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# Generalized Hardy-Cesàro operators between weighted spaces

Thomas Vils Pedersen

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## Abstract

We characterize those non-negative, measurable functions  $\psi$  on  $[0, 1]$  and positive, continuous functions  $\omega_1$  and  $\omega_2$  on  $\mathbb{R}^+$  for which the generalized Hardy-Cesàro operator

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t) dt$$

defines a bounded operator  $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$ . This generalizes a result of Xiao ([7]) to weighted spaces. Furthermore, we extend  $U_\psi$  to a bounded operator on  $M(\omega_1)$  with range in  $L^1(\omega_2) \oplus \mathbb{C}\delta_0$ , where  $M(\omega_1)$  is the weighted space of locally finite, complex Borel measures on  $\mathbb{R}^+$ . Finally, we show that the zero operator is the only weakly compact generalized Hardy-Cesàro operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ .

## 1 Introduction

A classical result of Hardy ([5]) shows that the Hardy-Cesàro operator

$$(Uf)(x) = \frac{1}{x} \int_0^x f(s) ds$$

defines a bounded linear operator on  $L^p(\mathbb{R}^+)$  with  $\|U\| = p/(p-1)$  for  $p > 1$ . Clearly,  $U$  is not bounded on  $L^1(\mathbb{R}^+)$ . Hardy's result has been generalized in various ways, of which we will mention some, which have inspired this paper.

For  $1 \leq p \leq q \leq \infty$  and non-negative measurable functions  $u$  and  $v$  on  $\mathbb{R}^+$ , Muckenhoupt ([6]) and Bradley ([3]) gave a necessary and sufficient condition for the existence of a constant  $C$  such that

$$\left( \int_0^\infty \left( u(x) \int_0^x f(t) dt \right)^q dx \right)^{1/q} \leq C \left( \int_0^\infty (v(x)f(x))^p dx \right)^{1/p}$$

for every positive, measurable function  $f$  on  $\mathbb{R}^+$ . This can be rephrased as a characterization of the weighted  $L^p$  and  $L^q$  spaces on  $\mathbb{R}^+$  between which the Hardy-Cesàro operator  $U$  is bounded.

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In a different direction, for a non-negative measurable function  $\psi$  on  $[0, 1]$ , Xiao ([7]) considered the generalized Hardy-Cesàro operators

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t) dt$$

for measurable functions  $f$  on  $\mathbb{R}^n$ . We remark that

$$(U_\psi f)(x) = \frac{1}{x} \int_0^x f(s)\psi(s/x) ds$$

for measurable functions  $f$  on  $\mathbb{R}$ . Xiao proved that  $U_\psi$  defines a bounded operator on  $L^p(\mathbb{R}^n)$  (for  $p \geq 1$ ) if and only if

$$\int_0^1 \frac{\psi(t)}{t^{n/p}} dt < \infty.$$

Xiao's result is the main motivation for this paper.

Finally, we mention that Albanese, Bonet and Ricker in a recent series of papers (see, for instance, [1] and [2]) have considered the spectrum, compactness and other properties of the Hardy-Cesàro operator on various spaces of continuous functions and discrete spaces.

In this paper we will study the generalized Hardy-Cesàro operators between weighted spaces of integrable functions, and we will obtain a generalization of Xiao's result in this context. Let  $\omega$  be a positive, continuous function on  $\mathbb{R}^+$  and let  $L^1(\omega)$  be the Banach space of (equivalence classes of) measurable functions  $f$  on  $\mathbb{R}^+$  for which

$$\|f\|_{L^1(\omega)} = \int_0^\infty |f(t)|\omega(t) dt < \infty.$$

In the usual way we identify the dual space of  $L^1(\omega)$  with the space  $L^\infty(1/\omega)$  of measurable functions  $h$  on  $\mathbb{R}^+$  for which

$$\|h\|_{L^\infty(1/\omega)} = \text{ess sup}_{t \in \mathbb{R}^+} |h(t)|/\omega(t) < \infty.$$

We denote by  $C_0(1/\omega)$  the closed subspace of  $L^\infty(1/\omega)$  consisting of the continuous functions  $g$  in  $L^\infty(1/\omega)$  for which  $g/\omega$  vanishes at infinity. Finally, we identify the dual space of  $C_0(1/\omega)$  with the space  $M(\omega)$  of locally finite, complex Borel measures  $\mu$  on  $\mathbb{R}^+$  for which

$$\|\mu\|_{M(\omega)} = \int_{\mathbb{R}^+} \omega(t) d|\mu|(t) < \infty.$$

We consider the space  $L^1(\omega)$  as a closed subspace of  $M(\omega)$ .

