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ON THE COMPARISON OF STABLE AND UNSTABLE $p$-COMPLETION

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Abstract. In this note we show that a $p$-complete nilpotent space $X$ has a $p$-complete suspension spectrum if and only if its homotopy groups $\pi_\ast X$ are bounded $p$-torsion. In contrast, if $\pi_\ast X$ is not all bounded $p$-torsion, we locate uncountable rational vector spaces in the integral homology and in the stable homotopy groups of $X$. To prove this, we establish a homological criterion for $p$-completeness of connective spectra. Moreover, we illustrate our results by studying the stable homotopy groups of $K(\mathbb{Z}_p, n)$ via Goodwillie calculus.

1. Introduction

The notion of $p$-completion plays a fundamental role in algebra and topology, for it provides effective means to isolate and study $p$-primary properties. Applied to homotopy theory by Bousfield and Kan [BK72] as well as Sullivan [Sul74] and developed further in [Bou75, Bou79], it has since become one of the standard tools in the hands of algebraic topologists. However, there appears to be no general account of the comparison between unstable and stable $p$-completion in the literature, which is the question we address in the present note.

Our main goal is to characterize $p$-complete spaces which have $p$-complete suspension spectra:

**Theorem 4.7.** If $X$ is a $p$-complete nilpotent space, then $\Sigma^\infty X$ is $p$-complete if and only if $\pi_n X$ is bounded $p$-torsion for each $n$.

In fact, we exhibit a sharp dichotomy of $p$-complete nilpotent spaces: if $X$ is a $p$-complete nilpotent space whose homotopy groups are not all bounded $p$-torsion, then the integral homology groups and stable homotopy groups of $X$ both contain an uncountable rational vector space. As a consequence, we deduce that a nilpotent space $X$ with derived $p$-complete integral homology and unstable homotopy must have both $H_n(X; \mathbb{Z})$ and $\pi_n X$ of bounded $p$-torsion for all $n$.

In a first step towards the proof of the theorem, we complement the second author’s characterization of $p$-complete spectra in terms of homotopy groups with an integral homological criterion, using a mild generalization of Serre classes appropriate for stable homotopy theory. This is in sharp contrast to the aforementioned fact that the integral homology of $p$-complete spaces is not well-behaved, and thus cannot be used to characterize $p$-completeness of spaces.

**Corollary 3.3.** A bounded below spectrum $X$ is $p$-complete if and only if $H_n(X; \mathbb{Z})$ is derived $p$-complete in each degree.

In order to use this result to prove the theorem, we need to detect rational classes in the homology of $p$-complete spaces whose homotopy is not bounded $p$-torsion. This rests on the study of the integral homology of $p$-complete spheres. We end this note with a sample computation, illustrating how Goodwillie calculus allows us to detect rational classes in the stable homotopy groups of the Eilenberg–MacLane space $K(\mathbb{Z}_p, n)$.

**Proposition 5.3.** For $n \geq 1$ and $k > 1$, the stable homotopy group $\pi_{nk} \Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not $p$-complete.

In fact, we also give a short alternative argument based on the integral homology of $K(\mathbb{Z}_p, n)$.

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Conventions. Throughout this paper, $p$ will be a fixed prime number and $\mathbb{Z}_p$ denotes the $p$-adic integers. We say that a nilpotent group $N$ is bounded $p$-torsion if there exists an $m$ such that for all $x \in N$, we have $x^{p^m} = 1$. A graded nilpotent group $N_*$ is said to be of bounded $p$-torsion if $N_k$ is bounded $p$-torsion for each $k$; however, we do not require a uniform bound. Whenever we are in a graded context, we indicate the degree of an abelian group $A$ by square brackets, i.e., $A[n]$ refers to $A$ placed in degree $n$. If $X$ is a topological space, then $H_*(X;A)$ is the reduced homology of $X$ with coefficients in $A$. For a space or spectrum $X$, we write $\tau_{\leq n}X = \tau_{\leq n}^{<}X$ for the $n$-th Postnikov section of $X$ and $\tau_{>n}X = \tau_{>n}^{>}X$ for the fiber of the canonical map $X \to \tau_{\leq n}X$.

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2. Preliminaries on $p$-completion

We briefly recall the basic properties of $p$-completion for nilpotent groups, topological spaces, and spectra. With the exceptions of Lemma 2.2 and Proposition 2.4, this material is mostly taken from [BKT72, Bou75, Bou79], and we refer to these sources as well as [HS99, MP12] for further references.

2.1. Algebraic $p$-completion for abelian groups. In general, the $p$-completion functor $M \mapsto \lim_i M/p^i M$ on the category of abelian groups is neither left nor right exact, so one studies its zeroth and first left derived functors $L_0$ and $L_1$, which may be expressed as $L_0 M = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$ and $L_1 M = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$ by [BK72, Ch. VI, 2.1]. An abelian group $M$ is called derived $p$-complete (or $\mathbb{Z}/p$-complete or $L$-complete) if the natural completion map $M \to L_0 M$ is an isomorphism. For each abelian group $M$, the map $M \to L_0 M$ will then be the universal homomorphism from $M$ to a derived $p$-complete abelian group by [BK72, Ch. VI, 3.2]. We will denote the full subcategory of derived $p$-complete abelian groups by $\mathcal{C}_p$.

