Generalized Hardy–Cesaro operators between weighted spaces

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Generalized Hardy-Cesàro operators between weighted spaces

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Abstract

We characterize those non-negative, measurable functions $\psi$ on $[0, 1]$ and positive, continuous functions $\omega_1$ and $\omega_2$ on $\mathbb{R}^+$ for which the generalized Hardy-Cesàro operator

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t) \, dt$$

defines a bounded operator $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$. This generalizes a result of Xiao ([7]) to weighted spaces. Furthermore, we extend $U_\psi$ to a bounded operator on $M(\omega_1)$ with range in $L^1(\omega_2) \oplus C_0$, where $M(\omega_1)$ is the weighted space of locally finite, complex Borel measures on $\mathbb{R}^+$. Finally, we show that the zero operator is the only weakly compact generalized Hardy-Cesàro operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

1 Introduction

A classical result of Hardy ([5]) shows that the Hardy-Cesàro operator

$$(Uf)(x) = \frac{1}{x} \int_0^x f(s) \, ds$$

defines a bounded linear operator on $L^p(\mathbb{R}^+)$ with $\|U\| = p/(p - 1)$ for $p > 1$. Clearly, $U$ is not bounded on $L^1(\mathbb{R}^+)$. Hardy’s result has been generalized in various ways, of which we will mention some, which have inspired this paper.

For $1 \leq p \leq q \leq \infty$ and non-negative measurable functions $u$ and $v$ on $\mathbb{R}^+$, Muckenhoupt ([6]) and Bradley ([3]) gave a necessary and sufficient condition for the existence of a constant $C$ such that

$$\left( \int_0^\infty \left( u(x) \int_0^x f(t) \, dt \right)^q \, dx \right)^{1/q} \leq C \left( \int_0^\infty (v(x)f(x))^p \, dx \right)^{1/p}$$

for every positive, measurable function $f$ on $\mathbb{R}^+$. This can be rephrased as a characterization of the weighted $L^p$ and $L^q$ spaces on $\mathbb{R}^+$ between which the Hardy-Cesàro operator $U$ is bounded.

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In a different direction, for a non-negative measurable function \( \psi \) on \([0, 1]\), Xiao [7] considered the generalized Hardy-Cesàro operators
\[
(U_\psi f)(x) = \int_0^1 f(tx)\psi(t) \, dt
\]
for measurable functions \( f \) on \( \mathbb{R}^n \). We remark that
\[
(U_\psi f)(x) = \frac{1}{x} \int_0^x f(s)\psi(s/x) \, ds
\]
for measurable functions \( f \) on \( \mathbb{R} \). Xiao proved that \( U_\psi \) defines a bounded operator on \( L^p(\mathbb{R}^n) \) (for \( p \geq 1 \)) if and only if
\[
\int_0^1 \psi(t) \, dt < \infty.
\]
Xiao’s result is the main motivation for this paper.

Finally, we mention that Albanese, Bonet and Ricker in a recent series of papers (see, for instance, [1] and [2]) have considered the spectrum, compactness and other properties of the Hardy-Cesàro operator on various spaces of continuous functions and discrete spaces.

In this paper we will study the generalized Hardy-Cesàro operators between weighted spaces of integrable functions, and we will obtain a generalization of Xiao’s result in this context. Let \( \omega \) be a positive, continuous function on \( \mathbb{R}^+ \) and let \( L^1(\omega) \) be the Banach space of (equivalence classes of) measurable functions \( f \) on \( \mathbb{R}^+ \) for which
\[
\|f\|_{L^1(\omega)} = \int_0^\infty |f(t)|\omega(t) \, dt < \infty.
\]
In the usual way we identify the dual space of \( L^1(\omega) \) with the space \( L^\infty(1/\omega) \) of measurable functions \( h \) on \( \mathbb{R}^+ \) for which
\[
\|h\|_{L^\infty(1/\omega)} = \text{ess sup}_{t \in \mathbb{R}^+} |h(t)|/\omega(t) < \infty.
\]
We denote by \( C_0(1/\omega) \) the closed subspace of \( L^\infty(1/\omega) \) consisting of the continuous functions \( g \) in \( L^\infty(1/\omega) \) for which \( g/\omega \) vanishes at infinity. Finally, we identify the dual space of \( C_0(1/\omega) \) with the space \( M(\omega) \) of locally finite, complex Borel measures \( \mu \) on \( \mathbb{R}^+ \) for which
\[
\|\mu\|_{M(\omega)} = \int_{\mathbb{R}^+} \omega(t) \, d|\mu|(t) < \infty.
\]
We consider the space \( L^1(\omega) \) as a closed subspace of \( M(\omega) \).