In Section 2 we characterize those functions  $\psi, \omega_1$  and  $\omega_2$  for which  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ . These operators are extended to bounded operators on  $M(\omega_1)$  in Section 3, where we also obtain results about their ranges. Finally, in Section 4 we show that there are no non-zero weakly compact generalized Hardy-Cesàro operators from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ .

## 2 A characterization of the generalized Hardy-Cesàro operators

For a non-negative, measurable function  $\psi$  on  $[0, 1]$  and positive, continuous functions  $\omega_1$  and  $\omega_2$  on  $\mathbb{R}^+$ , we say that condition (C) is satisfied if there exists a constant  $C$  such that

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \leq C\omega_1(s)$$

for every  $s \in \mathbb{R}^+$ .

**Theorem 2.1** *Let  $\psi$  be a non-negative, measurable function on  $[0, 1]$  and let  $\omega_1$  and  $\omega_2$  be positive, continuous functions on  $\mathbb{R}^+$ . Then  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$  if and only if condition (C) is satisfied.*

**Proof** Assume that condition (C) is satisfied and let  $f \in L^1(\omega_1)$ . Then

$$\int_0^\infty \int_0^1 |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) dt ds \leq C \int_0^\infty |f(s)| \omega_1(s) ds = C\|f\|_{L^1(\omega_1)} < \infty,$$

so it follows from Fubini's theorem that

$$\int_0^1 \int_0^\infty |f(tx)| \psi(t) \omega_2(x) dx dt = \int_0^1 \int_0^\infty |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) ds dt \leq C\|f\|_{L^1(\omega_1)} < \infty.$$

Another application of Fubini's theorem thus shows that  $(U_\psi f)(x)$  is defined for almost all  $x \in \mathbb{R}^+$  with

$$\begin{aligned} \|U_\psi f\|_{L^1(\omega_2)} &= \int_0^\infty |(U_\psi f)(x)| \omega_2(x) dx \leq \int_0^\infty \int_0^1 |f(tx)| \psi(t) \omega_2(x) dt dx \\ &= \int_0^1 \int_0^\infty |f(tx)| \psi(t) \omega_2(x) dx dt \leq C\|f\|_{L^1(\omega_1)} < \infty. \end{aligned}$$

Hence  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ .

Conversely, assume that  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ . Since  $L^1(\omega_2)$  is a closed subspace of  $M(\omega_2)$  which we identify with the dual space of  $C_0(1/\omega_2)$ , it follows from [4, Theorem VI.8.6] that there exists a map  $\rho$  from  $\mathbb{R}^+$  to  $M(\omega_2)$  for which the map  $s \mapsto \langle g, \rho(s) \rangle = \int_{\mathbb{R}^+} g(x) d\rho(s)(x)$  is measurable and essentially bounded on  $\mathbb{R}^+$  for every  $g \in C_0(1/\omega_2)$  with  $\|U_\psi\| = \text{ess sup}_{s \in \mathbb{R}^+} \|\rho(s)\|_{M(\omega_2)}$  and such that

$$\langle g, U_\psi f \rangle = \int_0^\infty \langle g, \rho(s) \rangle f(s) \omega_1(s) ds = \int_0^\infty \int_{\mathbb{R}^+} g(x) d\rho(s)(x) f(s) \omega_1(s) ds$$

for every  $g \in C_0(1/\omega_2)$  and  $f \in L^1(\omega_1)$ . On the other hand

$$\begin{aligned} \langle g, U_\psi f \rangle &= \int_0^\infty g(x) (U_\psi f)(x) dx \\ &= \int_0^\infty \int_0^x \frac{g(x)}{x} f(s) \psi(s/x) ds dx \\ &= \int_0^\infty \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) dx f(s) \omega_1(s) ds \end{aligned}$$

