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ON THE COMPARISON OF STABLE AND UNSTABLE $p$-COMPLETION

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Abstract. In this note we show that a $p$-complete nilpotent space $X$ has a $p$-complete suspension spectrum if and only if its homotopy groups $\pi_* X$ are bounded $p$-torsion. In contrast, if $\pi_* X$ is not all bounded $p$-torsion, we locate uncountable rational vector spaces in the integral homology and in the stable homotopy groups of $X$. To prove this, we establish a homological criterion for $p$-completeness of connective spectra. Moreover, we illustrate our results by studying the stable homotopy groups of $K(\mathbb{Z}_p, n)$ via Goodwillie calculus.

1. Introduction

The notion of $p$-completion plays a fundamental role in algebra and topology, for it provides effective means to isolate and study $p$-primary properties. Applied to homotopy theory by Bousfield and Kan [BK72] as well as Sullivan [Sul74] and developed further in [Bou75, Bou79], it has since become one of the standard tools in the hands of algebraic topologists. However, there appears to be no general account of the comparison between unstable and stable $p$-completion in the literature, which is the question we address in the present note.

Our main goal is to characterize $p$-complete spaces which have $p$-complete suspension spectra:

**Theorem 4.7.** If $X$ is a $p$-complete nilpotent space, then $\Sigma^\infty X$ is $p$-complete if and only if $\pi_n X$ is bounded $p$-torsion for each $n$.

In fact, we exhibit a sharp dichotomy of $p$-complete nilpotent spaces: if $X$ is a $p$-complete nilpotent space whose homotopy groups are not all bounded $p$-torsion, then the integral homology and stable homotopy groups of $X$ both contain an uncountable rational vector space. As a consequence, we deduce that a nilpotent space $X$ with derived $p$-complete integral homology and unstable homotopy must have both $H_*(X; \mathbb{Z})$ and $\pi_* X$ of bounded $p$-torsion for all $n$.

In a first step towards the proof of the theorem, we complement the second author’s characterization of $p$-complete spectra in terms of homotopy groups with an integral homological criterion, using a mild generalization of Serre classes appropriate for stable homotopy theory. This is in sharp contrast to the aforementioned fact that the integral homology of $p$-complete spaces is not well-behaved, and thus cannot be used to characterize $p$-completeness of spaces.

**Corollary 3.3.** A bounded below spectrum $X$ is $p$-complete if and only if $H_*(X; \mathbb{Z})$ is derived $p$-complete in each degree.

In order to use this result to prove the theorem, we need to detect rational classes in the homology of $p$-complete spaces whose homotopy is not bounded $p$-torsion. This rests on the study of the integral homology of $p$-complete spheres. We end this note with a sample computation, illustrating how Goodwillie calculus allows us to detect rational classes in the stable homotopy groups of the Eilenberg–MacLane space $K(\mathbb{Z}_p, n)$.

**Proposition 5.3.** For $n \geq 1$ and $k > 1$, the stable homotopy group $\pi_{nk} \Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not $p$-complete.

In fact, we also give a short alternative argument based on the integral homology of $K(\mathbb{Z}_p, n)$.

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Conventions. Throughout this paper, \( p \) will be a fixed prime number and \( \mathbb{Z}_p \) denotes the \( p \)-adic integers. We say that a nilpotent group \( N \) is bounded \( p \)-torsion if there exists an \( m \) such that for all \( x \in N \), we have \( x^{p^m} = 1 \). A graded nilpotent group \( N_* \) is said to be of bounded \( p \)-torsion if \( N_k \) is bounded \( p \)-torsion for each \( k \); however, we do not require a uniform bound. Whenever we are in a graded context, we indicate the degree of an abelian group \( A \) by square brackets, i.e., \( A[n] \) refers to \( A \) placed in degree \( n \). If \( X \) is a topological space, then \( H_* (X; A) \) is the reduced homology of \( X \) with coefficients in \( A \). For a space or spectrum \( X \), we write \( \tau_{\leq n} X = \tau_{\leq n+1} X \) for the \( n \)-th Postnikov section of \( X \) and \( \tau_{\geq n+1} X = \tau_{> n} X \) for the fiber of the canonical map \( X \to \tau_{\leq n} X \).

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2. Preliminaries on \( p \)-completion

We briefly recall the basic properties of \( p \)-completion for nilpotent groups, topological spaces, and spectra. With the exceptions of Lemma 2.2 and Proposition 2.4, this material is mostly taken from [BK72, Bou75, Bou79], and we refer to these sources as well as [HS99, MP12] for further references.

2.1. Algebraic \( p \)-completion for abelian groups. In general, the \( p \)-completion functor \( M \mapsto \lim_{\leftarrow} M/p^i M \) on the category of abelian groups is neither left nor right exact, so one studies its zeroth and first left derived functors \( L_0 \) and \( L_1 \), which may be expressed as \( L_0 M = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M) \) and \( L_1 M = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M) \) by [BK72, Ch. VI, 2.1]. An abelian group \( M \) is called derived \( p \)-complete (or \( \text{Ext} \)-complete or \( L \)-complete) if the natural completion map \( M \to L_0 M \) is an isomorphism. For each abelian group \( M \), the map \( M \to L_0 M \) will then be the universal homomorphism from \( M \) to a derived \( p \)-complete abelian group by [BK72, Ch. VI, 3.2]. We will denote the full subcategory of derived \( p \)-complete abelian groups by \( \mathcal{C}_p \).

