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Generalized Hardy-Cesàro operators between weighted spaces

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Abstract

We characterize those non-negative, measurable functions $\psi$ on $[0,1]$ and positive, continuous functions $\omega_1$ and $\omega_2$ on $\mathbb{R}^+$ for which the generalized Hardy-Cesàro operator

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t)\,dt$$

defines a bounded operator $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$. This generalizes a result of Xiao ([7]) to weighted spaces. Furthermore, we extend $U_\psi$ to a bounded operator on $M(\omega_1)$ with range in $L^1(\omega_2) \oplus \mathbb{C}\delta_0$, where $M(\omega_1)$ is the weighted space of locally finite, complex Borel measures on $\mathbb{R}^+$. Finally, we show that the zero operator is the only weakly compact generalized Hardy-Cesàro operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

1 Introduction

A classical result of Hardy ([5]) shows that the Hardy-Cesàro operator

$$(Uf)(x) = \frac{1}{x} \int_0^x f(s)\,ds$$

defines a bounded linear operator on $L^p(\mathbb{R}^+)$ with $\|U\| = p/(p-1)$ for $p > 1$. Clearly, $U$ is not bounded on $L^1(\mathbb{R}^+)$. Hardy’s result has been generalized in various ways, of which we will mention some, which have inspired this paper.

For $1 \leq p \leq q \leq \infty$ and non-negative measurable functions $u$ and $v$ on $\mathbb{R}^+$, Muckenhoupt ([6]) and Bradley ([3]) gave a necessary and sufficient condition for the existence of a constant $C$ such that

$$\left(\int_0^\infty \left(\int_0^x u(x)f(t)\,dt\right)^q\,dx\right)^{1/q} \leq C \left(\int_0^\infty (v(x)f(x))^p\,dx\right)^{1/p}$$

for every positive, measurable function $f$ on $\mathbb{R}^+$. This can be rephrased as a characterization of the weighted $L^p$ and $L^q$ spaces on $\mathbb{R}^+$ between which the Hardy-Cesàro operator $U$ is bounded.

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In a different direction, for a non-negative measurable function $\psi$ on $[0, 1]$, Xiao ([7]) considered the generalized Hardy-Cesàro operators

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t) \, dt$$

for measurable functions $f$ on $\mathbb{R}^n$. We remark that

$$(U_\psi f)(x) = \frac{1}{x} \int_0^x f(s)\psi(s/x) \, ds$$

for measurable functions $f$ on $\mathbb{R}$. Xiao proved that $U_\psi$ defines a bounded operator on $L^p(\mathbb{R}^n)$ (for $p \geq 1$) if and only if

$$\int_0^1 \frac{\psi(t)}{t^{n/p}} \, dt < \infty.$$ 

Xiao’s result is the main motivation for this paper.

Finally, we mention that Albanese, Bonet and Ricker in a recent series of papers (see, for instance, [1] and [2]) have considered the spectrum, compactness and other properties of the Hardy-Cesàro operator on various spaces of continuous functions and discrete spaces.

In this paper we will study the generalized Hardy-Cesàro operators between weighted spaces of integrable functions, and we will obtain a generalization of Xiao’s result in this context. Let $\omega$ be a positive, continuous function on $\mathbb{R}^+$ and let $L^1(\omega)$ be the Banach space of (equivalence classes of) measurable functions $f$ on $\mathbb{R}^+$ for which

$$\|f\|_{L^1(\omega)} = \int_0^\infty |f(t)|\omega(t) \, dt < \infty.$$ 

In the usual way we identify the dual space of $L^1(\omega)$ with the space $L^\infty(1/\omega)$ of measurable functions $h$ on $\mathbb{R}^+$ for which

$$\|h\|_{L^\infty(1/\omega)} = \text{ess sup}_{t \in \mathbb{R}^+} |h(t)|/\omega(t) < \infty.$$ 

We denote by $C_0(1/\omega)$ the closed subspace of $L^\infty(1/\omega)$ consisting of the continuous functions $g$ in $L^\infty(1/\omega)$ for which $g/\omega$ vanishes at infinity. Finally, we identify the dual space of $C_0(1/\omega)$ with the space $M(\omega)$ of locally finite, complex Borel measures $\mu$ on $\mathbb{R}^+$ for which

$$\|\mu\|_{M(\omega)} = \int_{\mathbb{R}^+} \omega(t) \, d|\mu|(t) < \infty.$$ 

We consider the space $L^1(\omega)$ as a closed subspace of $M(\omega)$.

