{\lambda-1} c_2^\lambda(f)(x) \geq 0$, $(x^{\lambda+1} f(x))' = x^\lambda c_1^\lambda(f)(x) + \lambda x^\lambda c_0^\lambda(f)(x) \geq 0$, and thus (i) is satisfied. Property (ii) is clearly satisfied, and (iii) follows since $(x^{\lambda+1} f(x))'$ is non-negative and increasing.

Next we assume that $f$ satisfies $c_j^\lambda(f) \geq 0$ for all $j \leq k+1$, and aim at verifying (i), (ii), and (iii) with $k$ replaced by $k+1$. For $j = k+1$ we get $(x^{\lambda+k} f(x))^{(j)} = x^{\lambda-1} c_{k+1}^\lambda(f)(x) \geq 0$. For $1 \leq j \leq k$ we use

$$(x^{\lambda+k} f(x))^{(j)} = x(x^{\lambda-1+k} f(x))^{(j)} + j(x^{\lambda-1+k} f(x))^{(j-1)} \geq 0,$$

and (i) is verified. To see (ii), notice that

$$(x^{\lambda+k} f(x))^{(k-1)} = x(x^{\lambda-1+k} f(x))^{(k-1)} + (k-1)(x^{\lambda-1+k} f(x))^{(k-2)},$$

The last term tends to zero by the induction hypothesis, and the first term equals $x$ times a non-negative and increasing function. Hence (ii) holds for $k+1$. Property (iii) for $k+1$ follows since $(x^{\lambda+k} f(x))^{(k)}$ is a positive and increasing function. This proves the lemma. □

**Lemma 3.5** *Let* $f \in C^\infty((0,\infty))$ *and suppose that* $c_0^\lambda(f), \ldots, c_{k-1}^\lambda(f)$ *are non-negative functions. If* $c_k^\lambda(f)$ *is completely monotonic then* $\gamma_k^\lambda(f)$ *is also completely monotonic and*

$$\gamma_k^\lambda(f)(x) = \frac{l_k}{x^\lambda} + \frac{b_k}{\lambda} + \int_0^\infty M_k(u) u^{\lambda-1} e^{-xu}\, du,$$

*where*

$$M_k(u) = \int_u^\infty s^{-\lambda} d\mu_k(s).$$

*Proof.* By the complete monotonicity we may write

$$(x^{\lambda-1+k}f(x))^{(k)} = x^{\lambda-1}c_k^\lambda(f)(x) = x^{\lambda-1}\int_0^\infty e^{-xs}\,d\mu_k(s) + b_k x^{\lambda-1},$$

where $b_k \geq 0$ and $\mu_k$ is a positive measure on $(0,\infty)$. The assumptions on non-negativity yield that the function $x^\lambda \gamma_k^\lambda(f)(x) = (x^{\lambda-1+k}f(x))^{(k-1)}$ is non-negative and increasing. Hence

$$l_k \equiv \lim_{x\to 0} x^\lambda \gamma_k^\lambda(f)(x) \geq 0.$$

Furthermore,

$$\begin{aligned}
x^\lambda\gamma_k^\lambda(f)(x) - l_k &= \int_0^x (t^{\lambda-1+k}f(t))^{(k)}\,dt \\
&= \int_0^x t^{\lambda-1}\left(\int_0^\infty e^{-ts}\,d\mu_k(s) + b_k\right)dt \\
&= \frac{b_k}{\lambda}x^\lambda + \int_0^\infty\int_0^x t^{\lambda-1}e^{-ts}\,dt\,d\mu_k(s) \\
&= \frac{b_k}{\lambda}x^\lambda + x^\lambda\int_0^\infty\int_0^s u^{\lambda-1}e^{-xu}\,du\,s^{-\lambda}d\mu_k(s) \\
&= \frac{b_k}{\lambda}x^\lambda + x^\lambda\int_0^\infty\int_u^\infty s^{-\lambda}d\mu_k(s)u^{\lambda-1}e^{-xu}\,du,
\end{aligned}$$

by Tonelli's theorem. Consequently,

$$M_k(u) = \int_u^\infty s^{-\lambda}d\mu_k(s)$$

is finite and $M_k(u)u^{\lambda-1}$ is integrable at 0.