In Section 2 we characterize those functions \( \psi, \omega_1 \) and \( \omega_2 \) for which \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). These operators are extended to bounded operators on \( M(\omega_1) \) in Section 3 where we also obtain results about their ranges. Finally, in Section 4 we show that there are no non-zero weakly compact generalized Hardy-Cesàro operators from \( L^1(\omega_1) \) to \( L^1(\omega_2) \).
2 A characterization of the generalized Hardy-Cesàro operators

For a non-negative, measurable function $\psi$ on $[0, 1]$ and positive, continuous functions $\omega_1$ and $\omega_2$ on $\mathbb{R}^+$, we say that condition (C) is satisfied if there exists a constant $C$ such that

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \leq C \omega_1(s)$$

for every $s \in \mathbb{R}^+$.

**Theorem 2.1** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if condition (C) is satisfied.

**Proof** Assume that condition (C) is satisfied and let $f \in L^1(\omega_1)$. Then

$$\int_0^\infty \int_0^1 |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) dt ds \leq C \int_0^\infty |f(s)| \omega_1(s) ds = C \|f\|_{L^1(\omega_1)} < \infty,$$

so it follows from Fubini’s theorem that

$$\int_0^1 \int_0^\infty |f(tx)| \psi(t) \omega_2(x) dx dt = \int_0^1 \int_0^\infty |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) dt ds \leq C \|f\|_{L^1(\omega_1)} < \infty.$$

Another application of Fubini’s theorem thus shows that $(U_\psi f)(x)$ is defined for almost all $x \in \mathbb{R}^+$ with

$$\|U_\psi f\|_{L^1(\omega_2)} = \int_0^\infty |(U_\psi f)(x)| \omega_2(x) dx \leq \int_0^\infty \int_0^1 |f(tx)| \psi(t) \omega_2(x) dx dt \leq C \|f\|_{L^1(\omega_1)} < \infty.$$

Hence $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

Conversely, assume that $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Since $L^1(\omega_2)$ is a closed subspace of $M(\omega_2)$ which we identify with the dual space of $C_0(1/\omega_2)$, it follows from [4, Theorem VI.8.6] that there exists a map $\rho$ from $\mathbb{R}^+$ to $M(\omega_2)$ for which the map $s \mapsto \langle g, \rho(s) \rangle = \int_{\mathbb{R}^+} g(x) d\rho(s)(x)$ is measurable and essentially bounded on $\mathbb{R}^+$ for every $g \in C_0(1/\omega_2)$ with $\|U_\psi\| = \text{ess sup}_{s \in \mathbb{R}^+} ||\rho(s)||_{M(\omega_2)}$ and such that

$$\langle g, U_\psi f \rangle = \int_0^\infty \langle g, \rho(s) \rangle f(s) \omega_1(s) ds = \int_0^\infty \int_{\mathbb{R}^+} g(x) d\rho(s)(x) f(s) \omega_1(s) ds$$

for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$. On the other hand

$$\langle g, U_\psi f \rangle = \int_0^\infty g(x) (U_\psi f)(x) dx$$

$$= \int_0^\infty \int_0^x \frac{g(x)}{x} f(s) \psi(s/x) ds dx$$

$$= \int_0^\infty \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) dx f(s) \omega_1(s) ds$$

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for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$, so it follows that

$$\int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) = \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx$$

for almost all $s \in \mathbb{R}^+$ and every $g \in C_0(1/\omega_2)$ (considering both sides as elements of $L^\infty(\mathbb{R}^+)$). Considered as elements of $M(\omega_2)$ we thus have

$$d\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s} \, dx$$

for almost all $s, x \in \mathbb{R}^+$. Hence $\rho(s) \in L^1(\omega_2)$ with

$$\|\rho(s)\|_{L^1(\omega_2)} = \int_0^\infty \omega_2(x) \, d\rho(s)(x)$$

$$= \frac{1}{\omega_1(s)} \int_0^\infty \frac{1}{x} \psi(s/x) 1_{x \geq s} \omega_2(x) \, dx$$

$$= \frac{1}{\omega_1(s)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx$$

$$= \frac{1}{\omega_1(s)} \int_0^1 \frac{\psi(t)}{t} \omega_2(s/t) \, dt$$

for almost all $s \in \mathbb{R}^+$. Therefore

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \|\rho(s)\|_{L^1(\omega_2)} \omega_1(s) \leq \|U_\psi\| \omega_1(s)$$

for almost all $s \in \mathbb{R}^+$. Since both sides of the inequality are continuous functions of $s$, the inequality holds for every $s \in \mathbb{R}^+$, so condition (C) holds.