Proposition 2.1. The category $\mathcal{C}_p$ is a full abelian subcategory of $\text{Mod}_{\mathbb{Z}}$ closed under extensions and limits. Furthermore, for any $M \in \text{Mod}_{\mathbb{Z}}$ there is a short exact sequence

$0 \longrightarrow \lim^1_1 \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^i, M) \longrightarrow L_0 M \longrightarrow \lim_i M/p^i M \longrightarrow 0$

relating derived $p$-completion to ordinary $p$-completion.

Proof. This is essentially proven in [BK72, Ch. VI, 2.1], but can also be deduced as a special case of [HS99, Thms. A.2 and A.6].

We will later make use of the following observation.

Lemma 2.2. If $A \in \mathcal{C}_p$ is torsion, then $A$ is bounded $p$-torsion.

Proof. We give two proofs, a conceptual one and an elementary argument. First, any derived $p$-complete group $A$ has $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/[1/p], A) = 0 = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/[1/p], A)$ by [BK72, Ch. VI, 3.4], and hence $A$ satisfies $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, A) = 0 = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, A)$ since $\mathbb{Q}$ is a quotient of free $\mathbb{Z}/[1/p]$-modules. Thus, $A$ is a cotorsion group with no nontrivial divisible subgroups, so the Baer–Fomin theorem [Bae36] implies that $A$ is a bounded $p$-torsion group.

Second, suppose that the conclusion of the lemma is false, i.e., that there exists a sequence $(a_i)_{i \in \mathbb{N}}$ of elements of $A$ such that the order of $a_i$ is $p^i$. Set $x_j = \sum_{i=0}^{j-1} a_{2i+1} p^i$, then the element
$x = (x_1, x_2, x_3, \ldots) \in \prod_{i \in \mathbb{N}} A$ lies in $\lim_j A/p^j$. By construction, $x$ is not $p$-torsion, which contradicts the fact that $A \to \lim_j A/p^j$ is surjective, forcing $\lim_j A/p^j$ to be $p$-torsion. \hfill \Box

Remark 2.3. By a theorem of Prüfer, the conclusion of the lemma implies that $A$ must in fact be a direct sum of cyclic $p$-groups.

2.2. Algebraic $p$-completion for nilpotent groups. Recall from [BK72, Ch. VI, §2] that the notion of derived $p$-completion can be extended to nilpotent groups, as follows: If $X^\wedge_p$ denotes the Bousfield–Kan $p$-completion of a nilpotent space $X$ as recalled in the next subsection, then we define the derived $p$-completion of the nilpotent group $N$ as $L_0N = \pi_1(K(N, 1)^\wedge_p)$ and $L_1N = \pi_2(K(N, 1)^\wedge_p)$. A nilpotent group $N$ is called derived $p$-complete if the completion map $N \to L_0N$ is an isomorphism; for each nilpotent group $N$, the map $N \to L_0N$ will then be the universal homomorphism from $N$ to a derived $p$-complete nilpotent group by [BK72, Ch. VI, 3.2]. We denote the category of derived $p$-complete nilpotent groups by $N_p$.

The inclusion functor $C_p \to N_p$ has a left adjoint given by taking a derived $p$-complete nilpotent group $N$ to the derived $p$-completion of its abelianization $L_0(N/[N, N])$. Note that the unit of this adjunction is surjective, i.e., for any derived $p$-complete nilpotent group $N$, the canonical map $N \to L_0(N/[N, N])$ is surjective. Indeed, since $L_0$ preserves epimorphisms of nilpotent groups, all maps in the following commutative diagram are surjective:

\[
\begin{array}{ccc}
N & \rightarrow & N/[N, N] \\
\downarrow & & \downarrow \\
L_0N & \rightarrow & L_0(N/[N, N]).
\end{array}
\]

We obtain the following generalization of Lemma 2.2:

Proposition 2.4. The following conditions are equivalent for $N \in N_p$:

1. $N$ is torsion.
2. $L_0(N/[N, N])$ is torsion.
3. $N$ is bounded $p$-torsion.

Proof. The surjectivity of the map $N \to L_0(N/[N, N])$ observed above immediately gives the implication (1) $\Rightarrow$ (2), while (3) $\Rightarrow$ (1) is trivial.

Assume that $L_0(N/[N, N])$ is torsion and thus bounded $p$-torsion by Lemma 2.2. Consider the lower central series of $N$,

$N = \gamma_1N \supseteq \gamma_2N \supseteq \ldots \supseteq \gamma_mN = 1,$

with successive abelian quotients $Q_i(N) = \gamma_iN/\gamma_{i+1}N$. We claim that, for each $i \geq 1$, $Q_i(N)$ is a direct sum of a $p$-divisible group and a bounded $p$-torsion group. Indeed, we start with the abelianization $Q_1(N) = N/[N, N]$ of $N$. Lemma 3.7 in [BK72, Ch. VI] implies that the kernel of the completion map $Q_1(N) \to L_0Q_1(N)$ is $p$-divisible, so the claim holds for $Q_1(N)$. The general case follows from this, because $\bigoplus_{i \geq 1} Q_i(N)$ is generated as a Lie algebra by $Q_1(N)$. By [BK72, Ch. VI, 2.5], there is an exact sequence

$0 \longrightarrow L_0Q_i(N) \longrightarrow L_0(N/\gamma_{i+1}N) \longrightarrow L_0(N/\gamma_iN) \longrightarrow 1$

for any $i \geq 1$. Using the previous claim, $L_0Q_i(N)$ is bounded $p$-torsion, so we see inductively that $L_0(N/\gamma_iN)$ is bounded $p$-torsion for all $i \geq 1$, hence (3) holds. \hfill \Box

