ON THE COMPARISON OF STABLE AND UNSTABLE $p$-COMPLETION

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Abstract. In this note we show that a $p$-complete nilpotent space $X$ has a $p$-complete suspension spectrum if and only if its homotopy groups $\pi_* X$ are bounded $p$-torsion. In contrast, if $\pi_* X$ is not all bounded $p$-torsion, we locate uncountable rational vector spaces in the integral homology and in the stable homotopy groups of $X$. To prove this, we establish a homological criterion for $p$-completeness of connective spectra. Moreover, we illustrate our results by studying the stable homotopy groups of $K(\mathbb{Z}_p, n)$ via Goodwillie calculus.

1. Introduction

The notion of $p$-completion plays a fundamental role in algebra and topology, for it provides effective means to isolate and study $p$-primary properties. Applied to homotopy theory by Bousfield and Kan [BK72] as well as Sullivan [Sul74] and developed further in [Bou75, Bou79], it has since become one of the standard tools in the hands of algebraic topologists. However, there appears to be no general account of the comparison between unstable and stable $p$-completion in the literature, which is the question we address in the present note.

Our main goal is to characterize $p$-complete spaces which have $p$-complete suspension spectra:

**Theorem 4.7.** If $X$ is a $p$-complete nilpotent space, then $\Sigma^\infty X$ is $p$-complete if and only if $\pi_n X$ is bounded $p$-torsion for each $n$.

In fact, we exhibit a sharp dichotomy of $p$-complete nilpotent spaces: if $X$ is a $p$-complete nilpotent space whose homotopy groups are not all bounded $p$-torsion, then the integral homology groups and stable homotopy groups of $X$ both contain an uncountable rational vector space. As a consequence, we deduce that a nilpotent space $X$ with derived $p$-complete integral homology and unstable homotopy must have both $H_*(X; \mathbb{Z})$ and $\pi_n X$ of bounded $p$-torsion for all $n$.

In a first step towards the proof of the theorem, we complement the second author’s characterization of $p$-complete spectra in terms of homotopy groups with an integral homological criterion, using a mild generalization of Serre classes appropriate for stable homotopy theory. This is in sharp contrast to the aforementioned fact that the integral homology of $p$-complete spaces is not well-behaved, and thus cannot be used to characterize $p$-completeness of spaces.

**Corollary 3.3.** A bounded below spectrum $X$ is $p$-complete if and only if $H_*(X; \mathbb{Z})$ is derived $p$-complete in each degree.

In order to use this result to prove the theorem, we need to detect rational classes in the homology of $p$-complete spaces whose homotopy is not bounded $p$-torsion. This rests on the study of the integral homology of $p$-complete spheres. We end this note with a sample computation, illustrating how Goodwillie calculus allows us to detect rational classes in the stable homotopy groups of the Eilenberg–MacLane space $K(\mathbb{Z}_p, n)$.

**Proposition 5.3.** For $n \geq 1$ and $k > 1$, the stable homotopy group $\pi_{nk} \Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not $p$-complete.

In fact, we also give a short alternative argument based on the integral homology of $K(\mathbb{Z}_p, n)$. 

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Conventions. Throughout this paper, $p$ will be a fixed prime number and $\mathbb{Z}_p$ denotes the $p$-adic integers. We say that a nilpotent group $N$ is bounded $p$-torsion if there exists an $m$ such that for all $x \in N$, we have $x^{p^m} = 1$. A graded nilpotent group $N_n$ is said to be of bounded $p$-torsion if $N_k$ is bounded $p$-torsion for each $k$; however, we do not require a uniform bound. Whenever we are in a graded context, we indicate the degree of an abelian group $A$ by square brackets, i.e., $A[n]$ refers to $A$ placed in degree $n$. If $X$ is a topological space, then $H_\ast(X;A)$ is the reduced homology of $X$ with coefficients in $A$. For a space or spectrum $X$, we write $\tau_{\leq n}X = \tau_{\leq n}X$ for the $n$-th Postnikov section of $X$ and $\tau_{\geq n+1}X = \tau_{\geq n}X$ for the fiber of the canonical map $X \to \tau_{\leq n}X$.

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2. Preliminaries on $p$-completion

We briefly recall the basic properties of $p$-completion for nilpotent groups, topological spaces, and spectra. With the exceptions of Lemma 2.2 and Proposition 2.4, this material is mostly taken from [BK72, Bou75, Bou79], and we refer to these sources as well as [HS99, MP12] for further references.

2.1. Algebraic $p$-completion for abelian groups. In general, the $p$-completion functor $M \mapsto \lim\limits_{\rightarrow} M/p^i M$ on the category of abelian groups is neither left nor right exact, so one studies its zeroth and first left derived functors $L_0$ and $L_1$, which may be expressed as $L_0 M = \text{Ext}_Z^1(\mathbb{Z}/p^\infty, M)$ and $L_1 M = \text{Hom}_{\mathbb{Z}}[\mathbb{Z}/p^\infty, M]$ by [BK72, Ch. VI, 2.1]. An abelian group $M$ is called derived $p$-complete (or $p$-complete or $L$-complete) if the natural completion map $M \to L_0 M$ is an isomorphism. For each abelian group $M$, the map $M \to L_0 M$ will then be the universal homomorphism from $M$ to a derived $p$-complete abelian group by [BK72, Ch. VI, 3.2]. We will denote the full subcategory of derived $p$-complete abelian groups by $C_p$.