In Section 2 we characterize those functions $\psi, \omega_1$ and $\omega_2$ for which $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. These operators are extended to bounded operators on $M(\omega_1)$ in Section 3 where we also obtain results about their ranges. Finally, in Section 4 we show that there are no non-zero weakly compact generalized Hardy-Cesàro operators from $L^1(\omega_1)$ to $L^1(\omega_2)$. 

2
2 A characterization of the generalized Hardy-Cesàro operators

For a non-negative, measurable function $\psi$ on $[0, 1]$ and positive, continuous functions $\omega_1$ and $\omega_2$ on $\mathbb{R}^+$, we say that condition (C) is satisfied if there exists a constant $C$ such that

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \leq C \omega_1(s)$$

for every $s \in \mathbb{R}^+$. 

**Theorem 2.1** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if condition (C) is satisfied.

**Proof** Assume that condition (C) is satisfied and let $f \in L^1(\omega_1)$. Then

$$\int_0^\infty \int_0^1 |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) \, dt \, ds \leq C \int_0^\infty |f(s)| \omega_1(s) \, ds = C \|f\|_{L^1(\omega_1)} < \infty,$$

so it follows from Fubini’s theorem that

$$\int_0^1 \int_0^\infty |f(tx)| \psi(t) \omega_2(x) \, dx \, dt = \int_0^1 \int_0^\infty |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) \, dt \, ds \leq C \|f\|_{L^1(\omega_1)} < \infty.$$

Another application of Fubini’s theorem thus shows that $(U_\psi f)(x)$ is defined for almost all $x \in \mathbb{R}^+$ with

$$\|U_\psi f\|_{L^1(\omega_2)} = \int_0^\infty |(U_\psi f)(x)| \omega_2(x) \, dx \leq \int_0^\infty \int_0^1 |f(tx)| \psi(t) \omega_2(x) \, dx \, dt \leq C \|f\|_{L^1(\omega_1)} < \infty.$$

Hence $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

Conversely, assume that $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Since $L^1(\omega_2)$ is a closed subspace of $M(\omega_2)$ which we identify with the dual space of $C_0(1/\omega_2)$, it follows from [4, Theorem VI.8.6] that there exists a map $\rho$ from $\mathbb{R}^+$ to $M(\omega_2)$ for which the map $s \mapsto \langle g, \rho(s) \rangle = \int_{\mathbb{R}^+} g(x) \, d\rho(s)(x)$ is measurable and essentially bounded on $\mathbb{R}^+$ for every $g \in C_0(1/\omega_2)$ with $\|U_\psi\| = \text{ess sup}_{s \in \mathbb{R}^+} \|\rho(s)\|_{M(\omega_2)}$ and such that

$$\langle g, U_\psi f \rangle = \int_0^\infty \langle g, \rho(s) \rangle f(s) \omega_1(s) \, ds = \int_0^\infty \int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) \, f(s) \omega_1(s) \, ds$$

for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$. On the other hand

$$\langle g, U_\psi f \rangle = \int_0^\infty g(x) (U_\psi f)(x) \, dx$$

$$= \int_0^\infty \int_0^x g(x) f(s) \psi(s/x) \, ds \, dx$$

$$= \int_0^\infty \frac{1}{\omega_1(s)} \int_s^\infty g(x) \psi(s/x) \, dx \, f(s) \omega_1(s) \, ds$$
for every \( g \in C_0(1/\omega_2) \) and \( f \in L^1(\omega_1) \), so it follows that
\[
\int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) = \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx
\]
for almost all \( s \in \mathbb{R}^+ \) and every \( g \in C_0(1/\omega_2) \) (considering both sides as elements of \( L^\infty(\mathbb{R}^+) \)). Considered as elements of \( M(\omega_2) \) we thus have
\[
d\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s} \, dx
\]
for almost all \( s, x \in \mathbb{R}^+ \). Hence \( \rho(s) \in L^1(\omega_2) \) with
\[
\|\rho(s)\|_{L^1(\omega_2)} = \int_0^\infty \omega_2(x) \, d\rho(s)(x)
= \frac{1}{\omega_1(s)} \int_0^\infty \frac{1}{x} \psi(s/x) 1_{x \geq s} \omega_2(x) \, dx
= \frac{1}{\omega_1(s)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx
= \frac{1}{\omega_1(s)} \int_0^1 \frac{\psi(t)}{t} \omega_2(s/t) \, dt
\]
for almost all \( s \in \mathbb{R}^+ \). Therefore
\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \|\rho(s)\|_{L^1(\omega_2)} \omega_1(s) \leq \|U_\psi\|_{L^1(\omega_1)}
\]
for almost all \( s \in \mathbb{R}^+ \). Since both sides of the inequality are continuous functions of \( s \), the inequality holds for every \( s \in \mathbb{R}^+ \), so condition (C) holds. \( \square \)