Proposition 2.1. The category \( \mathcal{C}_p \) is a full abelian subcategory of \( \text{Mod}_{\mathbb{Z}} \) closed under extensions and limits. Furthermore, for any \( M \in \text{Mod}_{\mathbb{Z}} \) there is a short exact sequence

\[
0 \longrightarrow \lim^1 \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^i, M) \longrightarrow L_0 M \longrightarrow \lim M/p^i M \longrightarrow 0
\]

relating derived \( p \)-completion to ordinary \( p \)-completion.

Proof. This is essentially proven in [BK72, Ch. VI, 2.1], but can also be deduced as a special case of [HS99, Thms. A.2 and A.6].

We will later make use of the following observation.

Lemma 2.2. If \( A \in \mathcal{C}_p \) is torsion, then \( A \) is bounded \( p \)-torsion.

Proof. We give two proofs, a conceptual one and an elementary argument. First, any derived \( p \)-complete group \( A \) has \( \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/[1/p], A) = 0 = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/[1/p], A) \) by [BK72, Ch. VI, 3.4], and hence \( A \) satisfies \( \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, A) = 0 = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, A) \) since \( \mathbb{Q} \) is a quotient of free \( \mathbb{Z}/[1/p] \)-modules. Thus, \( A \) is a cotorsion group with no nontrivial divisible subgroups, so the Baer–Fomin theorem [Bae36] implies that \( A \) is a bounded \( p \)-torsion group.

Second, suppose that the conclusion of the lemma is false, i.e., that there exists a sequence \( (a_i)_{i \in \mathbb{N}} \) of elements of \( A \) such that the order of \( a_i \) is \( p^i \). Set \( x_j = \sum_{i=0}^{j-1} a_{2i+1} p^i \), then the element
The lower central series of $N$ of the completion map $Q$ is a direct sum of a $p$-torsion group, which contradicts the fact that $A \to \lim_j A/p^j$ is surjective, forcing $\lim_j A/p^j$ to be $p$-torsion.

**Remark 2.3.** By a theorem of Prüfer, the conclusion of the lemma implies that $A$ must in fact be a direct sum of cyclic $p$-groups.

### 2.2. Algebraic $p$-completion for nilpotent groups

Recall from [BK72, Ch. VI, §2] that the notion of derived $p$-completion can be extended to nilpotent groups, as follows: If $X^p$ denotes the Bousfield–Kan $p$-completion of a nilpotent space $X$ as recalled in the next subsection, then we define the derived $p$-completion of the nilpotent group $N$ as $L_0 N = \pi_1 (K(N,1)^p)$ and $L_1 N = \pi_2 (K(N,1)^p)$. A nilpotent group $N$ is called derived $p$-complete if the completion map $N \to L_0 N$ is an isomorphism; for each nilpotent group $N$, the map $N \to L_0 N$ will then be the universal homomorphism from $N$ to a derived $p$-complete nilpotent group by [BK72, Ch. VI, 3.2]. We denote the category of derived $p$-complete nilpotent groups by $\mathcal{N}_p$.

The inclusion functor $C_p \to \mathcal{N}_p$ has a left adjoint given by taking a derived $p$-complete nilpotent group $N$ to the derived $p$-completion of its abelianization $L_0 (N/[N,N])$. Note that the unit of this adjunction is surjective, i.e., for any derived $p$-complete nilpotent group $N$, the canonical map $N \to L_0 (N/[N,N])$ is surjective. Indeed, since $L_0$ preserves epimorphisms of nilpotent groups, all maps in the following commutative diagram are surjective:

\[
\begin{array}{ccc}
N & \to & N/[N,N] \\
\downarrow & & \downarrow \\
L_0 N & \to & L_0 (N/[N,N]).
\end{array}
\]

We obtain the following generalization of Lemma 2.2:

**Proposition 2.4.** The following conditions are equivalent for $N \in \mathcal{N}_p$:

1. $N$ is torsion.
2. $L_0 (N/[N,N])$ is torsion.
3. $N$ is bounded $p$-torsion.

**Proof.** The surjectivity of the map $N \to L_0 (N/[N,N])$ observed above immediately gives the implication (1) $\Rightarrow$ (2), while (3) $\Rightarrow$ (1) is trivial.

Assume that $L_0 (N/[N,N])$ is torsion and thus bounded $p$-torsion by Lemma 2.2. Consider the lower central series of $N$,

\[N = \gamma_1 N \supseteq \gamma_2 N \supseteq \ldots \supseteq \gamma_m N = 1,\]

with successive abelian quotients $Q_i(N) = \gamma_i N/\gamma_{i+1} N$. We claim that, for each $i \geq 1$, $Q_i(N)$ is a direct sum of a $p$-divisible group and a bounded $p$-torsion group. Indeed, we start with the abelianization $Q_1(N) = N/[N,N]$ of $N$. Lemma 3.7 in [BK72, Ch. VI] implies that the kernel of the completion map $Q_1(N) \to L_0 Q_1(N)$ is $p$-divisible, so the claim holds for $Q_1(N)$. The general case follows from this, because $\bigoplus_{i \geq 1} Q_i(N)$ is generated as a Lie algebra by $Q_1(N)$. By [BK72, Ch. VI, 2.5], there is an exact sequence

\[L_0 Q_i(N) \to L_0 (N/\gamma_{i+1} N) \to L_0 (N/\gamma_i N) \to 1\]

for any $i \geq 1$. Using the previous claim, $L_0 Q_i(N)$ is bounded $p$-torsion, so we see inductively that $L_0 (N/\gamma_i N)$ is bounded $p$-torsion for all $i \geq 1$, hence (3) holds.