Proposition 2.1. The category $C_p$ is a full abelian subcategory of $\text{Mod}_\mathbb{Z}$ closed under extensions and limits. Furthermore, for any $M \in \text{Mod}_\mathbb{Z}$ there is a short exact sequence

\[ 0 \longrightarrow \text{lim}^1 \text{Hom}_\mathbb{Z}([p], M) \longrightarrow L_0 M \longrightarrow \text{lim} \text{Hom}_\mathbb{Z}([p^i], M) \longrightarrow 0 \]

relating derived $p$-completion to ordinary $p$-completion.

Proof. This is essentially proven in [BK72, Ch. VI, 2.1], but can also be deduced as a special case of [HS99, Thms. A.2 and A.6].

We will later make use of the following observation.

Lemma 2.2. If $A \in C_p$ is torsion, then $A$ is bounded $p$-torsion.

Proof. We give two proofs, a conceptual one and an elementary argument. First, any derived $p$-complete group $A$ has $\text{Hom}_\mathbb{Z}([1/p], A) = 0 = \text{Ext}_Z^1([1/p], A)$ by [BK72, Ch. VI, 3.4], and hence $A$ satisfies $\text{Hom}_\mathbb{Z}([p], A) = 0 = \text{Ext}_Z^1([p], A)$ since $Q$ is a quotient of free $\mathbb{Z}[1/p]$-modules. Thus, $A$ is a cotorsion group with no nontrivial divisible subgroups, so the Baer–Fomin theorem [Bae36] implies that $A$ is a bounded $p$-torsion group.

Second, suppose that the conclusion of the lemma is false, i.e., that there exists a sequence $(a_i)_{i \in \mathbb{N}}$ of elements of $A$ such that the order of $a_i$ is $p^i$. Set $x_j = \sum_{i=0}^{j-1} a_{2i+1}p^i$, then the element
The lower central series of $N$ is a direct sum of a $p$-torsion, which contradicts the fact that $A \to \lim_j A/p^j$ is surjective, forcing $\lim_j A/p^j$ to be $p$-torsion. □

Remark 2.3. By a theorem of Prüfer, the conclusion of the lemma implies that $A$ must in fact be a direct sum of cyclic $p$-groups.

2.2. Algebraic $p$-completion for nilpotent groups. Recall from [BK72, Ch. VI, §2] that the notion of derived $p$-completion can be extended to nilpotent groups, as follows: If $X^p$ denotes the Bousfield–Kan $p$-completion of a nilpotent space $X$ as recalled in the next subsection, then we define the derived $p$-completion of the nilpotent group $N$ as $L_0N = \pi_1(K(N,1)^p)$ and $L_1N = \pi_2(K(N,1)^p)$. A nilpotent group $N$ is called derived $p$-complete if the completion map $N \to L_0N$ is an isomorphism; for each nilpotent group $N$, the map $N \to L_0N$ will then be the universal homomorphism from $N$ to a derived $p$-complete nilpotent group by [BK72, Ch. VI, 3.2]. We denote the category of derived $p$-complete nilpotent groups by $\mathcal{N}_p$.

The inclusion functor $C_p \to \mathcal{N}_p$ has a left adjoint given by taking a derived $p$-complete nilpotent group $N$ to the derived $p$-completion of its abelianization $L_0(N/[N,N])$. Note that the unit of this adjunction is surjective, i.e., for any derived $p$-complete nilpotent group $N$, the canonical map $N \to L_0(N/[N,N])$ is surjective. Indeed, since $L_0$ preserves epimorphisms of nilpotent groups, all maps in the following commutative diagram are surjective:

\[
\begin{array}{ccc}
N & \to & N/[N,N] \\
\downarrow & & \downarrow \\
L_0N & \to & L_0(N/[N,N]).
\end{array}
\]

We obtain the following generalization of Lemma 2.2:

**Proposition 2.4.** The following conditions are equivalent for $N \in \mathcal{N}_p$:

1. $N$ is torsion.
2. $L_0(N/[N,N])$ is torsion.
3. $N$ is bounded $p$-torsion.

**Proof.** The surjectivity of the map $N \to L_0(N/[N,N])$ observed above immediately gives the implication (1) ⇒ (2), while (3) ⇒ (1) is trivial.

Assume that $L_0(N/[N,N])$ is torsion and thus bounded $p$-torsion by Lemma 2.2. Consider the lower central series of $N$,

$N = \gamma_1N \supseteq \gamma_2N \supseteq \ldots \supseteq \gamma_mN = 1,$

with successive abelian quotients $Q_i(N) = \gamma_iN/\gamma_{i+1}N$. We claim that, for each $i \geq 1$, $Q_i(N)$ is a direct sum of a $p$-divisible group and a bounded $p$-torsion group. Indeed, we start with the abelianization $Q_1(N) = N/[N,N]$ of $N$. Lemma 3.7 in [BK72, Ch. VI] implies that the kernel of the completion map $Q_1(N) \to L_0Q_1(N)$ is $p$-divisible, so the claim holds for $Q_1(N)$. The general case follows from this, because $\bigoplus_{i \geq 1} Q_i(N)$ is generated as a Lie algebra by $Q_1(N)$. By [BK72, Ch. VI, 2.5], there is an exact sequence

\[
L_0Q_i(N) \longrightarrow L_0(N/\gamma_{i+1}N) \longrightarrow L_0(N/\gamma_iN) \longrightarrow 1
\]

for any $i \geq 1$. Using the previous claim, $L_0Q_i(N)$ is bounded $p$-torsion, so we see inductively that $L_0(N/\gamma_iN)$ is bounded $p$-torsion for all $i \geq 1$, hence (3) holds. □

