ON THE COMPARISON OF STABLE AND UNSTABLE $p$-COMPLETION

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Abstract. In this note we show that a $p$-complete nilpotent space $X$ has a $p$-complete suspension spectrum if and only if its homotopy groups $\pi_\ast X$ are bounded $p$-torsion. In contrast, if $\pi_\ast X$ is not all bounded $p$-torsion, we locate uncountable rational vector spaces in the integral homology and in the stable homotopy groups of $X$. To prove this, we establish a homological criterion for $p$-completeness of connective spectra. Moreover, we illustrate our results by studying the stable homotopy groups of $K(\mathbb{Z}_p, n)$ via Goodwillie calculus.

1. Introduction

The notion of $p$-completion plays a fundamental role in algebra and topology, for it provides effective means to isolate and study $p$-primary properties. Applied to homotopy theory by Bousfield and Kan [BK72] as well as Sullivan [Sul74] and developed further in [Bou75, Bou79], it has since become one of the standard tools in the hands of algebraic topologists. However, there appears to be no general account of the comparison between unstable and stable $p$-completion in the literature, which is the question we address in the present note.

Our main goal is to characterize $p$-complete spaces which have $p$-complete suspension spectra:

Theorem 4.7. If $X$ is a $p$-complete nilpotent space, then $\Sigma^\infty X$ is $p$-complete if and only if $\pi_n X$ is bounded $p$-torsion for each $n$.

In fact, we exhibit a sharp dichotomy of $p$-complete nilpotent spaces: if $X$ is a $p$-complete nilpotent space whose homotopy groups are not all bounded $p$-torsion, then the integral homology groups and stable homotopy groups of $X$ both contain an uncountable rational vector space. As a consequence, we deduce that a nilpotent space $X$ with derived $p$-complete integral homology and unstable homotopy must have both $H_n(X; \mathbb{Z})$ and $\pi_n X$ of bounded $p$-torsion for all $n$.

In a first step towards the proof of the theorem, we complement the second author’s characterization of $p$-complete spectra in terms of homotopy groups with an integral homological criterion, using a mild generalization of Serre classes appropriate for stable homotopy theory. This is in sharp contrast to the aforementioned fact that the integral homology of $p$-complete spaces is not well-behaved, and thus cannot be used to characterize $p$-completeness of spaces.

Corollary 3.3. A bounded below spectrum $X$ is $p$-complete if and only if $H_\ast(X; \mathbb{Z})$ is derived $p$-complete in each degree.

In order to use this result to prove the theorem, we need to detect rational classes in the homology of $p$-complete spaces whose homotopy is not bounded $p$-torsion. This rests on the study of the integral homology of $p$-complete spheres. We end this note with a sample computation, illustrating how Goodwillie calculus allows us to detect rational classes in the stable homotopy groups of the Eilenberg–MacLane space $K(\mathbb{Z}_p, n)$.

Proposition 5.3. For $n \geq 1$ and $k > 1$, the stable homotopy group $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not $p$-complete.

In fact, we also give a short alternative argument based on the integral homology of $K(\mathbb{Z}_p, n)$.

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Conventions. Throughout this paper, $p$ will be a fixed prime number and $\mathbb{Z}_p$ denotes the $p$-adic integers. We say that a nilpotent group $N$ is bounded $p$-torsion if there exists an $m$ such that for all $x \in N$, we have $x^{p^m} = 1$. A graded nilpotent group $N_*$ is said to be of bounded $p$-torsion if $N_k$ is bounded $p$-torsion for each $k$; however, we do not require a uniform bound. Whenever we are in a graded context, we indicate the degree of an abelian group $A$ by square brackets, i.e., $A[n]$ refers to $A$ placed in degree $n$. If $X$ is a topological space, then $H_*(X; A)$ is the reduced homology of $X$ with coefficients in $A$. For a space or spectrum $X$, we write $\tau_{\leq n}X = \tau_{\leq n-1}X$ and $\tau_{\geq n+1}X = \tau_{\geq n}X$ for the $n$-th Postnikov section of $X$ and $\tau_{\geq n+1}X = \tau_{\geq n}X$ for the fiber of the canonical map $X \to \tau_{\leq n}X$.

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2. Preliminaries on $p$-completion

We briefly recall the basic properties of $p$-completion for nilpotent groups, topological spaces, and spectra. With the exceptions of Lemma 2.2 and Proposition 2.4, this material is mostly taken from [BK72, Bou75, Bou79], and we refer to these sources as well as [HS99, MP12] for further references.

2.1. Algebraic $p$-completion for abelian groups. In general, the $p$-completion functor $M \mapsto \lim^i M/p^i M$ on the category of abelian groups is neither left nor right exact, so one studies its zeroth and first left derived functors $L_0$ and $L_1$, which may be expressed as $L_0 M = \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$ and $L_1 M = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$ by [BK72, Ch. VI, 2.1]. An abelian group $M$ is called derived $p$-complete (or $\text{Ext}$-$p$-complete or $L$-complete) if the natural completion map $M \to L_0 M$ is an isomorphism. For each abelian group $M$, the map $M \to L_0 M$ will then be the universal homomorphism from $M$ to a derived $p$-complete abelian group by [BK72, Ch. VI, 3.2]. We will denote the full subcategory of derived $p$-complete abelian groups by $C_p$.