**Remark 2.5.** The implication (1) $\Rightarrow$ (3) in the previous proposition could also be proven more directly via the upper central series of $N$, whose quotients are known to be derived $p$-complete by [BK72, VI, 3.4(ii)], but this result would be insufficient for our later use.
2.3. Topological \( p \)-completion. In [BK72], Bousfield and Kan introduced the notion of \( p \)-completion for topological spaces, lifting the algebraic notion defined above to topology. In general, the \( p \)-completion of a space is difficult to describe, but the theory simplifies significantly for nilpotent spaces; in particular, in this case \( p \)-completion coincides with \( HP_p \)-localization \[Bou75\]. Furthermore, for nilpotent spaces with \( \mathbb{F}_p \)-homology of finite type, \( p \)-completion can be identified with \( p \)-profinite completion due to Sullivan \[Sul74\]. Similarly, the category of spectra admits (at least) two notions of \( p \)-completion, given either by \( HP_p \)-localization or, the one we will use here, localization at the mod \( p \) Moore spectrum \( S^0/p \), see \[Bou79\]. The next result summarizes the relation between these constructions and lists their basic properties.

**Theorem 2.6** (Bousfield, Kan).

(1) A nilpotent space \( X \) is \( p \)-complete if and only if \( \pi_n X \) is derived \( p \)-complete for all \( n \in \mathbb{N} \).

Moreover, the notions of \( p \)-completion and \( HP_p \)-localization coincide for nilpotent spaces.

(2) A spectrum \( X \) is \( p \)-complete if and only if \( \pi_n X \) is derived \( p \)-complete for all \( n \in \mathbb{Z} \). If \( X \) is bounded below, then \( X \) is \( p \)-complete if and only if \( X \) is \( HP_p \)-local.

Moreover, if \( X \) is a nilpotent space or spectrum, then there exists a splittable short exact sequence computing the unstable or stable homotopy groups of its \( p \)-completion, respectively:

\[
0 \longrightarrow L_0 \pi_n X \longrightarrow \pi_n(X_p^\infty) \longrightarrow L_1 \pi_{n-1} X \longrightarrow 0
\]

for any \( n \), where \( L_i (-) \cong \text{Ext}^{i-1}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, -) \) are the derived functors of \( p \)-completion.

3. Generalized Serre theory

The full subcategory \( C_p \) of \( \text{Mod}_{\mathbb{Z}} \) is not closed under subobjects or quotients, and thus does not form a Serre class in the usual sense. This necessitates a mild generalization of Serre’s mod \( C \) theory which we develop in this section.

**Definition 3.1.** A weak Serre class is a full subcategory \( C \subseteq \text{Mod}_{\mathbb{Z}} \) such that if

\[
A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5
\]

is an exact sequence in \( \text{Mod}_{\mathbb{Z}} \) with \( A_1, A_2, A_4, A_5 \in C \), then also \( A_3 \in C \).

More explicitly, this means that \( C \subseteq \text{Mod}_{\mathbb{Z}} \) is a full additive subcategory closed under kernels, cokernels, and extensions. It follows that \( C \) is also closed under tensoring and \( \text{Tor}^2 \) with respect to finitely generated abelian groups. For instance, any Serre subcategory of \( \text{Mod}_{\mathbb{Z}} \) is a weak Serre class, but the converse does not hold. The main example of interest to us here is the category \( C_p \) of derived \( p \)-complete abelian groups, see Proposition 2.1.

**Proposition 3.2.** Suppose \( C \) is a weak Serre class. If \( X \) is a bounded below spectrum, then the following two conditions are equivalent:

(1) \( \pi_n X \in C \) for all \( n \in \mathbb{Z} \).

(2) \( H_n(X; \mathbb{Z}) \in C \) for all \( n \in \mathbb{Z} \).

**Proof.** Assume the first condition holds; we will argue via the Postnikov tower \( \tau_{\leq n} X \) of \( X \). For simplicity, we will write \( H_*(-) \) for the integral homology of a spectrum \( Y \) throughout this proof.

To start with, we need to show that \( H_*(HA) \in C \) for \( A \in C \). Using the isomorphisms \( H_*(HA) \cong H_*(HZ; A) \), the universal coefficient theorem gives a short exact sequence

\[
0 \longrightarrow H_*(HZ) \otimes A \longrightarrow H_*(HA) \longrightarrow \text{Tor}^2(H_{*-1}(HZ), A) \longrightarrow 0.
\]

In each degree, the integral Steenrod algebra \( H_*(HZ) \) is finitely generated over \( \mathbb{Z} \), as follows from Serre theory for the class of finitely generated abelian groups. Therefore, the outer terms of this sequence are in \( C \). This shows \( H_*(HA) \in C \) as well.
Given $n \in \mathbb{Z}$, we will now prove that $H_n(X) \in \mathcal{C}$. Since $H_n(\tau_{\geq n}X) = 0 = H_{n-1}(\tau_{\geq n}X)$ by connectivity, we see that $H_n(X) \cong H_n(\tau_{\leq n}X)$. This reduces the claim to proving that $H_*(\tau_{\leq n}X) \in \mathcal{C}$. This follows inductively, using the exact sequence

$$H_{n+1}(\tau_{\leq n-1}X) \longrightarrow H_*(\Sigma^n H\pi_nX) \longrightarrow H_*(\tau_{\leq n}X) \longrightarrow H_*(\tau_{\leq n-1}X) \longrightarrow H_{n-1}(\Sigma^n H\pi_nX)$$

associated to the fiber sequence $\Sigma^n H(\pi_nX) \to \tau_{\leq n}X \to \tau_{\leq n-1}X$. Since $H_k(H\pi_nX) \in \mathcal{C}$ for all $k \in \mathbb{Z}$, this gives the implication $(1) \Rightarrow (2)$.

