Generalized Hardy–Cesaro operators between weighted spaces

Pedersen, Thomas Vils

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Abstract

We characterize those non-negative, measurable functions \( \psi \) on \([0, 1]\) and positive, continuous functions \( \omega_1 \) and \( \omega_2 \) on \( \mathbb{R}^+ \) for which the generalized Hardy-Cesàro operator

\[
(U \psi f)(x) = \int_0^1 f(tx) \psi(t) \, dt
\]

defines a bounded operator \( U_\psi : L^1(\omega_1) \to L^1(\omega_2) \). This generalizes a result of Xiao ([7]) to weighted spaces. Furthermore, we extend \( U_\psi \) to a bounded operator on \( M(\omega_1) \) with range in \( L^1(\omega_2) \oplus \mathbb{C}\delta_0 \), where \( M(\omega_1) \) is the weighted space of locally finite, complex Borel measures on \( \mathbb{R}^+ \). Finally, we show that the zero operator is the only weakly compact generalized Hardy-Cesàro operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \).

1 Introduction

A classical result of Hardy ([5]) shows that the Hardy-Cesàro operator

\[
(Uf)(x) = \frac{1}{x} \int_0^x f(s) \, ds
\]

defines a bounded linear operator on \( L^p(\mathbb{R}^+) \) with \( \|U\| = p/(p - 1) \) for \( p > 1 \). Clearly, \( U \) is not bounded on \( L^1(\mathbb{R}^+) \). Hardy’s result has been generalized in various ways, of which we will mention some, which have inspired this paper.

For \( 1 \leq p \leq q \leq \infty \) and non-negative measurable functions \( u \) and \( v \) on \( \mathbb{R}^+ \), Muckenhoupt ([6]) and Bradley ([3]) gave a necessary and sufficient condition for the existence of a constant \( C \) such that

\[
\left( \int_0^\infty \left( u(x) \int_0^x f(t) \, dt \right)^q dx \right)^{1/q} \leq C \left( \int_0^\infty (v(x)f(x))^p dx \right)^{1/p}
\]

for every positive, measurable function \( f \) on \( \mathbb{R}^+ \). This can be rephrased as a characterization of the weighted \( L^p \) and \( L^q \) spaces on \( \mathbb{R}^+ \) between which the Hardy-Cesàro operator \( U \) is bounded.

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In a different direction, for a non-negative measurable function $\psi$ on $[0, 1]$, Xiao [7] considered the generalized Hardy-Cesàro operators

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t) \, dt$$

for measurable functions $f$ on $\mathbb{R}^n$. We remark that

$$(U_\psi f)(x) = \frac{1}{x} \int_0^x f(s)x\psi(s/x) \, ds$$

for measurable functions $f$ on $\mathbb{R}$. Xiao proved that $U_\psi$ defines a bounded operator on $L^p(\mathbb{R}^n)$ (for $p \geq 1$) if and only if

$$\int_0^1 \psi(t) t^{n/p} \, dt < \infty.$$

Xiao’s result is the main motivation for this paper.

Finally, we mention that Albanese, Bonet and Ricker in a recent series of papers (see, for instance, [1] and [2]) have considered the spectrum, compactness and other properties of the Hardy-Cesàro operator on various spaces of continuous functions and discrete spaces.

In this paper we will study the generalized Hardy-Cesàro operators between weighted spaces of integrable functions, and we will obtain a generalization of Xiao’s result in this context. Let $\omega$ be a positive, continuous function on $\mathbb{R}^+$ and let $L^1(\omega)$ be the Banach space of (equivalence classes of) measurable functions $f$ on $\mathbb{R}^+$ for which

$$\|f\|_{L^1(\omega)} = \int_0^\infty |f(t)|\omega(t) \, dt < \infty.$$  

In the usual way we identify the dual space of $L^1(\omega)$ with the space $L^\infty(1/\omega)$ of measurable functions $h$ on $\mathbb{R}^+$ for which

$$\|h\|_{L^\infty(1/\omega)} = \text{ess sup}_{t \in \mathbb{R}^+} |h(t)|/\omega(t) < \infty.$$  