Letting \( s = 0 \) in condition (C) we see that Xiao’s condition is necessary in our situation.

**Corollary 2.2** Let \( \psi \) be a non-negative, measurable function on \([0, 1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). If \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \), then
\[
\int_0^1 \psi(t)/t \, dt < \infty.
\]

The following straightforward consequences can be deduced from Theorem 2.1.

**Corollary 2.3** Let \( \psi \) be a non-negative, measurable function on \([0, 1]\)

(a) Let \( \omega \) be a decreasing, positive, continuous function on \( \mathbb{R}^+ \), and assume that
\[
\int_0^1 \psi(t)/t \, dt < \infty.
\]
Then \( U_\psi \) defines a bounded operator from \( L^1(\omega) \) to \( L^1(\omega) \).

(b) Let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \), and assume that \( \omega_2 \) is increasing. If \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \), then there exists a constant \( C \) such that \( \omega_2(s) \leq C\omega_1(s) \) for every \( s \in \mathbb{R}^+ \).
(c) Let $\omega$ be an increasing, positive, continuous function on $\mathbb{R}^+$, and assume that there exists $a < 1$ and $K > 0$ such that $\psi(t) \geq K$ almost everywhere on $[a, 1]$. If $U_\psi$ defines a bounded operator from $L^1(\omega)$ to $L^1(\omega)$, then there exist positive constants $C_1$ and $C_2$ such that
\[
C_1 \omega(s) \leq \int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \leq C_2 \omega(s)
\]
for every $s \in \mathbb{R}^+$.

**Proof (a):** We have
\[
\int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \leq \int_0^1 \frac{\psi(t)}{t} \, dt \omega(s)
\]
for every $s \in \mathbb{R}^+$, so condition (C) is satisfied with $\omega_1 = \omega_2 = \omega$ and the result follows.

(b): We have
\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \geq \int_0^1 \frac{\psi(t)}{t} \, dt \omega_2(s)
\]
for every $s \in \mathbb{R}^+$. Since condition (C) is satisfied, the result follows.

(c): We have
\[
\int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \geq K \int_a^1 \omega(s/t) \, dt \geq K (1-a) \omega(s)
\]
for every $s \in \mathbb{R}^+$. The other inequality is just condition (C) with $\omega_1 = \omega_2 = \omega$.  

We finish the section with some examples of functions $\psi, \omega_1$ and $\omega_2$ for which $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

**Example 2.4**

(a) For $\alpha > 0$, let $\psi(t) = t^\alpha$ for $t \in [0, 1]$. Also, for $\beta_1, \beta_2 \in \mathbb{R}$, let $\omega_1(x) = (1 + x)^{\beta_i}$ for $x \in \mathbb{R}^+$ and $i = 1, 2$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if $\beta_2 \leq \beta_1$ and $\beta_2 < \alpha$.

(b) For $\alpha > 0$, let $\psi(t) = t^\alpha$ for $t \in [0, 1]$. Also, let $\omega_1(x) = e^{-x}/(1 + x)$ and $\omega_2(x) = e^{-x}$ for $x \in \mathbb{R}^+$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Moreover, it is not possible to replace $\omega_1(x)$ by a function tending faster to zero at infinity.

(c) Let $\psi(t) = e^{-1/t^2}$ for $t \in [0, 1]$. Also, let $\omega_1(x) = e^{x^2/4}/x$ and $\omega_2(x) = e^x$ for $x \in \mathbb{R}^+$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Moreover, it is not possible to replace $\omega_1(x)$ by a function tending slower to infinity at infinity.