for every  $g \in C_0(1/\omega_2)$  and  $f \in L^1(\omega_1)$ , so it follows that

$$\int_{\mathbb{R}^+} g(x) d\rho(s)(x) = \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) dx$$

for almost all  $s \in \mathbb{R}^+$  and every  $g \in C_0(1/\omega_2)$  (considering both sides as elements of  $L^\infty(\mathbb{R}^+)$ ). Considered as elements of  $M(\omega_2)$  we thus have

$$d\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s} dx$$

for almost all  $s, x \in \mathbb{R}^+$ . Hence  $\rho(s) \in L^1(\omega_2)$  with

$$\begin{aligned} \|\rho(s)\|_{L^1(\omega_2)} &= \int_0^\infty \omega_2(x) d\rho(s)(x) \\ &= \frac{1}{\omega_1(s)} \int_0^\infty \frac{1}{x} \psi(s/x) 1_{x \geq s} \omega_2(x) dx \\ &= \frac{1}{\omega_1(s)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) dx \\ &= \frac{1}{\omega_1(s)} \int_0^1 \frac{\psi(t)}{t} \omega_2(s/t) dt \end{aligned}$$

for almost all  $s \in \mathbb{R}^+$ . Therefore

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \|\rho(s)\|_{L^1(\omega_2)} \omega_1(s) \leq \|U_\psi\| \omega_1(s)$$

for almost all  $s \in \mathbb{R}^+$ . Since both sides of the inequality are continuous functions of  $s$ , the inequality holds for every  $s \in \mathbb{R}^+$ , so condition (C) holds.  $\square$

Letting  $s = 0$  in condition (C) we see that Xiao's condition is necessary in our situation.

**Corollary 2.2** *Let  $\psi$  be a non-negative, measurable function on  $[0, 1]$  and let  $\omega_1$  and  $\omega_2$  be positive, continuous functions on  $\mathbb{R}^+$ . If  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ , then*

$$\int_0^1 \frac{\psi(t)}{t} dt < \infty.$$

The following straightforward consequences can be deduced from Theorem 2.1.

**Corollary 2.3** *Let  $\psi$  be a non-negative, measurable function on  $[0, 1]$*

- (a) *Let  $\omega$  be a decreasing, positive, continuous function on  $\mathbb{R}^+$ , and assume that  $\int_0^1 \psi(t)/t dt < \infty$ . Then  $U_\psi$  defines a bounded operator from  $L^1(\omega)$  to  $L^1(\omega)$ .*
- (b) *Let  $\omega_1$  and  $\omega_2$  be positive, continuous functions on  $\mathbb{R}^+$ , and assume that  $\omega_2$  is increasing. If  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ , then there exists a constant  $C$  such that  $\omega_2(s) \leq C\omega_1(s)$  for every  $s \in \mathbb{R}^+$ .*

(c) Let  $\omega$  be an increasing, positive, continuous function on  $\mathbb{R}^+$ , and assume that there exists  $a < 1$  and  $K > 0$  such that  $\psi(t) \geq K$  almost everywhere on  $[a, 1]$ . If  $U_\psi$  defines a bounded operator from  $L^1(\omega)$  to  $L^1(\omega)$ , then there exist positive constants  $C_1$  and  $C_2$  such that

$$C_1\omega(s) \leq \int_0^1 \omega(s/t) \frac{\psi(t)}{t} dt \leq C_2\omega(s)$$

for every  $s \in \mathbb{R}^+$ .

**Proof** (a): We have

$$\int_0^1 \omega(s/t) \frac{\psi(t)}{t} dt \leq \int_0^1 \frac{\psi(t)}{t} dt \omega(s)$$

for every  $s \in \mathbb{R}^+$ , so condition (C) is satisfied with  $\omega_1 = \omega_2 = \omega$  and the result follows.

(b): We have

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \geq \int_0^1 \frac{\psi(t)}{t} dt \omega_2(s)$$

for every  $s \in \mathbb{R}^+$ . Since condition (C) is satisfied, the result follows.