Proposition 2.1. The category $C_p$ is a full abelian subcategory of $\text{Mod}_{\mathbb{Z}}$ closed under extensions and limits. Furthermore, for any $M \in \text{Mod}_{\mathbb{Z}}$ there is a short exact sequence

$$0 \longrightarrow \lim^1 \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^i, M) \longrightarrow L_0 M \longrightarrow \lim M/p^i M \longrightarrow 0$$

relating derived $p$-completion to ordinary $p$-completion.

Proof. This is essentially proven in [BK72, Ch. VI, 2.1], but can also be deduced as a special case of [HS99, Thms. A.2 and A.6].

We will later make use of the following observation.

Lemma 2.2. If $A \in C_p$ is torsion, then $A$ is bounded $p$-torsion.

Proof. We give two proofs, a conceptual one and an elementary argument. First, any derived $p$-complete group $A$ has $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/[1/p], A) = 0 = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/[1/p], A)$ by [BK72, Ch. VI, 3.4], and hence $A$ satisfies $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, A) = 0 = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, A)$ since $\mathbb{Q}$ is a quotient of free $\mathbb{Z}/[1/p]$-modules. Thus, $A$ is a cotorsion group with no nontrivial divisible subgroups, so the Baer–Fomin theorem [Bae36] implies that $A$ is a bounded $p$-torsion group.

Second, suppose that the conclusion of the lemma is false, i.e., that there exists a sequence $(a_i)_{i \in \mathbb{N}}$ of elements of $A$ such that the order of $a_i$ is $p^i$. Set $x_j = \sum_{i=0}^{j-1} a_{2i+1} p^i$, then the element
\(x = (x_1, x_2, x_3, \ldots) \in \prod_{i \in \mathbb{N}} A\) lies in \(\lim_j A/p^j\). By construction, \(x\) is not \(p\)-torsion, which contradicts the fact that \(A \to \lim_j A/p^j\) is surjective, forcing \(\lim_j A/p^j\) to be \(p\)-torsion. \(\Box\)

**Remark 2.3.** By a theorem of Prüfer, the conclusion of the lemma implies that \(A\) must in fact be a direct sum of cyclic \(p\)-groups.

### 2.2. Algebraic \(p\)-completion for nilpotent groups.

Recall from [BK72, Ch. VI, §2] that the notion of derived \(p\)-completion can be extended to nilpotent groups, as follows: If \(X^p\) denotes the Bousfield–Kan \(p\)-completion of a nilpotent space \(X\) as recalled in the next subsection, then we define the derived \(p\)-completion of the nilpotent group \(N\) as \(L_0N = \pi_1(K(N, 1)^p)\) and \(L_1N = \pi_2(K(N, 1)^p)\). A nilpotent group \(N\) is called derived \(p\)-complete if the completion map \(N \to L_0N\) is an isomorphism; for each nilpotent group \(N\), the map \(N \to L_0N\) will then be the universal homomorphism from \(N\) to a derived \(p\)-complete nilpotent group by [BK72, Ch. VI, 3.2]. We denote the category of derived \(p\)-complete nilpotent groups by \(N_p\).

The inclusion functor \(C_p \to N_p\) has a left adjoint given by taking a derived \(p\)-complete nilpotent group \(N\) to the derived \(p\)-completion of its abelianization \(L_0(N/[N, N])\). Note that the unit of this adjunction is surjective, i.e., for any derived \(p\)-complete nilpotent group \(N\), the canonical map \(N \to L_0(N/[N, N])\) is surjective. Indeed, since \(L_0\) preserves epimorphisms of nilpotent groups, all maps in the following commutative diagram are surjective:

\[
\begin{array}{ccc}
N & \longrightarrow & N/[N, N] \\
\downarrow & & \downarrow \\
L_0N & \longrightarrow & L_0(N/[N, N]).
\end{array}
\]

We obtain the following generalization of Lemma 2.2:

**Proposition 2.4.** The following conditions are equivalent for \(N \in N_p\):

1. \(N\) is torsion.
2. \(L_0(N/[N, N])\) is torsion.
3. \(N\) is bounded \(p\)-torsion.

**Proof.** The surjectivity of the map \(N \to L_0(N/[N, N])\) observed above immediately gives the implication (1) \(\Rightarrow\) (2), while (3) \(\Rightarrow\) (1) is trivial.

Assume that \(L_0(N/[N, N])\) is torsion and thus bounded \(p\)-torsion by Lemma 2.2. Consider the lower central series of \(N\):

\[N = \gamma_1 N \supseteq \gamma_2 N \supseteq \ldots \supseteq \gamma_m N = 1,\]

with successive abelian quotients \(Q_i(N) = \gamma_i N/\gamma_{i+1} N\). We claim that, for each \(i \geq 1\), \(Q_i(N)\) is a direct sum of a \(p\)-divisible group and a bounded \(p\)-torsion group. Indeed, we start with the abelianization \(Q_1(N) = N/[N, N]\) of \(N\). Lemma 3.7 in [BK72, Ch. VI] implies that the kernel of the completion map \(Q_1(N) \to L_0Q_1(N)\) is \(p\)-divisible, so the claim holds for \(Q_1(N)\). The general case follows from this, because \(\bigoplus_{i \geq 1} Q_i(N)\) is generated as a Lie algebra by \(Q_1(N)\). By [BK72, Ch. VI, 2.5], there is an exact sequence

\[
\begin{array}{cccc}
L_0Q_i(N) & \longrightarrow & L_0(N/\gamma_{i+1} N) & \longrightarrow & L_0(N/\gamma_i N) & \longrightarrow & 1
\end{array}
\]

for any \(i \geq 1\). Using the previous claim, \(L_0Q_i(N)\) is bounded \(p\)-torsion, so we see inductively that \(L_0(N/\gamma_i N)\) is bounded \(p\)-torsion for all \(i \geq 1\), hence (3) holds. \(\Box\)