Letting $s = 0$ in condition (C) we see that Xiao’s condition is necessary in our situation.

**Corollary 2.2** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. If $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then

$$\int_0^1 \psi(t) \frac{t}{\omega_2(t)} \, dt < \infty.$$

The following straightforward consequences can be deduced from Theorem 2.1.

**Corollary 2.3** Let $\psi$ be a non-negative, measurable function on $[0, 1]$

(a) Let $\omega$ be a decreasing, positive, continuous function on $\mathbb{R}^+$, and assume that

$$\int_0^1 \psi(t) \frac{t}{\omega(t)} \, dt < \infty.$$ 

Then $U_\psi$ defines a bounded operator from $L^1(\omega)$ to $L^1(\omega)$.

(b) Let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$, and assume that $\omega_2$ is increasing. If $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then there exists a constant $C$ such that $\omega_2(s) \leq C \omega_1(s)$ for every $s \in \mathbb{R}^+$.
(c) Let \( \omega \) be an increasing, positive, continuous function on \( \mathbb{R}^+ \), and assume that there exists \( a < 1 \) and \( K > 0 \) such that \( \psi(t) \geq K \) almost everywhere on \([a, 1]\). If \( U_\psi \) defines a bounded operator from \( L^1(\omega) \) to \( L^1(\omega) \), then there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \omega(s) \leq \int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \leq C_2 \omega(s)
\]

for every \( s \in \mathbb{R}^+ \).

**Proof** (a): We have

\[
\int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \leq \int_0^1 \frac{\psi(t)}{t} \, dt \omega(s)
\]

for every \( s \in \mathbb{R}^+ \), so condition (C) is satisfied with \( \omega_1 = \omega_2 = \omega \) and the result follows.

(b): We have

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \geq \int_0^1 \frac{\psi(t)}{t} \, dt \omega_2(s)
\]

for every \( s \in \mathbb{R}^+ \). Since condition (C) is satisfied, the result follows.

(c): We have

\[
\int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \geq K \int_a^1 \omega(s/t) \, dt \geq K(1 - a) \omega(s)
\]

for every \( s \in \mathbb{R}^+ \). The other inequality is just condition (C) with \( \omega_1 = \omega_2 = \omega \). \( \square \)

We finish the section with some examples of functions \( \psi, \omega_1 \) and \( \omega_2 \) for which \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \).

**Example 2.4**

(a) For \( \alpha > 0 \), let \( \psi(t) = t^\alpha \) for \( t \in [0, 1] \). Also, for \( \beta_1, \beta_2 \in \mathbb{R} \), let \( \omega_i(x) = (1 + x)^{\beta_i} \) for \( x \in \mathbb{R}^+ \) and \( i = 1, 2 \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \) if and only if \( \beta_2 \leq \beta_1 \) and \( \beta_2 < \alpha \).

(b) For \( \alpha > 0 \), let \( \psi(t) = t^\alpha \) for \( t \in [0, 1] \). Also, let \( \omega_1(x) = e^{-x}/(1 + x) \) and \( \omega_2(x) = e^{-x} \) for \( x \in \mathbb{R}^+ \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). Moreover, it is not possible to replace \( \omega_1(x) \) by a function tending faster to zero at infinity.

(c) Let \( \psi(t) = e^{-t^2} \) for \( t \in [0, 1] \). Also, let \( \omega_1(x) = e^{x^2/4}/x \) and \( \omega_2(x) = e^x \) for \( x \in \mathbb{R}^+ \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). Moreover, it is not possible to replace \( \omega_1(x) \) by a function tending slower to infinity at infinity.