(c): We have

$$\int_0^1 \omega(s/t) \frac{\psi(t)}{t} dt \geq K \int_a^1 \omega(s/t) dt \geq K(1-a)\omega(s)$$

for every  $s \in \mathbb{R}^+$ . The other inequality is just condition (C) with  $\omega_1 = \omega_2 = \omega$ .  $\square$

We finish the section with some examples of functions  $\psi, \omega_1$  and  $\omega_2$  for which  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ .

#### Example 2.4

(a) For  $\alpha > 0$ , let  $\psi(t) = t^\alpha$  for  $t \in [0, 1]$ . Also, for  $\beta_1, \beta_2 \in \mathbb{R}$ , let  $\omega_i(x) = (1+x)^{\beta_i}$  for  $x \in \mathbb{R}^+$  and  $i = 1, 2$ . Then  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$  if and only if  $\beta_2 \leq \beta_1$  and  $\beta_2 < \alpha$ .

(b) For  $\alpha > 0$ , let  $\psi(t) = t^\alpha$  for  $t \in [0, 1]$ . Also, let  $\omega_1(x) = e^{-x}/(1+x)$  and  $\omega_2(x) = e^{-x}$  for  $x \in \mathbb{R}^+$ . Then  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ . Moreover, it is not possible to replace  $\omega_1(x)$  by a function tending faster to zero at infinity.

(c) Let  $\psi(t) = e^{-1/t^2}$  for  $t \in [0, 1]$ . Also, let  $\omega_1(x) = e^{x^2/4}/x$  and  $\omega_2(x) = e^x$  for  $x \in \mathbb{R}^+$ . Then  $U_\psi$  defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ . Moreover, it is not possible to replace  $\omega_1(x)$  by a function tending slower to infinity at infinity.

**Proof** (a): For  $s \geq 1$  and  $t \in [0, 1]$  we have  $s/t < 1 + s/t \leq 2s/t$ , so

$$\begin{aligned} \int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt &= \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1} dt \\ &\simeq s^{\beta_2} \int_0^1 t^{\alpha-\beta_2-1} dt \\ &\simeq s^{\beta_2} \end{aligned}$$

for  $s \geq 1$  if  $\beta_2 < \alpha$  (where  $F(s) \simeq G(s)$  for positive functions  $F$  and  $G$  on  $[1, \infty)$  indicates the existence of positive constants  $C_1$  and  $C_2$  such that  $C_1 F(s) \leq G(s) \leq C_2 F(s)$  for all  $s \in [1, \infty)$ ), whereas the integrals diverge if  $\beta_2 \geq \alpha$ . Moreover, the expression

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1} dt$$

defines a positive, continuous function of  $s$  on  $\mathbb{R}^+$ , so it follows that condition (C) is satisfied if and only if  $\beta_2 \leq \beta_1$  and  $\beta_2 < \alpha$ .

(b): For  $s \geq 1$  we have

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_s^\infty \frac{\omega_2(x)}{x} \psi(s/x) dx = \int_s^\infty \frac{e^{-x}}{x} \frac{s^\alpha}{x^\alpha} dx \leq \int_s^\infty \frac{e^{-x}}{x} dx \leq \frac{e^{-s}}{s}.$$

Moreover,

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \leq \int_0^1 \frac{\psi(t)}{t} dt < \infty$$

for all  $s \in \mathbb{R}^+$ , so condition (C) is satisfied and  $U_\psi$  thus defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ . On the other hand, since

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \geq \int_s^{2s} \frac{e^{-x}}{x} \frac{s^\alpha}{x^\alpha} dx \geq \frac{1}{2^{\alpha+1}s} \int_s^{2s} e^{-x} dx \geq \frac{1}{2^{\alpha+2}} \frac{e^{-s}}{s}$$

for  $s \geq 1$ , it is not possible to replace  $\omega_1(x)$  by a function tending faster to zero at infinity.

