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Barthel, Tobias; Bousfield, A. K.

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ON THE COMPARISON OF STABLE AND UNSTABLE \( p \)-COMPLETION

TOBIAS BARTHEL AND A. K. BOUSFIELD

Abstract. In this note we show that a \( p \)-complete nilpotent space \( X \) has a \( p \)-complete suspension spectrum if and only if its homotopy groups \( \pi_* X \) are bounded \( p \)-torsion. In contrast, if \( \pi_* X \) is not all bounded \( p \)-torsion, we locate uncountable rational vector spaces in the integral homology and in the stable homotopy groups of \( X \). To prove this, we establish a homological criterion for \( p \)-completeness of connective spectra. Moreover, we illustrate our results by studying the stable homotopy groups of \( K(\mathbb{Z}_p, n) \) via Goodwillie calculus.

1. Introduction

The notion of \( p \)-completion plays a fundamental role in algebra and topology, for it provides effective means to isolate and study \( p \)-primary properties. Applied to homotopy theory by Bousfield and Kan [BK72] as well as Sullivan [Sul74] and developed further in [Bou75, Bou79], it has since become one of the standard tools in the hands of algebraic topologists. However, there appears to be no general account of the comparison between unstable and stable \( p \)-completion in the literature, which is the question we address in the present note.

Our main goal is to characterize \( p \)-complete spaces which have \( p \)-complete suspension spectra:

**Theorem 4.7.** If \( X \) is a \( p \)-complete nilpotent space, then \( \Sigma^\infty X \) is \( p \)-complete if and only if \( \pi_n X \) is bounded \( p \)-torsion for each \( n \).

In fact, we exhibit a sharp dichotomy of \( p \)-complete nilpotent spaces: if \( X \) is a \( p \)-complete nilpotent space whose homotopy groups are not all bounded \( p \)-torsion, then the integral homology groups and stable homotopy groups of \( X \) both contain an uncountable rational vector space. As a consequence, we deduce that a nilpotent space \( X \) with derived \( p \)-complete integral homology and unstable homotopy must have both \( H_n(X; \mathbb{Z}) \) and \( \pi_n X \) of bounded \( p \)-torsion for all \( n \).

In a first step towards the proof of the theorem, we complement the second author’s characterization of \( p \)-complete spectra in terms of homotopy groups with an integral homological criterion, using a mild generalization of Serre classes appropriate for stable homotopy theory. This is in sharp contrast to the aforementioned fact that the integral homology of \( p \)-complete spaces is not well-behaved, and thus cannot be used to characterize \( p \)-completeness of spaces.

**Corollary 3.3.** A bounded below spectrum \( X \) is \( p \)-complete if and only if \( H_*(X; \mathbb{Z}) \) is derived \( p \)-complete in each degree.

In order to use this result to prove the theorem, we need to detect rational classes in the homology of \( p \)-complete spaces whose homotopy is not bounded \( p \)-torsion. This rests on the study of the integral homology of \( p \)-complete spheres. We end this note with a sample computation, illustrating how Goodwillie calculus allows us to detect rational classes in the stable homotopy groups of the Eilenberg–MacLane space \( K(\mathbb{Z}_p, n) \).

**Proposition 5.3.** For \( n \geq 1 \) and \( k > 1 \), the stable homotopy group \( \pi_{nk} \Sigma^\infty K(\mathbb{Z}_p, n) \) contains an uncountable rational vector space. In particular, \( \Sigma^\infty K(\mathbb{Z}_p, n) \) is not \( p \)-complete.

In fact, we also give a short alternative argument based on the integral homology of \( K(\mathbb{Z}_p, n) \).
Conventions. Throughout this paper, $p$ will be a fixed prime number and $\mathbb{Z}_p$ denotes the $p$-adic integers. We say that a nilpotent group $N$ is bounded $p$-torsion if there exists an $m$ such that for all $x \in N$, we have $x^{p^m} = 1$. A graded nilpotent group $N_*$ is said to be of bounded $p$-torsion if $N_k$ is bounded $p$-torsion for each $k$; however, we do not require a uniform bound. Whenever we are in a graded context, we indicate the degree of an abelian group $A$ by square brackets, i.e., $A[n]$ refers to $A$ placed in degree $n$. If $X$ is a topological space, then $H_*(X;A)$ is the reduced homology of $X$ with coefficients in $A$. For a space or spectrum $X$, we write $\tau_{\leq n}X = \tau_{\leq n+1}X$ for the $n$-th Postnikov section of $X$ and $\tau_{n+1}X = \tau_{\leq n}X$ for the fiber of the canonical map $X \to \tau_{\leq n}X$.

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2. Preliminaries on $p$-completion

We briefly recall the basic properties of $p$-completion for nilpotent groups, topological spaces, and spectra. With the exceptions of Lemma 2.2 and Proposition 2.4, this material is mostly taken from [BK72, Bou75, Bou79], and we refer to these sources as well as [HS99, MP12] for further references.