**Proof** (a): For \( s \geq 1 \) and \( t \in [0, 1] \) we have \( s/t < 1 + s/t \leq 2s/t \), so

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1} \, dt 
\]

\[
\simeq s^{\beta_2} \int_0^1 t^{\alpha-\beta_2-1} \, dt 
\]

\[
\simeq s^{\beta_2}
\]
for $s \geq 1$ if $\beta_2 < \alpha$ (where $F(s) \simeq G(s)$ for positive functions $F$ and $G$ on $[1, \infty)$ indicates the existence of positive constants $C_1$ and $C_2$ such that $C_1 F(s) \leq G(s) \leq C_2 F(s)$ for all $s \in [1, \infty)$), whereas the integrals diverge if $\beta_2 \geq \alpha$. Moreover, the expression

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1} dt$$

defines a positive, continuous function of $s$ on $\mathbb{R}^+$, so it follows that condition (C) is satisfied if and only if $\beta_2 \leq \beta_1$ and $\beta_2 < \alpha$.

(b): For $s \geq 1$ we have

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_s^\infty \omega_2(x) \frac{\psi(s/x)}{x} dx = \int_s^\infty e^{-x} s^{\alpha} \frac{dx}{x^{\alpha}} \leq \int_s^\infty e^{-x} \frac{dx}{x} \leq \frac{e^{-s}}{s}.$$ 

Moreover,

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \leq \int_0^1 \frac{\psi(t)}{t} dt < \infty$$

for all $s \in \mathbb{R}^+$, so condition (C) is satisfied and $U_\psi$ thus defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. On the other hand, since

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \geq \int_s^\infty \frac{e^{-x} s^{\alpha}}{x^{\alpha}} dx \geq \frac{1}{2^{\alpha+1}s} \int_s^\infty e^{-x} dx = \frac{1}{2^{\alpha+2}} \frac{e^{-s}}{s}$$

for $s \geq 1$, it is not possible to replace $\omega_1(x)$ by a function tending faster to zero at infinity.

(c): For $s \in \mathbb{R}^+$ we have

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_s^\infty \omega_2(x) \frac{\psi(s/x)}{x} dx = \int_s^\infty e^{-x/s^2} \frac{dx}{x} = \int_1^\infty e^{sy-y^2} dy.$$ 

Moreover, for $s \geq 4$

$$\int_{s/4}^\infty e^{sy-y^2} \frac{dy}{y} \leq \frac{4}{s} \int_{s/4}^\infty e^{-(y-s/2)^2+s^2/4} dy = 4 \int_{-s/4}^\infty e^{-u^2} du \frac{e^{s^2/4}}{s}$$

and

$$\int_1^{s/4} e^{sy-y^2} \frac{dy}{y} \leq \int_1^{s/4} e^{sy} dy \leq \frac{e^{s^2/4}}{s},$$

so condition (C) is satisfied and $U_\psi$ thus defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

On the other hand, the estimate

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_1^\infty e^{sy-y^2} \frac{dy}{y} \geq \frac{1}{s} \int_{s/2}^{s+1} e^{-(y-s/2)^2+s^2/4} dy = \int_0^1 e^{-u^2} du \frac{e^{s^2/4}}{s}$$

for $s \geq 2$ shows that it is not possible to replace $\omega_1(x)$ by a function tending slower to infinity at infinity.

In Example 2.3(b) we have $\omega_2(x)/\omega_1(x) \to \infty$ as $x \to \infty$, which should be compared to the conclusion in Corollary 2.3(b). Conversely, Example 2.3(c) shows an example where we need $\omega_2(x)/\omega_1(x) \to 0$ rapidly as $x \to \infty$ in order for $U_\psi$ to be defined.
3 Extensions to weighted spaces of measures

Identifying the dual space of $L^1(\omega)$ with $L^\infty(1/\omega)$ as in the introduction, we have the following result about the adjoint of $U_\psi$.

**Proposition 3.1** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator, and consider the adjoint operator $U_\psi^* : L^\infty(1/\omega_2) \to L^\infty(1/\omega_1)$.

(a) For $h \in L^\infty(1/\omega_2)$ we have

$$(U_\psi^* h)(x) = \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt$$

for almost all $x \in \mathbb{R}^+$.

(b) $U_\psi^*$ maps $C_0(1/\omega_2)$ into $C_0(1/\omega_1)$.