Remark 2.5. The implication (1) ⇒ (3) in the previous proposition could also be proven more directly via the upper central series of $N$, whose quotients are known to be derived $p$-complete by [BK72, VI, 3.4(ii)], but this result would be insufficient for our later use.
2.3. Topological $p$-completion. In [BK72], Bousfield and Kan introduced the notion of $p$-completion for topological spaces, lifting the algebraic notion defined above to topology. In general, the $p$-completion of a space is difficult to describe, but the theory simplifies significantly for nilpotent spaces; in particular, in this case $p$-completion coincides with $HF_p$-localization [Bou75]. Furthermore, for nilpotent spaces with $F_p$-homology of finite type, $p$-completion can be identified with $p$-profinite completion due to Sullivan [Sul74]. Similarly, the category of spectra admits (at least) two notions of $p$-completion, given either by $HF_p$-localization or, the one we will use here, localization at the mod $p$ Moore spectrum $S^0/p$, see [Bou79]. The next result summarizes the relation between these constructions and lists their basic properties.

**Theorem 2.6** (Bousfield, Kan).

1. A nilpotent space $X$ is $p$-complete if and only if $\pi_n X$ is derived $p$-complete for all $n \in \mathbb{N}$. Moreover, the notions of $p$-completion and $HF_p$-localization coincide for nilpotent spaces.

2. A spectrum $X$ is $p$-complete if and only if $\pi_n X$ is derived $p$-complete for all $n \in \mathbb{Z}$. If $X$ is bounded below, then $X$ is $p$-complete if and only if $X$ is $HF_p$-local.

Moreover, if $X$ is a nilpotent space or spectrum, then there exists a splittable short exact sequence computing the unstable or stable homotopy groups of its $p$-completion, respectively:

$$0 \to L_0 \pi_n X \to \pi_n (X^p_n) \to L_1 \pi_{n-1} X \to 0$$

for any $n$, where $L_i (-) \cong \text{Ext}^{i-1}_Z(\mathbb{Z}/p^{\infty}, -)$ are the derived functors of $p$-completion.

3. Generalized Serre theory

The full subcategory $C_p$ of $\text{Mod}_\mathbb{Z}$ is not closed under subobjects or quotients, and thus does not form a Serre class in the usual sense. This necessitates a mild generalization of Serre’s mod $C$ theory which we develop in this section.

**Definition 3.1.** A weak Serre class is a full subcategory $C \subseteq \text{Mod}_\mathbb{Z}$ such that if

$$A_1 \to A_2 \to A_3 \to A_4 \to A_5$$

is an exact sequence in $\text{Mod}_\mathbb{Z}$ with $A_1, A_2, A_4, A_5 \in C$, then also $A_3 \in C$.

More explicitly, this means that $C \subseteq \text{Mod}_\mathbb{Z}$ is a full additive subcategory closed under kernels, cokernels, and extensions. It follows that $C$ is also closed under tensoring and $\text{Tor}_i^Z$ with respect to finitely generated abelian groups. For instance, any Serre subcategory of $\text{Mod}_\mathbb{Z}$ is a weak Serre class, but the converse does not hold. The main example of interest to us here is the category $C_p$ of derived $p$-complete abelian groups, see Proposition 2.1.

**Proposition 3.2.** Suppose $C$ is a weak Serre class. If $X$ is a bounded below spectrum, then the following two conditions are equivalent:

1. $\pi_n X \in C$ for all $n \in \mathbb{Z}$.

2. $H_n (X; \mathbb{Z}) \in C$ for all $n \in \mathbb{Z}$.

**Proof.** Assume the first condition holds; we will argue via the Postnikov tower $(\tau_{<n} X)$ of $X$. For simplicity, we will write $H_*(Y)$ for the integral homology of a spectrum $Y$ throughout this proof.

To start with, we need to show that $H_* (HA) \in C$ for $A \in C$. Using the isomorphisms $H_*(HA) \cong H_*(HZ; A)$, the universal coefficient theorem gives a short exact sequence

$$0 \to H_* (HZ) \otimes \mathbb{Z} A \to H_* (HA) \to \text{Tor}_1^\mathbb{Z} (H_{*-1} (HZ); A) \to 0.$$ 

In each degree, the integral Steenrod algebra $H_* (HZ)$ is finitely generated over $\mathbb{Z}$, as follows from Serre theory for the class of finitely generated abelian groups. Therefore, the outer terms of this sequence are in $C$. This shows $H_* (HA) \in C$ as well.
Given $n \in \mathbb{Z}$, we will now prove that $H_n(X) \in \mathcal{C}$. Since $H_n(\tau_{\geq n}X) = 0 = H_{n-1}(\tau_{\geq n}X)$ by connectivity, we see that $H_n(X) \cong H_n(\tau_{\leq n}X)$. This reduces the claim to proving that $H_*(\tau_{\leq n}X) \in \mathcal{C}$. This follows inductively, using the exact sequence

$$H_{n+1}(\tau_{\leq n}X) \rightarrow H_*(\Sigma^n H\pi_nX) \rightarrow H_*(\tau_{\leq n}X) \rightarrow H_{n-1}(\Sigma^n H\pi_nX)$$

associated to the fiber sequence $\Sigma^n H\pi_nX \rightarrow \tau_{\leq n}X \rightarrow \tau_{\leq n-1}X$. Since $H_k(H\pi_nX) \in \mathcal{C}$ for all $k \in \mathbb{Z}$, this gives the implication $(1) \Rightarrow (2)$.