We denote by $C_0(1/\omega)$ the closed subspace of $L^\infty(1/\omega)$ consisting of the continuous functions $g$ in $L^\infty(1/\omega)$ for which $g/\omega$ vanishes at infinity. Finally, we identify the dual space of $C_0(1/\omega)$ with the space $M(\omega)$ of locally finite, complex Borel measures $\mu$ on $\mathbb{R}^+$ for which

$$\|\mu\|_{M(\omega)} = \int_{\mathbb{R}^+} \omega(t) \, d|\mu|(t) < \infty.$$  

We consider the space $L^1(\omega)$ as a closed subspace of $M(\omega)$.

In Section 2 we characterize those functions $\psi, \omega_1$ and $\omega_2$ for which $U_{\psi}$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. These operators are extended to bounded operators on $M(\omega_1)$ in Section 3 where we also obtain results about their ranges. Finally, in Section 4 we show that there are no non-zero weakly compact generalized Hardy-Cesàro operators from $L^1(\omega_1)$ to $L^1(\omega_2)$.  

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2 A characterization of the generalized Hardy-Cesàro operators

For a non-negative, measurable function $\psi$ on $[0, 1]$ and positive, continuous functions $\omega_1$ and $\omega_2$ on $\mathbb{R}^+$, we say that condition (C) is satisfied if there exists a constant $C$ such that

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \leq C \omega_1(s)$$

for every $s \in \mathbb{R}^+.$

**Theorem 2.1** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if condition (C) is satisfied.

**Proof** Assume that condition (C) is satisfied and let $f \in L^1(\omega_1)$. Then

$$\int_0^\infty \int_0^1 |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) \, dt \, ds \leq C \int_0^\infty |f(s)| \omega_1(s) \, ds = C \|f\|_{L^1(\omega_1)} < \infty,$$

so it follows from Fubini’s theorem that

$$\int_0^1 \int_0^\infty |f(tx)| \psi(t) \omega_2(x) \, dx \, dt = \int_0^1 \int_0^\infty |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) \, dt \, ds \leq C \|f\|_{L^1(\omega_1)} < \infty.$$  

Another application of Fubini’s theorem thus shows that $(U_\psi f)(x)$ is defined for almost all $x \in \mathbb{R}^+$ with

$$\|U_\psi f\|_{L^1(\omega_2)} = \int_0^\infty |(U_\psi f)(x)| \omega_2(x) \, dx \leq \int_0^\infty \int_0^1 |f(tx)| \psi(t) \omega_2(x) \, dt \, dx \leq C \|f\|_{L^1(\omega_1)} < \infty.$$

Hence $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

Conversely, assume that $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Since $L^1(\omega_2)$ is a closed subspace of $M(\omega_2)$ which we identify with the dual space of $C_0(1/\omega_2)$, it follows from [1, Theorem VI.8.6] that there exists a map $\rho$ from $\mathbb{R}^+$ to $M(\omega_2)$ for which the map $s \mapsto \langle g, \rho(s) \rangle = \int_{\mathbb{R}^+} g(x) \, d\rho(s)(x)$ is measurable and essentially bounded on $\mathbb{R}^+$ for every $g \in C_0(1/\omega_2)$ with $\|U_\psi\| = \text{ess sup}_{x \in \mathbb{R}^+} \|\rho(s)\|_{M(\omega_2)}$ and such that

$$\langle g, U_\psi f \rangle = \int_0^\infty \langle g, \rho(s) \rangle f(s) \omega_1(s) \, ds = \int_0^\infty \int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) \ f(s) \omega_1(s) \, ds$$