**Proof (a):** For $s \geq 1$ and $t \in [0, 1]$ we have $s/t < 1 + s/t \leq 2s/t$, so
\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1} \, dt
\]
\[
\simeq s^{\beta_2} \int_0^1 t^{\alpha-\beta_2-1} \, dt
\]
\[
\simeq s^{\beta_2}
\]
for \( s \geq 1 \) if \( \beta_2 < \alpha \) (where \( F(s) \simeq G(s) \) for positive functions \( F \) and \( G \) on \([1, \infty)\) indicates the existence of positive constants \( C_1 \) and \( C_2 \) such that \( C_1 F(s) \leq G(s) \leq C_2 F(s) \) for all \( s \in [1, \infty) \)), whereas the integrals diverge if \( \beta_2 \geq \alpha \). Moreover, the expression

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_0^1 \left( 1 + \frac{s}{t} \right)^{\beta_2} t^{\alpha-1} dt
\]

defines a positive, continuous function of \( s \) on \( \mathbb{R}^+ \), so it follows that condition (C) is satisfied if and only if \( \beta_2 \leq \beta_1 \) and \( \beta_2 < \alpha \).

(b): For \( s \geq 1 \) we have

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_s^\infty \frac{\omega_2(x)}{x} \psi(s/x) dx = \int_s^\infty \frac{e^{-x} s^\alpha}{x} x^\alpha dx \leq \int_s^\infty \frac{e^{-x}}{x} dx \leq \frac{e^{-s}}{s}.
\]

Moreover,

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \leq \int_0^1 \frac{\psi(t)}{t} dt < \infty
\]

for all \( s \in \mathbb{R}^+ \), so condition (C) is satisfied and \( U_\psi \) thus defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). On the other hand, since

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \geq \int_s^\infty \frac{e^{-x} s^\alpha}{x} x^\alpha dx \geq \frac{1}{2^{\alpha+1} s} \int_s^\infty e^{-x} dx \geq \frac{1}{2^{\alpha+2} s} e^{-s}
\]

for \( s \geq 1 \), it is not possible to replace \( \omega_1(x) \) by a function tending faster to zero at infinity.

(c): For \( s \in \mathbb{R}^+ \) we have

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_s^\infty \frac{\omega_2(x)}{x} \psi(s/x) dx = \int_s^\infty \frac{e^{x-s^2/2}}{x} dx = \int_1^\infty \frac{e^{s/y-y^2}}{y} dy.
\]

Moreover, for \( s \geq 4 \)

\[
\int_{s/4}^\infty \frac{e^{s/y-y^2}}{y} dy \leq \frac{4}{s} \int_{s/4}^\infty e^{-(y-s^2/2)^2+s^2/4} dy = 4 \int_{-s/4}^\infty e^{-u^2} du \frac{e^{s^2/4}}{s}
\]

and

\[
\int_1^{s/4} \frac{e^{s/y-y^2}}{y} dy \leq \int_1^{s/4} e^{s/y} dy \leq \frac{e^{s^2/4}}{s},
\]

so condition (C) is satisfied and \( U_\psi \) thus defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). On the other hand, the estimate

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_1^\infty \frac{e^{s/y-y^2}}{y} dy \geq \frac{1}{s} \int_{s/2}^{s/2+1} e^{-(y-s^2/2)^2+s^2/4} dy = \int_0^1 e^{-u^2} du \frac{e^{s^2/4}}{s}
\]

for \( s \geq 2 \) shows that it is not possible to replace \( \omega_1(x) \) by a function tending slower to infinity at infinity.

In Example 2.3(b) we have \( \omega_2(x)/\omega_1(x) \to \infty \) as \( x \to \infty \), which should be compared to the conclusion in Corollary 2.3(b). Conversely, Example 2.3(c) shows an example where we need \( \omega_2(x)/\omega_1(x) \to 0 \) rapidly as \( x \to \infty \) in order for \( U_\psi \) to be defined.
3 Extensions to weighted spaces of measures

Identifying the dual space of $L^1(\omega)$ with $L^\infty(1/\omega)$ as in the introduction, we have the following result about the adjoint of $U_\psi$.

**Proposition 3.1** Let $\psi$ be a non-negative, measurable function on $[0,1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator, and consider the adjoint operator $U_\psi^* : L^\infty(1/\omega_2) \to L^\infty(1/\omega_1)$.

(a) For $h \in L^\infty(1/\omega_2)$ we have

$$
(U_\psi^* h)(x) = \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt
$$

for almost all $x \in \mathbb{R}^+$.