For the converse, consider the convergent Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} \cong H_s(X; \pi_t S^0) \implies \pi_{s+t}X.$$

Since $\pi_t S^0$ is finitely generated over $\mathbb{Z}$ for each $t \in \mathbb{Z}$, $H_s(X; \pi_t S^0) \in \mathcal{C}$ for each bidegree $(s,t)$, hence $\pi_nX$ is also in $\mathcal{C}$ for all $n \in \mathbb{Z}$.

When applied to the weak Serre class $\mathcal{C}_p$, we obtain a homological characterization of $p$-completeness for bounded below spectra.

**Corollary 3.3.** For a bounded below spectrum $X$, the following conditions are equivalent:

1. $X$ is $p$-complete.
2. $\pi_nX$ is derived $p$-complete for all $n$.
3. $H_*(X; \mathbb{Z})$ is derived $p$-complete for all $n$.

**Proof.** The equivalence of (1) and (2) is the content of Theorem 2.6(2), while (2) is equivalent to (3) by Proposition 3.2.

We deduce that the integral homology of $p$-complete spaces is well-behaved in the stable range.

**Corollary 3.4.** Suppose $X$ is $p$-complete space. If $X$ is $n$-connected, then $H_k(X; \mathbb{Z})$ is derived $p$-complete for all $k \leq 2n$.

**Proof.** Since $\pi_k \Sigma^\infty X \cong \pi_k X$ for $k \leq 2n$ by the Freudenthal suspension theorem, Theorem 2.6 implies that $\pi_\tau_{\leq 2n} \Sigma^\infty X$ is derived $p$-complete in each degree, hence so is $H_*(\tau_{\leq 2n} \Sigma^\infty X; \mathbb{Z})$ by Corollary 3.3. We thus get that $H_k(X; \mathbb{Z}) \cong H_k(\Sigma^\infty X; \mathbb{Z}) \cong H_k(\tau_{\leq 2n} \Sigma^\infty X; \mathbb{Z})$ is derived $p$-complete for $k \leq 2n$.

**Corollary 3.5.** For a bounded below spectrum $X$, there exists a splittable short exact sequence computing the integral homology groups of its $p$-completion:

$$0 \rightarrow L_0 H_n(X; \mathbb{Z}) \rightarrow H_n(X^p; \mathbb{Z}) \rightarrow L_1 H_{n-1}(X; \mathbb{Z}) \rightarrow 0$$

for any $n$.

**Proof.** Since the spectrum $H\mathbb{Z} \wedge X^p$ is $p$-complete by Corollary 3.3, there is a canonical map $(H\mathbb{Z} \wedge X^p) \rightarrow H\mathbb{Z} \wedge X^p$, and this map must be an equivalence because it is an $H\mathbb{F}_p$-equivalence of $p$-complete bounded below spectra. Hence, the claim follows from Theorem 2.6.

From Corollary 3.5, we obtain the following description of the $p$-complete sphere spectrum as a Moore spectrum.

**Example 3.6.** There is a canonical equivalence $S^0_p \overset{\sim}{\rightarrow} M\mathbb{Z}_p$.

4. **The comparison**

In this section, we first study the relation between $p$-completion for spectra and spaces under the infinite loop space functor $\Omega^\infty$, and then prove our main theorem.
4.1. Infinite loop spaces. It is easy to deduce from Theorem 2.6 the following relation between unstable and stable $p$-completion under $\Omega^\infty$.

**Proposition 4.1.** For 0-connected spectra $X$ and $Y$, we have:

1. $X$ is $p$-complete if and only if $\Omega^\infty X$ is $p$-complete.
2. A map $f: X \to Y$ is an $HF^p$-equivalence if and only if $\Omega^\infty f$ is an $HF^p$-equivalence.
3. The canonical comparison map $(\Omega^\infty X)^\wedge_p \to \Omega^\infty (X^\wedge_p)$ is an equivalence.

**Proof.** Since $\pi_*\Omega^\infty X \cong \pi_* X$ and $\Omega^\infty X$ is nilpotent, the first claim is a direct consequence of Theorem 2.6. In order to prove (2), note that $f$ is an $HF^p$-equivalence if and only if the homotopy groups $\pi_* \text{cof}(f)$ of the cofiber of $f$ are uniquely $p$-divisible. This is equivalent to the statement that the $\mathbb{F}_p$-homology $H_*(\Omega^\infty \text{cof}(f); \mathbb{F}_p)$ is trivial. The Serre spectral sequence associated to the fiber sequence

$$\Omega^\infty X \xrightarrow{\Omega^\infty f} \Omega^\infty Y \xrightarrow{\Omega^\infty \text{cof}(f)}$$

thus shows that this happens if and only if $\Omega^\infty f$ is an $HF^p$-equivalence.

Statement (1) implies that $\Omega^\infty (X^\wedge_p)$ is $p$-complete, so the map $\Omega^\infty X \to \Omega^\infty (X^\wedge_p)$ factors canonically through $\phi: (\Omega^\infty X)^\wedge_p \to \Omega^\infty (X^\wedge_p)$, making the following diagram commute:

$$\Omega^\infty X \xrightarrow{(\Omega^\infty f)^\wedge_p} (\Omega^\infty X)^\wedge_p \xrightarrow{\phi} \Omega^\infty (X^\wedge_p).$$

By Statement (2), both the horizontal and the diagonal map are $HF^p$-equivalences, hence so is the vertical comparison map. □

**Remark 4.2.** Let $\Omega^\infty_0$ be the 0-component of $\Omega^\infty$. The last part of the proposition can be strengthened to an equivalence $(\Omega^\infty_0 X)^\wedge_p \to \Omega^\infty_0 (X^\wedge_p)$ for any connective spectrum $X$ such that $\pi_0 X$ does not contain any copies of $\mathbb{Z}/p^\infty$. To prove this directly, one may use the short exact sequences displayed at the end of Theorem 2.6.