For the converse, consider the convergent Atiyah–Hirzebruch spectral sequence

$$E^2_{t+1} \cong H_*(X; \pi_tS^0) \Rightarrow \pi_{t+1}X.$$

Since $\pi_tS^0$ is finitely generated over $\mathbb{Z}$ for each $t \in \mathbb{Z}$, $H_*(X; \pi_tS^0) \in \mathcal{C}$ for each bidegree $(s, t)$, hence $\pi_nX$ is also in $\mathcal{C}$ for all $n \in \mathbb{Z}$. $\square$

When applied to the weak Serre class $\mathcal{C}_p$, we obtain a homological characterization of $p$-completeness for bounded below spectra.

**Corollary 3.3.** For a bounded below spectrum $X$, the following conditions are equivalent:

1. $X$ is $p$-complete.
2. $\pi_nX$ is derived $p$-complete for all $n$.
3. $H_n(X; \mathbb{Z})$ is derived $p$-complete for all $n$.

**Proof.** The equivalence of (1) and (2) is the content of Theorem 2.6(2), while (2) is equivalent to (3) by Proposition 3.2. $\square$

We deduce that the integral homology of $p$-complete spaces is well-behaved in the stable range.

**Corollary 3.4.** Suppose $X$ is $p$-complete space. If $X$ is $n$-connected, then $H_k(X; \mathbb{Z})$ is derived $p$-complete for all $k \leq 2n$.

**Proof.** Since $\pi_k \Sigma^\infty X \cong \pi_k X$ for $k \leq 2n$ by the Freudenthal suspension theorem, Theorem 2.6 implies that $\pi\tau_{\leq 2n} \Sigma^\infty X$ is derived $p$-complete in each degree, hence so is $H_*(\tau_{\leq 2n} \Sigma^\infty X; \mathbb{Z})$ by Corollary 3.3. We thus get that $H_k(X; \mathbb{Z}) \cong H_k(\Sigma^\infty X; \mathbb{Z}) \cong H_k(\tau_{\leq 2n} \Sigma^\infty X; \mathbb{Z})$ is derived $p$-complete for $k \leq 2n$. $\square$

**Corollary 3.5.** For a bounded below spectrum $X$, there exists a splittable short exact sequence computing the integral homology groups of its $p$-completion:

$$0 \longrightarrow L_0H_n(X; \mathbb{Z}) \longrightarrow H_n(X^\wedge_p; \mathbb{Z}) \longrightarrow L_1H_{n-1}(X; \mathbb{Z}) \longrightarrow 0$$

for any $n$.

**Proof.** Since the spectrum $HZ \wedge X^\wedge_p$ is $p$-complete by Corollary 3.3, there is a canonical map $(HZ \wedge X)^\wedge_p \to HZ \wedge X^\wedge_p$, and this map must be an equivalence because it is an $H\mathbb{F}_p$-equivalence of $p$-complete bounded below spectra. Hence, the claim follows from Theorem 2.6. $\square$

From Corollary 3.5, we obtain the following description of the $p$-complete sphere spectrum as a Moore spectrum.

**Example 3.6.** There is a canonical equivalence $S^0_p \simto M\mathbb{Z}_p$.

4. The comparison

In this section, we first study the relation between $p$-completion for spectra and spaces under the infinite loop space functor $\Omega^\infty$, and then prove our main theorem.
4.1. Infinite loop spaces. It is easy to deduce from Theorem 2.6 the following relation between unstable and stable p-completion under $\Omega^\infty$.

**Proposition 4.1.** For 0-connected spectra $X$ and $Y$, we have:

1. $X$ is $p$-complete if and only if $\Omega^\infty X$ is $p$-complete.
2. A map $f: X \to Y$ is an $HF_p$-equivalence if and only if $\Omega^\infty f$ is an $HF_p$-equivalence.
3. The canonical comparison map $(\Omega^\infty X)^\wedge_p \to \Omega^\infty (X^\wedge_p)$ is an equivalence.

**Proof.** Since $\pi_\ast \Omega^\infty X \cong \pi_\ast X$ and $\Omega^\infty X$ is nilpotent, the first claim is a direct consequence of Theorem 2.6. In order to prove (2), note that $f$ is an $HF_p$-equivalence if and only if the homotopy groups $\pi_\ast \operatorname{cof}(f)$ of the cofiber of $f$ are uniquely $p$-divisible. This is equivalent to the statement that the $F_p$-homology $H_\ast(\Omega^\infty \operatorname{cof}(f); F_p)$ is trivial. The Serre spectral sequence associated to the fiber sequence

$$
\Omega^\infty X \xrightarrow{\Omega^\infty f} \Omega^\infty Y \xrightarrow{} \Omega^\infty \operatorname{cof}(f)
$$

thus shows that this happens if and only if $\Omega^\infty f$ is an $HF_p$-equivalence.