2.1. Algebraic $p$-completion for abelian groups. In general, the $p$-completion functor $M \mapsto \lim_{j\to\infty} M/p^j M$ on the category of abelian groups is neither left nor right exact, so one studies its zeroth and first left derived functors $L_0$ and $L_1$, which may be expressed as $L_0M = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$ and $L_1M = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$ by [BK72, Ch. VI, 2.1]. An abelian group $M$ is called derived $p$-complete (or $\mathbb{Z}$-complete or $L$-complete) if the natural completion map $M \to L_0M$ is an isomorphism. For each abelian group $M$, the map $M \to L_0M$ will then be the universal homomorphism from $M$ to a derived $p$-complete abelian group by [BK72, Ch. VI, 3.2]. We will denote the full subcategory of derived $p$-complete abelian groups by $C_p$.

Proposition 2.1. The category $C_p$ is a full abelian subcategory of $\text{Mod}_{\mathbb{Z}}$ closed under extensions and limits. Furthermore, for any $M \in \text{Mod}_{\mathbb{Z}}$ there is a short exact sequence

$$0 \to \text{lim}^1_{\mathbb{Z}} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^j, M) \to L_0M \to \lim_{j\to\infty} M/p^j M \to 0$$

relating derived $p$-completion to ordinary $p$-completion.

Proof. This is essentially proven in [BK72, Ch. VI, 2.1], but can also be deduced as a special case of [HS99, Thms. A.2 and A.6].

We will later make use of the following observation.

Lemma 2.2. If $A \in C_p$ is torsion, then $A$ is bounded $p$-torsion.

Proof. We give two proofs, a conceptual one and an elementary argument. First, any derived $p$-complete group $A$ has $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/1/p, A) = 0 = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/1/p, A)$ by [BK72, Ch. VI, 3.4], and hence $A$ satisfies $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, A) = 0 = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, A)$ since $\mathbb{Q}$ is a quotient of free $\mathbb{Z}/1/p$-modules. Thus, $A$ is a cotorsion group with no nontrivial divisible subgroups, so the Baer–Fomin theorem [Bae36] implies that $A$ is a bounded $p$-torsion group.

Second, suppose that the conclusion of the lemma is false, i.e., that there exists a sequence $(a_i)_{i\in\mathbb{N}}$ of elements of $A$ such that the order of $a_i$ is $p^i$. Set $x_j = \sum_{i=0}^{j-1} a_{2i+1}p^i$, then the element...
The lower central series of $Q$ is a direct sum of cyclic groups. By construction, $x$ is not $p$-torsion, which contradicts the fact that $A \to \lim_j A/p^j$ is surjective, forcing $\lim_j A/p^j$ to be $p$-torsion. \hfill \Box

**Remark 2.3.** By a theorem of Prüfer, the conclusion of the lemma implies that $A$ must in fact be a direct sum of cyclic $p$-groups.

### 2.2. Algebraic $p$-completion for nilpotent groups

Recall from [BK72, Ch. VI, §2] that the notion of derived $p$-completion can be extended to nilpotent groups, as follows: If $X_p^\wedge$ denotes the Bousfield–Kan $p$-completion of a nilpotent space $X$ as recalled in the next subsection, then we define the derived $p$-completion of the nilpotent group $N$ as $L_0 N = \pi_1 (K(N, 1)_p^\wedge)$ and $L_1 N = \pi_2 (K(N, 1)_p^\wedge)$. A nilpotent group $N$ is called derived $p$-complete if the completion map $N \to L_0 N$ is an isomorphism; for each nilpotent group $N$, the map $N \to L_0 N$ will then be the universal homomorphism from $N$ to a derived $p$-complete nilpotent group by [BK72, Ch. VI, 3.2]. We denote the category of derived $p$-complete nilpotent groups by $N_p$.

The inclusion functor $C_p \to N_p$ has a left adjoint given by taking a derived $p$-complete nilpotent group $N$ to the derived $p$-completion of its abelianization $L_0 (N/[N, N])$. Note that the unit of this adjunction is surjective, i.e., for any derived $p$-complete nilpotent group $N$, the canonical map $N \to L_0 (N/[N, N])$ is surjective. Indeed, since $L_0$ preserves epimorphisms of nilpotent groups, all maps in the following commutative diagram are surjective:

$$
\begin{array}{ccc}
N & \to & N/[N, N] \\
\downarrow & & \downarrow \\
L_0 N & \to & L_0 (N/[N, N]).
\end{array}
$$

We obtain the following generalization of Lemma 2.2:

**Proposition 2.4.** The following conditions are equivalent for $N \in N_p$:

1. $N$ is torsion.
2. $L_0 ([N, N])$ is torsion.
3. $N$ is bounded $p$-torsion.

**Proof.** The surjectivity of the map $N \to L_0 (N/[N, N])$ observed above immediately gives the implication (1) $\Rightarrow$ (2), while (3) $\Rightarrow$ (1) is trivial.