(c): For  $s \in \mathbb{R}^+$  we have

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_s^\infty \frac{\omega_2(x)}{x} \psi(s/x) dx = \int_s^\infty \frac{e^{x-x^2/s^2}}{x} dx = \int_1^\infty \frac{e^{sy-y^2}}{y} dy.$$

Moreover, for  $s \geq 4$

$$\int_{s/4}^\infty \frac{e^{sy-y^2}}{y} dy \leq \frac{4}{s} \int_{s/4}^\infty e^{-(y-s/2)^2+s^2/4} dy = 4 \int_{-s/4}^\infty e^{-u^2} du \frac{e^{s^2/4}}{s}$$

and

$$\int_1^{s/4} \frac{e^{sy-y^2}}{y} dy \leq \int_1^{s/4} e^{sy} dy \leq \frac{e^{s^2/4}}{s},$$

so condition (C) is satisfied and  $U_\psi$  thus defines a bounded operator from  $L^1(\omega_1)$  to  $L^1(\omega_2)$ . On the other hand, the estimate

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_1^\infty \frac{e^{sy-y^2}}{y} dy \geq \frac{1}{s} \int_{s/2}^{s/2+1} e^{-(y-s/2)^2+s^2/4} dy = \int_0^1 e^{-u^2} du \frac{e^{s^2/4}}{s}$$

for  $s \geq 2$  shows that it is not possible to replace  $\omega_1(x)$  by a function tending slower to infinity at infinity.  $\square$

In Example 2.4(b) we have  $\omega_2(x)/\omega_1(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , which should be compared to the conclusion in Corollary 2.3(b). Conversely, Example 2.4(c) shows an example where we need  $\omega_2(x)/\omega_1(x) \rightarrow 0$  rapidly as  $x \rightarrow \infty$  in order for  $U_\psi$  to be defined.

### 3 Extensions to weighted spaces of measures

Identifying the dual space of  $L^1(\omega)$  with  $L^\infty(1/\omega)$  as in the introduction, we have the following result about the adjoint of  $U_\psi$ .

**Proposition 3.1** *Let  $\psi$  be a non-negative, measurable function on  $[0, 1]$  and let  $\omega_1$  and  $\omega_2$  be positive, continuous functions on  $\mathbb{R}^+$ . Assume that condition (C) is satisfied so that  $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$  is a bounded operator, and consider the adjoint operator  $U_\psi^* : L^\infty(1/\omega_2) \rightarrow L^\infty(1/\omega_1)$ .*

(a) For  $h \in L^\infty(1/\omega_2)$  we have

$$(U_\psi^*h)(x) = \int_0^1 h(x/t) \frac{\psi(t)}{t} dt$$

for almost all  $x \in \mathbb{R}^+$ .

(b)  $U_\psi^*$  maps  $C_0(1/\omega_2)$  into  $C_0(1/\omega_1)$ .

**Proof** (a): Let  $h \in L^\infty(1/\omega_2)$ . Since  $|h(x/t)| \leq \|h\|_{L^\infty(1/\omega_2)} \omega_2(x/t)$  for almost all  $x, t \in \mathbb{R}^+$ , it follows from condition (C) that  $\int_0^1 h(x/t)\psi(t)/t dt$  is defined and satisfies

$$\left| \int_0^1 h(x/t) \frac{\psi(t)}{t} dt \right| \leq \|h\|_{L^\infty(1/\omega_2)} \int_0^1 \omega_2(x/t) \frac{\psi(t)}{t} dt \leq C \|h\|_{L^\infty(1/\omega_2)} \omega_1(x)$$

for almost all  $x \in \mathbb{R}^+$ . Hence the function  $x \mapsto \int_0^1 h(x/t)\psi(t)/t dt$  belongs to  $L^\infty(1/\omega_1)$ . Also, for  $f \in L^1(\omega_1)$  we have

$$\begin{aligned} \langle f, U_\psi^*h \rangle &= \langle U_\psi f, h \rangle = \int_0^\infty (U_\psi f)(s) h(s) ds \\ &= \int_0^\infty \int_0^s \frac{1}{s} f(x) \psi(x/s) h(s) dx ds \\ &= \int_0^\infty \int_x^\infty \frac{h(s)}{s} \psi(x/s) ds f(x) dx \end{aligned}$$

from which it follows that

$$(U_\psi^*h)(x) = \int_x^\infty \frac{h(s)}{s} \psi(x/s) ds = \int_0^1 h(x/t) \frac{\psi(t)}{t} dt$$

for almost all  $x \in \mathbb{R}^+$ .