Statement (1) implies that $\Omega^\infty(X^\wedge_p)$ is $p$-complete, so the map $\Omega^\infty(X) \to \Omega^\infty(X^\wedge_p)$ factors canonically through $\phi: (\Omega^\infty X)^\wedge_p \to \Omega^\infty(X^\wedge_p)$, making the following diagram commute:

$$
\Omega^\infty X \xrightarrow{(\Omega^\infty X)^\wedge_p} \Omega^\infty(X^\wedge_p) \\
\downarrow \quad \quad \quad \downarrow
$$

By Statement (2), both the horizontal and the diagonal map are $HF_p$-equivalences, hence so is the vertical comparison map.

**Remark 4.2.** Let $\Omega^\infty_0$ be the 0-component of $\Omega^\infty$. The last part of the proposition can be strengthened to an equivalence $(\Omega^\infty_0 X)^\wedge_p \to \Omega^\infty(X^\wedge_p)$ for any connective spectrum $X$ such that $\pi_0 X$ does not contain any copies of $\mathbb{Z}/p^\infty$. To prove this directly, one may use the short exact sequences displayed at the end of Theorem 2.6.

4.2. Suspension spectra. We now turn to the comparison under $\Sigma^\infty$. In odd dimensions, the next result has also been observed in [BK72, Rem. VI.5.7], see also [MP12, Rem. 11.1.5].

**Lemma 4.3.** Let $n \geq 1$ and write $S^a_p$ for the $p$-completion of $S^n$. There exists an uncountable rational vector space in $H_{2n}(S^a_p; \mathbb{Z})$ which injects into $H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z})$ under the map $S^a_p \to \tau_{\leq n} S^a_p \cong K(\mathbb{Z}_p, n)$.

**Proof.** Consider the following segment of the Serre long exact sequence for the fibration $F \to S^a_p \to K(\mathbb{Z}_p, n)$:

$$
H_{2n}(F; \mathbb{Z}) \xrightarrow{} H_{2n}(S^a_p; \mathbb{Z}) \xrightarrow{} H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z}) \xrightarrow{} H_{2n-1}(F; \mathbb{Z}) \xrightarrow{} \ldots
$$

Corollary 3.4 implies that $H_{2n}(F; \mathbb{Z})$ and $H_{2n-1}(F; \mathbb{Z})$ are derived $p$-complete. Recalling that $\operatorname{Hom}_\mathbb{Z}(\mathbb{Q}, A) = 0 = \operatorname{Ext}^{1}_{\mathbb{Z}}(\mathbb{Q}, A)$ whenever $A$ is derived $p$-complete, we see that the natural map $\operatorname{Hom}_\mathbb{Z}(\mathbb{Q}, H_{2n}(S^a_p; \mathbb{Z})) \to \operatorname{Hom}_\mathbb{Z}(\mathbb{Q}, H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z}))$ is surjective. Thus, it will suffice to show that $H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z})$ contains an uncountable rational vector space, which will be verified in the homological proof of Proposition 5.3 below.

**Note.** Because $H_\ast(S^a_p; \mathbb{F}_p) \cong H_\ast(S^n; \mathbb{F}_p) \cong \mathbb{F}_p[n]$, an application of the universal coefficient theorem shows that $H_k(S^a_p; \mathbb{Z})$ is rational for all $k > n$. 


Lemma 4.4. Suppose $N$ is a derived $p$-complete nilpotent (abelian) group and $n = 1$ ($n \geq 1$). If $N$ is not bounded $p$-torsion, then there exists an element $x \in N$ of infinite order inducing a monomorphism $H_*(K(\mathbb{Z}_p, n); \mathbb{Q}) \to H_*(K(N, n); \mathbb{Q})$.

Proof. By assumption on $N$ and Proposition 2.4, $L_0(N/[N, N])$ contains elements of infinite order. Let $\pi$ be such an element and let $x \in N$ be a lift of $\pi$. For the remainder of the proof we assume $n = 1$; the (easier) case $n \geq 2$ and $N$ abelian is proven similarly. The element $x$ induces a map

$$K(\mathbb{Z}_p, 1) \to K(N, 1) \to K(L_0(N/[N, N]), 1)$$

such that the composite is injective on $\pi_1$. It follows that the rationalization $K(\mathbb{Z}_p, 1)_{\mathbb{Q}} \to K(L_0(N/[N, N]), 1)_{\mathbb{Q}}$ of this map is split, hence the composite

$$H_*(K(\mathbb{Z}_p, 1); \mathbb{Q}) \to H_*K(N, 1); \mathbb{Q}) \to H_*(K(L_0(N/[N, N]), 1); \mathbb{Q})$$

is a split monomorphism, which implies the claim. □

Proposition 4.5. If $X$ is a $p$-complete nilpotent space whose homotopy groups are not all bounded $p$-torsion, then the integral homology groups $H_*(X; \mathbb{Z})$ and the stable homotopy groups $\pi_\infty^sX$ both contain an uncountable rational vector space.