Assume that $L_0 (N/[N, N])$ is torsion and thus bounded $p$-torsion by Lemma 2.2. Consider the lower central series of $N$,

$$
N = \gamma_1 N \supseteq \gamma_2 N \supseteq \cdots \supseteq \gamma_m N = 1,
$$

with successive abelian quotients $Q_i (N) = \gamma_i N/\gamma_{i+1} N$. We claim that, for each $i \geq 1$, $Q_i (N)$ is a direct sum of a $p$-divisible group and a bounded $p$-torsion group. Indeed, we start with the abelianization $Q_1 (N) = N/[N, N]$ of $N$. Lemma 3.7 in [BK72, Ch. VI] implies that the kernel of the completion map $Q_1 (N) \to L_0 Q_1 (N)$ is $p$-divisible, so the claim holds for $Q_1 (N)$. The general case follows from this, because $\bigoplus_{i \geq 1} Q_i (N)$ is generated as a Lie algebra by $Q_1 (N)$. By [BK72, Ch. VI, 2.5], there is an exact sequence

$$
L_0 Q_i (N) \longrightarrow L_0 (N/\gamma_{i+1} N) \longrightarrow L_0 (N/\gamma_i N) \longrightarrow 1
$$

for any $i \geq 1$. Using the previous claim, $L_0 Q_i (N)$ is bounded $p$-torsion, so we see inductively that $L_0 (N/\gamma_i N)$ is bounded $p$-torsion for all $i \geq 1$, hence (3) holds. \hfill \Box

**Remark 2.5.** The implication (1) $\Rightarrow$ (3) in the previous proposition could also be proven more directly via the upper central series of $N$, whose quotients are known to be derived $p$-complete by [BK72, VI, 3.4(ii)], but this result would be insufficient for our later use.
2.3. Topological $p$-completion. In [BK72], Bousfield and Kan introduced the notion of $p$-completion for topological spaces, lifting the algebraic notion defined above to topology. In general, the $p$-completion of a space is difficult to describe, but the theory simplifies significantly for nilpotent spaces; in particular, in this case $p$-completion coincides with $\mathbb{F}_p$-localization [Bou75]. Furthermore, for nilpotent spaces with $\mathbb{F}_p$-homology of finite type, $p$-completion can be identified with $p$-profinite completion due to Sullivan [Sul74]. Similarly, the category of spectra admits (at least) two notions of $p$-completion, given either by $\mathbb{F}_p$-localization or, the one we will use here, localization at the mod $p$ Moore spectrum $S^0/p$, see [Bou79]. The next result summarizes the relation between these constructions and lists their basic properties.

Theorem 2.6 (Bousfield, Kan).

(1) A nilpotent space $X$ is $p$-complete if and only if $\pi_n X$ is derived $p$-complete for all $n \in \mathbb{N}$.
Moreover, the notions of $p$-completion and $\mathbb{F}_p$-localization coincide for nilpotent spaces.
(2) A spectrum $X$ is $p$-complete if and only if $\pi_n X$ is derived $p$-complete for all $n \in \mathbb{Z}$. If $X$ is bounded below, then $X$ is $p$-complete if and only if $X$ is $\mathbb{F}_p$-local.

Moreover, if $X$ is a nilpotent space or spectrum, then there exists a splittable short exact sequence computing the unstable or stable homotopy groups of its $p$-completion, respectively:

$$ 0 \longrightarrow L_0 \pi_n X \longrightarrow \pi_n (X^p_n) \longrightarrow L_1 \pi_{n-1} X \longrightarrow 0 $$

for any $n$, where $L_i (-) \cong \text{Ext}_{\mathbb{Z}}^{i-1} (\mathbb{Z}/p^\infty, -)$ are the derived functors of $p$-completion.

3. Generalized Serre theory

The full subcategory $\mathcal{C}_p$ of $\text{Mod}_\mathbb{Z}$ is not closed under subobjects or quotients, and thus does not form a Serre class in the usual sense. This necessitates a mild generalization of Serre’s mod $\mathcal{C}$ theory which we develop in this section.

Definition 3.1. A weak Serre class is a full subcategory $\mathcal{C} \subseteq \text{Mod}_\mathbb{Z}$ such that if

$$ A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5 $$

is an exact sequence in $\text{Mod}_\mathbb{Z}$ with $A_1, A_2, A_4, A_5 \in \mathcal{C}$, then also $A_3 \in \mathcal{C}$.

More explicitly, this means that $\mathcal{C} \subseteq \text{Mod}_\mathbb{Z}$ is a full additive subcategory closed under kernels, cokernels, and extensions. It follows that $\mathcal{C}$ is also closed under tensoring and $\text{Tor}_1^\mathbb{Z}$ with respect to finitely generated abelian groups. For instance, any Serre subcategory of $\text{Mod}_\mathbb{Z}$ is a weak Serre class, but the converse does not hold. The main example of interest to us here is the category $\mathcal{C}_p$ of derived $p$-complete abelian groups, see Proposition 2.1.

Proposition 3.2. Suppose $\mathcal{C}$ is a weak Serre class. If $X$ is a bounded below spectrum, then the following two conditions are equivalent:

(1) $\pi_n X \in \mathcal{C}$ for all $n \in \mathbb{Z}$.
(2) $H_n (X; \mathbb{Z}) \in \mathcal{C}$ for all $n \in \mathbb{Z}$.

Proof. Assume the first condition holds; we will argue via the Postnikov tower $(\tau_{\leq n} X)$ of $X$. For simplicity, we will write $H_\ast (Y)$ for the integral homology of a spectrum $Y$ throughout this proof.

