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Special classes of homomorphisms between generalized Verma modules for $\mathcal{U}_q(su(n,n))$

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Abstract. We study homomorphisms between quantized generalized Verma modules $M(V_\Lambda) \xrightarrow{\phi_{\Lambda_1}} M(V_{\Lambda_1})$ for $\mathcal{U}_q(su(n,n))$. There is a natural notion of degree for such maps, and if the map is of degree $k$, we write $\phi_{k\Lambda, \Lambda_1}$. We examine when one can have a series of such homomorphisms $\phi_1^{\Lambda_{n-1}}, \phi_1^{\Lambda_{n-2}}, \cdots, \phi_1^{\Lambda, \Lambda_1} = \text{Det}_q$, where $\text{Det}_q$ denotes the map $M(V_\Lambda) \ni p \to \det_q p \in M(V_{\Lambda_n})$. If, classically, $su(n,n)^C = p^- \oplus (su(n) \oplus su(n) \oplus \mathbb{C}) \oplus p^+$, then $\Lambda = (\Lambda_L, \Lambda_R, \lambda)$ and $\Lambda_n = (\Lambda_L, \Lambda_R, \lambda + 2)$. The answer is then that $\Lambda$ must be one-sided in the sense that either $\Lambda_L = 0$ or $\Lambda_R = 0$ (non-exclusively). There are further demands on $\lambda$ if we insist on $\mathcal{U}_q(g^C)$ homomorphisms. However, it is also interesting to loosen this to considering only $\mathcal{U}_q(g^C)$ homomorphisms, in which case the conditions on $\lambda$ disappear. By duality, there result have implications on covariant quantized differential operators. We finish by giving an explicit, though sketched, determination of the full set of $\mathcal{U}_q(g^C)$ homomorphisms $\phi_{k, \Lambda_1}$.

Dedicated to I.E. Segal (1918-1998) in commemoration of the centenary of his birth.

1. Introduction
Generalized and quantized Verma modules have physically attractive properties similar to the Fock space. There is a “vacuum vector”, here called a highest weight vector, which is annihilated by the “upper diagonal” operators, is an eigenvector for the “diagonal operators”, and which generate the whole space when acted upon by the algebra of “lower diagonal operators”. Since it may happen that there is a second vacuum vector, it is of interest to determine cases in which this may happen. This is further interesting because by duality, such cases correspond to quantized covariant differential operators such as the Maxwell equations. We give here a complete proof of the one-sidedness and we give a sketch of the case of an arbitrary first order. Further details as well as the dual picture will appear in a forthcoming article. For the “classical” analogue, see e.g. [2]. On a personal note: The explicitness presented here is in line with how mathematical physics was taught to me by Segal, my Ph.D. advisor.

2. Set-up

\[ g^C = su(n,n)^C = \mathfrak{t}^C \oplus \mathfrak{p} = p^- \oplus \mathfrak{t}^C \oplus p^+ = p^- \oplus \mathfrak{t}^C \oplus p^+, \]

\[ \mathfrak{t}^C = su(n)^C \oplus \mathbb{C} \oplus su(n)^C = \mathfrak{t}_L^C \oplus \zeta \oplus \mathfrak{t}_R^C. \]

$\zeta$ is the center, $p^\pm$ are abelian $\mathcal{U}(\mathfrak{t}^C)$ modules, and $\mathcal{U}(g^C) = \mathcal{P}(p^-) \cdot \mathcal{U}(\mathfrak{t}^C) \cdot \mathcal{P}(p^+)$. 

(1)

(2)
We let the simple roots be denoted \( \Pi = \{\mu_1, \ldots, \mu_{n-1}\} \cup \{\beta\} \cup \{\mu_1, \ldots, \mu_{n-1}\} \), where \( \beta \) is the unique non-compact roots and where the decomposition of simple roots corresponds to the decomposition of \( t^C \) above. In the quantum group \( \mathcal{U}_q(\mathfrak{g}^C) \), we denote the generators by \( E_\alpha, F_\alpha, K_\alpha^{\pm 1} \) for \( \alpha \in \Pi \). There are also decompositions

\[
\mathcal{U}_q(\mathfrak{g}^C) = \mathcal{A}_q^- \cdot \mathcal{U}_q(t^C) \cdot \mathcal{A}_q^+,
\]

(3)

\[
\mathcal{U}_q(t^C) = \mathcal{U}_q(t_t^C) \cdot \mathbb{C}[K_\beta^{\pm 1}] \cdot \mathcal{U}_q(t_t^C).
\]

(4)

Here, \( \mathcal{A}_q^\pm \) are quadratic algebras which are furthermore \( \mathcal{U}_q(t^C) \) modules. Specifically,

\[
\mathcal{A}_q^- = \mathbb{C}[W_{i,j} \mid i, j = 1, \ldots, n],
\]

(5)

\[
\mathcal{A}_q^+ = \mathbb{C}[Z_{i,j} \mid i, j = 1, \ldots, n],
\]

(6)

with relations

\[
Z_{ij}Z_{ik} = q^{-1}Z_{ik}Z_{ij} \quad \text{if} \quad j < k;
\]

(7)

\[
Z_{ij}Z_{kj} = q^{-1}Z_{kj}Z_{ij} \quad \text{if} \quad i < k;
\]

(8)

\[
Z_{ij}Z_{st} = Z_{st}Z_{ij} \quad \text{if} \quad i < s \quad \text{and} \quad t < j;
\]

(9)

\[
Z_{ij}Z_{st} = Z_{st}Z_{ij} - (q-q^{-1})Z_{st}Z_{sj} = \quad \text{if} \quad i < s \quad \text{and} \quad j < t.
\]

(10)

The algebra \( \mathcal{A}_q^- \) have the same relations, but the algebras \( \mathcal{A}_q^\pm \) are different as \( \mathcal{U}_q(t^C) \) modules. The elements \( Z_{ij} \) and \( W_{ij} \) are constructed by means of the Lusztig operators. References [4] and [3] are general references of much of this. Using the Serre relations one gets, setting \( \mu_0 = Id \).