(b) $U_\psi^*$ maps $C_0(1/\omega_2)$ into $C_0(1/\omega_1)$.

**Proof** (a): Let $h \in L^\infty(1/\omega_2)$. Since $|h(x/t)| \leq \|h\|_{L^\infty(1/\omega_2)}\omega_2(x/t)$ for almost all $x,t \in \mathbb{R}^+$, it follows from condition (C) that $\int_0^1 h(x/t)\psi(t)/t \, dt$ is defined and satisfies

$$
\left| \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt \right| \leq \|h\|_{L^\infty(1/\omega_2)} \int_0^1 \omega_2(x/t) \frac{\psi(t)}{t} \, dt \leq C\|h\|_{L^\infty(1/\omega_2)}\omega_1(x)
$$

for almost all $x \in \mathbb{R}^+$. Hence the function $x \mapsto \int_0^1 h(x/t)\psi(t)/t \, dt$ belongs to $L^\infty(1/\omega_1)$. Also, for $f \in L^1(\omega_1)$ we have

$$
\langle f, U_\psi^* h \rangle = \langle U_\psi f, h \rangle = \int_0^\infty (U_\psi f)(s)h(s) \, ds = \int_0^\infty \int_0^s \frac{1}{s} f(x)\psi(x/s)h(s) \, dx \, ds = \int_0^\infty \int_x^\infty \frac{h(s)}{s} \psi(x/s) \, ds f(x) \, dx
$$

from which it follows that

$$
(U_\psi^* h)(x) = \int_x^\infty \frac{h(s)}{s} \psi(x/s) \, ds = \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt
$$

for almost all $x \in \mathbb{R}^+$.

(b): It suffices to show that $U_\psi^*$ maps $C^c_c(\mathbb{R}^+)$ (the continuous functions on $\mathbb{R}^+$ with compact support) into $C_0(1/\omega_1)$. Let $g \in C^c_c(\mathbb{R}^+)$, let $x_0 \in \mathbb{R}^+$ and let $(x_n)$ be a sequence in $\mathbb{R}^+$ with $x_n \to x_0$ as $n \to \infty$. Then

$$
(U_\psi^* g)(x_n) - (U_\psi^* g)(x_0) = \int_0^1 (g(x_n/t) - g(x_0/t)) \frac{\psi(t)}{t} \, dt
$$

for $n \in \mathbb{N}$. Since $g$ is bounded on $\mathbb{R}^+$ and since $\int_0^1 \psi(t)/t \, dt < \infty$ by Corollary 2.2, it follows from Lebesgue’s dominated convergence theorem that $(U_\psi^* g)(x_n) \to (U_\psi^* g)(x_0)$ as $n \to \infty$. Hence $U_\psi^* g$ is continuous on $\mathbb{R}^+$. Finally, from the expression

$$
(U_\psi^* g)(x) = \int_x^\infty \frac{g(s)}{s} \psi(x/s) \, ds
$$

for almost all $x \in \mathbb{R}^+$.
it follows that $\text{supp} U_{\psi}^* g \subseteq \text{supp} g$, so we conclude that $U_{\psi}^* g \in C_c(\mathbb{R}^+) \subseteq C_0(1/\omega_1)$. □

Let $V_\psi$ be the restriction of $U_{\psi}^*$ to $C_0(1/\omega_2)$ considered as a map into $C_0(1/\omega_1)$. We then immediately have the following result.

**Corollary 3.2** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. The bounded operator $U_\psi = V_\psi^*$ from $M(\omega_1)$ to $M(\omega_2)$ is an extension of $U_\psi$.

Let $\psi$ be a non-negative, continuous function on $[0, 1]$ with $\psi(0) = 0$. For $\mu \in M(\omega_1)$ and $x > 0$ let

$$(W_\psi \mu)(x) = \frac{1}{x} \int_{(0, x)} \psi(s/x) \, d\mu(s).$$

**Proposition 3.3** Let $\psi$ be a non-negative, continuous function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. Then $W_\psi \mu \in L^1(\omega_2)$ and

$$\overline{U_\psi \mu} = W_\psi \mu + \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \mu(\{0\}) \delta_0$$

for $\mu \in M(\omega_1)$. In particular $\text{ran} \overline{U_\psi} \subseteq L^1(\omega_2) \oplus \mathbb{C} \delta_0$ and $\overline{U_\psi}$ maps $M((0, \infty), \omega_1)$ into $L^1(\omega_2)$.