Remark 2.5. The implication (1) $\Rightarrow$ (3) in the previous proposition could also be proven more directly via the upper central series of $N$, whose quotients are known to be derived $p$-complete by [BK72, VI. 3.4(ii)], but this result would be insufficient for our later use.
2.3. Topological \( p \)-completion. In [BK72], Bousfield and Kan introduced the notion of \( p \)-completion for topological spaces, lifting the algebraic notion defined above to topology. In general, the \( p \)-completion of a space is difficult to describe, but the theory simplifies significantly for nilpotent spaces; in particular, in this case \( p \)-completion coincides with \( H\mathbb{F}_p \)-localization [Bou75]. Furthermore, for nilpotent spaces with \( \mathbb{F}_p \)-homology of finite type, \( p \)-completion can be identified with \( p \)-profinite completion due to Sullivan [Sul74]. Similarly, the category of spectra admits (at least) two notions of \( p \)-completion, given either by \( H\mathbb{F}_p \)-localization or, the one we will use here, localization at the mod \( p \) Moore spectrum \( S^0/p \), see [Bou79]. The next result summarizes the relation between these constructions and lists their basic properties.

Theorem 2.6 (Bousfield, Kan).

(1) A nilpotent space \( X \) is \( p \)-complete if and only if \( \pi_n X \) is derived \( p \)-complete for all \( n \in \mathbb{N} \). Moreover, the notions of \( p \)-completion and \( H\mathbb{F}_p \)-localization coincide for nilpotent spaces.

(2) A spectrum \( X \) is \( p \)-complete if and only if \( \pi_n X \) is derived \( p \)-complete for all \( n \in \mathbb{Z} \). If \( X \) is bounded below, then \( X \) is \( p \)-complete if and only if \( X \) is \( H\mathbb{F}_p \)-local.

Moreover, if \( X \) is a nilpotent space or spectrum, then there exists a splittable short exact sequence computing the unstable or stable homotopy groups of its \( p \)-completion, respectively:

\[
\begin{array}{cccccc}
0 & \longrightarrow & L_0 \pi_n X & \longrightarrow & \pi_n(X^p_n) & \longrightarrow & L_1 \pi_{n-1} X & \longrightarrow & 0
\end{array}
\]

for any \( n \), where \( L_i(-) \cong \text{Ext}_{\mathbb{Z}}^{i-1}(\mathbb{Z}/p^\infty, -) \) are the derived functors of \( p \)-completion.

3. Generalized Serre theory

The full subcategory \( C_p \) of \( \text{Mod}_{\mathbb{Z}} \) is not closed under subobjects or quotients, and thus does not form a Serre class in the usual sense. This necessitates a mild generalization of Serre’s mod \( C \) theory which we develop in this section.

Definition 3.1. A weak Serre class is a full subcategory \( C \subseteq \text{Mod}_{\mathbb{Z}} \) such that if

\[
\begin{array}{cccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5
\end{array}
\]

is an exact sequence in \( \text{Mod}_{\mathbb{Z}} \) with \( A_1, A_2, A_4, A_5 \in C \), then also \( A_3 \in C \).

More explicitly, this means that \( C \subseteq \text{Mod}_{\mathbb{Z}} \) is a full additive subcategory closed under kernels, cokernels, and extensions. It follows that \( C \) is also closed under tensoring and \( \text{Tor}_{1}^\mathbb{Z} \) with respect to finitely generated abelian groups. For instance, any Serre subcategory of \( \text{Mod}_{\mathbb{Z}} \) is a weak Serre class, but the converse does not hold. The main example of interest to us here is the category \( C_p \) of derived \( p \)-complete abelian groups, see Proposition 2.1.

Proposition 3.2. Suppose \( C \) is a weak Serre class. If \( X \) is a bounded below spectrum, then the following two conditions are equivalent:

(1) \( \pi_n X \in C \) for all \( n \in \mathbb{Z} \).

(2) \( H_n(X; \mathbb{Z}) \in C \) for all \( n \in \mathbb{Z} \).

Proof. Assume the first condition holds; we will argue via the Postnikov tower \( (\tau_{\leq n}X) \) of \( X \). For simplicity, we will write \( H_*(Y) \) for the integral homology of a spectrum \( Y \) throughout this proof.

To start with, we need to show that \( H_*(HA) \in C \) for \( A \in C \). Using the isomorphisms \( H_*(HA) \cong H_*(HZ; A) \), the universal coefficient theorem gives a short exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & H_*(HZ) \otimes_{\mathbb{Z}} A & \longrightarrow & H_*(HA) & \longrightarrow & \text{Tor}_1^\mathbb{Z}(H_{*+1}(HZ); A) & \longrightarrow & 0.
\end{array}
\]

In each degree, the integral Steenrod algebra \( H_*(HZ) \) is finitely generated over \( \mathbb{Z} \), as follows from Serre theory for the class of finitely generated abelian groups. Therefore, the outer terms of this sequence are in \( C \). This shows \( H_*(HA) \in C \) as well.
Given \( n \in \mathbb{Z} \), we will now prove that \( H_n(X) \in \mathcal{C} \). Since \( H_n(\tau_n X) = 0 = H_{n-1}(\tau_n X) \) by connectivity, we see that \( H_n(X) \cong H_n(\tau_n X) \). This reduces the claim to proving that \( H_n(X) \in \mathcal{C} \). This follows inductively, using the exact sequence

\[
\cdots \rightarrow H_{n+1}(\tau_{\leq n} X) \rightarrow H_n(\Sigma^n H\pi_n X) \rightarrow H_n(\tau_{\leq n} X) \rightarrow H_n(\tau_{\leq n-1} X) \rightarrow H_{n-1}(\Sigma^n H\pi_n X) \rightarrow \cdots
\]

associated to the fiber sequence \( \Sigma^n H(\pi_n X) \to \tau_{\leq n} X \to \tau_{\leq n-1} X \). Since \( H_k(H\pi_n X) \in \mathcal{C} \) for all \( k \in \mathbb{Z} \), this gives the implication \((1) \Rightarrow (2)\).