**Remark 2.5.** The implication (1) \(\Rightarrow\) (3) in the previous proposition could also be proven more directly via the upper central series of \(N\), whose quotients are known to be derived \(p\)-complete by [BK72, VI. 3.4(ii)], but this result would be insufficient for our later use.
2.3. Topological $p$-completion. In [BK72], Bousfield and Kan introduced the notion of $p$-completion for topological spaces, lifting the algebraic notion defined above to topology. In general, the $p$-completion of a space is difficult to describe, but the theory simplifies significantly for nilpotent spaces; in particular, in this case $p$-completion coincides with $H\mathbb{F}_p$-localization [Bou75]. Furthermore, for nilpotent spaces with $\mathbb{F}_p$-homology of finite type, $p$-completion can be identified with $p$-profinite completion due to Sullivan [Sul74]. Similarly, the category of spectra admits (at least) two notions of $p$-completion, given either by $H\mathbb{F}_p$-localization or, the one we will use here, localization at the mod $p$ Moore spectrum $S^0/p$, see [Bou79]. The next result summarizes the relation between these constructions and lists their basic properties.

**Theorem 2.6** (Bousfield, Kan).

(1) A nilpotent space $X$ is $p$-complete if and only if $\pi_n X$ is derived $p$-complete for all $n \in \mathbb{N}$. Moreover, the notions of $p$-completion and $H\mathbb{F}_p$-localization coincide for nilpotent spaces.

(2) A spectrum $X$ is $p$-complete if and only if $\pi_n X$ is derived $p$-complete for all $n \in \mathbb{Z}$. If $X$ is bounded below, then $X$ is $p$-complete if and only if $X$ is $H\mathbb{F}_p$-local.

Moreover, if $X$ is a nilpotent space or spectrum, then there exists a splittable short exact sequence computing the unstable or stable homotopy groups of its $p$-completion, respectively:

\[ 0 \longrightarrow L_0 \pi_n X \longrightarrow \pi_n(X^p_n) \longrightarrow L_1 \pi_{n-1} X \longrightarrow 0 \]

for any $n$, where $L_i(-) \cong \text{Ext}_{\mathbb{Z}}^{i-1}(\mathbb{Z}/p^\infty, -)$ are the derived functors of $p$-completion.

3. Generalized Serre theory

The full subcategory $C_p$ of $\text{Mod}_{\mathbb{Z}}$ is not closed under subobjects or quotients, and thus does not form a Serre class in the usual sense. This necessitates a mild generalization of Serre’s mod $C$ theory which we develop in this section.

**Definition 3.1.** A weak Serre class is a full subcategory $C \subseteq \text{Mod}_{\mathbb{Z}}$ such that if

\[ A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5 \]

is an exact sequence in $\text{Mod}_{\mathbb{Z}}$ with $A_1, A_2, A_4, A_5 \in C$, then also $A_3 \in C$.

More explicitly, this means that $C \subseteq \text{Mod}_{\mathbb{Z}}$ is a full additive subcategory closed under kernels, cokernels, and extensions. It follows that $C$ is also closed under tensoring and $\text{Tor}^1$ with respect to finitely generated abelian groups. For instance, any Serre subcategory of $\text{Mod}_{\mathbb{Z}}$ is a weak Serre class, but the converse does not hold. The main example of interest to us here is the category $C_p$ of derived $p$-complete abelian groups, see Proposition 2.1.

**Proposition 3.2.** Suppose $C$ is a weak Serre class. If $X$ is a bounded below spectrum, then the following two conditions are equivalent:

(1) $\pi_n X \in C$ for all $n \in \mathbb{Z}$.

(2) $H_n(X; \mathbb{Z}) \in C$ for all $n \in \mathbb{Z}$.

**Proof.** Assume the first condition holds; we will argue via the Postnikov tower $(\tau_{\leq n} X)$ of $X$. For simplicity, we will write $H_*(Y)$ for the integral homology of a spectrum $Y$ throughout this proof.

To start with, we need to show that $H_*(HA) \in C$ for $A \in C$. Using the isomorphisms $H_*(HA) \cong H_*(HZ; A)$, the universal coefficient theorem gives a short exact sequence

\[ 0 \longrightarrow H_*(HZ) \otimes \mathbb{Z} A \longrightarrow H_*(HA) \longrightarrow \text{Tor}^1(H_{*-1}(HZ), A) \longrightarrow 0. \]