4.2. Suspension spectra. We now turn to the comparison under $\Sigma^\infty$. In odd dimensions, the next result has also been observed in [BK72, Rem. VI.5.7], see also [MP12, Rem. 11.1.5].

**Lemma 4.3.** Let $n \geq 1$ and write $S^\wedge_p^n$ for the $p$-completion of $S^n$. There exists an uncountable rational vector space in $H_{2n}(S^\wedge_p^n; \mathbb{Z})$ which injects into $H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z})$ under the map $S^\wedge_p^n \to \tau_{\leq n} S^\wedge_p^n \cong K(\mathbb{Z}_p, n)$.

**Proof.** Consider the following segment of the Serre long exact sequence for the fibration $F \to S^\wedge_p^n \to K(\mathbb{Z}_p, n)$:

$$H_{2n}(F; \mathbb{Z}) \xrightarrow{\tau_{\leq n}} H_{2n}(S^\wedge_p^n; \mathbb{Z}) \xrightarrow{\tau_{\leq n}} H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z}) \xrightarrow{\tau_{\leq n}} H_{2n-1}(F; \mathbb{Z}) \xrightarrow{\tau_{\leq n}} \ldots.$$

Corollary 3.4 implies that $H_{2n}(F; \mathbb{Z})$ and $H_{2n-1}(F; \mathbb{Z})$ are derived $p$-complete. Recalling that $\text{Hom}_\mathbb{Z}(\mathbb{Q}, A) = 0 = \text{Ext}_A^1(\mathbb{Q}, A)$ whenever $A$ is derived $p$-complete, we see that the natural map $\text{Hom}_\mathbb{Z}(\mathbb{Q}, H_{2n}(S^\wedge_p^n; \mathbb{Z})) \to \text{Hom}_\mathbb{Z}(\mathbb{Q}, H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z}))$ is surjective. Thus, it will suffice to show that $H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z})$ contains an uncountable rational vector space, which will be verified in the homological proof of Proposition 5.3 below. □

Note that, because $H_*(S^\wedge_p^n; \mathbb{F}_p) \cong H_*(S^n; \mathbb{F}_p) \cong \mathbb{F}_p[n]$, an application of the universal coefficient theorem shows that $H_k(S^\wedge_p^n; \mathbb{Z})$ is rational for all $k > n$. 

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Lemma 4.4. Suppose $N$ is a derived $p$-complete nilpotent (abelian) group and $n = 1$ ($n \geq 1$). If $N$ is not bounded $p$-torsion, then there exists an element $x \in N$ of infinite order inducing a monomorphism $H_* (K(\mathbb{Z}_p, n); \mathbb{Q}) \to H_* (K(N, n); \mathbb{Q})$.

Proof. By assumption on $N$ and Proposition 2.4, $L_0 (N/[N,N])$ contains elements of infinite order. Let $\pi$ be such an element and let $x \in N$ be a lift of $\pi$. For the remainder of the proof we assume $n = 1$; the (easier) case $n \geq 2$ and $N$ abelian is proven similarly. The element $x$ induces a map

$$K(\mathbb{Z}_p, 1) \xrightarrow{} K(N, 1) \xrightarrow{} K(L_0 (N/[N,N]), 1)$$

such that the composite is injective on $\pi_1$. It follows that the rationalization $K(\mathbb{Z}_p, 1)_{\mathbb{Q}} \to K(L_0 (N/[N,N]), 1)_{\mathbb{Q}}$ of this map is split, hence the composite

$$H_*(K(\mathbb{Z}_p, 1); \mathbb{Q}) \xrightarrow{} H_*(K(N, 1); \mathbb{Q}) \xrightarrow{} H_*(K(L_0 (N/[N,N]), 1); \mathbb{Q})$$

is a split monomorphism, which implies the claim. $\square$

Proposition 4.5. If $X$ is a $p$-complete nilpotent space whose homotopy groups are not all bounded $p$-torsion, then the integral homology groups $H_*(X; \mathbb{Z})$ and the stable homotopy groups $\pi_* \Sigma^\infty X$ both contain an uncountable rational vector space.

Proof. Assume that $\pi_0 X$ is not all bounded $p$-torsion, and let $\pi_0 X$ be the lowest such group. It then follows from Lemma 4.4 that $\pi_0 X$ contains a class $x$ of infinite order inducing a monomorphism $H_* (K(\mathbb{Z}_p, n); \mathbb{Q}) \to H_* (K(\pi_0 X, n); \mathbb{Q})$. Since the map $\tau_{\geq n} X \to X$ is a rational homology equivalence, any rational subgroup of $H_n (\tau_{\geq n} X; \mathbb{Q})$ must map monomorphically to $H_n (X; \mathbb{Q})$, so it suffices to prove the homological claim for $\tau_{\geq n} X$. The element $x$ yields a map $S^n_p \to \tau_{\geq n} X$ such that the composite $S^n_p \to \tau_{\geq n} X \to K(\pi_0 X, n)$ factors as

$$\begin{array}{ccc}
\tau_{\geq n} X & \xrightarrow{} & \tau_{\leq n} \tau_{\geq n} X \\ & & \simeq K(\pi_0 X, n) \\
S^n_p & \xrightarrow{} & \tau_{\leq n} S^n_p \\
& & \simeq K(\mathbb{Z}_p, n).
\end{array}$$

It follows from Lemma 4.3 and the choice of $x$ that the induced homomorphism in homology

$$H_{2n} (S^n_p; \mathbb{Z}) \xrightarrow{} H_{2n} (\tau_{\geq n} X; \mathbb{Z}) \xrightarrow{} H_{2n} (K(\pi_0 X, n); \mathbb{Z})$$

maps an uncountable rational vector space monomorphically to $H_{2n} (K(\pi_0 X, n); \mathbb{Z})$, hence so does the map $H_{2n} (S^n_p; \mathbb{Z}) \to H_{2n} (\tau_{\geq n} X; \mathbb{Z})$. This verifies the claim about the integral homology of $X$.