To start with, we need to show that $H_\ast (HA) \in \mathcal{C}$ for $A \in \mathcal{C}$. Using the isomorphisms $H_\ast (HA) \cong H_\ast (HZ; A)$, the universal coefficient theorem gives a short exact sequence

$$ 0 \longrightarrow H_\ast (HZ) \otimes \mathbb{Z} A \longrightarrow H_\ast (HA) \longrightarrow \text{Tor}_1^\mathbb{Z} (H_{\ast-1} (HZ), A) \longrightarrow 0. $$

In each degree, the integral Steenrod algebra $H_\ast (HZ)$ is finitely generated over $\mathbb{Z}$, as follows from Serre theory for the class of finitely generated abelian groups. Therefore, the outer terms of this sequence are in $\mathcal{C}$. This shows $H_\ast (HA) \in \mathcal{C}$ as well.
Given \( n \in \mathbb{Z} \), we will now prove that \( H_n(X) \in \mathcal{C} \). Since \( H_n(\tau_{\leq n}X) = 0 = H_{n-1}(\tau_{\leq n}X) \) by connectivity, we see that \( H_n(X) \cong H_n(\tau_{\leq n}X) \). This reduces the claim to proving that \( H_*(\tau_{\leq n}X) \in \mathcal{C} \). This follows inductively, using the exact sequence

\[
H_{n+1}(\tau_{\leq n-1}X) \longrightarrow H_*(\Sigma^nH\tau_{n}X) \longrightarrow H_*(\tau_{\leq n}X) \longrightarrow H_*(\tau_{\leq n-1}X) \longrightarrow H_{n-1}(\Sigma^nH\tau_{n}X)
\]

associated to the fiber sequence \( \Sigma^nH\tau_{n}X \rightarrow \tau_{\leq n}X \rightarrow \tau_{\leq n-1}X \). Since \( H_k(H\tau_{n}X) \in \mathcal{C} \) for all \( k \in \mathbb{Z} \), this gives the implication \((1) \Rightarrow (2)\).

For the converse, consider the convergent Atiyah–Hirzebruch spectral sequence.

Since \( \pi_0S^0 \) is finitely generated over \( \mathbb{Z} \) for each \( t \in \mathbb{Z} \), \( H_*X;\pi_0S^0) \in \mathcal{C} \) for each bidegree \((s,t)\), hence \( \pi_nX \) is also in \( \mathcal{C} \) for all \( n \in \mathbb{Z} \).

When applied to the weak Serre class \( \mathcal{C}_p \), we obtain a homological characterization of \( p \)-completeness for bounded below spectra.

**Corollary 3.3.** For a bounded below spectrum \( X \), the following conditions are equivalent:

1. \( X \) is \( p \)-complete.
2. \( \pi_nX \) is derived \( p \)-complete for all \( n \).
3. \( H_*(X;\mathbb{Z}) \) is derived \( p \)-complete for all \( n \).

**Proof.** The equivalence of (1) and (2) is the content of Theorem 2.6(2), while (2) is equivalent to (3) by Proposition 3.2.

We deduce that the integral homology of \( p \)-complete spaces is well-behaved in the stable range.

**Corollary 3.4.** Suppose \( X \) is \( p \)-complete space. If \( X \) is \( n \)-connected, then \( H_k(X;\mathbb{Z}) \) is derived \( p \)-complete for all \( k \leq 2n \).

**Proof.** Since \( \pi_k\Sigma^\infty X \cong \pi_kX \) for \( k \leq 2n \) by the Freudenthal suspension theorem, Theorem 2.6 implies that \( \pi_\tau\Sigma_{\leq 2n}\Sigma^\infty X \) is derived \( p \)-complete in each degree, hence so is \( H_*\tau\Sigma_{\leq 2n}\Sigma^\infty X;\mathbb{Z} \) by Corollary 3.3. We thus get that \( H_k(X;\mathbb{Z}) \cong H_k(\Sigma^\infty X;\mathbb{Z}) \cong H_k(\tau\Sigma_{\leq 2n}\Sigma^\infty X;\mathbb{Z}) \) is derived \( p \)-complete for \( k \leq 2n \).

**Corollary 3.5.** For a bounded below spectrum \( X \), there exists a splittable short exact sequence computing the integral homology groups of its \( p \)-completion:

\[
0 \longrightarrow L_0H_n(X;\mathbb{Z}) \longrightarrow H_n(X;\mathbb{Z}) \longrightarrow L_1H_{n-1}(X;\mathbb{Z}) \longrightarrow 0
\]

for any \( n \).

**Proof.** Since the spectrum \( H\Sigma^\infty X_p \) is \( p \)-complete by Corollary 3.3, there is a canonical map \( (H\Sigma^\infty X)_p \rightarrow H\Sigma^\infty X_p \), and this map must be an equivalence because it is an \( H\Sigma^\infty \)-equivalence of \( p \)-complete bounded below spectra. Hence, the claim follows from Theorem 2.6.

From Corollary 3.5, we obtain the following description of the \( p \)-complete sphere spectrum as a Moore spectrum.

**Example 3.6.** There is a canonical equivalence \( S^0_p \sim M\mathbb{Z}_p \).

4. The comparison

In this section, we first study the relation between \( p \)-completion for spectra and spaces under the infinite loop space functor \( \Omega^\infty \), and then prove our main theorem.
4.1. Infinite loop spaces. It is easy to deduce from Theorem 2.6 the following relation between unstable and stable p-completion under Ω∞.

