DIMENSION OF NONCOMMUTATIVE PLANE CURVES

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ABSTRACT. In this note we prove that an algebra of the form \( k\langle x, y \rangle/(f) \) is never right (or left) artinian in case \((f)\) is a proper ideal and \( k \) is an uncountable, algebraically closed field of characteristic 0.

1. Introduction

We consider \( k\langle x, y \rangle/(f) \), where \( k \) is an uncountable, algebraically closed field of characteristic 0. Here \( k\langle x, y \rangle \) denotes the free algebra in \( x \) and \( y \) and \( 0 \neq f \in k\langle x, y \rangle \).

We denote by \( f_0 \) the homomorphic image of \( f \) in \( k[x, y] \).

The classical principal ideal theorem implies that in case \((f_0)\) is a proper ideal, then the corresponding factor ring, \( k[x, y]/(f_0) \), is never artinian.

We prove (with the assumptions as above) that \( k\langle x, y \rangle/(f) \) is never a left (or right) artinian ring, whenever \((f)\) is a proper ideal.

The proof is by contradiction. In fact our arguments lead to a contradiction to the Krull Generalized Principal Ideal Theorem for commutative rings.

We first show that in case \( A = k\langle x, y \rangle/(f) \) is right artinian \( \dim_k A \) is finite. From this we get that \((f)\) is finitely generated as a left and as right ideal by [4, 9.1.7, Proposition] and [2].

As noted above \( f_0 = k[x, y] \) hence \( f_0 = u, u \in k^* \) and

\[ u^{-1} f_0 = 1 + g, \]

where \( g \in ([x, y]). \) Consequently

\[ (f) = (u^{-1} f) = (1 + g). \]

Remark 1.1. When we have shown that \( \dim_k A \) is finite, it follows that we have a relation

\[ (x^n + \cdots + a_1 x + a_0) \in (1 + g), \text{ where } a_j \in k \text{ and } g \in ([x, y]). \]

It seems to be unknown whether or not such a relation can hold for a nonzero \( g \) as above. A negative answer will shorten our proof considerably.

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Our argument for the main result is not combinatorical, but makes heavy use of structure theory.

2. Main result

In this section we prove our main result:

**Theorem 2.1.** Let $k$ be an uncountable, algebraically closed field of characteristic 0. Let $(f)$ be a proper principal two-ideal of $k\langle x, y \rangle$. Then $k\langle x, y \rangle/(f)$ is not right artinian.

Before proving Theorem 2.1 we list a couple of results which might be known to experts. Our aim is to reduce the problem to a question on Azumaya algebras and then to a problem on commutative rings.

We shall need the following result concerning Azumaya algebras:

**Lemma 2.2.** Let $A$ be an Azumaya algebra of constant rank $n^2$ with a local center $C$. Then for $f \in A$, $(f) \cap C$ can be generated by $n^2$ elements.

*Proof.* This is probably well-known, but for the sake of completeness we provide an argument.

Since $A$ is a finitely generated projective $C$-module and $C$ is a local ring, it follows that $A$ is a free $C$-module.

We let $\{e_j|1 \leq j \leq n^2\}$ be a free basis for $A$ and $\{e_j^*|1 \leq j \leq n^2\}$ the dual base. This means that

$$e_j^* \in A^* = \text{Hom}_C(A, C) \subseteq \text{Hom}_C(A, A)$$

and $e_j^*(e_i) = \delta_{ij}$.

We let $E$ denote $\text{End}(A_C)$. Since $A$ is an Azumaya algebra the natural homomorphism

$$\Theta : A \otimes_C A^{op} \to E$$

is a ring isomorphism.

As $A^* \subseteq E \simeq A \otimes_C A^{op}$, we can identify $A^*$ with its image in $A \otimes_C A^{op}$.

For each $j$ we write

$$e_j^* = \Theta(\sum_i \alpha_{ij} \otimes \beta_{ij}).$$

Thus,

$$e_j^*(f) = (\sum_i \alpha_{ij} f \beta_{ij}) \in C \cap (f),$$

hence $\sum_j e_j^*(f) C \subseteq C \cap (f)$.