Recall that, for any connective spectrum $Y$, the Hurewicz map $\pi_* Y \to H_* (Y; \mathbb{Z})$ has kernel and cokernel of bounded torsion in each degree. Indeed, the fiber sequence $Y \wedge \tau_{>0} S^0 \to Y \to Y \wedge H\mathbb{Z}$ reduces this claim to showing that $\pi_* (Y \wedge \tau_{>0} S^0)$ is bounded torsion in each degree. This follows from the convergent Atiyah–Hirzebruch spectral sequence

$$H_* (Y; \pi_* \tau_{>0} S^0) \Rightarrow \pi_{*+t} (Y \wedge \tau_{>0} S^0),$$

because $H_* (Y; \pi_* \tau_{>0} S^0)$ is bounded torsion for all $s$ and $t$. Therefore, any rational vector space in $H_* (Y; \mathbb{Z})$ may be lifted back to $\pi_* Y$. In particular, an uncountable rational vector space in $H_{2n} (X; \mathbb{Z})$ may be lifted back to $\pi_{2n} (\Sigma^\infty X)$ after suspension. $\square$

Remark 4.6. Suppose $X$ is a $p$-complete nilpotent space such that $\pi_0 X$ is the lowest homotopy group not of bounded $p$-torsion. The above argument shows that $H_{2n} (X; \mathbb{Z})$ contains an uncountable rational vector space. With more work, we can also show that $H_k (X; \mathbb{Z})$ is derived $p$-complete for $k \leq 2n - 2$ and thus cannot contain any rational classes. Note that when $X$ is
(n − 1)-connected, this follows immediately from Corollary 3.4 since \( H_k(X; \mathbb{Z}) \) is in the stable range.

We can now prove our main theorem.

**Theorem 4.7.** If \( X \) is a \( p \)-complete nilpotent space, then \( \Sigma^\infty X \) is \( p \)-complete if and only if \( \pi_n X \) is \( p \)-torsion for each \( n \).

Note that the torsion exponent of \( \pi_n X \) may vary with \( n \) and does not need to be bounded uniformly for all \( n \).

**Proof.** First assume that \( X \) is a \( p \)-complete nilpotent space with \( \pi_n X \) of bounded \( p \)-torsion for each \( n \); we can apply [BK72, Ch. II, 4.7] to see that the Postnikov tower of \( X \) can be refined to a tower of principal fibrations whose fibers are Eilenberg–MacLane spaces for bounded \( p \)-torsion abelian groups. The category of bounded \( p \)-torsion abelian groups forms a Serre class, so Serre theory implies that \( H_*(X; \mathbb{Z}) \cong H_*(\Sigma^\infty X; \mathbb{Z}) \) is degreewise bounded \( p \)-torsion. Hence, \( \Sigma^\infty X \) is \( p \)-complete as a spectrum by Corollary 3.3.

The converse is a consequence of Proposition 4.5: if \( \pi_n X \) is not all bounded torsion, then \( H_*(\Sigma^\infty X; \mathbb{Z}) \) contains rational classes and thus cannot be derived \( p \)-complete, hence \( \Sigma^\infty X \) is not \( p \)-complete by Corollary 3.3. \( \square \)

The next result generalizes [PSS17, Prop. 2.4].

**Corollary 4.8.** If \( X \) is a pointed connected space with degreewise finite homotopy groups, then the canonical map \( (\Sigma^\infty X)_p^\wedge \to \Sigma^\infty X_p^\wedge \) is an equivalence.

**Proof.** By [BK72, Ch. VII, 4.3], \( X \) is a \( \mathbb{Z}/p \)-good space and \( X_p^\wedge \) is a \( p \)-complete nilpotent space whose homotopy groups are all finite \( p \)-groups. Hence \( \Sigma^\infty X_p^\wedge \) is \( p \)-complete by Theorem 4.7. It follows that the natural map \( (\Sigma^\infty X)_p^\wedge \to \Sigma^\infty X_p^\wedge \) is an \( HF_p \)-equivalence between \( HF_p \)-local spectra, which implies the claim. \( \square \)

**Corollary 4.9.** If \( X \) is a nilpotent space with \( H_n(X; \mathbb{Z}) \) and \( \pi_n X \) derived \( p \)-complete for all \( n \), then \( H_n(X; \mathbb{Z}) \) and \( \pi_n X \) are bounded \( p \)-torsion for all \( n \).

**Proof.** The assumption on \( \pi_n X \) implies that \( X \) is \( p \)-complete by Theorem 2.6, while the assumption on \( H_*(X; \mathbb{Z}) \) shows that \( \Sigma^\infty X \) is \( p \)-complete, using Corollary 3.3. It thus follows from Theorem 4.7 that \( \pi_n X \) is degreewise bounded \( p \)-torsion, hence so is \( H_*(X; \mathbb{Z}) \) by the proof of Theorem 4.7. \( \square \)

The analogue of this corollary does not hold stably, as the following example demonstrates.