In each degree, the integral Steenrod algebra $H_*(HZ)$ is finitely generated over $\mathbb{Z}$, as follows from Serre theory for the class of finitely generated abelian groups. Therefore, the outer terms of this sequence are in $C$. This shows $H_*(HA) \in C$ as well.
Given \( n \in \mathbb{Z} \), we will now prove that \( H_n(X) \in C \). Since \( H_n(\tau_{\leq n}X) = 0 = H_{n-1}(\tau_{\leq n}X) \) by connectivity, we see that \( H_n(X) \cong H_n(\tau_{\leq n}X) \). This reduces the claim to proving that \( H_n(\tau_{\leq n}X) \in C \). This follows inductively, using the exact sequence

\[
H_{n+1}(\tau_{\leq n-1}X) \rightarrow H_n(\Sigma^nH\pi_nX) \rightarrow H_n(\tau_{\leq n}X) \rightarrow H_n(\tau_{\leq n-1}X) \rightarrow H_{n+1}(\Sigma^nH\pi_nX)
\]

associated to the fiber sequence \( \Sigma^nH\pi_nX \rightarrow \tau_{\leq n}X \rightarrow \tau_{\leq n-1}X \). Since \( H_k(H\pi_nX) \in C \) for all \( k \in \mathbb{Z} \), this gives the implication \((1) \Rightarrow (2)\).

For the converse, consider the convergent Atiyah–Hirzebruch spectral sequence

\[
E^2_{s,t} \cong H_s(X; \pi_tS^0) \Rightarrow \pi_{s+t}X.
\]

Since \( \pi_tS^0 \) is finitely generated over \( \mathbb{Z} \) for each \( t \in \mathbb{Z} \), \( H_s(X; \pi_tS^0) \in C \) for each bidegree \((s,t)\), hence \( \pi_nX \) is also in \( C \) for all \( n \in \mathbb{Z} \).

When applied to the weak Serre class \( C_p \), we obtain a homological characterization of \( p \)-completeness for bounded below spectra.

**Corollary 3.3.** For a bounded below spectrum \( X \), the following conditions are equivalent:

1. \( X \) is \( p \)-complete.
2. \( \pi_nX \) is derived \( p \)-complete for all \( n \).
3. \( H_n(X; \mathbb{Z}) \) is derived \( p \)-complete for all \( n \).

**Proof.** The equivalence of (1) and (2) is the content of Theorem 2.6(2), while (2) is equivalent to (3) by Proposition 3.2.

We deduce that the integral homology of \( p \)-complete spaces is well-behaved in the stable range.

**Corollary 3.4.** Suppose \( X \) is \( p \)-complete space. If \( X \) is \( n \)-connected, then \( H_k(X; \mathbb{Z}) \) is derived \( p \)-complete for all \( k \leq 2n \).

**Proof.** Since \( \pi_k\Sigma^\infty X \cong \pi_kX \) for \( k \leq 2n \) by the Freudenthal suspension theorem, Theorem 2.6 implies that \( \pi_{n-2n}\Sigma^\infty X \) is derived \( p \)-complete in each degree, hence so is \( H_{n-2n}\Sigma^\infty X; \mathbb{Z} \) by Corollary 3.3. We thus get that \( H_k(X; \mathbb{Z}) \cong H_k(\Sigma^\infty X; \mathbb{Z}) \cong H_k(\tau_{\leq 2n}\Sigma^\infty X; \mathbb{Z}) \) is derived \( p \)-complete for \( k \leq 2n \).

**Corollary 3.5.** For a bounded below spectrum \( X \), there exists a splittable short exact sequence computing the integral homology groups of its \( p \)-completion:

\[
0 \rightarrow L_0H_n(X; \mathbb{Z}) \rightarrow H_n(X^{\wedge}_p; \mathbb{Z}) \rightarrow L_1H_{n-1}(X; \mathbb{Z}) \rightarrow 0
\]

for any \( n \).

**Proof.** Since the spectrum \( H\mathbb{Z} \wedge X^{\wedge}_p \) is \( p \)-complete by Corollary 3.3, there is a canonical map \( (H\mathbb{Z} \wedge X)_p^{\wedge} \rightarrow H\mathbb{Z} \wedge X^{\wedge}_p \), and this map must be an equivalence because it is an \( HF_p \)-equivalence of \( p \)-complete bounded below spectra. Hence, the claim follows from Theorem 2.6.

From Corollary 3.5, we obtain the following description of the \( p \)-complete sphere spectrum as a Moore spectrum.

**Example 3.6.** There is a canonical equivalence \( S^0_p \sim \rightarrow M\mathbb{Z}_p \).

4. The comparison

In this section, we first study the relation between \( p \)-completion for spectra and spaces under the infinite loop space functor \( \Omega^\infty \), and then prove our main theorem.
4.1. Infinite loop spaces. It is easy to deduce from Theorem 2.6 the following relation between unstable and stable $p$-completion under $Ω^∞$.

**Proposition 4.1.** For 0-connected spectra $X$ and $Y$, we have:

1. $X$ is $p$-complete if and only if $Ω^∞ X$ is $p$-complete.
2. A map $f: X → Y$ is an $H^p_F$-equivalence if and only if $Ω^∞ f$ is an $H^p_F$-equivalence.
3. The canonical comparison map $(Ω^∞ X) ≃_p Ω^∞(X) ≃_p$ is an equivalence.