For the converse, consider the convergent Atiyah–Hirzebruch spectral sequence

\[
E^2_{s,t} \cong H_s(X; \pi_t S^0) \Rightarrow \pi_{s+t} X.
\]

Since \( \pi_t S^0 \) is finitely generated over \( \mathbb{Z} \) for each \( t \in \mathbb{Z} \), \( H_n(X; \pi_t S^0) \in \mathcal{C} \) for each bidegree \((s,t)\), hence \( \pi_n X \) is also in \( \mathcal{C} \) for all \( n \in \mathbb{Z} \).

When applied to the weak Serre class \( \mathcal{C}_p \), we obtain a homological characterization of \( p \)-completeness for bounded below spectra.

**Corollary 3.3.** For a bounded below spectrum \( X \), the following conditions are equivalent:

1. \( X \) is \( p \)-complete.
2. \( \pi_n X \) is derived \( p \)-complete for all \( n \).
3. \( H_n(X; \mathbb{Z}) \) is derived \( p \)-complete for all \( n \).

**Proof.** The equivalence of (1) and (2) is the content of Theorem 2.6(2), while (2) is equivalent to (3) by Proposition 3.2.

We deduce that the integral homology of \( p \)-complete spaces is well-behaved in the stable range.

**Corollary 3.4.** Suppose \( X \) is \( p \)-complete space. If \( X \) is \( n \)-connected, then \( H_k(X; \mathbb{Z}) \) is derived \( p \)-complete for all \( k \leq 2n \).

**Proof.** Since \( \pi_k \Sigma^\infty X \cong \pi_k X \) for \( k \leq 2n \) by the Freudenthal suspension theorem, Theorem 2.6 implies that \( \pi_k \Sigma^\infty X \) is derived \( p \)-complete in each degree, hence so is \( H_k(\tau_{\leq 2n} \Sigma^\infty X; \mathbb{Z}) \) by Corollary 3.3. We thus get that \( H_k(X; \mathbb{Z}) \cong H_k(\Sigma^\infty X; \mathbb{Z}) \cong H_k(\tau_{\leq 2n} \Sigma^\infty X; \mathbb{Z}) \) is derived \( p \)-complete for \( k \leq 2n \).

**Corollary 3.5.** For a bounded below spectrum \( X \), there exists a splittable short exact sequence computing the integral homology groups of its \( p \)-completion:

\[
0 \rightarrow L_0 H_n(X; \mathbb{Z}) \rightarrow H_n(X^p; \mathbb{Z}) \rightarrow L_1 H_n-1(X; \mathbb{Z}) \rightarrow 0
\]

for any \( n \).

**Proof.** Since the spectrum \( H\mathbb{Z} \wedge X^p \) is \( p \)-complete by Corollary 3.3, there is a canonical map \( (H\mathbb{Z} \wedge X)^p \rightarrow H\mathbb{Z} \wedge X^p \), and this map must be an equivalence because it is an \( HF_p \)-equivalence of \( p \)-complete bounded below spectra. Hence, the claim follows from Theorem 2.6.

From Corollary 3.5, we obtain the following description of the \( p \)-complete sphere spectrum as a Moore spectrum.

**Example 3.6.** There is a canonical equivalence \( S^0_p \cong M\mathbb{Z}_p \).

4. **The comparison**

In this section, we first study the relation between \( p \)-completion for spectra and spaces under the infinite loop space functor \( \Omega^\infty \), and then prove our main theorem.
4.1. Infinite loop spaces. It is easy to deduce from Theorem 2.6 the following relation between unstable and stable $p$-completion under $\Omega^\infty$.

**Proposition 4.1.** For 0-connected spectra $X$ and $Y$, we have:

1. $X$ is $p$-complete if and only if $\Omega^\infty X$ is $p$-complete.
2. A map $f: X \to Y$ is an $HF^p_\infty$-equivalence if and only if $\Omega^\infty f$ is an $HF^p_\infty$-equivalence.
3. The canonical comparison map $(\Omega^\infty X)^p_\infty \to \Omega^\infty (X^p_\infty)$ is an equivalence.

**Proof.** Since $\pi_0 \Omega^\infty X \cong \pi_0 X$ and $\Omega^\infty X$ is nilpotent, the first claim is a direct consequence of Theorem 2.6. In order to prove (2), note that $f$ is an $HF^p_\infty$-equivalence if and only if the homotopy groups $\pi_n \operatorname{cof}(f)$ of the cofiber of $f$ are uniquely $p$-divisible. This is equivalent to the statement that the $\mathbb{F}_p$-homology $H_\ast(\Omega^\infty \operatorname{cof}(f); \mathbb{F}_p)$ is trivial. The Serre spectral sequence associated to the fiber sequence

$$\Omega^\infty X \xrightarrow{\Omega^\infty f} \Omega^\infty Y \xrightarrow{\Omega^\infty \operatorname{cof}(f)}$$

thus shows that this happens if and only if $\Omega^\infty f$ is an $HF^p_\infty$-equivalence.