**Lemma 2.1.**

\[
Z_{i,j} = T_{\nu_{i-1}}T_{\nu_{j-2}} \cdots T_{\nu_0} \cdot T_{\mu_{i-1}}T_{\mu_{i-2}} \cdots T_{\mu_0} (E_\beta),
\]

(11)

\[
W_{i,j} = T_{\nu_{i-1}}T_{\nu_{j-2}} \cdots T_{\nu_0} \cdot T_{\mu_{i-1}}T_{\mu_{i-2}} \cdots T_{\mu_0} (F_\beta).
\]

(12)

For later use, we give the relations in the full algebra:

\[
E_{\mu_k} W_{i,j} = W_{i,j} E_{\mu_k} \quad \text{if} \quad k \neq i - 1,
\]

(13)

\[
E_{\mu_k} W_{i,j}^a = (-q)[a] W_{i-1,j} W_{i,j}^{a-1} K_{\mu_k} + W_{i,j}^a E_{\mu_k} \quad \text{if} \quad k = i - 1,
\]

(14)

\[
F_{\mu_k} W_{i,j} = W_{i,j} F_{\mu_k} \quad \text{if} \quad k \neq i, i - 1,
\]

(15)

\[
F_{\mu_k} W_{i,j}^a = -q^{-1}[a] W_{i-1,j} W_{i,j}^{a-1} K_{\mu_k}^a + q^{-a} W_{i,j}^a F_{\mu_k} \quad \text{if} \quad k = i,
\]

(16)

\[
F_{\mu_k} W_{i,j} = qW_{i,j} F_{\mu_k} \quad \text{if} \quad k = i - 1,
\]

(17)

\[
F_{\mu_k} Z_{i,j} = Z_{i,j} F_{\mu_k} \quad \text{if} \quad k \neq i - 1,
\]

(18)

\[
F_{\mu_k} Z_{i,j}^a = [a] Z_{i-1,j} Z_{i,j}^{a-1} K_{\mu_k}^a + W_{i,j}^a E_{\mu_k} \quad \text{if} \quad k = i - 1,
\]

(19)

\[
E_{\mu_k} Z_{i,j} = Z_{i,j} E_{\mu_k} \quad \text{if} \quad k \neq i, i - 1,
\]

(20)

\[
E_{\mu_k} Z_{i,j}^a = [a] Z_{i-1,j}^{a-1} Z_{i+1,j} + q^{-a} Z_{i,j}^a E_{\mu_k} \quad \text{if} \quad k = i,
\]

(21)

\[
E_{\mu_k} Z_{i,j} = qZ_{i,j} E_{\mu_k} \quad \text{if} \quad k = i - 1.
\]

(22)

There are similar formulas for the commutators involving \( E_{\nu_k} \) and \( F_{\nu_k} \). If e.g. \( S \) denotes the obvious automorphism defined on generators by \( W_{ij} \rightarrow W_{j,i} \), and similarly, \( Z_{ij} \rightarrow Z_{j,i} \) then \( E_{\nu_k} = SE_{\nu_k} S \) and \( F_{\nu_k} = SF_{\nu_k} S \).
3. Finite dimensional $\mathcal{U}_q(\mathfrak{g})$ modules

A non-zero vector $v_\lambda$ of a finite dimensional module $V_\lambda$ of $\mathcal{U}_q(\mathfrak{g})$ is a highest weight vector of highest weight $\lambda$, and $V_\lambda$ is a highest weight module of highest weight $\lambda$, if

$$\forall i = 1, \ldots, n - 1 : K_{\mu_i}^{\pm 1} = q^{\pm \mu_i} v_\lambda, \quad K_{\nu_i}^{\pm 1} = q^{\pm \nu_i} v_\lambda, \quad \text{and} \quad K_{\beta}^{\pm 1} = q^{\pm \lambda} v_\lambda.$$  \hspace{1cm} (23)

Finally, $\mathcal{U}_q^+(\mathfrak{g}) v_\lambda = 0$, and $\mathcal{U}_q^-(\mathfrak{g}) v_\lambda = V$.

We set $\Lambda = ((\lambda_1^L, \ldots, \lambda_{n-1}^L), (\lambda_1^R, \ldots, \lambda_{n-1}^R); \lambda) = (\Lambda_1, \Lambda_2, \lambda)$.

As a vector space, $V_{\Lambda} = V_{\Lambda_1} \otimes V_{\Lambda_2}$ where $V_{\Lambda_1}$ and $V_{\Lambda_2}$ are highest weight representations of $\mathcal{U}_q^1(\mathfrak{g})$ and $\mathcal{U}_q^2(\mathfrak{g})$, respectively, of highest weights $\Lambda_1 = (\lambda_1^L, \ldots, \lambda_{n-1}^L)$ and $\Lambda_2 = (\lambda_1^R, \ldots, \lambda_{n-1}^R)$, respectively. The highest weight vector can then be written as $v_{\Lambda} = v_{\Lambda_1} \otimes v_{\Lambda_2}$ with the stipulation that $K_{\beta}^{\pm 1} v_{\Lambda_1} \otimes v_{\Lambda_2} = q^{\pm \lambda} v_{\Lambda_1} \otimes v_{\Lambda_2}$.