for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$. On the other hand

$$\langle g, U_\psi f \rangle = \int_0^\infty g(x) (U_\psi f)(x) \, dx$$

$$= \int_0^\infty \int_0^x \frac{g(x)}{x} f(s) \psi(s/x) \, ds \, dx$$

$$= \int_0^\infty \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx \ f(s) \omega_1(s) \, ds$$
for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$, so it follows that

$$\int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) = \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx$$

for almost all $s \in \mathbb{R}^+$ and every $g \in C_0(1/\omega_2)$ (considering both sides as elements of $L^\infty(\mathbb{R}^+)$). Considered as elements of $M(\omega_2)$ we thus have

$$d\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s} \, dx$$

for almost all $s, x \in \mathbb{R}^+$. Hence $\rho(s) \in L^1(\omega_2)$ with

$$\|\rho(s)\|_{L^1(\omega_2)} = \int_0^\infty \omega_2(x) \, d\rho(s)(x)$$

$$= \frac{1}{\omega_1(s)} \int_0^\infty \frac{1}{x} \psi(s/x) 1_{x \geq s} \omega_2(x) \, dx$$

$$= \frac{1}{\omega_1(s)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx$$

$$= \frac{1}{\omega_1(s)} \int_0^1 \psi(t) \, \omega_2(s/t) \, dt$$

for almost all $s \in \mathbb{R}^+$. Therefore

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \|\rho(s)\|_{L^1(\omega_2)} \omega_1(s) \leq \|U_\psi\| \omega_1(s)$$

for almost all $s \in \mathbb{R}^+$. Since both sides of the inequality are continuous functions of $s$, the inequality holds for every $s \in \mathbb{R}^+$, so condition (C) holds.

Letting $s = 0$ in condition (C) we see that Xiao’s condition is necessary in our situation.

**Corollary 2.2** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. If $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then

$$\int_0^1 \psi(t) \, dt < \infty.$$

The following straightforward consequences can be deduced from Theorem 2.1.

**Corollary 2.3** Let $\psi$ be a non-negative, measurable function on $[0, 1]$

(a) Let $\omega$ be a decreasing, positive, continuous function on $\mathbb{R}^+$, and assume that

$$\int_0^1 \psi(t)/t \, dt < \infty.$$ Then $U_\psi$ defines a bounded operator from $L^1(\omega)$ to $L^1(\omega)$.

(b) Let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$, and assume that $\omega_2$ is increasing. If $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then there exists a constant $C$ such that $\omega_2(s) \leq C \omega_1(s)$ for every $s \in \mathbb{R}^+$. 

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(c) Let $\omega$ be an increasing, positive, continuous function on $\mathbb{R}^+$, and assume that there exists $a < 1$ and $K > 0$ such that $\psi(t) \geq K$ almost everywhere on $[a, 1]$. If $U_\psi$ defines a bounded operator from $L^1(\omega)$ to $L^1(\omega)$, then there exist positive constants $C_1$ and $C_2$ such that

$$C_1 \omega(s) \leq \int_0^1 \omega(s/t) \frac{\psi(t)}{t} dt \leq C_2 \omega(s)$$

for every $s \in \mathbb{R}^+$.

Proof (a): We have

$$\int_0^1 \omega(s/t) \frac{\psi(t)}{t} dt \leq \int_0^1 \psi(t) dt \omega(s)$$

for every $s \in \mathbb{R}^+$, so condition (C) is satisfied with $\omega_1 = \omega_2 = \omega$ and the result follows.

(b): We have

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \geq \int_0^1 \psi(t) dt \omega_2(s)$$

for every $s \in \mathbb{R}^+$. Since condition (C) is satisfied, the result follows.