**Proof** (a): Let $h \in L^\infty(1/\omega_2)$. Since $|h(x/t)| \leq \| h \|_{L^\infty(1/\omega_2)\omega_2}(x/t)$ for almost all $x, t \in \mathbb{R}^+$, it follows from condition (C) that $\int_0^1 h(x/t)\psi(t)/t \, dt$ is defined and satisfies

$$\left| \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt \right| \leq \| h \|_{L^\infty(1/\omega_2)} \int_0^1 \omega_2(x/t) \frac{\psi(t)}{t} \, dt \leq C \| h \|_{L^\infty(1/\omega_2)\omega_1}(x)$$

for almost all $x \in \mathbb{R}^+$. Hence the function $x \mapsto \int_0^1 h(x/t)\psi(t)/t \, dt$ belongs to $L^\infty(1/\omega_1)$. Also, for $f \in L^1(\omega_1)$ we have

$$\langle f, U_\psi^* h \rangle = \langle U_\psi f, h \rangle = \int_0^\infty (U_\psi f)(s) h(s) \, ds$$

$$= \int_0^\infty \int_0^s \frac{1}{s} f(x) \psi(x/s) h(s) \, dx \, ds$$

$$= \int_0^\infty \int_x^\infty \frac{h(s)}{s} \psi(x/s) \, ds f(x) \, dx$$

from which it follows that

$$(U_\psi^* h)(x) = \int_x^\infty \frac{h(s)}{s} \psi(x/s) \, ds = \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt$$

for almost all $x \in \mathbb{R}^+$.

(b): It suffices to show that $U_\psi^*$ maps $C_c(\mathbb{R}^+)$ (the continuous functions on $\mathbb{R}^+$ with compact support) into $C_0(1/\omega_1)$. Let $g \in C_c(\mathbb{R}^+)$, let $x_0 \in \mathbb{R}^+$ and let $(x_n)$ be a sequence in $\mathbb{R}^+$ with $x_n \to x_0$ as $n \to \infty$. Then

$$(U_\psi^* g)(x_n) - (U_\psi^* g)(x_0) = \int_0^1 (g(x_n/t) - g(x_0/t)) \frac{\psi(t)}{t} \, dt$$

for $n \in \mathbb{N}$. Since $g$ is bounded on $\mathbb{R}^+$ and since $\int_0^1 \psi(t)/t \, dt < \infty$ by Corollary 2.2, it follows from Lebesgue’s dominated convergence theorem that $(U_\psi^* g)(x_n) \to (U_\psi^* g)(x_0)$ as $n \to \infty$. Hence $U_\psi^* g$ is continuous on $\mathbb{R}^+$. Finally, from the expression

$$(U_\psi^* g)(x) = \int_x^\infty \frac{g(s)}{s} \psi(x/s) \, ds$$
it follows that \( \text{supp} U^*_x g \subseteq \text{supp} g \), so we conclude that \( U^*_x g \in C_0(\mathbb{R}^+) \subseteq C_0(1/\omega_1) \). \( \square \)

Let \( V_\psi \) be the restriction of \( U^*_x \) to \( C_0(1/\omega_2) \) considered as a map into \( C_0(1/\omega_1) \). We then immediately have the following result.

**Corollary 3.2** Let \( \psi \) be a non-negative, measurable function on \([0,1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition \( (C) \) is satisfied so that \( U_\psi : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. The bounded operator \( \overline{U}_\psi = V_\psi^* \) from \( M(\omega_1) \) to \( M(\omega_2) \) is an extension of \( U_\psi \).

Let \( \psi \) be a non-negative, continuous function on \([0,1]\) with \( \psi(0) = 0 \). For \( \mu \in M(\omega_1) \) and \( x > 0 \) let
\[
(W_\psi \mu)(x) = \frac{1}{x} \int_{(0,x)} \psi(s/x) \, d\mu(s).
\]

**Proposition 3.3** Let \( \psi \) be a non-negative, continuous function on \([0,1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition \( (C) \) is satisfied so that \( U_\psi : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. Then \( W_\psi \mu \in L^1(\omega_2) \) and
\[
\overline{U}_\psi \mu = W_\psi \mu + \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \mu(\{0\}) \delta_0
\]
for \( \mu \in M(\omega_1) \). In particular \( \text{ran} \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C} \delta_0 \) and \( \overline{U}_\psi \) maps \( M((0,\infty),\omega_1) \) into \( L^1(\omega_2) \).