4. Generalized quantized Verma modules and their homomorphisms

Consider a finite dimensional module $V_\lambda = V_{\Lambda_1,\Lambda_2,\lambda}$ over $\mathcal{U}_q(\mathfrak{g})$ with highest weight is defined by $\Lambda = (\Lambda_1, \Lambda_2, \lambda)$ where $\Lambda_1 = (\lambda_1^L, \ldots, \lambda_{n-1}^L, 0)$, $\Lambda_2 = (\lambda_1^R, \ldots, \lambda_{n-1}^R, 0)$, and $\lambda \in \mathbb{C}$.

We extend such a module to a $\mathcal{U}_q(\mathfrak{g}) A_q^+$ module, by the same name, by letting $A_q^+$ act trivially.

**Definition 4.1.** The quantized generalized Verma module $M(V_\lambda)$ is given by

$$M(V_\lambda) = \mathcal{U}_q(\mathfrak{g}) \bigotimes_{\mathcal{U}_q(\mathfrak{g}). A_q^+} V_\lambda$$  \hspace{1cm} (24)

with the natural action from the left.

As a vector space,

$$M(V_\lambda) = A_q^+ \otimes V_\lambda.$$  \hspace{1cm} (25)

We are interested in structure preserving homomorphisms between quantized generalized Verma modules. We call such maps intertwiners, covariants, or equivariants, indiscriminately. Dually, they will be quantized covariant differential operators. In abstract notation, the structure under investigation is

$$Hom_{\mathcal{U}_q(\mathfrak{g})}(M(V_\lambda), M(V_{\lambda_1})).$$  \hspace{1cm} (26)

However, for the time being we will consider

$$Hom_{A_q^+ \mathcal{U}_q(\mathfrak{g})}(M(V_\lambda), M(V_{\lambda_1})).$$  \hspace{1cm} (27)

An element $\phi_{\Lambda_1,\Lambda_2}$ in the latter space is completely determined by the $\mathcal{U}_q(\mathfrak{g})$ equivariant map, denoted by the same symbol:

$$V_\lambda \xrightarrow{\phi_{\Lambda_1,\Lambda_2}} A_q^+ \otimes V_{\lambda_1} \text{ leads to } A_q^+ \otimes V_\lambda \xrightarrow{\phi_{\Lambda_1,\Lambda_2}} A_q^+ \otimes V_{\lambda_1}. $$  \hspace{1cm} (28)

Specifically, $\phi_{\Lambda_1,\Lambda_2}$ does not depend on $\lambda$ and is completely given by the condition that the image of the highest weight vector $\phi_{\Lambda_1,\Lambda_2}(v_\lambda)$ is a highest weight vector for $\mathcal{U}_q(\mathfrak{g})$. For the map $\phi_{\Lambda_1,\Lambda_2}$ to belong to the former space (27) it is necessary, and sufficient that, additionally, ($Z_\beta$ acting in $M(V_{\lambda_1})$)

$$Z_\beta (\phi_{\Lambda_1,\Lambda_2}(v_\lambda)) = 0.$$  \hspace{1cm} (29)

This equation depends heavily on $\lambda$. It is clear that such maps, whether of the first or second kind, can be combined:

$$\phi_{\Lambda_1,\Lambda_2} \circ \phi_{\Lambda_1,\Lambda_2} = \phi_{\Lambda_1,\Lambda_2}.$$  \hspace{1cm} (30)
though it may happen that the composite is zero.

We use the terminology of degree of elements of $A_q^-$ in the obvious way, and we let, for $k = 1, \ldots, A_q^-(k)$ denote the $U_q(\mathfrak{g})$ module spanned by homogeneous elements of degree $k$. If the elements $p_{ij}$ all belong to $A_q^-(k)$, we write $\phi_{\Lambda, A_i}^k$.

**General Problem:** When is it possible to have $\phi_{\Lambda_{n-1}, \Lambda_n}^1 \circ \phi_{\Lambda_{n-2}, \Lambda_{n-1}}^1 \circ \cdots \circ \phi_{\Lambda, \Lambda_1}^1 = \text{Det}_q$?

In this case, if $\Lambda = (\Lambda_L, \Lambda_R, \lambda)$, then $\Lambda_n = (\Lambda_L, \Lambda_R, \lambda + 2)$.

5. Laplace expansion

If $m = n$, one may define the quantum determinant $\text{det}_q$ in $A_q^-$ as follows:

$$\text{det}_q(n) = \text{det}_q = \sum_{\sigma \in S_n} (-q^{-1})^{\ell(\sigma)} W_{1, \sigma(1)} W_{2, \sigma(2)} \cdots W_{n, \sigma(n)}$$

$$= \sum_{\delta \in S_n} (-q^{-1})^{\ell(\delta)} W_{\delta(1), 1} W_{\delta(2), 2} \cdots W_{\delta(n), n}.$$  (31)

If $m = n$ and $I = \{i_1 < i_2 < \cdots < i_{n-1}\} = \{1, 2, \cdots, n\} \setminus \{i\}, J = \{j_1 < j_2 < \cdots < j_{n-1}\} = \{1, 2, \cdots, n\} \setminus \{j\}$, we set

$$A(i, j) = \sum_{\sigma \in S_{n-1}} (-q^{-1})^{\ell(\sigma)} W_{i, J, \sigma(1)} W_{i_2, J, \sigma(2)} \cdots W_{i_{n-1}, J, \sigma(n-1)}$$

$$= \sum_{\tau \in S_{n-1}} (-q^{-1})^{\ell(\tau)} W_{\tau(1), i} W_{\tau(2), j_1} \cdots W_{\tau(n-1), j_{n-1}}.$$  (33)

These elements are quantum $(n - 1) \times (n - 1)$ minors. The following was proved by Parshall and Wang [6]:

**Proposition 5.1.** $\text{det}_q$ is central. Furthermore, let $i, k \leq n$ be fixed integers. Then

$$\delta_{i,k} \text{det}_q = \sum_{j=1}^n (-q^{-1})^{j-k} W_{i, j} A(k, j) = \sum_j (-q^{-1})^{i-j} A(i, j) W_{k, j}$$

$$= \sum_j (-q^{-1})^{j-k} W_{j, i} A(j, k) = \sum_j (-q^{-1})^{i-j} A(j, i) W_{j, k}.$$  (35)

6. 1. order

Any finite dimensional highest weight representation of $U_q(\mathfrak{g})$ of the form $\Lambda = (\Lambda_L, \Lambda_R, \lambda)$ in which either $\Lambda_L = 0$ or $\Lambda_R = 0$ will be called one-sided. We will now give an explicit form for a highest weight vector $v_1$ of an irreducible sub-representation of $A^-(1) \otimes V_{\Lambda=(\lambda, 0, \lambda)}$. Specifically, consider

$$v_1 = W_{N+1,1} v_0 + W_{N,1} u_N v_0 + W_{N-1,1} u_N u_{N-1,1} v_0 + W_{N-2,1} u_N u_{N-2,1} v_0 + \cdots + W_{1,1} u_1 v_0,$$  (37)

where $\forall i = 1, \ldots, N : u_i \in U_q^{-\mu_N - \mu_{N-1} - \cdots - \mu_1}(\mathfrak{g})$.

Because of this, we first want to consider a basis of $U_q^{-\mu_N - \cdots - \mu_1}(\mathfrak{g})$.

Set $E_{\ell,N} = \{\ell, \ell + 1, \cdots, N\} \subseteq \{1, 2, \ldots, n - 1\}$. Any sequence $I_{\ell,N} = (i_\ell, i_{\ell+1}, \cdots, i_N)$ made up of pairwise different elements of $E_{\ell,N}^0$ defines a non-zero element

$$F_{\mu_1} F_{\mu_{i_{\ell+1}}} \cdots F_{\mu_N} = F_{\mu}(I_{\ell,N}) \in U_q^{-\mu_N - \cdots - \mu_1}(\mathfrak{g}).$$  (38)
We will call such a sequence **allowed**. We reserve the name $E_{\ell,N}$ for the special sequence $(\ell, \ell + 1, \cdots, N)$.

We will say that a transposition $(i_\ell, i_{\ell+1}, \ldots, i_k, i_{k+1}, \ldots, i_N) \rightarrow (i_\ell, i_{\ell+1}, \ldots, i_k, i_{k+1}, \ldots, i_N)$ is legal if $|i_{k+1} - i_k| > 1$.

Recall that $F_{\mu_j}F_{\mu_j} = F_{\mu_j}F_{\mu_j}$ if $|i - j| > 1$. We will say that two allowed sequences $I^{(1)}$ and $I^{(2)}$ are equivalent if one can be obtained from the other by a series of legal transpositions. It is clear that any allowed sequence $I$ can be brought, uniquely, and by legal transpositions, into the form $J_1J_2\cdots J_r$ which is the concatenation of sequences $J_i$ that are either descending or ascending, and such that the following are satisfied: Firstly, the elements of $J_s$ are smaller than the elements of $J_t$ if $s < t$, and $\cup_s J_s = \{ \ell, \ell + 1, \ldots, N \}$. Secondly, two neighboring sequences cannot both be ascending (maximality), and thirdly, singletons are ascending.

We denote by $\mathcal{J}_{\ell,N}$ the set of such sequences. The following is then obvious:

**Proposition 6.1.**

$$\{ F^\mu(I_{\ell,N}) \mid I_{\ell,N} \in \mathcal{J}_{\ell,N} \} \quad (39)$$

is a basis of $U_q^{-\mu_1-\cdots-\mu_N}(t_L^C)$.

We furthermore have from e.g. [1, lemma 6.27]:

**Proposition 6.2.** Let $V = V(\Lambda_L)$ be a finite dimensional highest weight representation of $U_q(t_L^C)$ with $\Lambda_L = (\lambda_1^L, \lambda_2^L, \cdots, \lambda_{n-1}^L)$ satisfying: $\lambda_1^L > 0, \lambda_{\ell+1}^L > 0, \ldots, \lambda_N^L > 0$. Let $v_0$ denote a highest weight vector (unique up to a non zero constant). Then

$$\{ F^\mu(I_{\ell,N})v_0 \mid I_{\ell,N} \in \mathcal{J}_{\ell,N} \} \quad (40)$$

is a basis of $V^{\Lambda_L-\mu_1-\cdots-\mu_N}$.

If $I_{\ell,N} = J_1J_2\cdots J_s \in \mathcal{J}_{\ell,N}$ as above, we attach to it a sequence $O^\mu(I_{\ell,N}) = (c_{i_\ell}, c_{i_{\ell+1}}, \cdots, c_{i_N})$ where $c_k = a_k$ if either $i_k$ belongs to an ascending sub-sequence $J_x$ of $I_{\ell,N}$ or if $i_k$ is the biggest element in a descending sub-sequence $J_y$ of $I_{\ell,N}$. Here, $x, y \in \{ 1, 2, \ldots, s \}$. In the remaining cases, $c_k = b_k$. We furthermore set $f^\mu(O^\mu(I_{\ell,N})) = \prod_{i=\ell}^N c_i$.

We can then maintain the assumptions from Lemma 6.2:

**Proposition 6.3.** If the vector $v_1$ in (37) is a highest weight vector in $\mathcal{A}_-^\ell \otimes V(\Lambda_L)$ then

$$\forall \ell = 1, \ldots, N : u_\ell = \sum_{I_{\ell,N} \in \mathcal{J}_{\ell,N}} f^\mu(O^\mu(I_{\ell,N}))F^\mu(I_{\ell,N})v_0. \quad (41)$$

Later, we shall find it convenient to set $\mathcal{J}_{N+1,N} = \emptyset$ and $f^\mu(O^\mu(\emptyset)) = 1 = F^\mu(\emptyset)$. Likewise, $E_{N+1,N} = \emptyset$.