**Proof.** Since $π_∗Ω^∞ X ≃ π_∗ X$ and $Ω^∞ X$ is nilpotent, the first claim is a direct consequence of Theorem 2.6. In order to prove (2), note that $f$ is an $H^p_F$-equivalence if and only if the homotopy groups $π_∗ cof(f)$ of the cofiber of $f$ are uniquely $p$-divisible. This is equivalent to the statement that the $F_p$-homology $H_*(Ω^∞ cof(f); F_p)$ is trivial. The Serre spectral sequence associated to the fiber sequence

$$Ω^∞ X → Ω^∞ f → Ω^∞ Y → Ω^∞ cof(f)$$

thus shows that this happens if and only if $Ω^∞ f$ is an $H^p_F$-equivalence.

Statement (1) implies that $Ω^∞ (X) ≃_p Ω^∞(X_p) ≃_p$ factors canonically through $φ: (Ω^∞ X) ≃_p → Ω^∞(X_p)$, making the following diagram commute:

$$Ω^∞ X → (Ω^∞ X) ≃_p$$

By Statement (2), both the horizontal and the diagonal map are $H^p_F$-equivalences, hence so is the vertical comparison map. □

**Remark 4.2.** Let $Ω^∞_0$ be the 0-component of $Ω^∞$. The last part of the proposition can be strengthened to an equivalence $(Ω^∞_0 X) ≃_p Ω^∞_0(Ω^∞ X_p)$ for any connective spectrum $X$ such that $π_0 X$ does not contain any copies of $Z/p^∞$. To prove this directly, one may use the short exact sequences displayed at the end of Theorem 2.6.

4.2. Suspension spectra. We now turn to the comparison under $Σ^∞$. In odd dimensions, the next result has also been observed in [BK72, Rem. VI.5.7], see also [MP12, Rem. 11.1.5].

**Lemma 4.3.** Let $n ≥ 1$ and write $S^n_p$ for the $p$-completion of $S^n$. There exists an uncountable rational vector space in $H_{2n}(S^n_p; Z)$ which injects into $H_{2n}(K(Z_p, n); Z)$ under the map $S^n_p → τ_{≥ n} S^n_p ≃ K(Z_p, n)$.

**Proof.** Consider the following segment of the Serre long exact sequence for the fibration $F → S^n_p → K(Z_p, n)$:

$$H_{2n}(F; Z) → H_{2n}(S^n_p; Z) → H_{2n}(K(Z_p, n); Z) → H_{2n-1}(F; Z) → \ldots$$

Corollary 3.4 implies that $H_{2n}(F; Z)$ and $H_{2n-1}(F; Z)$ are derived $p$-complete. Recalling that $Hom_Z(Q, A) = 0 = Ext^1_Z(Q, A)$ whenever $A$ is derived $p$-complete, we see that the natural map $Hom_Z(Q, H_{2n}(S^n_p; Z)) → Hom_Z(Q, H_{2n}(K(Z_p, n); Z))$ is surjective. Thus, it will suffice to show that $H_{2n}(K(Z_p, n); Z)$ contains an uncountable rational vector space, which will be verified in the homological proof of Proposition 5.3 below. □

Note that, because $H_∗(S^n_p; F_p) ≃ H_∗(S^n; F_p) ≃ F_p [n]$, an application of the universal coefficient theorem shows that $H_k(S^n_p; Z)$ is rational for all $k > n$. 


Lemma 4.4. Suppose $N$ is a derived $p$-complete nilpotent (abelian) group and $n = 1$ ($n \geq 1$). If $N$ is not bounded $p$-torsion, then there exists an element $x \in N$ of infinite order inducing a monomorphism $H_0(K(\mathbb{Z}_p, n); \mathbb{Q}) \to H_0(K(N, n); \mathbb{Q})$.

Proof. By assumption on $N$ and Proposition 2.4, $L_0(N/[N, N])$ contains elements of infinite order. Let $\pi$ be such an element and let $x \in N$ be a lift of $\pi$. For the remainder of the proof we assume $n = 1$; the (easier) case $n \geq 2$ and $N$ abelian is proven similarly. The element $x$ induces a map

$$K(\mathbb{Z}_p, 1) \longrightarrow K(N, 1) \longrightarrow K(L_0(N/[N, N]), 1)$$

such that the composite is injective on $\pi_1$. It follows that the rationalization $K(\mathbb{Z}_p, 1)_{\mathbb{Q}} \to K(L_0(N/[N, N]), 1)_{\mathbb{Q}}$ of this map is split, hence the composite

$$H_*(K(\mathbb{Z}_p, 1); \mathbb{Q}) \longrightarrow H_*(K(N, 1); \mathbb{Q}) \longrightarrow H_*(K(L_0(N/[N, N]), 1); \mathbb{Q})$$

is a split monomorphism, which implies the claim. □

Proposition 4.5. If $X$ is a $p$-complete nilpotent space whose homotopy groups are not all bounded $p$-torsion, then the integral homology groups $H_*(X; \mathbb{Z})$ and the stable homotopy groups $\pi_*\Sigma^\infty X$ both contain an uncountable rational vector space.