The formulas above also show that

$$\gamma_k^\lambda(f)(x) = \frac{l_k}{x^\lambda} + \frac{b_k}{\lambda} + \int_0^\infty M_k(u)u^{\lambda-1}e^{-xu}\,du$$

is completely monotonic. □

*Proof of Proposition 3.1:* From Lemma 3.5,

$$(x^{\lambda-1+k}f(x))^{(k-1)} = l_k + \frac{b_k}{\lambda}x^\lambda + x^\lambda\int_0^\infty M_k(u)u^{\lambda-1}e^{-xu}\,du,$$

where $l_k, b_k \geq 0$ and $M_k(u) = \int_u^\infty s^{-\lambda}\,d\mu_k(s)$. Notice that

$$u^\lambda M_k(u) \to 0, \quad u \to 0. \tag{4}$$

(To see this, rewrite as follows

$$u^{\lambda} M_k(u) = \int_u^1 \left(\frac{u}{s}\right)^{\lambda} d\mu_k(s) + u^{\lambda} \int_1^{\infty} \frac{d\mu_k(s)}{s^{\lambda}},$$

and use the dominated convergence theorem on the first term.)

Integrating this relation from $\epsilon$ to $x$, and letting $\epsilon$ tend to 0 we get, using (ii) of Lemma 3.4, that

$$\begin{aligned}(x^{\lambda-1+k} f(x))^{(k-2)} = l_k x + \frac{b_k}{\lambda(\lambda+1)} x^{\lambda+1} \\ + x^{\lambda+1} \int_0^{\infty} M_{k,k-1}(u) u^{\lambda} e^{-xu}\, du,\end{aligned}$$

where

$$M_{k,k-1}(u) = \int_u^{\infty} \frac{M_k(s)}{s^2}\, ds.$$

Continuing this process (using in each step (ii) of Lemma 3.4) we get

$$\begin{aligned}x^{\lambda-1+k} f(x) = \frac{l_k}{(k-1)!} x^{k-1} + \frac{b_k}{(\lambda)_k} x^{\lambda+k-1} \\ + x^{\lambda+k-1} \int_0^{\infty} M_{k,1}(u) u^{\lambda+k-2} e^{-xu}\, du,\end{aligned} \tag{5}$$

where

$$M_{k,k}(u) = M_k(u), \quad M_{k,j}(u) = \int_u^{\infty} \frac{M_{k,j+1}(s)}{s^2}\, ds, \quad j = 1, \ldots, k-1.$$

To ease notation we write $M_j(u) = M_{k,j}(u)$. (We remark that $M_j$ does not have the same meaning here as in Lemma 3.5.) Division by $x^{\lambda-1+k}$ in (5) shows that $f$ is completely monotonic, and has the representation

$$\begin{aligned}f(x) &= \frac{l_k}{(k-1)!} x^{-\lambda} + \frac{b_k}{(\lambda)_k} + \int_0^{\infty} M_1(u) u^{\lambda+k-2} e^{-xu}\, du \\ &= \frac{b_k}{(\lambda)_k} + \int_0^{\infty} \left( M_1(u) u^{k-1} + \frac{l_k}{(k-1)!\Gamma(\lambda)} \right) e^{-xu} u^{\lambda-1}\, du.\end{aligned} \tag{6}$$

In order to show that the functions $c_1^{\lambda}(f), \ldots, c_{k-1}^{\lambda}(f)$ are completely monotonic it suffices (Theorem 1.6) to verify that $(-1)^j \partial^j \left(M_1(u) u^{k-1}\right) \geq 0$ for $j = 1, \ldots, k-1$. Now,

$$M_1(u) u^{k-1} = u^{k-1} \int_u^{\infty} \frac{M_2(s)}{s^2}\, ds = \int_1^{\infty} \frac{M_2(ut)(ut)^{k-2}}{t^k}\, dt,$$

so it is enough to verify that $(-1)^j\partial^j\left(M_2(u)u^{k-2}\right)\geq 0$ in order to obtain that $(-1)^j\partial^j\left(M_1(u)u^{k-1}\right)\geq 0$. Repeating this argument we end up having to verify that

$$(-1)^j\partial^j\left(M_{k-j+1}(u)u^{j-1}\right)\geq 0,\quad 1\leq j\leq k-1. \tag{7}$$