Our general case of interest is where we only assume $\lambda_1^L \neq 0$. Bear in mind that in the sequence $C(I_{\ell,N})$, $c_0 = b_0$ signals that the corresponding $\mu$, taking part in $F(I_{\ell,N})$, can be moved all the way to the right without changing $F(I_{\ell,N})$. If we allow $\lambda_1^L = 0$ this means that such elements, when applied to $v_0$, give zero. Hence if we let $\mathcal{Z}_{\ell,N} = \{ i = \ell, \cdots, N \mid \lambda_i^L = 0 \}$ and if we let $\mathcal{J}_{\ell,N}^2$ denote those sequences $I$ in $\mathcal{J}_{\ell,N}$ for which any index $i$ from $\mathcal{Z}_{\ell,N}$ either belongs to an increasing sequence or is the biggest index in a decreasing sequence, then we have:

**Proposition 6.4.**

$$\{ F^\mu(I_{\ell,N})v_0 \mid I_{\ell,N} \in \mathcal{J}_{\ell,N}^2 \} \quad (42)$$

is a basis of $V^{\Lambda_L-\mu_1-\cdots-\mu_N}$.
Clearly there is an analogue to Proposition 6.3 for this general case (just as long as $\lambda_i^\mu > 0$).

There is yet another helpful way to view the various sets $J_{\ell,N}$, $\ell = N, N - 1, \ldots, 1$, namely as a labeled, directed rooted tree with root at $F_{\mu_1}$:

$$F^\mu(I_{\ell,N}) L_{\ell-1} \leftarrow R_{\ell-1} F_{\mu_{\ell-1}} F^\mu(I_{\ell,N}) F^\mu(I_{\ell,N})$$

(43)

Here, it is really only the relative positions of $F_{\mu_1}$ and $F_{\mu_{\ell-1}}$ that matter.

If we have $\lambda_i^\mu = 0$ we just modify the tree by removing all branches labeled by $R_i$ - as well as everything above these branches - from the tree. (In this picture, the root is lowest.)

In this way, there is an obvious bijection between the paths in the modified tree and the basis.

We now return to (37). To obtain the following equations, it is used that $E_{\mu_{\ell-1}}(W_{i,j}) = -qW_{i-1,j}K_{\mu_{\ell-1}} + (W_{i,j})E_{\mu_{\ell-1}}$, which follows from Lemma 2.1. Furthermore, for the vector in (37) to be a $U_q(T^C)$ highest weight vector we clearly only need to look at $U_q(T^C)$. Here we must have:

$$\forall i = 1, \ldots, N : (-q)W_{i-1}K_{\mu_1}u_{i+1}v_0 + W_{i,1}E_{\mu_1}u_i v_0 = 0$$

(44)

$$\forall i, j = 1, \ldots, N : E_{\mu_j} u_i v_0 = 0 \text{ if } i \neq j.$$  

(45)

We assume throughout that $\lambda_i^\mu \neq 0$.

Using Proposition 6.3, we set $u_{N+1} = 1$ and

$$\forall i = 1, \ldots, N, u_i := a_i F_{\mu_i} u_{i+1} + b_i u_{i+1} F_{\mu_i} \text{ (except } b_N := 0).$$

(46)

**Lemma 6.5.** The vector $v_1$ in (37) is a highest weight vector if and only if

$$a_N = q^{1+\lambda_N^\mu}/[\lambda_N^\mu],$$

(47)

$$(a_{N-1}[\lambda_{N-1}^\mu + 1] + b_{N-1}[\lambda_{N-1}^\mu])u_N v_0 = q^{\lambda_{N-1}^\mu+2}u_N v_0,$$

(48)

$$(a_{N-1}[\lambda_N^\mu] + b_{N-1}[\lambda_N^\mu + 1])F_{\mu_{N-1}} u_{N+1} v_0 = 0.$$  

(49)

For $i < N - 1$:

$$a_i[a_i + 1][\lambda_i^\mu + 1] + b_i[\lambda_i^\mu]u_{i+1}v_0 = q^{\lambda_i^\mu+2}u_{i+1}v_0,$$

(50)

$$(a_i[a_i + 1][\lambda_i^\mu + 1] + b_i[\lambda_i^\mu])F_{\mu_i} u_{i+2}v_0 + (b_i[a_i + 1][\lambda_i^\mu + 2] + b_i[\lambda_i^\mu]) F_{\mu_i} u_{i+2}v_0 = 0.$$  

(51)

(52)

In continuation of the discussion following Proposition 6.4, notice that if $\lambda_i^\mu = 0$ then equation (53) should be stricken, $b_i = 0$, and $a_i = q^2$.

Returning to the general case: If all $\lambda_i^\mu \neq 0$:

$$a_N = q^{\lambda_N^\mu+1}/[\lambda_N^\mu],$$

(53)

$$\forall k = 1, \ldots, N - 1 :$$

$$a_{N-k} = q^{\lambda_{N-k}^\mu+2}/[\lambda_{N-k}^\mu + \lambda_{N-k+1}^\mu + \lambda_{N-k}^\mu + k],$$

(54)

$$b_{N-k} = -q^{\lambda_{N-k}^\mu+2}/[\lambda_{N-k}^\mu + \lambda_{N-k+1}^\mu + \lambda_{N-k}^\mu + k].$$

(55)

(56)
If \( \lambda^\mu_{N-1} = 0 = \cdots = \lambda^\mu_{N-R} \) the \( a_{N-k} \) just become \( q^k \) for \( k = 1, \ldots, R \). This is just the limit of the equations (55). The corresponding \( b_{N-k} = 0 \) seemingly do not have a nice limit, but recall that instead, we just cut all branches of the tree marked by \( R_{N-i}, \ i = 1, \ldots, R \). Actually, in this sense there is a nice limit for any case in which \( \lambda^\mu_i = 0 \) for some values of \( i = 1, \ldots, N-1 \).