(c): We have

$$\int_0^1 \omega(s/t) \frac{\psi(t)}{t} dt \geq K \int_a^1 \omega(s/t) dt \geq K(1-a) \omega(s)$$

for every $s \in \mathbb{R}^+$. The other inequality is just condition (C) with $\omega_1 = \omega_2 = \omega$. □

We finish the section with some examples of functions $\psi, \omega_1$ and $\omega_2$ for which $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

Example 2.4

(a) For $\alpha > 0$, let $\psi(t) = t^\alpha$ for $t \in [0, 1]$. Also, for $\beta_1, \beta_2 \in \mathbb{R}$, let $\omega_i(x) = (1 + x)^{\beta_i}$ for $x \in \mathbb{R}^+$ and $i = 1, 2$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if $\beta_2 \leq \beta_1$ and $\beta_2 < \alpha$.

(b) For $\alpha > 0$, let $\psi(t) = t^\alpha$ for $t \in [0, 1]$. Also, let $\omega_1(x) = e^{-x}/(1+x)$ and $\omega_2(x) = e^{-x}$ for $x \in \mathbb{R}^+$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Moreover, it is not possible to replace $\omega_1(x)$ by a function tending faster to zero at infinity.

(c) Let $\psi(t) = e^{-1/t^2}$ for $t \in [0, 1]$. Also, let $\omega_1(x) = e^{x^2/4}/x$ and $\omega_2(x) = e^x$ for $x \in \mathbb{R}^+$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Moreover, it is not possible to replace $\omega_1(x)$ by a function tending slower to infinity at infinity.

Proof (a): For $s \geq 1$ and $t \in [0, 1]$ we have $s/t < 1 + s/t \leq 2s/t$, so

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1} dt$$

$$\simeq s^{\beta_2} \int_0^1 t^{\alpha-\beta_2-1} dt$$

$$\simeq s^{\beta_2}$$
for \(s \geq 1\) if \(\beta_2 < \alpha\) (where \(F(s) \simeq G(s)\) for positive functions \(F\) and \(G\) on \([1, \infty)\) indicates the existence of positive constants \(C_1\) and \(C_2\) such that \(C_1 F(s) \leq G(s) \leq C_2 F(s)\) for all \(s \in [1, \infty)\)), whereas the integrals diverge if \(\beta_2 \geq \alpha\). Moreover, the expression

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1} \, dt
\]
defines a positive, continuous function of \(s\) on \(\mathbb{R}^+\), so it follows that condition (C) is satisfied if and only if \(\beta_2 \leq \beta_1\) and \(\beta_2 < \alpha\).

(b): For \(s \geq 1\) we have

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_s^\infty \omega_2(x) \frac{\psi(s/x)}{x} \, dx = \int_s^\infty \frac{e^{-x}}{x} \frac{s^\alpha}{x^\alpha} \, dx \leq \int_s^\infty \frac{e^{-x}}{x} \, dx \leq \frac{e^{-s}}{s}.
\]

Moreover,

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \leq \int_0^1 \frac{\psi(t)}{t} \, dt < \infty
\]

for all \(s \in \mathbb{R}^+\), so condition (C) is satisfied and \(U_\psi\) thus defines a bounded operator from \(L^1(\omega_1)\) to \(L^1(\omega_2)\). On the other hand, since

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \geq \int_s^\infty \frac{e^{-x}}{x} \frac{s^\alpha}{x^\alpha} \, dx \geq \frac{1}{2^{\alpha+1}} \int_s^\infty \frac{e^{-x}}{x} \, dx \geq \frac{1}{2^{\alpha+1}} \frac{e^{-s}}{s}
\]

for \(s \geq 1\), it is not possible to replace \(\omega_1(x)\) by a function tending faster to zero at infinity.