**Proof** By Corollary 2.2 we have $\int_0^1 \psi(t)/t \, dt < \infty$, so it follows that $\psi(0) = 0$. Let $\mu \in M(\omega_1)$ with $\mu(\{0\}) = 0$. By condition (C) we have

$$\int_{(0, \infty)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx \, d|\mu|(s) = \int_{(0, \infty)} \int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \, d|\mu|(s) \leq C \int_{(0, \infty)} \omega_1(s) \, d|\mu|(s) = C \|\mu\|_{M(\omega_1)} < \infty,$$

so it follows from Fubini’s theorem that

$$\int_0^\infty \frac{1}{x} \int_{(0, x)} \psi(s/x) \, d|\mu|(s) \omega_2(x) \, dx < \infty.$$

Hence $W_\psi \mu \in L^1(\omega_2)$. Moreover, for $g \in C_0(1/\omega_2)$ we have

$$\langle g, \overline{U_\psi \mu} \rangle = \langle V_\psi g, \mu \rangle = \int_{(0, \infty)} \int_0^1 g(s/t) \frac{\psi(t)}{t} \, dt \, d\mu(s)$$

$$= \int_{(0, \infty)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx \, d\mu(s)$$

$$= \int_0^\infty \frac{1}{x} \int_{(0, x)} \psi(s/x) \, d\mu(s) \, g(x) \, dx$$

$$= \int_0^\infty (W_\psi \mu)(x) g(x) \, dx = \langle g, W_\psi \mu \rangle,$$
so we conclude that $\mathcal{U}_\psi \mu = W_\psi \mu$. Finally, for $g \in C_0(1/\omega_2)$ we have

$$\langle g, \mathcal{U}_\psi \delta_0 \rangle = \langle V_\psi g, \delta_0 \rangle = (V_\psi g)(0) = g(0) \int_0^1 \frac{\psi(t)}{t} \, dt = \langle g, \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \delta_0 \rangle.$$  

Since $W_\psi \delta_0 = 0$ this finishes the proof. \hfill $\Box$

The conclusion about the range of $\mathcal{U}_\psi$ can be generalized to the case, where $\psi$ is not assumed to be continuous.

**Proposition 3.4** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. Then $\text{ran} \mathcal{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$.

**Proof** Choose a sequence of non-negative, continuous functions $(\psi_n)$ on $[0, 1]$ with $\psi_n \leq \psi$ and

$$\int_0^1 \frac{\psi(t) - \psi_n(t)}{t} \, dt \to 0 \quad \text{as } n \to \infty.$$ 

For $\mu \in M(\omega_1)$ and $g \in C_0(1/\omega_2)$ we have

$$|\langle g, (\mathcal{U}_\psi - \mathcal{U}_{\psi_n})\mu \rangle| = |\langle (V_\psi - V_{\psi_n})g, \mu \rangle| = \left| \int_{\mathbb{R}^+} \int_0^1 g(x/t) \frac{\psi(t) - \psi_n(t)}{t} \, dt \, d\mu(x) \right| \leq \|g\|_{C_0(1/\omega_2)} \int_{\mathbb{R}^+} \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} \, dt \, d\mu(x).$$

Let

$$p_n(x) = \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} \, dt$$

for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. By condition (C) there exists a constant $C$ such that $p_n(x) \leq C\omega_1(x)$ for every $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Moreover, for every $x \in \mathbb{R}^+$ we have $p_n(x) \to 0$ as $n \to \infty$ by Lebesgue’s dominated convergence theorem. Hence

$$\| (\mathcal{U}_\psi - \mathcal{U}_{\psi_n})\mu \|_{M(\omega_2)} \leq \sup_{\|g\|_{C_0(1/\omega_2)} \leq 1} |\langle g, (\mathcal{U}_\psi - \mathcal{U}_{\psi_n})\mu \rangle| \leq \int_{\mathbb{R}^+} p_n(x) \, d\mu(x) \to 0$$

as $n \to \infty$ again by Lebesgue’s dominated convergence theorem. Consequently, $\mathcal{U}_{\psi_n} \to \mathcal{U}_\psi$ strongly as $n \to \infty$. Since $\text{ran} \mathcal{U}_{\psi_n} \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$ for $n \in \mathbb{N}$ by Proposition 3.3 the same thus holds for $\text{ran} \mathcal{U}_\psi$. \hfill $\Box$

**Corollary 3.5** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. For $s > 0$ we then have $(\mathcal{U}_\psi \delta_s)(x) = \psi(s/x)/x$ for almost all $x \geq s$ and $(\mathcal{U}_\psi \delta_s)(x) = 0$ for almost all $x < s$. 