Proof. Assume that $\pi_0X$ is not all bounded $p$-torsion, and let $\pi_0X$ be the lowest such group. It then follows from Lemma 4.4 that $\pi_0X$ contains a class $x$ of infinite order inducing a monomorphism $H_*(K(\mathbb{Z}_p, n); \mathbb{Q}) \to H_*(K(\pi_0X, n); \mathbb{Q})$. Since the map $\tau_{\geq n}X \to X$ is a rational homology equivalence, any rational subgroup of $H_*(\tau_{\geq n}X; \mathbb{Q})$ must map monomorphically to $H_*(X; \mathbb{Q})$, so it suffices to prove the homological claim for $\tau_{\geq n}X$. The element $x$ yields a map $S^n_p \to \tau_{\geq n}X$ such that the composite $S^n_p \to \tau_{\geq n}X \to K(\pi_0X, n)$ factors as

$$\tau_{\geq n}X \xrightarrow{x} \tau_{\leq n}\tau_{\geq n}X \cong K(\pi_0X, n)$$

It follows from Lemma 4.3 and the choice of $x$ that the induced homomorphism in homology

$$H_{2n}(S^n_p; \mathbb{Z}) \longrightarrow H_{2n}(\tau_{\geq n}X; \mathbb{Z}) \longrightarrow H_{2n}(K(\pi_0X, n); \mathbb{Z})$$

maps an uncountable rational vector space monomorphically to $H_{2n}(K(\pi_0X, n); \mathbb{Z})$, hence so does the map $H_{2n}(S^n_p; \mathbb{Z}) \to H_{2n}(\tau_{\geq n}X; \mathbb{Z})$. This verifies the claim about the integral homology of $X$.

Recall that, for any connective spectrum $Y$, the Hurewicz map $\pi_*Y \to H_*(Y; \mathbb{Z})$ has kernel and cokernel of bounded torsion in each degree. Indeed, the fiber sequence $Y \wedge \tau_{\geq 0}S^0 \to Y \to Y \wedge \mathbb{H}Z$ reduces this claim to showing that $\pi_*(Y \wedge \tau_{\geq 0}S^0)$ is bounded torsion in each degree. This follows from the convergent Atiyah–Hirzebruch spectral sequence

$$H_*Y; \pi_*\tau_{\geq 0}S^0) \Rightarrow \pi_{*+t}(Y \wedge \tau_{\geq 0}S^0),$$

because $H_*Y; \pi_*\tau_{\geq 0}S^0)$ is bounded torsion for all $s$ and $t$. Therefore, any rational vector space in $H_*(Y; \mathbb{Z})$ may be lifted back to $\pi_*Y$. In particular, an uncountable rational vector space in $H_{2n}(X; \mathbb{Z})$ may be lifted back to $\pi_{2n}(\Sigma^\infty X)$ after suspension. □

Remark 4.6. Suppose $X$ is a $p$-complete nilpotent space such that $\pi_0X$ is the lowest homotopy group not of bounded $p$-torsion. The above argument shows that $H_{2n}(X; \mathbb{Z})$ contains an uncountable rational vector space. With more work, we can also show that $H_k(X; \mathbb{Z})$ is derived $p$-complete for $k \leq 2n - 2$ and thus cannot contain any rational classes. Note that when $X$ is
It follows that the natural map $\Sigma^\infty_+$ whose homotopy groups are all finite $p$-torsion for each $n$.

Proof. First assume that $X$ is a $p$-complete nilpotent space with $\pi_n X$ of bounded $p$-torsion for each $n$; we can apply [BK72, Ch. VII, 4.7] to see that the Postnikov tower of $X$ can be refined to a tower of principal fibrations whose fibers are Eilenberg–MacLane spaces for bounded $p$-torsion abelian groups. The category of bounded $p$-torsion abelian groups forms a Serre class, so Serre theory implies that $H_*(X; \mathbb{Z}) \cong H_*(\Sigma^\infty_+ X; \mathbb{Z})$ is degreewise bounded $p$-torsion. Hence, $\Sigma^\infty_+ X$ is $p$-complete as a spectrum by Corollary 3.3.

The converse is a consequence of Proposition 4.5: if $\pi_* X$ is not all bounded torsion, then $H_*(\Sigma^\infty_+ X; \mathbb{Z})$ contains rational classes and thus cannot be derived $p$-complete, hence $\Sigma^\infty_+ X$ is not $p$-complete by Corollary 3.3.

The next result generalizes [PSS17, Prop. 2.4].

Corollary 4.8. If $X$ is a pointed connected space with degreewise finite homotopy groups, then the canonical map $(\Sigma^\infty X)_p^\wedge \to \Sigma^\infty X_p^\wedge$ is an equivalence.

Proof. By [BK72, Ch. VII, 4.3], $X$ is a $\mathbb{Z}/p$-good space and $X_p^\wedge$ is a $p$-complete nilpotent space whose homotopy groups are all finite $p$-groups. Hence $\Sigma^\infty X_p^\wedge$ is $p$-complete by Theorem 4.7. It follows that the natural map $(\Sigma^\infty X)_p^\wedge \to \Sigma^\infty X_p^\wedge$ is an $H\mathbb{F}_p$-equivalence between $H\mathbb{F}_p$-local spectra, which implies the claim.

Corollary 4.9. If $X$ is a nilpotent space with $H_n(X; \mathbb{Z})$ and $\pi_n X$ derived $p$-complete for all $n$, then $H_n(X; \mathbb{Z})$ and $\pi_n X$ are bounded $p$-torsion for all $n$.

Proof. The assumption on $\pi_n X$ implies that $X$ is $p$-complete by Theorem 2.6, while the assumption on $H_*(X; \mathbb{Z})$ shows that $\Sigma^\infty X$ is $p$-complete, using Corollary 3.3. It thus follows from Theorem 4.7 that $\pi_* X$ is degreewise bounded $p$-torsion, hence so is $H_*(X; \mathbb{Z})$ by the proof of Theorem 4.7.