(c): For \(s \in \mathbb{R}^+\) we have

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_s^\infty \omega_2(x) \frac{\psi(s/x)}{x} \, dx = \int_s^\infty \frac{e^{-x}}{x} \frac{s^\alpha}{x^\alpha} \, dx = \int_1^\infty e^{sy-y^2} \, dy.
\]

Moreover, for \(s \geq 4\)

\[
\int_s^\infty \frac{e^{sy-y^2}}{y} \, dy \leq \frac{4}{s} \int_s^\infty e^{-(y-s/2)^2+s/4} \, dy = 4 \int_{-s/4}^\infty e^{-y^2} \, dy \frac{e^{s/4}}{s}
\]

and

\[
\int_1^s \frac{e^{sy-y^2}}{y} \, dy \leq \int_1^{s/4} e^{sy} \, dy \leq \frac{e^{s^2/4}}{s},
\]

so condition (C) is satisfied and \(U_\psi\) thus defines a bounded operator from \(L^1(\omega_1)\) to \(L^1(\omega_2)\). On the other hand, the estimate

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_1^\infty \frac{e^{sy-y^2}}{y} \, dy \geq \frac{1}{s} \int_{s/2}^{s/2+1} e^{-(y-s/2)^2+s/4} \, dy = \int_0^1 e^{-y^2} \, dy \frac{e^{s^2/4}}{s}
\]

for \(s \geq 2\) shows that it is not possible to replace \(\omega_1(x)\) by a function tending slower to infinity at infinity.

In Example 2.4(b) we have \(\omega_2(x)/\omega_1(x) \to \infty\) as \(x \to \infty\), which should be compared to the conclusion in Corollary 2.3(b). Conversely, Example 2.4(c) shows an example where we need \(\omega_2(x)/\omega_1(x) \to 0\) rapidly as \(x \to \infty\) in order for \(U_\psi\) to be defined.
3 Extensions to weighted spaces of measures

Identifying the dual space of \(L^1(\omega)\) with \(L^\infty(1/\omega)\) as in the introduction, we have the following result about the adjoint of \(U_\psi\).

**Proposition 3.1** Let \(\psi\) be a non-negative, measurable function on \([0,1]\) and let \(\omega_1\) and \(\omega_2\) be positive, continuous functions on \(\mathbb{R}^+\). Assume that condition (C) is satisfied so that \(U_\psi : L^1(\omega_1) \to L^1(\omega_2)\) is a bounded operator, and consider the adjoint operator \(U_\psi^* : L^\infty(1/\omega_2) \to L^\infty(1/\omega_1)\).

(a) For \(h \in L^\infty(1/\omega_2)\) we have

\[
(U_\psi^* h)(x) = \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt
\]

for almost all \(x \in \mathbb{R}^+\).

(b) \(U_\psi^*\) maps \(C_0(1/\omega_2)\) into \(C_0(1/\omega_1)\).

**Proof** (a): Let \(h \in L^\infty(1/\omega_2)\). Since \(|h(x/t)| \leq \|h\|_{L^\infty(1/\omega_2)} \omega_2(x/t)\) for almost all \(x,t \in \mathbb{R}^+\), it follows from condition (C) that \(\int_0^1 h(x/t)\psi(t)/t \, dt\) is defined and satisfies

\[
\left| \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt \right| \leq \|h\|_{L^\infty(1/\omega_2)} \int_0^1 \omega_2(x/t) \frac{\psi(t)}{t} \, dt \leq C \|h\|_{L^\infty(1/\omega_2)} \omega_1(x)
\]

for almost all \(x \in \mathbb{R}^+\). Hence the function \(x \mapsto \int_0^1 h(x/t)\psi(t)/t \, dt\) belongs to \(L^\infty(1/\omega_1)\). Also, for \(f \in L^1(\omega_1)\) we have

\[
\langle f, U_\psi^* h \rangle = (U_\psi f, h) = \int_0^\infty (U_\psi f)(s) h(s) \, ds
\]

\[
= \int_0^\infty \int_0^s \frac{1}{s} f(x) \psi(x/s) h(s) \, dx \, ds
\]

\[
= \int_0^\infty \int_x^\infty \frac{h(s)}{s} \psi(x/s) \, ds f(x) \, dx
\]

from which it follows that

\[
(U_\psi^* h)(x) = \int_x^\infty \frac{h(s)}{s} \psi(x/s) \, ds = \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt
\]

for almost all \(x \in \mathbb{R}^+\).