Statement (1) implies that $\Omega^\infty (X^p_\infty)$ is $p$-complete, so the map $\Omega^\infty (X) \to \Omega^\infty (X^p_\infty)$ factors canonically through $\phi: (\Omega^\infty X)^p_\infty \to \Omega^\infty (X^p_\infty)$, making the following diagram commute:

$$\begin{array}{ccc}
\Omega^\infty X & \xrightarrow{\phi} & (\Omega^\infty X)^p_\infty \\
& \downarrow & \\
& \Omega^\infty (X^p_\infty). & \\
\end{array}$$

By Statement (2), both the horizontal and the diagonal map are $HF^p_\infty$-equivalences, hence so is the vertical comparison map. \qed

**Remark 4.2.** Let $\Omega^\infty_0 X$ be the 0-component of $\Omega^\infty$. The last part of the proposition can be strengthened to an equivalence $(\Omega^\infty_0 X)^p_\infty \to \Omega^\infty_0 (X^p_\infty)$ for any connective spectrum $X$ such that $\pi_0 X$ does not contain any copies of $\mathbb{Z}/p^\infty$. To prove this directly, one may use the short exact sequences displayed at the end of Theorem 2.6.

4.2. Suspension spectra. We now turn to the comparison under $\Sigma^\infty$. In odd dimensions, the next result has also been observed in [BK72, Rem. VI.5.7], see also [MP12, Rem. 11.1.5].

**Lemma 4.3.** Let $n \geq 1$ and write $S^p_n$ for the $p$-completion of $S^n$. There exists an uncountable rational vector space in $H_{2n}(S^p_n; \mathbb{Z})$ which injects into $H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z})$ under the map $S^p_n \to \tau_{\leq n} S^p_n \cong K(\mathbb{Z}_p, n)$.

**Proof.** Consider the following segment of the Serre long exact sequence for the fibration $F \to S^p_n \to K(\mathbb{Z}_p, n)$:

$$H_{2n}(F; \mathbb{Z}) \to H_{2n}(S^p_n; \mathbb{Z}) \to H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z}) \to H_{2n-1}(F; \mathbb{Z}) \to \ldots$$

Corollary 3.4 implies that $H_{2n}(F; \mathbb{Z})$ and $H_{2n-1}(F; \mathbb{Z})$ are derived $p$-complete. Recalling that $\operatorname{Hom}_\mathbb{Z}(\mathbb{Q}, A) = 0 = \operatorname{Ext}_\mathbb{Z}^1(\mathbb{Q}, A)$ whenever $A$ is derived $p$-complete, we see that the natural map $\operatorname{Hom}_\mathbb{Z}(\mathbb{Q}, H_{2n}(S^p_n; \mathbb{Z})) \to \operatorname{Hom}_\mathbb{Z}(\mathbb{Q}, H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z}))$ is surjective. Thus, it will suffice to show that $H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z})$ contains an uncountable rational vector space, which will be verified in the homological proof of Proposition 5.3 below. \qed

Note that, because $H_\ast(S^p_n; \mathbb{F}_p) \cong H_\ast(S^n; \mathbb{F}_p) \cong \mathbb{F}_p[n]$, an application of the universal coefficient theorem shows that $H_k(S^p_n; \mathbb{Z})$ is rational for all $k > n$. 

Lemma 4.4. Suppose $N$ is a derived $p$-complete nilpotent (abelian) group and $n = 1$ ($n \geq 1$). If $N$ is not bounded $p$-torsion, then there exists an element $x \in N$ of infinite order inducing a monomorphism $H_\ast(K(Z_p, n); \mathbb{Q}) \to H_\ast(K(N, n); \mathbb{Q})$.

Proof. By assumption on $N$ and Proposition 2.4, $L_0(N/[N, N])$ contains elements of infinite order. Let $\pi$ be such an element and let $x \in N$ be a lift of $\pi$. For the remainder of the proof we assume $n = 1$; the (easier) case $n \geq 2$ and $N$ abelian is proven similarly. The element $x$ induces a map

$$K(Z_p, 1) \longrightarrow K(N, 1) \longrightarrow K(L_0(N/[N, N]), 1)$$

such that the composite is injective on $\pi_1$. It follows that the rationalization $K(Z_p, 1)_\mathbb{Q} \to K(L_0(N/[N, N]), 1)_\mathbb{Q}$ of this map is split, hence the composite

$$H_\ast(K(Z_p, 1); \mathbb{Q}) \longrightarrow H_\ast(K(N, 1); \mathbb{Q}) \longrightarrow H_\ast(K(L_0(N/[N, N]), 1); \mathbb{Q})$$

is a split monomorphism, which implies the claim. \qed

Proposition 4.5. If $X$ is a $p$-complete nilpotent space whose homotopy groups are not all bounded $p$-torsion, then the integral homology groups $H_\ast(X; \mathbb{Z})$ and the stable homotopy groups $\pi_\ast\Sigma^\infty X$ both contain an uncountable rational vector space.