**Example 4.10.** Let \( M(\mathbb{Z}_p, n) \) be the Moore space for \( \mathbb{Z}_p \) in degree \( n \geq 2 \). As \( H_*(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}_p[n] \), we see that \( \Sigma^\infty M(\mathbb{Z}_p, n) \) is \( p \)-complete and consequently has derived \( p \)-complete stable homotopy groups and integral homology groups. However, \( H_n(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z}) \equiv \mathbb{Z}_p \) is clearly not bounded \( p \)-torsion. In particular, \( M(\mathbb{Z}_p, n) \) is not \( p \)-complete, so this also shows that the assumption that \( X \) be \( p \)-complete cannot be dropped in Theorem 4.7.

5. **Rational classes in the stable homotopy groups of \( K(\mathbb{Z}_p, n) \)**

In this section, we present an example that illustrates how the rational classes in the stable homotopy groups of \( p \)-complete spaces arise. In fact, we present two different approaches: One using the integral homology of \( K(\mathbb{Z}_p, n) \), and one using Goodwillie calculus. The latter derivation is entirely stable and might be of independent interest.

First, we need a well-known auxiliary result; we outline a proof because we were unable to find a published reference for it. For an abelian group \( A \) and any \( k \geq 0 \), let \( \text{Sym}^k_\mathbb{Z}(A) \) and \( \Lambda^k_\mathbb{Z}(A) \) be the \( k \)th symmetric power and the \( k \)th exterior power on \( A \), respectively.
Lemma 5.1. If \( k > 1 \), then \( A_n^k(\mathbb{Z}_p) \) and the kernel of the multiplication map \( \text{Sym}_2^k(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p \) are uncountable rational vector spaces.

**Proof.** Since both symmetric and exterior power commute with base-change along \( \mathbb{Z} \rightarrow \mathbb{Z}/l \) for any prime \( l \), the indicated maps are isomorphisms mod \( l \). Moreover, \( \text{Sym}_2^k(A) \) and \( A_n^k(A) \) are torsion-free whenever \( A \) is, so both \( \ker(\text{Sym}_2^k(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p) \) and \( A_n^k(\mathbb{Z}_p) \) are rational vector spaces. We may therefore base-change to \( \mathbb{Q} \), where it is easy to verify that the \( \mathbb{Q} \)-dimension of the groups under consideration is that of \( \mathbb{Q}_p \). \( \square \)

**Remark 5.2.** A similar argument also shows that \( \mathbb{Z}_p/\mathbb{Z}(p) \) is a rational vector space with the same \( \mathbb{Q} \)-dimension as \( \mathbb{Q}_p \).

**Proposition 5.3.** For \( n \geq 1 \) and all \( k > 1 \), the stable homotopy group \( \pi_{nk} \Sigma^\infty K(\mathbb{Z}_p, n) \) contains an uncountable rational vector space. In particular, \( \Sigma^\infty K(\mathbb{Z}_p, n) \) is not \( p \)-complete.

**First proof.** Let \( A \) be an abelian group and recall that \( H_*(K(A, n); \mathbb{Z}) \) equipped with the Pontryagin product is a graded commutative algebra such that squares of odd dimensional elements are zero; in fact, it has the structure of a graded divided power algebra, see [EML54, Car56] or more recently [Ric09]. With notation as in the previous lemma, the canonical isomorphism \( A \rightarrow H_n(K(A, n); \mathbb{Z}) \) thus extends to a natural homomorphism

\[
\begin{align*}
\phi^k(A, n) \colon A_n^k(A) & \xrightarrow{\cong} H_{kn}(K(A, n); \mathbb{Z}), & \text{if } n \text{ odd} \\
\phi^k(A, n) \colon \text{Sym}_2^k(A) & \xrightarrow{\cong} H_{kn}(K(A, n); \mathbb{Z}), & \text{if } n \text{ even}
\end{align*}
\]

for any \( n, k > 0 \). Moreover, we know that \( \phi^k(A, n) \otimes \mathbb{Q} \) is a rational isomorphism. It then follows from Lemma 5.1 that, for \( k > 1 \), there exists an uncountable rational vector space which is mapped monomorphically to \( H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z}) \) via \( \phi^k(\mathbb{Z}_p, n) \). We thus obtain an uncountable rational vector space in \( H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z}) \) that may be lifted back to give the desired uncountable rational vector space in \( \pi_{nk} \Sigma^\infty K(\mathbb{Z}_p, n) \) for \( k > 1 \), as in the proof of Proposition 4.5. \( \square \)

**Second proof.** We will compute the homotopy groups of \( \Sigma^\infty K(\mathbb{Z}_p, n) \simeq \Sigma^\infty \Omega^\infty \Sigma^n \mathbb{H}\mathbb{Z}_p \) using Goodwillie calculus [Goo03]. To this end, recall that the Goodwillie tower \( (P_k)_{k \geq 1} \) associated to the functor \( \Sigma^\infty \Omega^\infty \colon \text{Sp} \rightarrow \text{Sp} \) is assembled from fiber sequences of functors

\[
D_k \xrightarrow{\delta_k} P_k \xrightarrow{\delta_{k-1}} P_{k-1}
\]

with layers \( D_kX \cong X^{\wedge k}_{\Sigma k} \), where the homotopy orbits are formed with respect to the permutation action of \( \Sigma k \) (see for example [KM13] and the references given therein). Moreover, the Goodwillie tower \( (P_k)_{k \geq 0} \) converges for connective spectra, i.e., there is a canonical equivalence

\[
\Sigma^\infty \Omega^\infty X \xrightarrow{\sim} \text{lim}_k P_k X
\]

for any connective \( X \in \text{Sp} \). We will apply this in the case \( X = \Sigma^n \mathbb{H}\mathbb{Z}_p \).