Proposition 4.1. For 0-connected spectra X and Y, we have:

(1) X is p-complete if and only if Ω∞X is p-complete.
(2) A map f: X → Y is an HFp-equivalence if and only if Ω∞f is an HFp-equivalence.
(3) The canonical comparison map (Ω∞X)p → Ω∞(X∞p) is an equivalence.

Proof. Since π∗Ω∞X ≃ π∗X and Ω∞X is nilpotent, the first claim is a direct consequence of Theorem 2.6. In order to prove (2), note that f is an HFp-equivalence if and only if the homotopy groups π∗ cof(f) of the cofiber of f are uniquely p-divisible. This is equivalent to the statement that the Fp-homology H∗(Ω∞ cof(f); Fp) is trivial. The Serre spectral sequence associated to the fiber sequence

\[ \Omega^\infty X \xrightarrow{\Omega^\infty f} \Omega^\infty Y \xrightarrow{\Omega^\infty \text{cof}(f)} \Omega^\infty \text{cof}(f) \]

thus shows that this happens if and only if Ω∞f is an HFp-equivalence.

Statement (1) implies that Ω∞(X∞p) is p-complete, so the map Ω∞(X) → Ω∞(X∞p) factors canonically through φ: (Ω∞X)p → Ω∞(X∞p), making the following diagram commute:

\[ \Omega^\infty X \xrightarrow{(\Omega^\infty X)p} (\Omega^\infty X)p \]

\[ \Omega^\infty(X^\infty p) \]

By Statement (2), both the horizontal and the diagonal map are HFp-equivalences, hence so is the vertical comparison map. \qed

Remark 4.2. Let Ω0∞ be the 0-component of Ω∞. The last part of the proposition can be strengthened to an equivalence (Ω0∞X)p → Ω0∞(X∞p) for any connective spectrum X such that π0X does not contain any copies of Z/p∞. To prove this directly, one may use the short exact sequences displayed at the end of Theorem 2.6.

4.2. Suspension spectra. We now turn to the comparison under Σ∞. In odd dimensions, the next result has also been observed in [BK72, Rem. VI.5.7], see also [MP12, Rem. 11.1.5].

Lemma 4.3. Let n ≥ 1 and write Snp for the p-completion of Sn. There exists an uncountable rational vector space in H2n(Snp; Z) which injects into H2n(K(Zp, n); Z) under the map Snp → τ≤nSn ≥ K(Zp, n).

Proof. Consider the following segment of the Serre long exact sequence for the fibration F → Snp → K(Zp, n):

\[ H_{2n}(F; Z) \xrightarrow{H_{2n}(Snp; Z)} H_{2n}(K(Zp, n); Z) \xrightarrow{H_{2n-1}(F; Z)} \ldots \]

Corollary 3.4 implies that H2n(F; Z) and H2n-1(F; Z) are derived p-complete. Recalling that HomZ(Q, A) = 0 = ExtZ∕λ(Q, A) whenever A is derived p-complete, we see that the natural map HomZ(Q, H2n(Snp; Z)) → HomZ(Q, H2n(K(Zp, n); Z)) is surjective. Thus, it will suffice to show that H2n(K(Zp, n); Z) contains an uncountable rational vector space, which will be verified in the homological proof of Proposition 5.3 below. \qed

Note that, because H∗(Snp; Fp) ≃ H∗(Sn; Fp) ≃ Fp[n], an application of the universal coefficient theorem shows that Hk(Snp; Z) is rational for all k > n.
Lemma 4.4. Suppose $N$ is a derived $p$-complete nilpotent (abelian) group and $n = 1 \ (n \geq 1)$. If $N$ is not bounded $p$-torsion, then there exists an element $x \in N$ of infinite order inducing a monomorphism $H_\ast(K(Z_p, n); \mathbb{Q}) \to H_\ast(K(N, n); \mathbb{Q})$.

Proof. By assumption on $N$ and Proposition 2.4, $L_0(N/[N, N])$ contains elements of infinite order. Let $\pi$ be such an element and let $x \in N$ be a lift of $\pi$. For the remainder of the proof we assume $n = 1$; the (easier) case $n \geq 2$ and $N$ abelian is proven similarly. The element $x$ induces a map

$$K(Z_p, 1) \longrightarrow K(N, 1) \longrightarrow K(L_0(N/[N, N]), 1)$$

such that the composite is injective on $\pi_1$. It follows that the rationalization $K(Z_p, 1)_\mathbb{Q} \to K(L_0(N/[N, N]), 1)_\mathbb{Q}$ of this map is split, hence the composite

$$H_\ast(K(Z_p, 1); \mathbb{Q}) \longrightarrow H_\ast(K(N, 1); \mathbb{Q}) \longrightarrow H_\ast(K(L_0(N/[N, N]), 1); \mathbb{Q})$$

is a split monomorphism, which implies the claim.

Proposition 4.5. If $X$ is a $p$-complete nilpotent space whose homotopy groups are not all bounded $p$-torsion, then the integral homology groups $H_\ast(X; \mathbb{Z})$ and the stable homotopy groups $\pi_\ast\Sigma^\infty X$ both contain an uncountable rational vector space.