Proof. Assume that $\pi_n X$ is not all bounded $p$-torsion, and let $\pi_n X$ be the lowest such group. It then follows from Lemma 4.4 that $\pi_n X$ contains a class $x$ of infinite order inducing a monomorphism $H_\ast(K(Z_p, n); \mathbb{Q}) \to H_\ast(K(\pi_n X, n); \mathbb{Q})$. Since the map $\tau_{\geq n} X \to X$ is a rational homology equivalence, any rational subgroup of $H_\ast(\tau_{\geq n} X; \mathbb{Q})$ must map monomorphically to $H_\ast(X; \mathbb{Q})$, so it suffices to prove the homological claim for $\tau_{\geq n} X$. The element $x$ yields a map $S^n_p \to \tau_{\geq n} X$ such that the composite $S^n_p \to \tau_{\geq n} X \to K(\pi_n X, n)$ factors as

$$\tau_{\geq n} X \longrightarrow \tau_{\leq n} \tau_{\geq n} X \simeq K(\pi_n X, n)$$

$$S^n_p \longrightarrow \tau_{\leq n} S^n_p \simeq K(Z_p, n).$$

It follows from Lemma 4.3 and the choice of $x$ that the induced homomorphism in homology

$$H_{2n}(S^n_p; \mathbb{Z}) \longrightarrow H_{2n}(\tau_{\geq n} X; \mathbb{Z}) \longrightarrow H_{2n}(K(\pi_n X, n); \mathbb{Z})$$

maps an uncountable rational vector space monomorphically to $H_{2n}(K(\pi_n X, n); \mathbb{Z})$, hence so does the map $H_{2n}(S^n_p; \mathbb{Z}) \to H_{2n}(\tau_{\geq n} X; \mathbb{Z})$. This verifies the claim about the integral homology of $X$.

Recall that, for any connective spectrum $Y$, the Hurewicz map $\pi_\ast Y \to H_\ast(Y; \mathbb{Z})$ has kernel and cokernel of bounded torsion in each degree. Indeed, the fiber sequence $Y \wedge \tau_{\geq 0} S^0 \to Y \to Y \wedge H\mathbb{Z}$ reduces this claim to showing that $\pi_n (Y \wedge \tau_{\geq 0} S^0)$ is bounded torsion in each degree. This follows from the convergent Atiyah–Hirzebruch spectral sequence

$$H_\ast(Y; \pi_t \tau_{\geq 0} S^0) \Longrightarrow \pi_{s+t} (Y \wedge \tau_{\geq 0} S^0),$$

because $H_\ast(Y; \pi_t \tau_{\geq 0} S^0)$ is bounded torsion for all $s$ and $t$. Therefore, any rational vector space in $H_\ast(Y; \mathbb{Z})$ may be lifted back to $\pi_\ast Y$. In particular, an uncountable rational vector space in $H_{2n}(X; \mathbb{Z})$ may be lifted back to $\pi_{2n}(\Sigma^\infty X)$ after suspension. \qed

Remark 4.6. Suppose $X$ is a $p$-complete nilpotent space such that $\pi_n X$ is the lowest homotopy group not of bounded $p$-torsion. The above argument shows that $H_{2n}(X; \mathbb{Z})$ contains an uncountable rational vector space. With more work, we can also show that $H_k(X; \mathbb{Z})$ is derived $p$-complete for $k \leq 2n - 2$ and thus cannot contain any rational classes. Note that when $X$ is
(n − 1)-connected, this follows immediately from Corollary 3.4 since \(H_k(X; \mathbb{Z})\) is in the stable range.

We can now prove our main theorem.

**Theorem 4.7.** If \(X\) is a \(p\)-complete nilpotent space, then \(\Sigma^\infty X\) is \(p\)-complete if and only if \(\pi_n X\) is bounded \(p\)-torsion for each \(n\).

Note that the torsion exponent of \(\pi_n X\) may vary with \(n\) and does not need to be bounded uniformly for all \(n\).

**Proof.** First assume that \(X\) is a \(p\)-complete nilpotent space with \(\pi_n X\) of bounded \(p\)-torsion for each \(n\); we can apply [BK72, Ch. VII, 4.3] to see that the Postnikov tower of \(X\) can be refined to a tower of principal fibrations whose fibers are Eilenberg–MacLane spaces for bounded \(p\)-torsion abelian groups. The category of bounded \(p\)-torsion abelian groups forms a Serre class, so Serre theory implies that \(H_*(X; \mathbb{Z}) \cong H_*(\Sigma^\infty X; \mathbb{Z})\) is degreewise bounded \(p\)-torsion. Hence, \(\Sigma^\infty X\) is \(p\)-complete by Theorem 3.3.

The converse is a consequence of Proposition 4.5: if \(\pi_n X\) is not all bounded torsion, then \(H_*(\Sigma^\infty X; \mathbb{Z})\) contains rational classes and thus cannot be derived \(p\)-complete, hence \(\Sigma^\infty X\) is not \(p\)-complete by Corollary 3.3. □

The next result generalizes [PSS17, Prop. 2.4].

**Corollary 4.8.** If \(X\) is a pointed connected space with degreewise finite homotopy groups, then the canonical map \((\Sigma^\infty X)\wedge_p \to \Sigma^\infty X^\wedge_p\) is an equivalence.

**Proof.** By [BK72, Ch. VII, 4.3], \(X\) is a \(\mathbb{Z}/p\)-good space and \(X^\wedge_p\) is a \(p\)-complete nilpotent space whose homotopy groups are all finite \(p\)-groups. Hence \(\Sigma^\infty X^\wedge_p\) is \(p\)-complete by Theorem 4.7. It follows that the natural map \((\Sigma^\infty X)\wedge_p \to \Sigma^\infty X^\wedge_p\) is an \(HF_p\)-equivalence between \(HF_p\)-local spectra, which implies the claim. □

**Corollary 4.9.** If \(X\) is a nilpotent space with \(H_n(X; \mathbb{Z})\) and \(\pi_n X\) derived \(p\)-complete for all \(n\), then \(H_n(X; \mathbb{Z})\) and \(\pi_n X\) are bounded \(p\)-torsion for all \(n\).