The analogue of this corollary does not hold stably, as the following example demonstrates.

Example 4.10. Let $M(\mathbb{Z}_p, n)$ be the Moore space for $\mathbb{Z}_p$ in degree $n \geq 2$. As $H_*(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z})$ is isomorphic to $\mathbb{Z}_p[n]$, we see that $\Sigma^\infty M(\mathbb{Z}_p, n)$ is $p$-complete and consequently has derived $p$-complete stable homotopy groups and integral homology groups. However, $H_n(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z}) \cong \mathbb{Z}_p$ is clearly not bounded $p$-torsion. In particular, $M(\mathbb{Z}_p, n)$ is not $p$-complete, so this also shows that the assumption that $X$ be $p$-complete cannot be dropped in Theorem 4.7.

5. Rational classes in the stable homotopy groups of $K(\mathbb{Z}_p, n)$

In this section, we present an example that illustrates how the rational classes in the stable homotopy groups of $p$-complete spaces arise. In fact, we present two different approaches: One using the integral homology of $K(\mathbb{Z}_p, n)$, and one using Goodwillie calculus. The latter derivation is entirely stable and might be of independent interest.

First, we need a well-known auxiliary result; we outline a proof because we were unable to find a published reference for it. For an abelian group $A$ and any $k \geq 0$, let $\text{Sym}^k_\mathbb{Z}(A)$ and $\Lambda^k_\mathbb{Z}(A)$ be the $k$th symmetric power and the $k$th exterior power on $A$, respectively.
Lemma 5.1. If \( k > 1 \), then \( A^k_\mathbb{F}(\mathbb{Z}_p) \) and the kernel of the multiplication map \( \text{Sym}^k_\mathbb{Z}(\mathbb{Z}_p) \to \mathbb{Z}_p \) are uncountable rational vector spaces.

Proof. Since both symmetric and exterior power commute with base-change along \( \mathbb{Z} \to \mathbb{Z}/l \) for any prime \( l \), the indicated maps are isomorphisms mod \( l \). Moreover, \( \text{Sym}^k_\mathbb{Z}(A) \) and \( A^k_\mathbb{F}(A) \) are torsion-free whenever \( A \) is, so both \( \text{ker}(\text{Sym}^k_\mathbb{Z}(\mathbb{Z}_p) \to \mathbb{Z}_p) \) and \( A^k_\mathbb{F}(\mathbb{Z}_p) \) are rational vector spaces. We may therefore base-change to \( \mathbb{Q} \), where it is easy to verify that the \( \mathbb{Q} \)-dimension of the groups under consideration is that of \( \mathbb{Q}_p \).

Remark 5.2. A similar argument also shows that \( \mathbb{Z}_p/\mathbb{Z}(p) \) is a rational vector space with the same \( \mathbb{Q} \)-dimension as \( \mathbb{Q}_p \).

Proposition 5.3. For \( n \geq 1 \) and all \( k > 1 \), the stable homotopy group \( \pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n) \) contains an uncountable rational vector space. In particular, \( \Sigma^\infty K(\mathbb{Z}_p, n) \) is not \( p \)-complete.

First proof. Let \( A \) be an abelian group and recall that \( H_*(K(A, n); \mathbb{Z}) \) equipped with the Pontryagin product is a graded commutative algebra such that squares of odd dimensional elements are zero; in fact, it has the structure of a graded divided power algebra, see [EML54, Car56] or more recently [Ric09]. With notation as in the previous lemma, the canonical isomorphism \( A \to H_n(K(A, n); \mathbb{Z}) \) thus extends to a natural homomorphism

\[
\begin{align*}
\phi^k(A, n) &: \Lambda^n_\mathbb{Z}(A) \to H_{kn}(K(A, n); \mathbb{Z}), \\
\phi^k(A, n) &: \text{Sym}^n_\mathbb{Z}(A) \to H_{kn}(K(A, n); \mathbb{Z}),
\end{align*}
\]

for any \( n, k > 0 \). Moreover, we know that \( \phi^k(A, n) \otimes \mathbb{Q} \) is a rational isomorphism. It then follows from Lemma 5.1 that, for \( k > 1 \), there exists an uncountable rational vector space which is mapped monomorphically to \( H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z}) \) via \( \phi^k(\mathbb{Z}_p, n) \). We thus obtain an uncountable rational vector space in \( H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z}) \) that may be lifted back to give the desired uncountable rational vector space in \( \pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n) \) for \( k > 1 \), as in the proof of Proposition 4.5.

Second proof. We will compute the homotopy groups of \( \Sigma^\infty K(\mathbb{Z}_p, n) \simeq \Sigma^\infty \Omega^\infty \Sigma^n H\mathbb{Z}_p \) using Goodwillie calculus [Goo03]. To this end, recall that the Goodwillie tower \((P_k)_{k \geq 1}\) associated to the functor \( \Sigma^\infty \Omega^\infty \): \( \text{Sp} \to \text{Sp} \) is assembled from fiber sequences of functors

\[
\begin{array}{cccc}
D_k & \longrightarrow & P_k & \longrightarrow & P_{k-1} \\
\end{array}
\]

with layers \( D_k X \simeq X^\wedge k_{\Sigma k} \), where the homotopy orbits are formed with respect to the permutation action of \( \Sigma k \) (see for example [KM13] and the references given therein). Moreover, the Goodwillie tower \((P_k)_{k \geq 0}\) converges for connective spectra, i.e., there is a canonical equivalence

\[
\Sigma^\infty \Omega^\infty X \longrightarrow \text{lim}_k P_k X
\]

for any connective \( X \in \text{Sp} \). We will apply this in the case \( X = \Sigma^n H\mathbb{Z}_p \).