Proof. Assume that $\pi_nX$ is not all bounded $p$-torsion, and let $\pi_nX$ be the lowest such group. It then follows from Lemma 4.4 that $\pi_nX$ contains a class $x$ of infinite order inducing a monomorphism $H_*(K(\mathbb{Z}_p, n); \mathbb{Q}) \to H_*(K(\pi_nX, n); \mathbb{Q})$. Since the map $\tau_{\geq n}X \to X$ is a rational homology equivalence, any rational subgroup of $H_*(\tau_{\geq n}X; \mathbb{Z})$ must map monomorphically to $H_*(X; \mathbb{Z})$, so it suffices to prove the homological claim for $\tau_{\geq n}X$. The element $x$ yields a map $S_p^n \to \tau_{\geq n}X$ such that the composite $S_p^n \to \tau_{\geq n}X \to K(\pi_nX, n)$ factors as

$$\tau_{\geq n}X \to \tau_{\leq n}\tau_{\geq n}X \simeq K(\pi_nX, n) \to S_p^n \simeq K(\mathbb{Z}_p, n).$$

It follows from Lemma 4.3 and the choice of $x$ that the induced homomorphism in homology

$$H_2n(S_p^n; \mathbb{Z}) \to H_2n(\tau_{\geq n}X; \mathbb{Z}) \to H_2n(K(\pi_nX, n); \mathbb{Z})$$

maps an uncountable rational vector space monomorphically to $H_2n(K(\pi_nX, n); \mathbb{Z})$, hence so does the map $H_2n(S_p^n; \mathbb{Z}) \to H_2n(\tau_{\geq n}X; \mathbb{Z})$. This verifies the claim about the integral homology of $X$.

Recall that, for any connective spectrum $Y$, the Hurewicz map $\pi_*Y \to H_*(Y; \mathbb{Z})$ has kernel and cokernel of bounded torsion in each degree. Indeed, the fiber sequence $Y \wedge \tau_{\geq 0}S^0 \to Y \to Y \wedge HQ \mathbb{Z}$ reduces this claim to showing that $\pi_*(Y \wedge \tau_{\geq 0}S^0)$ is bounded torsion in each degree. This follows from the convergent Atiyah–Hirzebruch spectral sequence

$$H_*(Y; \pi_\tau\tau_{\geq 0}S^0) \Rightarrow \pi_{*+t}(Y \wedge \tau_{\geq 0}S^0),$$

because $H_*(Y; \pi_\tau\tau_{\geq 0}S^0)$ is bounded torsion for all $s$ and $t$. Therefore, any rational vector space in $H_*(Y; \mathbb{Z})$ may be lifted back to $\pi_*Y$. In particular, an uncountable rational vector space in $H_2n(X; \mathbb{Z})$ may be lifted back to $\pi_2n(\Sigma^\infty X)$ after suspension. □

Remark 4.6. Suppose $X$ is a $p$-complete nilpotent space such that $\pi_nX$ is the lowest homotopy group not of bounded $p$-torsion. The above argument shows that $H_2n(X; \mathbb{Z})$ contains an uncountable rational vector space. With more work, we can also show that $H_k(X; \mathbb{Z})$ is derived $p$-complete for $k \leq 2n - 2$ and thus cannot contain any rational classes. Note that when $X$ is
It follows that the natural map \(\Sigma^\infty X\) whose homotopy groups are all finite \(p\)-groups is \(p\)-complete by Corollary 3.3.

**Theorem 4.7.** If \(X\) is a \(p\)-complete nilpotent space, then \(\Sigma^\infty X\) is \(p\)-complete if and only if \(\pi_n X\) is \(p\)-torsion for each \(n\).

Note that the torsion exponent of \(\pi_n X\) may vary with \(n\) and does not need to be bounded uniformly for all \(n\).

**Proof.** First assume that \(X\) is a \(p\)-complete nilpotent space with \(\pi_n X\) of bounded \(p\)-torsion for each \(n\); we can apply [BK72, Ch. II, 4.7] to see that the Postnikov tower of \(X\) can be refined to a tower of principal fibrations whose fibers are Eilenberg–MacLane spaces for bounded \(p\)-torsion abelian groups. The category of bounded \(p\)-torsion abelian groups forms a Serre class, so Serre theory implies that \(H_*(X;\mathbb{Z}) \cong H_*(\Sigma^\infty X;\mathbb{Z})\) is degreewise bounded \(p\)-torsion. Hence, \(\Sigma^\infty X\) is \(p\)-complete as a spectrum by Corollary 3.3.

The converse is a consequence of Proposition 4.5: if \(\pi_n X\) is not all bounded torsion, then \(H_*(\Sigma^\infty X;\mathbb{Z})\) contains rational classes and thus cannot be derived \(p\)-complete, hence \(\Sigma^\infty X\) is not \(p\)-complete by Corollary 3.3.

The next result generalizes [PSS17, Prop. 2.4].

**Corollary 4.8.** If \(X\) is a pointed connected space with degreewise finite homotopy groups, then the canonical map \((\Sigma^\infty X)_p^\wedge \to \Sigma^\infty X_p^\wedge\) is an equivalence.

**Proof.** By [BK72, Ch. VII, 4.3], \(X\) is a \(\mathbb{Z}/p\)-good space and \(X_p^\wedge\) is a \(p\)-complete nilpotent space whose homotopy groups are all finite \(p\)-groups. Hence \(\Sigma^\infty X_p^\wedge\) is \(p\)-complete by Theorem 4.7.

It follows that the natural map \((\Sigma^\infty X)^\wedge_p \to \Sigma^\infty X_p^\wedge\) is an \(HF_p\)-equivalence between \(HF_p\)-local spectra, which implies the claim.

**Corollary 4.9.** If \(X\) is a nilpotent space with \(H_n(X;\mathbb{Z})\) and \(\pi_n X\) derived \(p\)-complete for all \(n\), then \(H_n(X;\mathbb{Z})\) and \(\pi_n X\) are bounded \(p\)-torsion for all \(n\).