(b): It suffices to show that \(U_\psi^*\) maps \(C_c(\mathbb{R}^+)\) (the continuous functions on \(\mathbb{R}^+\) with compact support) into \(C_0(1/\omega_1)\). Let \(g \in C_c(\mathbb{R}^+)\), let \(x_0 \in \mathbb{R}^+\) and let \((x_n)\) be a sequence in \(\mathbb{R}^+\) with \(x_n \to x_0\) as \(n \to \infty\). Then

\[
(U_\psi^* g)(x_n) - (U_\psi^* g)(x_0) = \int_0^1 (g(x_n/t) - g(x_0/t)) \frac{\psi(t)}{t} \, dt
\]

for \(n \in \mathbb{N}\). Since \(g\) is bounded on \(\mathbb{R}^+\) and since \(\int_0^1 \psi(t)/t \, dt < \infty\) by Corollary 2.2, it follows from Lebesgue’s dominated convergence theorem that \((U_\psi^* g)(x_n) \to (U_\psi^* g)(x_0)\) as \(n \to \infty\). Hence \(U_\psi^* g\) is continuous on \(\mathbb{R}^+\). Finally, from the expression

\[
(U_\psi^* g)(x) = \int_x^\infty \frac{g(s)}{s} \psi(x/s) \, ds
\]
it follows that \( \text{supp} \, U_\psi^* g \subseteq \text{supp} \, g \), so we conclude that \( U_\psi^* g \in C_\varepsilon(\mathbb{R}^+) \subseteq C_0(1/\omega_1) \). \qed

Let \( V_\psi \) be the restriction of \( U_\psi^* \) to \( C_0(1/\omega_2) \) considered as a map into \( C_0(1/\omega_1) \). We then immediately have the following result.

**Corollary 3.2** Let \( \psi \) be a non-negative, measurable function on \([0,1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( U_\psi : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. The bounded operator \( \overline{U}_\psi = U_\psi^* \) from \( M(\omega_1) \) to \( M(\omega_2) \) is an extension of \( U_\psi \).

Let \( \psi \) be a non-negative, continuous function on \([0,1]\) with \( \psi(0) = 0 \). For \( \mu \in M(\omega_1) \) and \( x > 0 \) let

\[
(W_\psi \mu)(x) = \frac{1}{x} \int_{(0,x)} \psi(s/x) \, d\mu(s).
\]

**Proposition 3.3** Let \( \psi \) be a non-negative, continuous function on \([0,1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( U_\psi : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. Then \( W_\psi \mu \in L^1(\omega_2) \) and

\[
\overline{U}_\psi \mu = W_\psi \mu + \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \mu(\{0\}) \delta_0
\]

for \( \mu \in M(\omega_1) \). In particular \( \text{ran} \, \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C} \delta_0 \) and \( \overline{U}_\psi \) maps \( M((0,\infty),\omega_1) \) into \( L^1(\omega_2) \).

**Proof** By Corollary 2.2 we have \( \int_0^1 \psi(t)/t \, dt < \infty \), so it follows that \( \psi(0) = 0 \). Let \( \mu \in M(\omega_1) \) with \( \mu(\{0\}) = 0 \). By condition (C) we have

\[
\int_{(0,\infty)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx \, d\mu(\{s\}) = \int_{(0,\infty)} \int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \, d\mu(\{s\})
\]

\[
\leq C \int_{(0,\infty)} \omega_1(s) \, d\mu(\{s\}) = C \| \mu \|_{M(\omega_1)} < \infty,
\]

so it follows from Fubini’s theorem that

\[
\int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x) \, d\mu(\{s\}) \omega_2(x) \, dx < \infty.
\]