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Proof For \( \psi \) continuous, this follows from Proposition 3.3. For general \( \psi \) it follows from the approach in the proof of Proposition 3.4 using \( \overline{U}_\psi n \to \overline{U}_\psi \) strongly as \( n \to \infty \).

It follows from Corollary 3.5 that
\[
\| \overline{U}_\psi \delta_s \|_{M(\omega_2)} = \int_{s}^{\infty} \frac{\omega_2(x)}{x} \psi(s/x) \, dx = \int_{0}^{1} \frac{\omega_2(s/t)}{t} \psi(t) \, dt,
\]
whereas \( \| \delta_s \|_{M(\omega_1)} = \omega_1(s) \). Since \( \overline{U}_\psi \) is bounded we thus recover condition (C). If we without using Theorem 2.1 could show that if \( \overline{U}_\psi : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator, then it has a bounded extension \( \overline{U}_\psi : M(\omega_1) \to M(\omega_2) \) for which Corollary 3.5 holds, then we would in this way obtain an alternative proof of condition (C).

4 Weakly compact operators

We finish the paper by showing that there are no non-zero, weakly compact generalized Hardy-Cesàro operators between \( L^1(\omega_1) \) and \( L^1(\omega_2) \).

Proposition 4.1 Let \( \psi \) be a non-negative, measurable function on \([0, 1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( \overline{U}_\psi : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. If \( \psi \neq 0 \), then \( \overline{U}_\psi \) is not weakly compact.

Proof For \( f \in L^1(\omega_1) \) and \( x \in \mathbb{R}^+ \) we have
\[
(U_\psi f)(x) = \frac{1}{x} \int_{0}^{x} f(s) \psi(s/x) \, ds = \int_{0}^{\infty} f(s) \rho(s)(x) \omega_1(s) \, ds,
\]
where (with a slight change of notation compared to the proof of Theorem 2.1)
\[
\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s}
\]
for \( x, s \in \mathbb{R}^+ \). In the proof of Theorem 2.1 we saw that \( \rho(s) \in L^1(\omega_2) \) with \( \| \rho(s) \|_{L^1(\omega_2)} \leq C \) for a constant \( C \) for almost all \( s \in \mathbb{R}^+ \). It thus follows from [4, Theorem VI.8.10] that \( \overline{U}_\psi \) is weakly compact if and only if \( \{ \rho(s) : s \in \mathbb{R}^+ \} \) is contained in a weakly compact set of \( L^1(\omega_2) \) (except possibly for \( s \) belonging to a null-set). Consider \( \rho(s) \) as an element of \( C_0(1/\omega_2)^* \) for \( s \in \mathbb{R}^+ \) and let \( g \in C_0(1/\omega_2) \). Then
\[
\langle g, \rho(s) \rangle = \int_{0}^{\infty} g(x) \rho(s)(x) \, dx
\]
\[
= \frac{1}{\omega_1(s)} \int_{s}^{\infty} \frac{g(x)}{x} \psi(s/x) \, dx
\]
\[
= \frac{1}{\omega_1(s)} \int_{0}^{1} g(s/t) \frac{\psi(t)}{t} \, dt.
\]
Since \( g(s/t) \to g(0) \) as \( s \to 0_+ \) for all \( t > 0 \), it follows from Lebesgue’s dominated convergence theorem that
\[
\langle g, \rho(s) \rangle \to \frac{1}{\omega_1(0)} g(0) \int_{0}^{1} \frac{\psi(t)}{t} \, dt
\]
as $s \to 0_+$. We therefore conclude that

$$
\rho(s) \to \frac{1}{\omega_1(0)} \int_0^1 \frac{\psi(t)}{t} dt \cdot \delta_0
$$

weak-star in $M(\omega_2)$ as $s \to 0_+$. Since $\delta_0 \notin L^1(\omega_2)$, it follows that $\{\rho(s) : s \in \mathbb{R}^+\}$ is not contained in a weakly compact set of $L^1(\omega_2)$ (even excepting null sets), and the result follows.

$\square$

References


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