(b): It suffices to show that  $U_\psi^*$  maps  $C_c(\mathbb{R}^+)$  (the continuous functions on  $\mathbb{R}^+$  with compact support) into  $C_0(1/\omega_1)$ . Let  $g \in C_c(\mathbb{R}^+)$ , let  $x_0 \in \mathbb{R}^+$  and let  $(x_n)$  be a sequence in  $\mathbb{R}^+$  with  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . Then

$$(U_\psi^*g)(x_n) - (U_\psi^*g)(x_0) = \int_0^1 (g(x_n/t) - g(x_0/t)) \frac{\psi(t)}{t} dt$$

for  $n \in \mathbb{N}$ . Since  $g$  is bounded on  $\mathbb{R}^+$  and since  $\int_0^1 \psi(t)/t dt < \infty$  by Corollary 2.2, it follows from Lebesgue's dominated convergence theorem that  $(U_\psi^*g)(x_n) \rightarrow (U_\psi^*g)(x_0)$  as  $n \rightarrow \infty$ . Hence  $U_\psi^*g$  is continuous on  $\mathbb{R}^+$ . Finally, from the expression

$$(U_\psi^*g)(x) = \int_x^\infty \frac{g(s)}{s} \psi(x/s) ds$$



it follows that  $\text{supp } U_\psi^* g \subseteq \text{supp } g$ , so we conclude that  $U_\psi^* g \in C_c(\mathbb{R}^+) \subseteq C_0(1/\omega_1)$ .  $\square$

Let  $V_\psi$  be the restriction of  $U_\psi^*$  to  $C_0(1/\omega_2)$  considered as a map into  $C_0(1/\omega_1)$ . We then immediately have the following result.

**Corollary 3.2** *Let  $\psi$  be a non-negative, measurable function on  $[0, 1]$  and let  $\omega_1$  and  $\omega_2$  be positive, continuous functions on  $\mathbb{R}^+$ . Assume that condition (C) is satisfied so that  $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$  is a bounded operator. The bounded operator  $\overline{U}_\psi = V_\psi^*$  from  $M(\omega_1)$  to  $M(\omega_2)$  is an extension of  $U_\psi$ .*

Let  $\psi$  be a non-negative, continuous function on  $[0, 1]$  with  $\psi(0) = 0$ . For  $\mu \in M(\omega_1)$  and  $x > 0$  let

$$(W_\psi \mu)(x) = \frac{1}{x} \int_{(0,x)} \psi(s/x) d\mu(s).$$

**Proposition 3.3** *Let  $\psi$  be a non-negative, continuous function on  $[0, 1]$  and let  $\omega_1$  and  $\omega_2$  be positive, continuous functions on  $\mathbb{R}^+$ . Assume that condition (C) is satisfied so that  $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$  is a bounded operator. Then  $W_\psi \mu \in L^1(\omega_2)$  and*

$$\overline{U}_\psi \mu = W_\psi \mu + \int_0^1 \frac{\psi(t)}{t} dt \cdot \mu(\{0\}) \delta_0$$

for  $\mu \in M(\omega_1)$ . In particular  $\text{ran } \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C} \delta_0$  and  $\overline{U}_\psi$  maps  $M((0, \infty), \omega_1)$  into  $L^1(\omega_2)$ .

**Proof** By Corollary 2.2 we have  $\int_0^1 \psi(t)/t dt < \infty$ , so it follows that  $\psi(0) = 0$ . Let  $\mu \in M(\omega_1)$  with  $\mu(\{0\}) = 0$ . By condition (C) we have

$$\begin{aligned} \int_{(0,\infty)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) dx d|\mu|(s) &= \int_{(0,\infty)} \int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt d|\mu|(s) \\ &\leq C \int_{(0,\infty)} \omega_1(s) d|\mu|(s) = C \|\mu\|_{M(\omega_1)} < \infty, \end{aligned}$$

so it follows from Fubini's theorem that

$$\int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x) d|\mu|(s) \omega_2(x) dx < \infty.$$