Proof. Assume that $\pi_\ast X$ is not all bounded $p$-torsion, and let $\pi_\ast X$ be the lowest such group. It then follows from Lemma 4.4 that $\pi_\ast X$ contains a class $x$ of infinite order inducing a monomorphism $H_\ast(K(Z_p, n); \mathbb{Q}) \to H_\ast(K(\pi_\ast X, n); \mathbb{Q})$. Since the map $\tau_{\geq n}X \to X$ is a rational homology equivalence, any rational subgroup of $H_\ast(\tau_{\geq n}X; \mathbb{Q})$ must map monomorphically to $H_\ast(X; \mathbb{Q})$, so it suffices to prove the homological claim for $\tau_{\geq n}X$. The element $x$ yields a map $S^n_p \to \tau_{\geq n}X$ such that the composite $S^n_p \to \tau_{\geq n}X \to K(\pi_\ast X, n)$ factors as

$$\tau_{\geq n}X \longrightarrow \tau_{\leq n}\tau_{\geq n}X \simeq K(\pi_\ast X, n) \downarrow \downarrow$$

$$S^n_p \longrightarrow \tau_{\leq n}S^n_p \simeq K(Z_p, n).$$

It follows from Lemma 4.3 and the choice of $x$ that the induced homomorphism in homology

$$H_\ast(S^n_p; \mathbb{Q}) \longrightarrow H_\ast(\tau_{\geq n}X; \mathbb{Z}) \longrightarrow H_\ast(K(\pi_\ast X, n); \mathbb{Z})$$

maps an uncountable rational vector space monomorphically to $H_\ast(X; \mathbb{Q})$, hence so does the map $H_\ast(S^n_p; \mathbb{Q}) \to H_\ast(X; \mathbb{Q})$. This verifies the claim about the integral homology of $X$.

Recall that, for any connective spectrum $Y$, the Hurewicz map $\pi_\ast Y \to H_\ast(Y; \mathbb{Z})$ has kernel and cokernel of bounded torsion in each degree. Indeed, the fiber sequence $Y \wedge \tau_{>0}S^0 \to Y \to Y \wedge H\mathbb{Z}$ reduces this claim to showing that $\pi_\ast (Y \wedge \tau_{>0}S^0)$ is bounded torsion in each degree. This follows from the convergent Atiyah–Hirzebruch spectral sequence

$$H_\ast(Y; \pi_\ast \tau_{>0}S^0) \Rightarrow \pi_{s+t}(Y \wedge \tau_{>0}S^0),$$

because $H_\ast(Y; \pi_\ast \tau_{>0}S^0)$ is bounded torsion for all $s$ and $t$. Therefore, any rational vector space in $H_\ast(Y; \mathbb{Z})$ may be lifted back to $\pi_\ast Y$. In particular, an uncountable rational vector space in $H_\ast(X; \mathbb{Z})$ may be lifted back to $\pi_\ast(\Sigma^\infty X)$ after suspension.

Remark 4.6. Suppose $X$ is a $p$-complete nilpotent space such that $\pi_\ast X$ is the lowest homotopy group not of bounded $p$-torsion. The above argument shows that $H_\ast(X; \mathbb{Q})$ contains an uncountable rational vector space. With more work, we can also show that $H_\ast(X; \mathbb{Z})$ is derived $p$-complete for $k \leq 2n - 2$ and thus cannot contain any rational classes. Note that when $X$ is
It follows that the natural map \( \Sigma^\infty \) whose homotopy groups are all finite.

**Proof.** By [BK72, Ch. VII, 4.3], the canonical map \( \Sigma^\infty \rightarrow \Sigma^\infty \) is an equivalence.

**Corollary 4.8.** If \( X \) is a pointed connected space with degreewise finite homotopy groups, then the canonical map \( (\Sigma^\infty X)_p \rightarrow \Sigma^\infty X^\wedge_p \) is an equivalence.

**Proof.** By [BK72, Ch. VII, 4.3], \( X \) is a \( \mathbb{Z}/p \)-good space and \( X^\wedge_p \) is a \( p \)-complete nilpotent space whose homotopy groups are all finite \( p \)-groups. Hence \( \Sigma^\infty X^\wedge_p \) is \( p \)-complete by Theorem 4.7. It follows that the natural map \( (\Sigma^\infty X)_p \rightarrow \Sigma^\infty X^\wedge_p \) is an \( H\mathbb{F}_p \)-equivalence between \( H\mathbb{F}_p \)-local spectra, which implies the claim.

**Corollary 4.9.** If \( X \) is a nilpotent space with \( H_n(X; \mathbb{Z}) \) and \( \pi_n X \) derived \( p \)-complete for all \( n \), then \( H_n(X; \mathbb{Z}) \) and \( \pi_n X \) are bounded \( p \)-torsion for all \( n \).

**Proof.** The assumption on \( \pi_n X \) implies that \( X \) is \( p \)-complete by Theorem 2.6, while the assumption on \( H_n(X; \mathbb{Z}) \) shows that \( \Sigma^\infty X \) is \( p \)-complete, using Corollary 3.3. It thus follows from Theorem 4.7 that \( \pi_n X \) is degreewise bounded \( p \)-torsion, hence so is \( H_n(X; \mathbb{Z}) \) by the proof of Theorem 4.7.

The analogue of this corollary does not hold stably, as the following example demonstrates.