7. One-sidedness

Recall that \( \det_q \) is central in \( \mathcal{A}^- \).

**Proposition 7.1** (One-sidedness). If \( \Lambda = (\Lambda_L, \Lambda_R, \lambda) \) and if

\[
\phi^1_{\Lambda_{n-1}, \Lambda_n} \circ \phi^1_{\Lambda_{n-2}, \Lambda_{n-1}} \circ \cdots \circ \phi^1_{\Lambda_1, \Lambda_0} = \det_q, \tag{57}
\]

where \( \det_q \) denotes the operator \( M(V_\lambda) \ni p \rightarrow \det_q \cdot p \in M(V_{\Lambda_n}) \), then at least one of the pair \( \Lambda_L, \Lambda_R \) is 0.

We call such a representation one-sided. We shall see later that there is a converse to this.

**Proof.** The proof (sketched) is obtained in 10 installments:

1. We shall need the following elementary result:

**Lemma 7.2.** Let \( a, b \in \mathbb{N} \) with \( b \leq a \). Then

\[
[a]_q [b]_q = [a + b - 1]_q + [a + b - 3]_q + \cdots + [a - b + 1]_q.
\]

**Proof of Lemma:** Using that \( [a + 1]_q = q^{-a} + q^{-a+2} + \cdots + q^a \), this follows easily by counting \( q \) exponents. \( \square \)

2. We have that \( \det_q \otimes V \subseteq \mathcal{A}^+_1 \otimes \mathcal{A}^-_{n-1} \otimes V = \mathcal{A}^+_1 \otimes (\mathcal{A}^-_{n-1} \otimes V) = (\mathcal{A}^+_1 \otimes \mathcal{A}^-_{n-1}) \otimes V \) and \( \mathcal{A}^-_{n-1} \) is a sum of double tableaux of box size \( (n-1) \times (n-1) \) and similarly \( \mathcal{A}^-_n \) is a sum of double tableaux of box size \( (n) \times (n) \). By the Littlewood-Richardson rule, to get \( \det_q \) we need to use the invariant subspace \( \mathcal{A}^+_1(n-1) \) of \( (n-1) \times (n-1) \) minors in \( \mathcal{A}^-_{n-1} \otimes V \). We can ignore contributions from other minors.

3. We now extend the notation used in Proposition 6.3 to also cover the cases of representations of \( \mathcal{U}_q(\mathfrak{f}^C_R) \) in the obvious way. We then have the following extension of said proposition:

If

\[
v_1 = \sum_{k=1, \ell=1}^{i+1, j+1} W_{k, \ell} u_{k, \ell}, \tag{58}
\]

is a highest weight vector and \( u_{i+1, j+1} = 1 \), then

\[
u_{k, \ell} = \sum_{I_{k, i} \in \mathcal{J}_{k, i}, I_{\ell, j} \in \mathcal{J}_{\ell, j}} f^\mu((C^\mu(I_{k, i}))) f^\nu((C^\nu(I_{\ell, j}))) F^\mu(I_{k, i}) F^\nu(I_{\ell, j}) v_0. \tag{59}
\]

4. Let \( \mathcal{A}^-_{n-1}(n-1) \) denote the space generated by the \( (n-1) \times (n-1) \) minors in \( \mathcal{A}^- \). This is a \( \mathcal{U}_q(\mathfrak{f}^C) \) module of highest weight \( \Lambda^\mu = (0, 0, \ldots, 0, 1) = \Lambda^\nu \). The same kind of reasoning can be applied to \( \mathcal{A}^-_{n-1}(n-1) \otimes V \). (Notice that \( \mathcal{A}^-_{n-1}(n-1) \) is the dual to \( \mathcal{A}^+_1 \)). A highest weight vector \( v_0 \) in an irreducible submodule \( V_0 \subseteq \mathcal{A}^-_{n-1}(n-1) \otimes V \) has the form

\[
v_0 = \sum_{k, \ell} A(n + k, b + \ell) \tilde{u}_{a+k, b+\ell} v_0, \tag{60}
\]

7
where the vectors $\tilde{u}_{q+k,b+\ell}\tilde{v}_0$, if $k + \ell > 0$, have weights strictly smaller than $\tilde{v}_0$.

5. If we insert (60) into (58) and isolate the $\tilde{v}_0$ terms, we get in particular, using (57, (59), and since clearly here $(a, b) = (i + 1, j + 1)$ that

$$\sum_{k=1}^{i+j+1} \sum_{\ell=1}^{j+1} W_{k,\ell}(I_{k,i})f^\mu(C^\nu(I_{\ell,j}))f^\nu(C^\sigma(I_{\ell,j}))F^\sigma(I_{k,i})F^\nu(I_{\ell,j})A(i+1,j+1) = \kappa \cdot \det_q$$

for some constant $\kappa \neq 0$. It is easy to see that $F^\mu(I_{k,i})F^\nu(I_{\ell,j})A(i+1,j+1) = 0$ unless $(I_{k,i}), (I_{\ell,j}) = (E_{k,i}, E_{\ell,j})$. In the latter case we get, by (74) in Chapter 1, $(-q^{-1})^{i+j-k-\ell}A(k,\ell)$. So

$$\sum_{k=1}^{i+j+1} \sum_{\ell=1}^{j+1} W_{k,\ell}(-q^{-1})^{i+j-k-\ell}f^\mu(C^\nu(E_{k,i}))f^\nu(C^\sigma(E_{\ell,j}))A(k,\ell) = \kappa \cdot \det_q.$$  

6. If both $i + 1 < n$ and $j - 1 < n$ we can apply $F^\nu_{\mu n-1} \cdots F^\nu_{\nu j+1}F^\mu_{\mu n-1} \cdots F^\mu_{\mu s+1}$ to both sides of (62) and get that $W_{n,n}A(i+1,j+1) = 0$; a contradiction.