**Proof.** The assumption on \(\pi_n X\) implies that \(X\) is \(p\)-complete by Theorem 2.6, while the assumption on \(H_*(X;\mathbb{Z})\) shows that \(\Sigma^\infty X\) is \(p\)-complete, using Corollary 3.3. It thus follows from Theorem 4.7 that \(\pi_n X\) is degreewise bounded \(p\)-torsion, hence so is \(H_*(X;\mathbb{Z})\) by the proof of Theorem 4.7.

The analogue of this corollary does not hold stably, as the following example demonstrates.

**Example 4.10.** Let \(M(\mathbb{Z}_p, n)\) be the Moore space for \(\mathbb{Z}_p\) in degree \(n \geq 2\). As \(H_*(\Sigma^\infty M(\mathbb{Z}_p, n);\mathbb{Z})\) is isomorphic to \(\mathbb{Z}_p[n]\), we see that \(\Sigma^\infty M(\mathbb{Z}_p, n)\) is \(p\)-complete and consequently has derived \(p\)-complete stable homotopy groups and integral homology groups. However, \(H_n(\Sigma^\infty M(\mathbb{Z}_p, n);\mathbb{Z}) \cong \mathbb{Z}_p\) is clearly not bounded \(p\)-torsion. In particular, \(M(\mathbb{Z}_p, n)\) is not \(p\)-complete, so this also shows that the assumption that \(X\) be \(p\)-complete cannot be dropped in Theorem 4.7.

5. **Rational classes in the stable homotopy groups of \(K(\mathbb{Z}_p, n)\)**

In this section, we present an example that illustrates how the rational classes in the stable homotopy groups of \(p\)-complete spaces arise. In fact, we present two different approaches: One using the integral homology of \(K(\mathbb{Z}_p, n)\), and one using Goodwillie calculus. The latter derivation is entirely stable and might be of independent interest.

First, we need a well-known auxiliary result; we outline a proof because we were unable to find a published reference for it. For an abelian group \(A\) and any \(k \geq 0\), let \(\text{Sym}_k^p(A)\) and \(\Lambda_k^p(A)\) be the \(k\)th symmetric power and the \(k\)th exterior power on \(A\), respectively.
Lemma 5.1. If $k > 1$, then $\Lambda^k_*(\mathbb{Z}_p)$ and the kernel of the multiplication map $\text{Sym}_2^k(\mathbb{Z}_p) \to \mathbb{Z}_p$ are uncountable rational vector spaces.

**Proof.** Since both symmetric and exterior power commute with base-change along $\mathbb{Z} \to \mathbb{Z}/l$ for any prime $l$, the indicated maps are isomorphisms mod $l$. Moreover, $\text{Sym}_2^k(A)$ and $\Lambda^k_0(A)$ are torsion-free whenever $A$ is, so both $\ker(\text{Sym}_2^k(\mathbb{Z}_p) \to \mathbb{Z}_p)$ and $\Lambda^k_0(\mathbb{Z}_p)$ are rational vector spaces. We may therefore base-change to $\mathbb{Q}$, where it is easy to verify that the $\mathbb{Q}$-dimension of the groups under consideration is that of $\mathbb{Q}_p$. □

**Remark 5.2.** A similar argument also shows that $\mathbb{Z}_p/\mathbb{Z}(p)$ is a rational vector space with the same $\mathbb{Q}$-dimension as $\mathbb{Q}_p$.

**Proposition 5.3.** For $n \geq 1$ and all $k > 1$, the stable homotopy group $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not $p$-complete.

**First proof.** Let $A$ be an abelian group and recall that $H_*(K(A, n); \mathbb{Z})$ equipped with the Pontrjagin product is a graded commutative algebra such that squares of odd dimensional elements are zero; in fact, it has the structure of a graded divided power algebra, see [EML54, Car56] or more recently [Ric09]. With notation as in the previous lemma, the canonical isomorphism $A \to H_n(K(A, n); \mathbb{Z})$ thus extends to a natural homomorphism

\[
\begin{align*}
\phi^k(A, n): \Lambda^k_2(A) & \longrightarrow H_{kn}(K(A, n); \mathbb{Z}), & \text{if } n \text{ odd} \\
\phi^k(A, n): \text{Sym}_2^k(A) & \longrightarrow H_{kn}(K(A, n); \mathbb{Z}), & \text{if } n \text{ even}
\end{align*}
\]

for any $n, k > 0$. Moreover, we know that $\phi^k(A, n) \otimes \mathbb{Q}$ is a rational isomorphism. It then follows from Lemma 5.1 that, for $k > 1$, there exists an uncountable rational vector space which is mapped monomorphically to $H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z})$ via $\phi^k(\mathbb{Z}_p, n)$. We thus obtain an uncountable rational vector space in $H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z})$ that may be lifted back to give the desired uncountable rational vector space in $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n)$ for $k > 1$, as in the proof of Proposition 4.5. □

**Second proof.** We will compute the homotopy groups of $\Sigma^\infty K(\mathbb{Z}_p, n) \simeq \Sigma^\infty \Omega^\infty \Sigma^n H\mathbb{Z}_p$ using Goodwillie calculus [Goo03]. To this end, recall that the Goodwillie tower $(P_k)_{k \geq 1}$ associated to the functor $\Sigma^\infty \Omega^\infty: \text{Sp} \to \text{Sp}$ is assembled from fiber sequences of functors

\[
D_k \longrightarrow P_k \longrightarrow P_{k-1}
\]

with layers $D_k X \simeq X^{\wedge, k}$, where the homotopy orbits are formed with respect to the permutation action of $\Sigma_k$ (see for example [KM13] and the references given therein). Moreover, the Goodwillie tower $(P_k)_{k \geq 0}$ converges for connective spectra, i.e., there is a canonical equivalence

\[
\Sigma^\infty \Omega^\infty X \longrightarrow \lim_k P_k X
\]

for any connective $X \in \text{Sp}$. We will apply this in the case $X = \Sigma^n H\mathbb{Z}_p$.