**Example 4.10.** Let \( M(\mathbb{Z}_p, n) \) be the Moore space for \( \mathbb{Z}_p \) in degree \( n \geq 2 \). As \( H_*(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}_p[n] \), we see that \( \Sigma^\infty M(\mathbb{Z}_p, n) \) is \( p \)-complete and consequently has derived \( p \)-complete stable homotopy groups and integral homology groups. However, \( H_n(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z}) \cong \mathbb{Z}_p \) is clearly not bounded \( p \)-torsion. In particular, \( M(\mathbb{Z}_p, n) \) is not \( p \)-complete, so this also shows that the assumption that \( X \) be \( p \)-complete cannot be dropped in Theorem 4.7.

### 5. Rational classes in the stable homotopy groups of \( K(\mathbb{Z}_p, n) \)

In this section, we present an example that illustrates how the rational classes in the stable homotopy groups of \( p \)-complete spaces arise. In fact, we present two different approaches: One using the integral homology of \( K(\mathbb{Z}_p, n) \), and one using Goodwillie calculus. The latter derivation is entirely stable and might be of independent interest.

First, we need a well-known auxiliary result; we outline a proof because we were unable to find a published reference for it. For an abelian group \( A \) and any \( k \geq 0 \), let \( \text{Sym}_k^\mathbb{Z}(A) \) and \( \Lambda_k^\mathbb{Z}(A) \) be the \( k \)th symmetric power and the \( k \)th exterior power on \( A \), respectively.
Lemma 5.1. If $k > 1$, then $\Lambda^k_\mathbb{Z}(\mathbb{Z}_p)$ and the kernel of the multiplication map $\text{Sym}^k_\mathbb{Z}(\mathbb{Z}_p) \to \mathbb{Z}_p$ are uncountable rational vector spaces.

Proof. Since both symmetric and exterior power commute with base-change along $\mathbb{Z} \to \mathbb{Z}/l$ for any prime $l$, the indicated maps are isomorphisms mod $l$. Moreover, $\text{Sym}^k_\mathbb{Z}(A)$ and $\Lambda^k_\mathbb{Z}(A)$ are torsion-free whenever $A$ is, so both $\ker(\text{Sym}^k_\mathbb{Z}(\mathbb{Z}_p) \to \mathbb{Z}_p)$ and $\Lambda^k_\mathbb{Z}(\mathbb{Z}_p)$ are rational vector spaces. We may therefore base-change to $\mathbb{Q}$, where it is easy to verify that the $\mathbb{Q}$-dimension of the groups under consideration is that of $\mathbb{Q}_p$. □

Remark 5.2. A similar argument also shows that $\mathbb{Z}_p/\mathbb{Z}(\mathbb{Z}_p)$ is a rational vector space with the same $\mathbb{Q}$-dimension as $\mathbb{Q}_p$.

Proposition 5.3. For $n \geq 1$ and all $k > 1$, the stable homotopy group $\pi_n \Sigma^n K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^n K(\mathbb{Z}_p, n)$ is not $p$-complete.

First proof. Let $A$ be an abelian group and recall that $H_*(K(A, n); \mathbb{Z})$ equipped with the Pontryagin product is a graded commutative algebra such that squares of odd dimensional elements are zero; in fact, it has the structure of a graded divided power algebra, see [EML54, Car56] or more recently [Ric09]. With notation as in the previous lemma, the canonical isomorphism $A \to H_n(K(A, n); \mathbb{Z})$ thus extends to a natural homomorphism

$$
\begin{align*}
\phi^k(A, n): \Lambda^k_\mathbb{Z}(A) &\longrightarrow H_{kn}(K(A, n); \mathbb{Z}), &\text{if } n \text{ odd} \\
\phi^k(A, n): \text{Sym}^k_\mathbb{Z}(A) &\longrightarrow H_{kn}(K(A, n); \mathbb{Z}), &\text{if } n \text{ even}
\end{align*}
$$

for any $n, k > 0$. Moreover, we know that $\phi^k(A, n) \otimes \mathbb{Q}$ is a rational isomorphism. It then follows from Lemma 5.1 that, for $k > 1$, there exists an uncountable rational vector space which is mapped monomorphically to $H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z})$ via $\phi^k(\mathbb{Z}_p, n)$. We thus obtain an uncountable rational vector space in $H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z})$ that may be lifted back to give the desired uncountable rational vector space in $\pi_n \Sigma^n K(\mathbb{Z}_p, n)$ for $k > 1$, as in the proof of Proposition 4.5. □

Second proof. We will compute the homotopy groups of $\Sigma^n K(\mathbb{Z}_p, n) \simeq \Sigma^n \Omega^n \Sigma^n H\mathbb{Z}_p$ using Goodwillie calculus [Goo03]. To this end, recall that the Goodwillie tower $(P_k)_{k \geq 1}$ associated to the functor $\Sigma^n \Omega^n : \text{Sp} \to \text{Sp}$ is assembled from fiber sequences of functors

$$
D_k \longrightarrow P_k \longrightarrow P_{k-1}
$$

with layers $D_k X \simeq X_{\Sigma^k}$, where the homotopy orbits are formed with respect to the permutation action of $\Sigma_k$ (see for example [KM13] and the references given therein). Moreover, the Goodwillie tower $(P_k)_{k \geq 0}$ converges for connective spectra, i.e., there is a canonical equivalence

$$
\Sigma^n \Omega^n X \longrightarrow \lim_k P_k X
$$

for any connective $X \in \text{Sp}$. We will apply this in the case $X = \Sigma^n H\mathbb{Z}_p$.