Hence \( W_\psi \mu \in L^1(\omega_2) \). Moreover, for \( g \in C_0(1/\omega_2) \) we have

\[
\langle g, \overline{U}_\psi \mu \rangle = \langle V_\psi g, \mu \rangle = \int_{(0,\infty)} \int_0^1 g(s/t) \frac{\psi(t)}{t} \, dt \, d\mu(\{s\})
\]

\[
= \int_{(0,\infty)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx \, d\mu(\{s\})
\]

\[
= \int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x) \, d\mu(\{s\}) \, g(x) \, dx
\]

\[
= \int_0^\infty (W_\psi \mu)(x) g(x) \, dx = \langle g, W_\psi \mu \rangle,
\]

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so we conclude that $U_\psi \mu = W_\psi \mu$. Finally, for $g \in C_0(1/\omega_2)$ we have
\[
\langle g, U_\psi \delta_0 \rangle = \langle V_\psi g, \delta_0 \rangle = (V_\psi g)(0) = g(0) \int_0^1 \frac{\psi(t)}{t} \, dt = \langle g, \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \delta_0 \rangle.
\]
Since $W_\psi \delta_0 = 0$ this finishes the proof. \qed

The conclusion about the range of $U_\psi$ can be generalized to the case, where $\psi$ is not assumed to be continuous.

**Proposition 3.4** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition $(C)$ is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. Then $\text{ran} U_\psi \subseteq L^1(\omega_2) \oplus C \delta_0$.

**Proof** Choose a sequence of non-negative, continuous functions $(\psi_n)$ on $[0, 1]$ with $\psi_n \leq \psi$ and
\[
\int_0^1 \frac{\psi(t) - \psi_n(t)}{t} \, dt \to 0 \quad \text{as } n \to \infty.
\]
For $\mu \in M(\omega_1)$ and $g \in C_0(1/\omega_2)$ we have
\[
|\langle g, (U_\psi - U_{\psi_n})\mu \rangle| = |\langle (V_\psi - V_{\psi_n})g, \mu \rangle| = \left| \int_{\mathbb{R}^+} \int_0^1 g(x/t) \frac{\psi(t) - \psi_n(t)}{t} \, dt \, d\mu(x) \right| \leq \|g\|_{C_0(1/\omega_2)} \int_{\mathbb{R}^+} \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} \, dt \, d\mu|(x).
\]
Let
\[
p_n(x) = \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} \, dt
\]
for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. By condition $(C)$ there exists a constant $C$ such that $p_n(x) \leq C\omega_1(x)$ for every $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Moreover, for every $x \in \mathbb{R}^+$ we have $p_n(x) \to 0$ as $n \to \infty$ by Lebesgue’s dominated convergence theorem. Hence
\[
\| (U_\psi - U_{\psi_n})\mu \|_{M(\omega_2)} = \sup_{\|g\|_{C_0(1/\omega_2)} \leq 1} |\langle g, (U_\psi - U_{\psi_n})\mu \rangle| \leq \int_{\mathbb{R}^+} p_n(x) \, d\mu|(x) \to 0
\]
as $n \to \infty$ again by Lebesgue’s dominated convergence theorem. Consequently, $U_{\psi_n} \to U_\psi$ strongly as $n \to \infty$. Since $\text{ran} U_{\psi_n} \subseteq L^1(\omega_2) \oplus C \delta_0$ for $n \in \mathbb{N}$ by Proposition 3.3 the same thus holds for $\text{ran} U_\psi$. \qed

**Corollary 3.5** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition $(C)$ is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. For $s > 0$ we then have $(U_\psi \delta_s)(x) = \psi(s/x)/x$ for almost all $x \geq s$ and $(U_\psi \delta_s)(x) = 0$ for almost all $x < s$. 