7. Let us first assume that $i = j = n$. If we set $d_{k,\ell} = f^\mu(C^\nu(E_{k,i}))f^\nu(C^\sigma(E_{\ell,j}))$, (62) becomes

$$\sum_{k=1}^{n} \sum_{\ell=1}^{n} W_{k,\ell}d_{k,\ell}(-q^{-1})^{2n-k-\ell}A(k,\ell) = \kappa \cdot \det_q.$$  

Using (35) we can subtract a certain multiple of $\det_q$ in each row such that in the resulting equations

$$\sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} W_{n-k,n-\ell}b_{k,\ell}A(n-k, n-\ell) = \bar{\kappa} \cdot \det_q,$$

we may assume: $\forall k : b_{k,n} = 0$. Of course, this may change the constant into $\bar{\kappa}$. A) If all the remaining $b_{k,\ell}$s are zero then, naturally, the resulting $\bar{\kappa}$ is zero but that will also imply that each row of the original system satisfies, up to a constant non-zero multiple, equation (35). In particular,

$$\sum_{k=1}^{n} W_{k,n}d_{k,n}(-q^{-1})^{n-k}A(k,n) = \bar{\kappa} \cdot \det_q.$$  

B) If a non-zero system remains, we can subtract using column equations (36) to remove the terms $W_{n,j}A(n,j)$: $j = 1, \ldots, n-1$ (the term with $j = n$ has already been removed. If there still remains an equation

$$\sum_{k=0}^{i-1} \sum_{\ell=0}^{j-1} W_{i-k,j-\ell}b_{k,\ell}A(i-k, j-\ell) = \kappa \cdot \det_q,$$

we reach a contradiction as in 6.

In conclusion:

There is either a column equation

$$\sum_{k=1}^{n} W_{k,n}d_{k,n}(-q^{-1})^{n-k}A(k,n) = \kappa \cdot \det_q,$$

or an analogous row equation

$$\sum_{\ell=1}^{n} W_{n,\ell}d_{n,\ell}(-q^{-1})^{n-\ell}A(n,\ell) = \bar{\kappa} \cdot \det_q.$$
8. Suppose that we have a row equation

Lemma 7.3.

\[ \sum_{\ell=1}^{n} W_{n,\ell} d_{n,\ell} (-q^{-1})^{n-\ell} A(n, \ell) = \kappa \cdot \det q. \] (69)

Then \( \lambda_{n-1}^\mu = 1 \) and \( \forall i = 1, \ldots, n - 2 : \lambda_{i}^\mu = 0. \)

Proof. We have a PBW basis made up of monomials \( W_{n,jn,W_{n,jn-1}, \ldots, W_{1,j1}}. \) It follows that \( \kappa = 1 \) and it follows from (69) and (35) that \( \forall k = d_{k,n} = q^{2(n-k)}. \) It is easy to see (see 7.) that \( d_{k,n} = a_{n-1}a_{n-2} \cdots a_{k}. \) This clearly implies that \( a_{k} = q^{2} \) for all \( k = 1, \ldots, n - 1. \)

In particular, \( a_{n-1} = q^{2}, \) hence

\[ q^{2} = q^{1+\lambda_{n-1}^\mu} \left[ \lambda_{n-1}^\mu \right] \Rightarrow q^{2\lambda_{n-1}^\mu - 2} = 1 \Rightarrow \lambda_{n-1}^\mu = 1. \]

Inductively, it follows from (55) that

\[ q^{\lambda_{N-k}^\mu} \left[ \frac{1 + k}{\lambda_{N-k}^\mu + 1 + k} \right] = 1 \Rightarrow \lambda_{N-k}^\mu = 0. \] (70)

9. If there is a column equation, it follows in the same way that \( \Lambda_{R} = (0,0, \ldots, 0,1). \)

10. By 6, 7 what remains are the cases \( i < n, j = n \) and \( i = n, j < n. \) However, it is clear that they, by inspection, are covered by the arguments of the case \( i = j = n \) simply by eliminating one possibility, so that if \( j = n, \) we must have \( \Lambda_{R} = 0 \) and if \( i = n \) we must have \( \lambda_{L} = 0. \)

We have the following converse which is quite straightforward:

Proposition 7.4. Let \( \Lambda_{i} = V(\Lambda_{i}, 0, \lambda). \) Set \( \Lambda_{0} = \lambda. \) Then there exist \( A_{q}^{-1} \mathcal{U}_{q}(\mathfrak{e}^{C}) \) intertwining maps \( \psi_{i}^{1} : \Lambda_{i} \rightarrow \Lambda_{i+1} \subset \Lambda_{i} \otimes A_{q}^{-1}(1), \) for \( i = 0, 1, \ldots, n - 1, \) independent of \( \lambda, \) such that, with \( \Lambda_{i} = (\Lambda_{i}, 0, \lambda + 2), \)

\[ \psi_{\Lambda_{n-1}, \Lambda_{n}} \circ \psi_{\Lambda_{n-2}, \Lambda_{n-1}} \circ \cdots \circ \psi_{\Lambda, \Lambda_{1}} = \det q. \]

This decomposition is not unique. Furthermore the maps may be grouped together to form maps of higher degrees, defined by means of minors of the given degree.