**Proof** By Corollary 2.2 we have \( \int_0^1 \psi(t)/t \, dt < \infty \), so it follows that \( \psi(0) = 0 \). Let \( \mu \in M(\omega_1) \) with \( \mu(\{0\}) = 0 \). By condition \( (C) \) we have
\[
\int_{(0,\infty)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx \, d|\mu|(s) = \int_{(0,\infty)} \int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \, d|\mu|(s)
\]
\[
\leq C \int_{(0,\infty)} \omega_1(s) \, d|\mu|(s) = C\|\mu\|_{M(\omega_1)} < \infty,
\]
so it follows from Fubini’s theorem that
\[
\int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x) \, d\mu(s) \, \omega_2(x) \, dx < \infty.
\]
Hence \( W_\psi \mu \in L^1(\omega_2) \). Moreover, for \( g \in C_0(1/\omega_2) \) we have
\[
\langle g, \overline{U}_\psi \mu \rangle = \langle V_\psi g, \mu \rangle = \int_{(0,\infty)} \int_0^1 g(s/t) \frac{\psi(t)}{t} \, dt \, d\mu(s)
\]
\[
= \int_{(0,\infty)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx \, d\mu(s)
\]
\[
= \int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x) \, d\mu(s) \, g(x) \, dx
\]
\[
= \int_0^\infty (W_\psi \mu)(x) g(x) \, dx = \langle g, W_\psi \mu \rangle,
\]
8
so we conclude that \( U_{\psi} \mu = W_{\psi} \mu \). Finally, for \( g \in C_0(1/\omega_2) \) we have
\[
\langle g, U_{\psi} \delta_0 \rangle = \langle V_{\psi} g, \delta_0 \rangle = (V_{\psi} g)(0) = g(0) \int_0^1 \frac{\psi(t)}{t} dt = \langle g, \int_0^1 \frac{\psi(t)}{t} dt \cdot \delta_0 \rangle.
\]
Since \( W_{\psi} \delta_0 = 0 \) this finishes the proof. \( \Box \)

The conclusion about the range of \( U_{\psi} \) can be generalized to the case, where \( \psi \) is not assumed to be continuous.

**Proposition 3.4** Let \( \psi \) be a non-negative, measurable function on \([0,1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( U_{\psi} : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. Then \( \text{ran} U_{\psi} \subseteq L^1(\omega_2) \oplus C \delta_0 \).

**Proof** Choose a sequence of non-negative, continuous functions \((\psi_n)\) on \([0,1]\) with \( \psi_n \leq \psi \) and
\[
\int_0^1 \frac{\psi(t) - \psi_n(t)}{t} dt \to 0 \quad \text{as} \quad n \to \infty.
\]
For \( \mu \in M(\omega_1) \) and \( g \in C_0(1/\omega_2) \) we have
\[
|\langle g, (U_{\psi} - U_{\psi_n}) \mu \rangle| = \left| \langle (V_{\psi} - V_{\psi_n}) g, \mu \rangle \right|
\]
\[
= \left| \int_{\mathbb{R}^+} \int_0^1 g(\frac{x}{t}) \frac{\psi(t) - \psi_n(t)}{t} dt d\mu(x) \right|
\]
\[
\leq \|g\|_{C_0(1/\omega_2)} \int_{\mathbb{R}^+} \int_0^1 \omega_2(\frac{x}{t}) \frac{\psi(t) - \psi_n(t)}{t} dt d\mu(x).
\]
Let
\[
p_n(x) = \int_0^1 \omega_2(\frac{x}{t}) \frac{\psi(t) - \psi_n(t)}{t} dt
\]
for \( x \in \mathbb{R}^+ \) and \( n \in \mathbb{N} \). By condition (C) there exists a constant \( C \) such that \( p_n(x) \leq C \omega_1(x) \) for every \( x \in \mathbb{R}^+ \) and \( n \in \mathbb{N} \). Moreover, for every \( x \in \mathbb{R}^+ \) we have \( p_n(x) \to 0 \) as \( n \to \infty \) by Lebesgue’s dominated convergence theorem. Hence
\[
\| (U_{\psi} - U_{\psi_n}) \mu \|_{M(\omega_2)} \leq \sup_{\|g\|_{C_0(1/\omega_2)} \leq 1} |\langle g, (U_{\psi} - U_{\psi_n}) \mu \rangle| \leq \int_{\mathbb{R}^+} p_n(x) d\mu(x) \to 0
\]
as \( n \to \infty \) again by Lebesgue’s dominated convergence theorem. Consequently, \( U_{\psi_n} \to U_{\psi} \) strongly as \( n \to \infty \). Since \( \text{ran} U_{\psi_n} \subseteq L^1(\omega_2) \oplus C \delta_0 \) for \( n \in \mathbb{N} \) by Proposition 3.3 the same thus holds for \( \text{ran} U_{\psi} \). \( \Box \)