In order to understand the layers, we start by analyzing \( \pi_*(\Sigma^n \mathbb{H}\mathbb{Z}_p)^{\wedge k} \) via the universal coefficient theorem. We claim that, for all \( k \geq 1 \), the homotopy groups have the following form

\[
\pi_*(\Sigma^n \mathbb{H}\mathbb{Z}_p)^{\wedge k} \cong \begin{cases} 0 & * < nk \\ \mathbb{Z} \otimes_{\mathbb{Z}_p}^{* = nk} \\ \text{finite} & * > nk \end{cases}
\]

By the universal coefficient theorem, we have an isomorphism

\[
\pi_*(\Sigma^n \mathbb{H}\mathbb{Z}_p)^{\wedge k} \cong (\pi_*(\Sigma^n \mathbb{H}Z)^{\wedge k}) \otimes_{\mathbb{Z}} \mathbb{Z}_p^{\otimes k}.
\]

In degrees \( * > nk \), the groups \( \pi_*(\Sigma^n \mathbb{H}Z)^{\wedge k} \) are torsion, so the only torsion-free summand appears in degree \( nk \). Since \( \pi_*(\Sigma^n \mathbb{H}Z)^{\wedge k} \) is finitely generated over \( \mathbb{Z} \) in each degree, the claim follows.
We now plug the formula (5.5) into the convergent homotopy orbit spectral sequence

\[ H_s(\Sigma_k, \pi_l(\Sigma^n H\mathbb{Z}_p)^{\wedge k}) \to \pi_{s+l} D_k(\Sigma^n H\mathbb{Z}_p). \]

There are two cases: If \( t > nk \) or \( t < nk \), then the groups \( H_s(\Sigma_k, \pi_l(\Sigma^n H\mathbb{Z}_p)^{\wedge k}) \) are finite or trivial for all \( s \), respectively. Let \( t = nk \). By Lemma 5.1 and (5.5), there is an isomorphism \( H_s(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^{\wedge k}) \cong H_s(\Sigma_k, \mathbb{Z}_p) \) for \( s > 0 \) and \( H_0(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^{\wedge k}) \) contains an uncountable rational vector space \( V_k \) if \( k > 1 \). To see the last statement, it suffices to compute the coinvariants on the rational submodule of \( \pi_{nk}\mathbb{Z}_p \) by choosing a \( \mathbb{Q} \)-bases, as in the proof of Lemma 5.1. Furthermore, since the integral homology of \( \Sigma_k \) is finitely generated over \( \mathbb{Z} \) in each degree and rationally trivial in positive degrees, \( H_s(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^{\wedge k}) \) is finite for all \( s > 0 \). Combining all this information, we obtain \( D_1 \Sigma^n H\mathbb{Z}_p \simeq \Sigma^n H\mathbb{Z}_p \) and for \( k > 1 \):

\[
\pi_* D_k(\Sigma^n H\mathbb{Z}_p) \cong \begin{cases} 
0 & * < nk \\
V_k \oplus W_k & * = nk \\
\text{finite} & * > nk,
\end{cases}
\]  

(5.6)

where \( V_k \) is an uncountable rational vector space and \( W_k \) is some abelian group.

This allows us to derive a structural formula for \( \pi_* P_k \Sigma^n H\mathbb{Z}_p \). Consider the following segment of the long exact sequence of homotopy groups associated to the fiber sequence (5.4):

\[
\ldots \to \pi_{nk+1} P_{k-1} \Sigma^n H\mathbb{Z}_p \to \pi_{nk} D_k \Sigma^n H\mathbb{Z}_p \to \pi_{nk} P_k \Sigma^n H\mathbb{Z}_p \to \ldots.
\]

Because \( n \geq 1 \), it follows inductively from (5.6) that the term on the left is finite, hence \( V_k \) must be a summand in \( \pi_{nk} P_k \Sigma^n H\mathbb{Z}_p \). This yields for all \( k \geq 1 \):

\[
\pi_* P_k \Sigma^n H\mathbb{Z}_p \cong \begin{cases} 
0 & * < n \\
V_l \oplus W'_l & * = nl \text{ with } 1 \leq l \leq k \\
\text{finite} & \text{otherwise},
\end{cases}
\]  

(5.7)

where \( V_l \) is as above for \( l \geq 2 \), and \( V_1 \) and \( W'_1 \) are some abelian groups.

Finally, since \( D_k \Sigma^n H\mathbb{Z}_p \) is \( nk \)-connective for all \( k \), the tower \( (\pi_* P_k \Sigma^n H\mathbb{Z}_p)_{k \geq 0} \) stabilizes after finally many steps in each degree and hence is Mittag-Leffler. The corresponding Milnor sequence thus degenerates to an isomorphism

\[
\pi_* \Sigma^\infty K(\mathbb{Z}_p, n) \cong \pi_* \Sigma^\infty \Omega^\infty \Sigma^n H\mathbb{Z}_p \cong \lim_k \pi_* P_k \Sigma^n H\mathbb{Z}_p.
\]

Therefore, the claim follows from (5.7). \( \square \)

References