By [3, Chapter II, 3.7 Proposition] and [3, Chapter II, 3.8 Theorem]

$$f = \sum_j e_j e_j^*(f).$$
This implies that
\[(f) \subseteq A(\sum_j e_j^*(f))C \subseteq A(C \cap (f)) \subseteq (f),\]
and thus \(A(\sum_j e_j^*(f))C = A(C \cap (f)).\)
Apply [3, Chapter II, 3.7 Corollary] to conclude that
\[\sum_j e_j^*(f)C = C \cap (f),\]
and the proof of the lemma is finished.

\[\square\]

**Remark 2.3.** Assume the situation above that \(A\) is an Azumaya algebra of constant rank \(n^2\) with local center \(C\).
Given \(l\) elements \(f_1, \ldots, f_l \in C, 1 \leq l \leq n^2\), then there exists an \(f \in k\langle x, y \rangle\) such that \((f) \cap C = (f_1, \ldots, f_l)\).

**Proof.** Define \(f = \sum_{i=1}^k e_if_i\). Then \(e_j^*(f) = f_j\) and the result follows by repeating the argument above.

\[\square\]

In the rest of this section \(k\) is an uncountable, algebraically closed field of characteristic 0.

**Lemma 2.4.** Suppose \(A = k\langle x, y \rangle/(f)\) is right artinian. Then \(A/\text{rad}(A)\) is a finite direct product of matrix rings with coefficients in \(k\). Moreover, \(\dim_k A\) is finite.

**Proof.** Since \(A\) is an artinian ring, \(A/\text{rad}(A)\) is a semisimple artinian algebra, thus isomorphic to a finite product of matrix rings over skew fields, \(D_1, \ldots, D_l\):
\[A/\text{rad}(A) \simeq \text{Mat}_{n_1}(D_1) \oplus \ldots \oplus \text{Mat}_{n_l}(D_l).\]

Since \(A/\text{rad}(A)\) is \(k\)-affine, it has at most countable \(k\)-dimension. Consequently for each \(j\), \(D_j\) has at most countable \(k\)-dimension.

It is well-known that \(D_j = k\), see the proof of [4, 9.1.7, Proposition].
Thus
\[A/\text{rad}(A) \simeq \text{Mat}_{n_1}(k) \oplus \ldots \oplus \text{Mat}_{n_l}(k),\]
where \(n_1 \leq n_2 \leq \ldots \leq n_l\).

We now prove that \(\dim_k A\) is finite. To do so we show for each \(j\) that \(\dim_k(\text{rad}(A)^j/\text{rad}(A)^{j+1})\) is finite. Since \(\text{rad}(A)\) is nilpotent our claim has then been established.

As \(A\) is right artinian \(A\) is right noetherian. Thus \(\text{rad}(A)^j/\text{rad}(A)^{j+1}\) is finitely generated as \(A\)-module and hence as an \(A/\text{rad}(A)\)-module.

\[\square\]

**Remark 2.5.** The argument shows in fact the following result (with \(k\) as above):
Let \(A\) be a semisimple \(k\)-affine algebra. Then \(A\) is a finite product of matrix algebras with coefficients in \(k\).
Before we prove the main result we need one more lemma, which will be applied in the proof of the main theorem.

**Lemma 2.6.** Let $A$ be a $k$-affine Azumaya algebra. Then $A$ and the center $Z$ of $A$ are right and left noetherian.

**Proof.** Since $A$ is a finite module over $Z$, the Artin-Tate lemma [4, 13.9.10] gives that there exists a $k$-affine subalgebra $Z_1$ of $Z$ such that $A$ is a finite $Z_1$-module. Since $Z_1$ is commutative ring it must be noetherian and so is $A$ as a $Z_1$-module, hence $A$ is left and right noetherian.

Also $Z$ must be finitely generated over $Z_1$, therefore the center $Z$ is noetherian. This last claim also follows from the fact that $A$ is an Azumaya algebra. □

We assume now that $A := k\langle x, y \rangle/(f)$ is artinian of finite $k$-dimension. Below we list some consequences and introduce some further notation.