**Proof.** The assumption on \(\pi_n X\) implies that \(X\) is \(p\)-complete by Theorem 2.6, while the assumption on \(H_n(X; \mathbb{Z})\) shows that \(\Sigma^\infty X\) is \(p\)-complete, using Corollary 3.3. It thus follows from Theorem 4.7 that \(\pi_n X\) is degreewise bounded \(p\)-torsion, hence so is \(H_n(X; \mathbb{Z})\) by the proof of Theorem 4.7. □

The analogue of this corollary does not hold stably, as the following example demonstrates.

**Example 4.10.** Let \(M(\mathbb{Z}_p, n)\) be the Moore space for \(\mathbb{Z}_p\) in degree \(n \geq 2\). As \(H_*(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z})\) is isomorphic to \(\mathbb{Z}/p[n]\), we see that \(\Sigma^\infty M(\mathbb{Z}_p, n)\) is \(p\)-complete and consequently has derived \(p\)-complete stable homotopy groups and integral homology groups. However, \(H_n(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z}) \cong \mathbb{Z}_p\) is clearly not bounded \(p\)-torsion. In particular, \(M(\mathbb{Z}_p, n)\) is not \(p\)-complete, so this also shows that the assumption that \(X\) be \(p\)-complete cannot be dropped in Theorem 4.7.

5. **Rational classes in the stable homotopy groups of \(K(\mathbb{Z}_p, n)\)**

In this section, we present an example that illustrates how the rational classes in the stable homotopy groups of \(p\)-complete spaces arise. In fact, we present two different approaches: One using the integral homology of \(K(\mathbb{Z}_p, n)\), and one using Goodwillie calculus. The latter derivation is entirely stable and might be of independent interest.

First, we need a well-known auxiliary result; we outline a proof because we were unable to find a published reference for it. For an abelian group \(A\) and any \(k \geq 0\), let \(\text{Sym}^k_\mathbb{Z}(A)\) and \(\Lambda^k_\mathbb{Z}(A)\) be the \(k\)th symmetric power and the \(k\)th exterior power on \(A\), respectively.
Lemma 5.1. If $k > 1$, then $\Lambda_k^0(\mathbb{Z}_p)$ and the kernel of the multiplication map $\text{Sym}_k^1(\mathbb{Z}_p) \to \mathbb{Z}_p$ are uncountable rational vector spaces.

Proof. Since both symmetric and exterior power commute with base-change along $\mathbb{Z} \to \mathbb{Z}/l$ for any prime $l$, the indicated maps are isomorphisms mod $l$. Moreover, $\text{Sym}_k^1(A)$ and $\Lambda_k^0(A)$ are torsion-free whenever $A$ is, so both $\ker(\text{Sym}_k^1(\mathbb{Z}_p) \to \mathbb{Z}_p)$ and $\Lambda_k^0(\mathbb{Z}_p)$ are rational vector spaces. We may therefore base-change to $\mathbb{Q}$, where it is easy to verify that the $\mathbb{Q}$-dimension of the groups under consideration is that of $\mathbb{Q}_p$. □

Remark 5.2. A similar argument also shows that $\mathbb{Z}_p/\mathbb{Z}(p)$ is a rational vector space with the same $\mathbb{Q}$-dimension as $\mathbb{Q}_p$.

Proposition 5.3. For $n \geq 1$ and all $k > 1$, the stable homotopy group $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not $p$-complete.

First proof. Let $A$ be an abelian group and recall that $H_*(K(A,n);\mathbb{Z})$ equipped with the Pontryagin product is a graded commutative algebra such that squares of odd dimensional elements are zero; in fact, it has the structure of a graded divided power algebra, see [EML54, Car56] or more recently [Ric09]. With notation as in the previous lemma, the canonical isomorphism $A \to H_n(K(A,n);\mathbb{Z})$ thus extends to a natural homomorphism

$$
\begin{align*}
\phi^k(A,n) : & \Lambda_k^0(A) \to H_{kn}(K(A,n);\mathbb{Z}), & \text{if } n \text{ odd} \\
\phi^k(A,n) : & \text{Sym}_k^1(A) \to H_{kn}(K(A,n);\mathbb{Z}), & \text{if } n \text{ even}
\end{align*}
$$

for any $n,k > 0$. Moreover, we know that $\phi^k(A,n) \otimes \mathbb{Q}$ is a rational isomorphism. It then follows from Lemma 5.1 that, for $k > 1$, there exists an uncountable rational vector space which is mapped monomorphically to $H_{kn}(K(\mathbb{Z}_p,n);\mathbb{Z})$ via $\phi^k(\mathbb{Z}_p,n)$. We thus obtain an uncountable rational vector space in $H_{kn}(K(\mathbb{Z}_p,n);\mathbb{Z})$ that may be lifted back to give the desired uncountable rational vector space in $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p,n)$ for $k > 1$, as in the proof of Proposition 4.5. □

Second proof. We will compute the homotopy groups of $\Sigma^\infty K(\mathbb{Z}_p, n) \simeq \Sigma^\infty \Omega^\infty \Sigma^n H\mathbb{Z}_p$ using Goodwillie calculus [Goo03]. To this end, recall that the Goodwillie tower $(P_k)_{k \geq 1}$ associated to the functor $\Sigma^\infty \Omega^\infty : \text{Sp} \to \text{Sp}$ is assembled from fiber sequences of functors

$$
\begin{align*}
D_k & \longrightarrow P_k \longrightarrow P_{k-1}
\end{align*}
$$

with layers $D_k X \simeq X^{\wedge,k}_{\Sigma_k}$, where the homotopy orbits are formed with respect to the permutation action of $\Sigma_k$ (see for example [KM13] and the references given therein). Moreover, the Goodwillie tower $(P_k)_{k \geq 0}$ converges for connective spectra, i.e., there is a canonical equivalence

$$
\Sigma^\infty \Omega^\infty X \longrightarrow \lim_k P_k X
$$

for any connective $X \in \text{Sp}$. We will apply this in the case $X = \Sigma^n H\mathbb{Z}_p$.