8. First order intertwiners

It is clear that any submodule \( V_{\Lambda_{i}} \) of \( A_{q}^{-1}(1) \otimes \Lambda_{i} \) defines a \( A_{q}^{-1} \mathcal{U}_{q}(\mathfrak{e}^{C}) \) equivariant map \( M(V_{\Lambda_{i}}) \rightarrow M(V_{\Lambda_{i}}). \) We shall now see that there is a unique \( \lambda = \lambda(\Lambda_{L}, \Lambda_{R}) \) for which this becomes a \( \mathcal{U}_{q}(\mathfrak{g}^{C}) \) equivariant map. See our forthcoming article for details. Notice also that the integrality assumption on \( (\Lambda_{L}, \Lambda_{R}) \) is not used.

We need the following extra information. Modulo \( A_{q}^{-1} E_{\beta} \) it holds:

\[ Z_{\beta}(W_{i,j}) = T_{\mu_{i-1}, \mu_{i-2}} \cdots T_{\mu_{2}}(F_{\mu_{1}}) K_{\beta}^{-1}, \] (71)

\[ Z_{\beta}(W_{i,j}) = -(q - q^{-1}) T_{\nu_{i-1}, \nu_{i-2}} \cdots T_{\nu_{2}}(F_{\nu_{1}}) T_{\mu_{i-1}, \mu_{i-2}} \cdots T_{\mu_{2}}(F_{\mu_{1}}) K_{\beta}^{-1} \] if \( i, j \geq 2. \) (72)

Proposition 8.1. To any \( \mathcal{U}_{q}(\mathfrak{e}^{C}) \) homomorphism \( V_{\Lambda_{1}} \rightarrow A_{q}^{-1}(1) \otimes \Lambda_{1} \) there corresponds a unique \( \lambda \) such that \( \psi_{\Lambda_{1}, \Lambda} \in \text{Hom}_{\mathcal{U}_{q}(\mathfrak{e}^{C})}(M(V_{\Lambda_{1}}), M(V_{\Lambda})) \).
We focus on the case where \( \Lambda = (\Lambda_L, 0, \lambda) \). Recall (29) and consider

\[
Z_\beta(W_{N+1}v_0 + W_Nu_Nv_0 + W_{N-1}u_Nv_0 + W_{N-2}u_{N-2}v_0 + \cdots + W_1u_1v_0) = q^{-\lambda} \sum_{k=0}^{N-1} T_{\mu_{N-k}} \cdots T_{\mu_2} T_{\mu_1} (F_{\mu_1}) u_{N-k+1}v_0 + [\lambda + 1]u_1v_0 = 0. \tag{74}
\]

We may expand the equation into equations for each vector \( F^{\mu}(I_{1,N}) \) in the basis. We claim that the general case can be reduced by contraction of trees to just the equation for \( F^{\mu}(E_{1,N}) = F_{\mu_1} F_{\mu_2} \cdots F_{\mu_N} v_0 \). Here we get

\[
q^{-\lambda}(1 - a_N + a_N a_{N-1} + a_N a_{N-2} + \cdots + a_N a_{N-1} a_{N-2} \cdots a_2 + [\lambda + 1] a_N a_{N-1} a_{N-2} \cdots a_2 a_1 = 0. \tag{75}
\]

\[
1 + a_N = q^\frac{[\lambda_{N}^\mu + 1]}{[\lambda_{N}^\nu + 1]}, \tag{76}
\]

\[
1 + a_N + a_N a_{N-1} = q^2 \frac{[\lambda_{N}^\mu + 1] [\lambda_{N-1}^\mu + 2]}{[\lambda_{N}^\nu + 1] [\lambda_{N-1}^\nu + 1]}, \tag{77}
\]

\[
1 + a_N + a_N a_{N-1} + a_N a_{N-1} a_{N-2} = q^3 \frac{[\lambda_{N}^\mu + 1] [\lambda_{N-1}^\mu + 2] [\lambda_{N-2}^\mu + 1]}{[\lambda_{N}^\nu + 1] [\lambda_{N-1}^\nu + 1] [\lambda_{N-2}^\nu + 1]}. \tag{78}
\]

Comparing to

\[
S := 1 + a_N + a_N a_{N-1} + a_N a_{N-1} a_{N-2} + \cdots + a_N a_{N-1} a_{N-2} \cdots a_2 = q^{N-1} \frac{[\lambda_{N}^\mu + 1]}{[\lambda_{N}^\nu + 1]} \cdots \frac{[\lambda_{N}^\mu + \cdots + \lambda_{N-k}^\mu + k + 1]}{[\lambda_{N}^\nu + \cdots + \lambda_{N-k}^\nu + k]} \cdots \frac{[\lambda_{N}^\mu + \cdots + \lambda_{2}^\mu + N-1]}{[\lambda_{N}^\nu + \cdots + \lambda_{2}^\nu + N-2]}. \tag{79}
\]

one easily obtains

\[
q^{-\lambda} S + [\lambda + 1] T = 0, \tag{80}
\]

which upon division becomes

\[
q^{-\lambda} + [\lambda + 1] q^{\lambda_{N}^\mu + \lambda_{2}^\mu + \cdots + \lambda_{N}^\mu + N} \frac{1}{[\lambda_{2}^\mu + \cdots + \lambda_{N}^\mu + N-1]} = 0. \tag{81}
\]

Using the equation \([a + b] = q^{-a}[b] + q^b[a]\), one easily concludes:

\[
[\lambda + \lambda_{2}^\mu + \lambda_{3}^\mu + \cdots + \lambda_{N}^\mu + N] = 0. \tag{82}
\]

This result can easily be generalized to the general first order case. It is related to the \( q \)-Shapovalov form [5].

9. References


[5] Mudrov A 2015 Orthogonal basis for the Shapovalov form on \( U_q((n+1) \) Reviews in Mathematical Physics 27 Issue 2 1550004