Hence  $W_\psi \mu \in L^1(\omega_2)$ . Moreover, for  $g \in C_0(1/\omega_2)$  we have

$$\begin{aligned} \langle g, \overline{U}_\psi \mu \rangle &= \langle V_\psi g, \mu \rangle = \int_{(0,\infty)} \int_0^1 g(s/t) \frac{\psi(t)}{t} dt d\mu(s) \\ &= \int_{(0,\infty)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) dx d\mu(s) \\ &= \int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x) d\mu(s) g(x) dx \\ &= \int_0^\infty (W_\psi \mu)(x) g(x) dx = \langle g, W_\psi \mu \rangle, \end{aligned}$$

so we conclude that  $\overline{U}_\psi \mu = W_\psi \mu$ . Finally, for  $g \in C_0(1/\omega_2)$  we have

$$\langle g, \overline{U}_\psi \delta_0 \rangle = \langle V_\psi g, \delta_0 \rangle = (V_\psi g)(0) = g(0) \int_0^1 \frac{\psi(t)}{t} dt = \langle g, \int_0^1 \frac{\psi(t)}{t} dt \cdot \delta_0 \rangle.$$

Since  $W_\psi \delta_0 = 0$  this finishes the proof.  $\square$

The conclusion about the range of  $\overline{U}_\psi$  can be generalized to the case, where  $\psi$  is not assumed to be continuous.

**Proposition 3.4** *Let  $\psi$  be a non-negative, measurable function on  $[0, 1]$  and let  $\omega_1$  and  $\omega_2$  be positive, continuous functions on  $\mathbb{R}^+$ . Assume that condition (C) is satisfied so that  $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$  is a bounded operator. Then  $\text{ran } \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$ .*

**Proof** Choose a sequence of non-negative, continuous functions  $(\psi_n)$  on  $[0, 1]$  with  $\psi_n \leq \psi$  and

$$\int_0^1 \frac{\psi(t) - \psi_n(t)}{t} dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For  $\mu \in M(\omega_1)$  and  $g \in C_0(1/\omega_2)$  we have

$$\begin{aligned} |\langle g, (\overline{U}_\psi - \overline{U}_{\psi_n})\mu \rangle| &= |\langle (V_\psi - V_{\psi_n})g, \mu \rangle| \\ &= \left| \int_{\mathbb{R}^+} \int_0^1 g(x/t) \frac{\psi(t) - \psi_n(t)}{t} dt d\mu(x) \right| \\ &\leq \|g\|_{C_0(1/\omega_2)} \int_{\mathbb{R}^+} \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} dt d|\mu|(x). \end{aligned}$$

Let

$$p_n(x) = \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} dt$$

for  $x \in \mathbb{R}^+$  and  $n \in \mathbb{N}$ . By condition (C) there exists a constant  $C$  such that  $p_n(x) \leq C\omega_1(x)$  for every  $x \in \mathbb{R}^+$  and  $n \in \mathbb{N}$ . Moreover, for every  $x \in \mathbb{R}^+$  we have  $p_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  by Lebesgue's dominated convergence theorem. Hence

$$\|(\overline{U}_\psi - \overline{U}_{\psi_n})\mu\|_{M(\omega_2)} = \sup_{\|g\|_{C_0(1/\omega_2)} \leq 1} |\langle g, (\overline{U}_\psi - \overline{U}_{\psi_n})\mu \rangle| \leq \int_{\mathbb{R}^+} p_n(x) d|\mu|(x) \rightarrow 0$$

as  $n \rightarrow \infty$  again by Lebesgue's dominated convergence theorem. Consequently,  $\overline{U}_{\psi_n} \rightarrow \overline{U}_\psi$  strongly as  $n \rightarrow \infty$ . Since  $\text{ran } \overline{U}_{\psi_n} \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$  for  $n \in \mathbb{N}$  by Proposition 3.3, the same thus holds for  $\text{ran } \overline{U}_\psi$ .  $\square$

**Corollary 3.5** *Let  $\psi$  be a non-negative, measurable function on  $[0, 1]$  and let  $\omega_1$  and  $\omega_2$  be positive, continuous functions on  $\mathbb{R}^+$ . Assume that condition (C) is satisfied so that  $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$  is a bounded operator. For  $s > 0$  we then have  $(\overline{U}_\psi \delta_s)(x) = \psi(s/x)/x$  for almost all  $x \geq s$  and  $(\overline{U}_\psi \delta_s)(x) = 0$  for almost all  $x < s$ .*