In order to understand the layers, we start by analyzing \( \pi_* (\Sigma^n H\mathbb{Z}_p)^\wedge k \) via the universal coefficient theorem. We claim that, for all \( k \geq 1 \), the homotopy groups have the following form

\[
\pi_* (\Sigma^n H\mathbb{Z}_p)^\wedge k \cong \begin{cases} 
0 & * < nk \\
\mathbb{Z}^\otimes k & * = nk \\
\text{finite} & * > nk.
\end{cases}
\]

(5.5)

By the universal coefficient theorem, we have an isomorphism

\[
\pi_* (\Sigma^n H\mathbb{Z}_p)^\wedge k \cong (\pi_* (\Sigma^n H\mathbb{Z})^\wedge k) \otimes \mathbb{Z}^\otimes k.
\]

In degrees \( * > nk \), the groups \( \pi_* (\Sigma^n H\mathbb{Z})^\wedge k \) are torsion, so the only torsion-free summand appears in degree \( nk \). Since \( \pi_* (\Sigma^n H\mathbb{Z})^\wedge k \) is finitely generated over \( \mathbb{Z} \) in each degree, the claim follows.
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We now plug the formula (5.5) into the convergent homotopy orbit spectral sequence

$$H_*(\Sigma_k, \pi_*(\Sigma^n H\mathbb{Z}_p)^\wedge k) \Longrightarrow \pi_{s+t} D_k(\Sigma^n H\mathbb{Z}_p).$$

There are two cases: If $t > nk$ or $t < nk$, then the groups $H_*(\Sigma_k, \pi_*(\Sigma^n H\mathbb{Z}_p)^\wedge k)$ are finite or trivial for all $s$, respectively. Let $t = nk$. By Lemma 5.1 and (5.5), there is an isomorphism $H_*(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^\wedge k) \cong H_*(\Sigma_k, \mathbb{Z}_p)$ for $s > 0$ and $H_0(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^\wedge k)$ contains an uncountable rational vector space $V_k$ if $k > 1$. To see the last statement, it suffices to compute the coinvariants on the rational submodule of $\pi_{n^k}^\wedge k$ by choosing a $\mathbb{Q}$-bases, as in the proof of Lemma 5.1. Furthermore, since the integral homology of $\Sigma_k$ is finitely generated over $\mathbb{Z}$ in each degree and rationally trivial in positive degrees, $H_*(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^\wedge k)$ is finite for all $s > 0$. Combining all this information, we obtain $D_1 \Sigma^n H\mathbb{Z}_p \simeq \Sigma^n H\mathbb{Z}_p$ and for $k > 1$:

$$\pi_* D_k(\Sigma^n H\mathbb{Z}_p) \cong \begin{cases} 0 & * < nk \\ \mathbb{V}_k \oplus W_k & * = nk \\ \text{finite} & * > nk, \end{cases}$$

(5.6)

where $V_k$ is an uncountable rational vector space and $W_k$ is some abelian group.

This allows us to derive a structural formula for $\pi_* P_k \Sigma^n H\mathbb{Z}_p$. Consider the following segment of the long exact sequence of homotopy groups associated to the fiber sequence (5.4):

$$\cdots \longrightarrow \pi_{nk+1} P_{k-1} \Sigma^n H\mathbb{Z}_p \longrightarrow \pi_{nk} D_k \Sigma^n H\mathbb{Z}_p \longrightarrow \pi_{nk} P_k \Sigma^n H\mathbb{Z}_p \longrightarrow \cdots$$

Because $n \geq 1$, it follows inductively from (5.6) that the term on the left is finite, hence $V_k$ must be a summand in $\pi_{nk} P_k \Sigma^n H\mathbb{Z}_p$. This yields for all $k > 1$:

$$\pi_* P_k \Sigma^n H\mathbb{Z}_p \cong \begin{cases} 0 & * < n \\ \mathbb{V}_l \oplus W'_l & * = nl \text{ with } 1 \leq l \leq k \\ \text{finite} & \text{otherwise}, \end{cases}$$

(5.7)

where $V_l$ is as above for $l \geq 2$, and $V_l$ and $W'_l$ are some abelian groups.

Finally, since $D_k \Sigma^n H\mathbb{Z}_p$ is $nk$-connective for all $k$, the tower $(\pi_* P_k \Sigma^n H\mathbb{Z}_p)_{k \geq 0}$ stabilizes after finally many steps in each degree and hence is Mittag-Leffler. The corresponding Milnor sequence thus degenerates to an isomorphism

$$\pi_* \Sigma^\infty K(\mathbb{Z}_p, n) \cong \pi_* \Sigma^\infty \Omega^\infty \Sigma^n H\mathbb{Z}_p \cong \lim_k \pi_* P_k \Sigma^n H\mathbb{Z}_p.$$

Therefore, the claim follows from (5.7). \hfill \square

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