In order to understand the layers, we start by analyzing $\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge, k}$ via the universal coefficient theorem. We claim that, for all $k \geq 1$, the homotopy groups have the following form

\[
\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge, k} \cong \begin{cases} 
0 & * < nk \\
\mathbb{Z}^{\otimes k} & * = nk \\
\text{finite} & * > nk.
\end{cases}
\]

By the universal coefficient theorem, we have an isomorphism

\[
\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge, k} \cong (\pi_*(\Sigma^n H\mathbb{Z})^{\wedge}) \otimes \mathbb{Z}^{\otimes k}.
\]

In degrees $* > nk$, the groups $\pi_*(\Sigma^n H\mathbb{Z})^{\wedge}$ are torsion, so the only torsion-free summand appears in degree $nk$. Since $\pi_*(\Sigma^n H\mathbb{Z})^{\wedge}$ is finitely generated over $\mathbb{Z}$ in each degree, the claim follows.
We now plug the formula (5.5) into the convergent homotopy orbit spectral sequence
\[ H_s(\Sigma_k, \pi_t(\Sigma^nH\mathbb{Z}_p)^{\wedge k}) \Rightarrow \pi_{s+t}D_k(\Sigma^nH\mathbb{Z}_p). \]

There are two cases: If \( t > nk \) or \( t < nk \), then the groups \( H_s(\Sigma_k, \pi_t(\Sigma^nH\mathbb{Z}_p)^{\wedge k}) \) are finite or trivial for all \( s \), respectively. Let \( t = nk \). By Lemma 5.1 and (5.5), there is an isomorphism \( H_s(\Sigma_k, \pi_{nk}(\Sigma^nH\mathbb{Z}_p)^{\wedge k}) \cong H_s(\Sigma_k, \mathbb{Z}_p) \) for \( s > 0 \) and \( H_0(\Sigma_k, \pi_{nk}(\Sigma^nH\mathbb{Z}_p)^{\wedge k}) \) contains an uncountable rational vector space \( V_k \) if \( k > 1 \). To see the last statement, it suffices to compute the coinvariants on the rational submodule of \( \pi_{nk}^{\wedge k} \) by choosing a \( \mathbb{Q} \)-bases, as in the proof of Lemma 5.1. Furthermore, since the integral homology of \( \Sigma_k \) is finitely generated over \( \mathbb{Z} \) in each degree and rationally trivial in positive degrees, \( H_s(\Sigma_k, \pi_{nk}(\Sigma^nH\mathbb{Z}_p)^{\wedge k}) \) is finite for all \( s > 0 \). Combining all this information, we obtain \( D_1\Sigma^nH\mathbb{Z}_p \cong \Sigma^nH\mathbb{Z}_p \) and for \( k > 1 \):

\[ \pi_\ast D_k(\Sigma^nH\mathbb{Z}_p) \cong \begin{cases} 0 & * < nk \\ V_k \oplus W_k & n = nk \\ \text{finite} & * > nk, \end{cases} \quad (5.6) \]

where \( V_k \) is an uncountable rational vector space and \( W_k \) is some abelian group.

This allows us to derive a structural formula for \( \pi_\ast P_k\Sigma^nH\mathbb{Z}_p \). Consider the following segment of the long exact sequence of homotopy groups associated to the fiber sequence (5.4):

\[ \ldots \longrightarrow \pi_{nk+1}P_{k-1}\Sigma^nH\mathbb{Z}_p \longrightarrow \pi_{nk}D_k\Sigma^nH\mathbb{Z}_p \longrightarrow \pi_{nk}P_k\Sigma^nH\mathbb{Z}_p \longrightarrow \ldots \]

Because \( n \geq 1 \), it follows inductively from (5.6) that the term on the left is finite, hence \( V_k \) must be a summand in \( \pi_{nk}P_k\Sigma^nH\mathbb{Z}_p \). This yields for all \( k \geq 1 \):

\[ \pi_\ast P_k\Sigma^nH\mathbb{Z}_p \cong \begin{cases} 0 & * < n \\ V_l \oplus W'_l & n = nl \text{ with } 1 \leq l \leq k \\ \text{finite} & \text{otherwise}, \end{cases} \quad (5.7) \]

where \( V_l \) is as above for \( l \geq 2 \), and \( V_1 \) and \( W'_1 \) are some abelian groups.

Finally, since \( D_k\Sigma^nH\mathbb{Z}_p \) is \( nk \)-connective for all \( k \), the tower \( (\pi_\ast P_k\Sigma^nH\mathbb{Z}_p)_{k \geq 0} \) stabilizes after finally many steps in each degree and hence is Mittag-Leffler. The corresponding Milnor sequence thus degenerates to an isomorphism

\[ \pi_\ast \Sigma^{\infty}K(\mathbb{Z}_p, n) \cong \pi_\ast \Sigma^{\infty}\Omega^{\infty}\Sigma^nH\mathbb{Z}_p \cong \lim_k \pi_\ast P_k\Sigma^nH\mathbb{Z}_p. \]

Therefore, the claim follows from (5.7). \( \square \)

REFERENCES


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