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Proof For \( \psi \) continuous, this follows from Proposition 3.3. For general \( \psi \) it follows from the approach in the proof of Proposition 3.4 using \( \mathcal{U}_n \rightarrow \mathcal{U}_\psi \) strongly as \( n \rightarrow \infty \). \( \square \)

It follows from Corollary 3.5 that

\[
\| \mathcal{U}_\psi \delta_s \|_{M(\omega_2)} = \frac{1}{\omega_1(s)} \int_0^\infty \frac{\omega_2(s)}{x} \psi(s/x) \, dx = \frac{1}{\omega_1(s)} \int_0^1 \frac{\omega_2(s/t)}{t} \psi(t) \, dt,
\]

whereas \( \| \delta_s \|_{M(\omega_1)} = \omega_1(s) \). Since \( \mathcal{U}_\psi \) is bounded we thus recover condition (C). If we without using Theorem 2.1 could show that if \( \mathcal{U}_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2) \) is a bounded operator, then is has a bounded extension \( \mathcal{U}_\psi : M(\omega_1) \rightarrow M(\omega_2) \) for which Corollary 3.5 holds, then we would in this way obtain an alternative proof of condition (C).

4 Weakly compact operators

We finish the paper by showing that there are no non-zero, weakly compact generalized Hardy-Cesàro operators between \( L^1(\omega_1) \) and \( L^1(\omega_2) \).

Proposition 4.1 Let \( \psi \) be a non-negative, measurable function on \([0,1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( \mathcal{U}_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2) \) is a bounded operator. If \( \psi \neq 0 \), then \( \mathcal{U}_\psi \) is not weakly compact.

Proof For \( f \in L^1(\omega_1) \) and \( x \in \mathbb{R}^+ \) we have

\[
(U_\psi f)(x) = \frac{1}{x} \int_0^x f(s) \psi(s/x) \, ds = \int_0^\infty f(s) \rho(s)(x) \omega_1(s) \, ds,
\]

where (with a slight change of notation compared to the proof of Theorem 2.1)

\[
\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s},
\]

for \( x, s \in \mathbb{R}^+ \). In the proof of Theorem 2.1 we saw that \( \rho(s) \in L^1(\omega_2) \) with \( \| \rho(s) \|_{L^1(\omega_2)} \leq C \) for a constant \( C \) for almost all \( s \in \mathbb{R}^+ \). It thus follows from [4, Theorem VI.8.10] that \( \mathcal{U}_\psi \) is weakly compact if and only if \{\( \rho(s) : s \in \mathbb{R}^+ \)\} is contained in a weakly compact set of \( L^1(\omega_2) \) (except possibly for \( s \) belonging to a null-set). Consider \( \rho(s) \) as an element of \( C_0(1/\omega_2)^* \) for \( s \in \mathbb{R}^+ \) and let \( g \in C_0(1/\omega_2) \). Then

\[
\langle g, \rho(s) \rangle = \int_0^\infty g(x) \rho(s)(x) \, dx = \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx = \frac{1}{\omega_1(s)} \int_0^1 g(s/t) \frac{\psi(t)}{t} \, dt.
\]

Since \( g(s/t) \rightarrow g(0) \) as \( s \rightarrow 0_+ \) for all \( t > 0 \), it follows from Lebesgue’s dominated convergence theorem that

\[
\langle g, \rho(s) \rangle \rightarrow \frac{1}{\omega_1(0)} g(0) \int_0^1 \frac{\psi(t)}{t} \, dt
\]
as $s \to 0_+$. We therefore conclude that

$$
\rho(s) \to \frac{1}{\omega_1(0)} \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \delta_0
$$

weak-star in $M(\omega_2)$ as $s \to 0_+$. Since $\delta_0 \notin L^1(\omega_2)$, it follows that $\{\rho(s) : s \in \mathbb{R}^+\}$ is not contained in a weakly compact set of $L^1(\omega_2)$ (even excepting null sets), and the result follows. 

\[
\square
\]

References


Thomas Vils Pedersen
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark
vils@math.ku.dk