For short, let $m = n_l$, where $n_l$ is defined in the proof of Lemma 2.4.

We have the following setup:

Here $k$ is an uncountable algebraically closed field of characteristic 0 and $0 \neq f \in k\langle x, y \rangle$ is such that $k\langle x, y \rangle/(f)$ is a right artinian algebra.

Then $k\langle x, y \rangle/(f, T_m)$ is a nonzero right artinian algebra, where $T_m$ denotes the ideal of identities for $(m \times m)$-matrices.

It is well-known that $k\langle x, y \rangle/T_m$ is isomorphic to the algebra $k\{u, v\}$ generated by two $m \times m$ generic matrices $u$ and $v$ [1, Introduction]. Thus we have a nonzero element $g$ (f’s image) in $k\{u, v\}$ such that the corresponding factor ring is a non-zero right artinian algebra with an $m$-dimensional simple representation.

By $c$ we denote a non-zero evaluation of a central polynomial for $M_m(k)$.

It is well-known that $k\{u, v\}{[c^{-1}]}$ is an Azumaya algebra of constant rank $m^2$ (Artin-Procesi) [4, 13.7.14] and moreover

$$B = k\{u, v\}{[c^{-1}]}/(g)$$

is a non-zero artinian $k$-algebra.

Next we list a couple of classical results, some holding in general and some for our more specific situation:

1) By Lemma 2.6 we get that $R = k\{u, v\}{[c^{-1}]}$ is noetherian.

2) ([4, Proposition 13.10.6]) Let $R$ be a prime affine $k$-algebra. Then $\dim R = \text{trdeg}Z$, where $Z$ is the center of the quotient ring of $R$.

3) From 2) and ([5, Chapter IV, Theorem 6.3]) we get that

$$\dim k\{u, v\} = m^2 + 1.$$  

4) (Schelter’s theorem, cf. [4, Proposition 13.10.12]). Let $R$ be a prime PI affine $k$-algebra and $\mathcal{P} \in \text{Spec}R$. Then

$$\dim R = \text{ht}(\mathcal{P}) + \dim R/\mathcal{P}.$$
Lemma 2.7. The ring $R = k\{u, v\}[c^{-1}]$ has dimension $m^2 + 1$.

Proof. This is an easy consequence of 2 and 3. □

Since $R$ is Azumaya and noetherian we get from the correspondence between the ideals of $R$ and the ideals of the center:

Lemma 2.8. The center $C$ of $R$ is noetherian with $\dim C = m^2 + 1$.

We finally conclude the proof of the Main Theorem. The notation and assumptions are as above.

Proof. First notice that $C$ is noetherian and all maximal ideals have the same height $(m^2 + 1)$ (cf. 4).

Let $\mathcal{M}$ be a maximal ideal in $C$ containing $RfR \cap C$. If we localize with respect to $\mathcal{M}$, we get an Azumaya algebra $R_{\mathcal{M}} = R_1$ with center $C_{\mathcal{M}} = C_1$. By previous results $\dim C_1 = m^2 + 1$ and $R_1/(f)$ is artinian.

By the Azumaya property every prime ideal containing $R_1fR_1 \cap C_1$ is maximal, therefore $C_1/R_1fR_1 \cap C_1$ is artinian.

By Lemma 2.2, $R_1fR_1 \cap C_1$ can be generated by $m^2$ elements, which contradicts Krull’s Generalized Principal Ideal Theorem. □

Lemma 2.2 can be generalized to ideals generated by more than one element. Moreover using 2) above and that the Krull dimension of the $k$-algebra generated by $n$-generic $m \times m$ matrices is $(n - 1)m^2 + 1$ ([5, Chapter IV, Theorem 6.3]) one can prove:

Theorem 2.9. Let $A = k\langle x_1, \ldots, x_n \rangle$, where $k$ is as above. Then for a proper ideal $I = (f_1, \ldots, f_{n-1})$, $A/I$ is not artinian.