**Corollary 3.5** Let \( \psi \) be a non-negative, measurable function on \([0,1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( U_{\psi} : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. For \( s > 0 \) we then have \( \langle U_{\psi} \delta_s \rangle (x) = \psi(s/x)/x \) for almost all \( x \geq s \) and \( \langle U_{\psi} \delta_s \rangle (x) = 0 \) for almost all \( x < s \).
Proof For $\psi$ continuous, this follows from Proposition 3.3. For general $\psi$ it follows from the approach in the proof of Proposition 3.4 using $U_{\psi_n} \to U_{\psi}$ strongly as $n \to \infty$. 

It follows from Corollary 3.5 that 

$$\|U_{\psi}\delta_s\|_{M(\omega_2)} = \int_{-\infty}^{\infty} \frac{\omega_2(x)}{x} \psi(s/x) \, dx = \int_{0}^{1} \omega_2(s/t) \frac{\psi(t)}{t} \, dt,$$

whereas $\|\delta_s\|_{M(\omega_1)} = \omega_1(s)$. Since $U_{\psi}$ is bounded we thus recover condition (C). If we without using Theorem 2.1 could show that if $U_{\psi} : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator, then is has a bounded extension $U_{\psi} : M(\omega_1) \to M(\omega_2)$ for which Corollary 3.5 holds, then we would in this way obtain an alternative proof of condition (C).

4 Weakly compact operators

We finish the paper by showing that there are no non-zero, weakly compact generalized Hardy-Cesàro operators between $L^1(\omega_1)$ and $L^1(\omega_2)$.

Proposition 4.1 Let $\psi$ be a non-negative, measurable function on $[0,1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_{\psi} : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. If $\psi \neq 0$, then $U_{\psi}$ is not weakly compact.

Proof For $f \in L^1(\omega_1)$ and $x \in \mathbb{R}^+$ we have 

$$(U_{\psi}f)(x) = \frac{1}{x} \int_{0}^{x} f(s) \psi(s/x) \, ds = \int_{0}^{\infty} f(s) \rho(s)(x) \omega_1(s) \, ds,$$

where (with a slight change of notation compared to the proof of Theorem 2.1) 

$$\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s}$$

for $x, s \in \mathbb{R}^+$. In the proof of Theorem 2.1 we saw that $\rho(s) \in L^1(\omega_2)$ with $\|\rho(s)\|_{L^1(\omega_2)} \leq C$ for a constant $C$ for almost all $s \in \mathbb{R}^+$. It thus follows from [4, Theorem VI.8.10] that $U_{\psi}$ is weakly compact if and only if $\{\rho(s) : s \in \mathbb{R}^+\}$ is contained in a weakly compact set of $L^1(\omega_2)$ (except possibly for $s$ belonging to a null-set). Consider $\rho(s)$ as an element of $C_0(1/\omega_2)^*$ for $s \in \mathbb{R}^+$ and let $g \in C_0(1/\omega_2)$. Then 

$$\langle g, \rho(s) \rangle = \int_{0}^{\infty} g(x) \rho(s)(x) \, dx = \frac{1}{\omega_1(s)} \int_{s}^{\infty} \frac{g(x)}{x} \psi(s/x) \, dx = \frac{1}{\omega_1(s)} \int_{0}^{1} g(s/t) \frac{\psi(t)}{t} \, dt.$$

Since $g(s/t) \to g(0)$ as $s \to 0_+$ for all $t > 0$, it follows from Lebesgue’s dominated convergence theorem that 

$$\langle g, \rho(s) \rangle \to \frac{1}{\omega_1(0)} g(0) \int_{0}^{1} \frac{\psi(t)}{t} \, dt.$$
as $s \to 0_+$. We therefore conclude that

$$
\rho(s) \to \frac{1}{\omega_1(0)} \int_0^1 \psi(t) \frac{dt}{t} \cdot \delta_0
$$

weak-star in $M(\omega_2)$ as $s \to 0_+$. Since $\delta_0 \notin L^1(\omega_2)$, it follows that $\{\rho(s) : s \in \mathbb{R}^+\}$ is not contained in a weakly compact set of $L^1(\omega_2)$ (even excepting null sets), and the result follows.

\[ \square \]

## References