In order to understand the layers, we start by analyzing $\pi_*(\Sigma^n H\mathbb{Z}_p)^\wedge,k$ via the universal coefficient theorem. We claim that, for all $k \geq 1$, the homotopy groups have the following form

$$
\pi_*(\Sigma^n H\mathbb{Z}_p)^\wedge,k \simeq \begin{cases} 
0 & * < nk \\
\mathbb{Z}_{p}^{\otimes,k} & * = nk \\
\text{finite} & * > nk.
\end{cases}
$$

(5.5)

By the universal coefficient theorem, we have an isomorphism

$$
\pi_*(\Sigma^n H\mathbb{Z}_p)^\wedge,k \cong (\pi_*(\Sigma^n H\mathbb{Z})^\wedge,k) \otimes \mathbb{Z}_{p}^{\otimes,k}.
$$

In degrees $* > nk$, the groups $\pi_*(\Sigma^n H\mathbb{Z})^\wedge,k$ are torsion, so the only torsion-free summand appears in degree $nk$. Since $\pi_*(\Sigma^n H\mathbb{Z})^\wedge,k$ is finitely generated over $\mathbb{Z}$ in each degree, the claim follows.
We now plug the formula (5.5) into the convergent homotopy orbit spectral sequence
\[ H_*(\Sigma_k, \pi_*(\Sigma^nHZ_p)^{\wedge k}) \implies \pi_*D_k(\Sigma^nHZ_p). \]
There are two cases: If \( t > nk \) or \( t < nk \), then the groups \( H_*(\Sigma_k, \pi_*(\Sigma^nHZ_p)^{\wedge k}) \) are finite or trivial for all \( s \), respectively. Let \( t = nk \). By Lemma 5.1 and (5.5), there is an isomorphism
\[ H_*(\Sigma_k, \pi_{nk}(\Sigma^nHZ_p)^{\wedge k}) \cong H_*(\Sigma_k, Z_p) \]
for \( s > 0 \) and \( H_0(\Sigma_k, \pi_{nk}(\Sigma^nHZ_p)^{\wedge k}) \) contains an uncountable rational vector space \( V_k \) if \( k > 1 \). To see the last statement, it suffices to compute the coinvariants on the rational submodule of \( \pi_{nk} \) by choosing a \( \mathbb{Q} \)-bases, as in the proof of Lemma 5.1. Furthermore, since the integral homology of \( \Sigma_k \) is finitely generated over \( \mathbb{Z} \) in each degree and rationally trivial in positive degrees, \( H_*(\Sigma_k, \pi_{nk}(\Sigma^nHZ_p)^{\wedge k}) \) is finite for all \( s > 0 \). Combining all this information, we obtain \( D_1\Sigma^nHZ_p \cong \Sigma^nHZ_p \) and for \( k > 1 \):
\[ \pi_*D_k(\Sigma^nHZ_p) \cong \begin{cases} 0 & * < nk \\ V_k \oplus W_k & * = nk \\ \text{finite} & * > nk, \end{cases} \tag{5.6} \]
where \( V_k \) is an uncountable rational vector space and \( W_k \) is some abelian group.

This allows us to derive a structural formula for \( \pi_*P_k\Sigma^nHZ_p \). Consider the following segment of the long exact sequence of homotopy groups associated to the fiber sequence (5.4):
\[ \cdots \implies \pi_{nk+1}P_k-1\Sigma^nHZ_p \overset{\pi_{nk}}{\longrightarrow} \pi_{nk}D_k\Sigma^nHZ_p \overset{\pi_{nk}}{\longrightarrow} \pi_{nk}P_k\Sigma^nHZ_p \implies \cdots \]
Because \( n \geq 1 \), it follows inductively from (5.6) that the term on the left is finite, hence \( V_k \) must be a summand in \( \pi_{nk}P_k\Sigma^nHZ_p \). This yields for all \( k \geq 1 \):
\[ \pi_*P_k\Sigma^nHZ_p \cong \begin{cases} 0 & * < n \\ V_l \oplus W_l' & * = nl \text{ with } 1 \leq l \leq k \\ \text{finite} & \text{otherwise}, \end{cases} \tag{5.7} \]
where \( V_l \) is as above for \( l \geq 2 \), and \( V_1 \) and \( W_l' \) are some abelian groups.

Finally, since \( D_k\Sigma^nHZ_p \) is \( nk \)-connective for all \( k \), the tower \( (\pi_*P_k\Sigma^nHZ_p)_{k \geq 0} \) stabilizes after finally many steps in each degree and hence is Mittag-Leffler. The corresponding Milnor sequence thus degenerates to an isomorphism
\[ \pi_*\Sigma^\infty K(\mathbb{Z}_p, n) \cong \pi_*\Sigma^\infty \Omega^\infty \Sigma^nHZ_p \cong \lim_k \pi_*P_k\Sigma^nHZ_p. \]
Therefore, the claim follows from (5.7). \( \square \)

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