3. Examples

In this section we consider the algebra $k\langle x, y \rangle$ and factors of the form $k\langle x, y \rangle/(f)$.

The Weyl algebra $k\langle x, y \rangle/(xy - yx - 1)$ shows that factors can have Krull dimension 0.

The algebra $k\langle x, y \rangle$ maps onto the ring of two $(m \times m)$ generic matrices $k\{u, v\}$, hence $k\langle x, y \rangle/(f)$ maps onto $k\{u, v\}/(f_1)$ and we make some remarks concerning possible values of the Krull dimensions of these two algebras. The argument proving the main result shows that 0 is not a possible dimension for the latter algebra by [5, Chapter V, Theorem 5.4].

Let us remark that generic matrix rings are Jacobson rings, so their Krull dimension and the dimension of the maximal spectrum coincide [4, 13.10.3].

Remark 3.1. Let $m \in \mathbb{N}$, let $A = k\{u, v\}$ (the ring of two $m \times m$ generic matrices) and let $c$ be an evaluation of a central polynomial for $M_m(k)$. Consider the ring $B$ of 2 generic $m \times m$-matrices localized at
powers of \( c \). Furthermore let \( 1 \leq l \leq m^2 + 1 \) be given. Suppose \( \mathcal{M} \) is a maximal ideal in the center \( C \) of \( B \). In \( C_{\mathcal{M}} \) we take a prime \( \mathcal{P}_{\mathcal{M}} \) of dimension \( l \). Then take a system of parameters \( (f_1, \ldots, f_l) \in \mathcal{P} \) i.e. \( \mathcal{P} \) is the only minimal prime containing \( (f_1, \ldots, f_l) \). Then by Remark 2.3 there is an \( f \in B_{\mathcal{M}} \) and a prime \( \tilde{\mathcal{P}} \subseteq B \) such that \( \tilde{\mathcal{P}} \) is the only prime being minimal over \( (f) \).

By replacing \( f \) by another generator of \( (f) \), \( f \) is the image of an element \( g \in A \). By 4) above \( \dim A/(g) = l \).

This implies that the \( \dim k\langle x, y \rangle/(g_0) \geq l \), where \( g_0 \) maps to \( g \) by the canonical homomorphism.

We don’t know if all integer dimensions larger than 1 can be obtained as the dimension of \( k\langle x, y \rangle/(f) \). However 1 and 2 are always possible by 1) and 2) below.

1) For \( f = x \) the Krull-dimension is 1.
2) For \( f = xy - yx \) the Krull-dimension is 2.

For \( m = 2 \) we give concrete examples of elements \( f \) with given dimension of the corresponding factor ring\( k\{u, v\}/(f) \).

For \( f = x^2 - 1 \) the dimension is 3. Below we briefly sketch the argument and leave out most details. It is similar to the original argument showing that the algebra of two \( 2 \times 2 \)-generic matrices has dimension 5.

First notice that the algebra is isomorphic to \( k\{U, V\} \), where

\[
U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}
\]

For short we denote the trace of a matrix \( A \) by \( t_A \) and the determinant by \( d_A \). Let \( L \) denote the quotient field of \( k[v_1, v_2, v_3, v_4] \). We let \( F \) denote the subfield \( k(t_V, d_V, t_{UV}) \). It is straightforward to check that \( t_V, d_V \) and \( t_{UV} \) are \( k \)-algebraic independent variables and hence \( F \) is a rational function field of degree 3.

Moreover it is readily checked that \( I, U, V, UV \) are linearly independent over the field \( L \), hence also linearly independent over \( F \).

By \( Q \) we denote the vectorspace

\[
Q = FI \oplus FU \oplus FV \oplus FUV
\]

One then shows that \( Q \) is an algebra and and that \( Q \) is the classical quotientring of \( k\{U, V\} \). This will establish our claim thanks to [4, Proposition 13.10.6].

Finally for \( f = [u, v]^2 - 1 \) the dimension is 4. This is a result in commutative algebra, since \( f \) is a central element and by factoring \( f \) out the dimension drops by 1.
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References


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