**Proof** For  $\psi$  continuous, this follows from Proposition 3.3. For general  $\psi$  it follows from the approach in the proof of Proposition 3.4 using  $\overline{U}_{\psi_n} \rightarrow \overline{U}_\psi$  strongly as  $n \rightarrow \infty$ .  $\square$

It follows from Corollary 3.5 that

$$\|\overline{U}_\psi \delta_s\|_{M(\omega_2)} = \int_s^\infty \frac{\omega_2(x)}{x} \psi(s/x) dx = \int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt,$$

whereas  $\|\delta_s\|_{M(\omega_1)} = \omega_1(s)$ . Since  $\overline{U}_\psi$  is bounded we thus recover condition (C). If we without using Theorem 2.1 could show that if  $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$  is a bounded operator, then it has a bounded extension  $\overline{U}_\psi : M(\omega_1) \rightarrow M(\omega_2)$  for which Corollary 3.5 holds, then we would in this way obtain an alternative proof of condition (C).

## 4 Weakly compact operators

We finish the paper by showing that there are no non-zero, weakly compact generalized Hardy-Cesàro operators between  $L^1(\omega_1)$  and  $L^1(\omega_2)$ .

**Proposition 4.1** *Let  $\psi$  be a non-negative, measurable function on  $[0, 1]$  and let  $\omega_1$  and  $\omega_2$  be positive, continuous functions on  $\mathbb{R}^+$ . Assume that condition (C) is satisfied so that  $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$  is a bounded operator. If  $\psi \neq 0$ , then  $U_\psi$  is not weakly compact.*

**Proof** For  $f \in L^1(\omega_1)$  and  $x \in \mathbb{R}^+$  we have

$$(U_\psi f)(x) = \frac{1}{x} \int_0^x f(s) \psi(s/x) ds = \int_0^\infty f(s) \rho(s)(x) \omega_1(s) ds,$$

where (with a slight change of notation compared to the proof of Theorem 2.1)

$$\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s}$$

for  $x, s \in \mathbb{R}^+$ . In the proof of Theorem 2.1 we saw that  $\rho(s) \in L^1(\omega_2)$  with  $\|\rho(s)\|_{L^1(\omega_2)} \leq C$  for a constant  $C$  for almost all  $s \in \mathbb{R}^+$ . It thus follows from [4, Theorem VI.8.10] that  $U_\psi$  is weakly compact if and only if  $\{\rho(s) : s \in \mathbb{R}^+\}$  is contained in a weakly compact set of  $L^1(\omega_2)$  (except possibly for  $s$  belonging to a null-set). Consider  $\rho(s)$  as an element of  $C_0(1/\omega_2)^*$  for  $s \in \mathbb{R}^+$  and let  $g \in C_0(1/\omega_2)$ . Then

$$\begin{aligned} \langle g, \rho(s) \rangle &= \int_0^\infty g(x) \rho(s)(x) dx \\ &= \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) dx \\ &= \frac{1}{\omega_1(s)} \int_0^1 g(s/t) \frac{\psi(t)}{t} dt. \end{aligned}$$

Since  $g(s/t) \rightarrow g(0)$  as  $s \rightarrow 0_+$  for all  $t > 0$ , it follows from Lebesgue's dominated convergence theorem that

$$\langle g, \rho(s) \rangle \rightarrow \frac{1}{\omega_1(0)} g(0) \int_0^1 \frac{\psi(t)}{t} dt$$

as  $s \rightarrow 0_+$ . We therefore conclude that

$$\rho(s) \rightarrow \frac{1}{\omega_1(0)} \int_0^1 \frac{\psi(t)}{t} dt \cdot \delta_0$$

weak-star in  $M(\omega_2)$  as  $s \rightarrow 0_+$ . Since  $\delta_0 \notin L^1(\omega_2)$ , it follows that  $\{\rho(s) : s \in \mathbb{R}^+\}$  is not contained in a weakly compact set of  $L^1(\omega_2)$  (even excepting null sets), and the result follows.  $\square$

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