In order to understand the layers, we start by analyzing $\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge^k}$ via the universal coefficient theorem. We claim that, for all $k \geq 1$, the homotopy groups have the following form

$$
\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge^k} \cong \begin{cases} 
0 & \ast < nk \\
\mathbb{Z}_p^{\wedge^k} & \ast = nk \\
finito \ast > nk.
\end{cases}
$$

By the universal coefficient theorem, we have an isomorphism

$$
\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge^k} \cong (\pi_*(\Sigma^n H\mathbb{Z})^{\wedge^k}) \otimes \mathbb{Z}_p^{\wedge^k}.
$$

In degrees $\ast > nk$, the groups $\pi_*(\Sigma^n H\mathbb{Z})^{\wedge^k}$ are torsion, so the only torsion-free summand appears in degree $nk$. Since $\pi_*(\Sigma^n H\mathbb{Z})^{\wedge^k}$ is finitely generated over $\mathbb{Z}$ in each degree, the claim follows.
We now plug the formula (5.5) into the convergent homotopy orbit spectral sequence

$$H_\ast(\Sigma_k, \pi_\ast(\Sigma^n H\mathbb{Z}_p)\wedge^k) \Longrightarrow \pi_{s+t}D_k(\Sigma^n H\mathbb{Z}_p).$$

There are two cases: If $t > nk$ or $t < nk$, then the groups $H_\ast(\Sigma_k, \pi_\ast(\Sigma^n H\mathbb{Z}_p)\wedge^k)$ are finite or trivial for all $s$, respectively. Let $t = nk$. By Lemma 5.1 and (5.5), there is an isomorphism $H_\ast(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)\wedge^k) \cong H_\ast(\Sigma_k, \mathbb{Z}_p)$ for $s > 0$ and $H_\ast(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)\wedge^k)$ contains an uncountable rational vector space $V_k$ if $k > 1$. To see the last statement, it suffices to compute the coinvariants on the rational submodule of $\pi_{\ast\wedge^k}$ by choosing a $\mathbb{Q}$-bases, as in the proof of Lemma 5.1. Furthermore, since the integral homology of $\Sigma_k$ is finitely generated over $\mathbb{Z}$ in each degree and rationally trivial in positive degrees, $H_\ast(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)\wedge^k)$ is finite for all $s > 0$. Combining all this information, we obtain $D_1\Sigma^n H\mathbb{Z}_p \simeq \Sigma^n H\mathbb{Z}_p$ and for $k > 1$:

$$\pi_\ast D_k(\Sigma^n H\mathbb{Z}_p) \cong \begin{cases} 0 & * < nk \\ V_k \oplus W_k & * = nk \\ \text{finite} & * > nk, \end{cases}$$

(5.6)

where $V_k$ is an uncountable rational vector space and $W_k$ is some abelian group.

This allows us to derive a structural formula for $\pi_\ast P_k \Sigma^n H\mathbb{Z}_p$. Consider the following segment of the long exact sequence of homotopy groups associated to the fiber sequence (5.4):

$$\cdots \longrightarrow \pi_{nk+l} P_{k-1} \Sigma^n H\mathbb{Z}_p \longrightarrow \pi_{nk} D_k \Sigma^n H\mathbb{Z}_p \longrightarrow \pi_{nk} P_k \Sigma^n H\mathbb{Z}_p \longrightarrow \pi_{nk} \Sigma^n H\mathbb{Z}_p \longrightarrow \cdots$$

Because $n \geq 1$, it follows inductively from (5.6) that the term on the left is finite, hence $V_k$ must be a summand in $\pi_{nk} P_k \Sigma^n H\mathbb{Z}_p$. This yields for all $k \geq 1$:

$$\pi_\ast P_k \Sigma^n H\mathbb{Z}_p \cong \begin{cases} 0 & * < n \\ V_l \oplus W'_l & * = nl \text{ with } 1 \leq l \leq k \\ \text{finite} & \text{otherwise}, \end{cases}$$

(5.7)

where $V_l$ is as above for $l \geq 2$, and $V_l$ and $W'_l$ are some abelian groups.

Finally, since $D_k \Sigma^n H\mathbb{Z}_p$ is $nk$-connective for all $k$, the tower $(\pi_\ast P_k \Sigma^n H\mathbb{Z}_p)_{k \geq 0}$ stabilizes after finally many steps in each degree and hence is Mittag-Leffler. The corresponding Milnor sequence thus degenerates to an isomorphism

$$\pi_\ast \Sigma^\infty K(\mathbb{Z}_p, n) \cong \pi_\ast \Sigma^\infty \Omega^\infty \Sigma^n H\mathbb{Z}_p \cong \lim_k \pi_\ast P_k \Sigma^n H\mathbb{Z}_p.$$

Therefore, the claim follows from (5.7). \qed

References


ON THE COMPARISON OF STABLE AND UNSTABLE $p$-COMPLETION


Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 København Ø, Denmark

*E-mail address: tbarthel@math.ku.dk*

Department of Mathematics, Statistics and Computer Sciences, University of Illinois at Chicago, 851 S. Morgan St. (M/C 249), Chicago, IL 60607-7045, USA

*E-mail address: bous@uic.edu*