These inequalities are verified using induction. For $j=1$ it reads $\partial M_k(u)\leq 0$ which is true since $M_k$ is a decreasing function. Next assuming that (7) holds for some $j\leq k-2$ we aim at verifying it for $j+1$. We rewrite the expression $(-1)^{j+1}\partial^{j+1}\left(M_{k-j}(u)u^{j+1}\right)$ in two ways:

$$\begin{aligned}(-1)^{j+1}\partial^{j+1}\left(M_{k-j}(u)u^{j+1}\right) &= (-1)^{j+1}u\partial^{j+1}\left(M_{k-j}(u)u^{j}\right)\\ &\quad + (-1)^{j+1}(j+1)\partial^{j}\left(M_{k-j}(u)u^{j}\right);\\ (-1)^{j+1}\partial^{j+1}\left(M_{k-j}(u)u^{j+1}\right) &= (-1)^{j+1}\partial^{j}\left(-\frac{M_{k-j+1}(u)}{u^2}u^{j+1}\right)\\ &\quad + (-1)^{j+1}(j+1)\partial^{j}\left(M_{k-j}(u)u^{j}\right).\end{aligned}$$

Comparing these two identities we infer that

$$(-1)^{j+1}u\partial^{j+1}\left(M_{k-j}(u)u^{j}\right)=(-1)^{j}\partial^{j}\left(M_{k-j+1}(u)u^{j-1}\right),$$

and thus (7) holds for $j+1$. □

**Remark 3.6** *Introducing the functions* $N_j(u)\equiv M_j(1/u)$ *for* $j=1,\dots,k$ *where* $M_1,\dots,M_k$ *are those considered in the proof of Proposition 3.1 it follows that*

$$N_k(u)=\int_{1/u}^{\infty}s^{-\lambda}\,d\mu_k(s)=\int_0^u t^{\lambda}\,d\widehat{\mu_k}(t),$$

*where* $\widehat{\mu_k}$ *denotes the image measure* $\phi(\mu_k)$*, with* $\phi(x)=1/x$*. For* $j\leq k-1$ *the relation between* $N_j$ *and* $N_{j+1}$ *is*

$$N_j(u)=\int_{1/u}^{\infty}\frac{M_{j+1}(s)}{s^2}\,ds=\int_0^u N_{j+1}(t)\,dt.$$

*Consequently we see that the derivatives* $N_1^{(j)}(u)$ *for* $j\leq k-1$ *are all non-negative, and take the value* $0$ *at* $u=0$*. In terms of these functions the representation* (6) *can be rewritten as*

$$\begin{aligned}f(x) &= \frac{b_k}{(\lambda)_k}+\frac{l_k}{(k-1)!x^{\lambda}}+\int_0^{\infty}M_1(u)u^{k-1}e^{-xu}u^{\lambda-1}\,du\\ &= \frac{b_k}{(\lambda)_k}+\frac{l_k}{(k-1)!x^{\lambda}}+\int_0^{\infty}N_1(s)s^{-k-\lambda}e^{-x/s}\,ds.\end{aligned}$$

The next proposition shows that for any given $N$ the classes $\mathcal{C}_N^\lambda$ become larger as $\lambda$ increases. As remarked in [9] it is not clear how to verify this even for $N=\infty$ only considering the operators $T^\lambda_{n,k}$.

**Proposition 3.7** *If $\lambda_1<\lambda_2$ then $\mathcal{C}_N^{\lambda_1}\subset\mathcal{C}_N^{\lambda_2}$ for all $N$.*

*Proof.* This follows from Leibniz' formula. Assume $f\in\mathcal{C}_N^{\lambda_1}$ and let $\lambda_2>\lambda_1$. Then

$$f(x)=c+\int_0^\infty e^{-xs}s^{\lambda_1-1}\,d\mu(s)=c+\int_0^\infty e^{-xs}s^{\lambda_2-1}\,s^{\lambda_1-\lambda_2}d\mu(s),$$

where $(-1)^js^j\partial^j\mu\geq 0$ for all $j\leq N$. Hence, for $k\leq N$,

$$\begin{aligned}&(-1)^ks^k\partial^k(s^{\lambda_1-\lambda_2}\mu)\\ &=(-1)^ks^k\sum_{j=0}^k\binom{k}{j}(-1)^{k-j}\frac{\Gamma(k-j+\lambda_2-\lambda_1)}{\Gamma(\lambda_2-\lambda_1)}s^{\lambda_1-\lambda_2+j-k}\partial^j\mu\\ &=s^{\lambda_1-\lambda_2}\sum_{j=0}^k\binom{k}{j}\frac{\Gamma(k-j+\lambda_2-\lambda_1)}{\Gamma(\lambda_2-\lambda_1)}(-1)^js^j\partial^j\mu\geq 0.\end{aligned}$$

□

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