Generalized Hardy–Cesaro operators between weighted spaces

Pedersen, Thomas Vils

Published in:
Glasgow Mathematical Journal

DOI:
10.1017/S0017089517000398

Publication date:
2019

Document version
Peer reviewed version

Citation for published version (APA):
Generalized Hardy-Cesàro operators between weighted spaces

Thomas Vils Pedersen

December 1, 2017

Abstract

We characterize those non-negative, measurable functions \( \psi \) on \([0, 1]\) and positive, continuous functions \( \omega_1 \) and \( \omega_2 \) on \( \mathbb{R}^+ \) for which the generalized Hardy-Cesàro operator

\[
(U_{\psi} f)(x) = \int_0^1 f(tx) \psi(t) \, dt
\]

defines a bounded operator \( U_{\psi} : L^1(\omega_1) \to L^1(\omega_2) \). This generalizes a result of Xiao \((\text{[7]})\) to weighted spaces. Furthermore, we extend \( U_{\psi} \) to a bounded operator on \( M(\omega_1) \) with range in \( L^1(\omega_2) \oplus \mathbb{C}d_0 \), where \( M(\omega_1) \) is the weighted space of locally finite, complex Borel measures on \( \mathbb{R}^+ \). Finally, we show that the zero operator is the only weakly compact generalized Hardy-Cesàro operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \).

1 Introduction

A classical result of Hardy \((\text{[5]})\) shows that the Hardy-Cesàro operator

\[
(Uf)(x) = \frac{1}{x} \int_0^x f(s) \, ds
\]

defines a bounded linear operator on \( L^p(\mathbb{R}^+) \) with \( \|U\| = p/(p - 1) \) for \( p > 1 \). Clearly, \( U \) is not bounded on \( L^1(\mathbb{R}^+) \). Hardy’s result has been generalized in various ways, of which we will mention some, which have inspired this paper.

For \( 1 \leq p \leq q \leq \infty \) and non-negative measurable functions \( u \) and \( v \) on \( \mathbb{R}^+ \), Muckenhoupt \((\text{[6]})\) and Bradley \((\text{[3]})\) gave a necessary and sufficient condition for the existence of a constant \( C \) such that

\[
\left( \int_0^\infty \left( u(x) \int_0^x f(t) \, dt \right)^q \, dx \right)^{1/q} \leq C \left( \int_0^\infty (v(x)f(x))^p \, dx \right)^{1/p}
\]

for every positive, measurable function \( f \) on \( \mathbb{R}^+ \). This can be rephrased as a characterization of the weighted \( L^p \) and \( L^q \) spaces on \( \mathbb{R}^+ \) between which the Hardy-Cesàro operator \( U \) is bounded.

---

\( ^0 \)2010 Mathematics Subject Classification: 44A15, 47B34, 47B38, 47G10

\( ^0 \)Keywords: Generalized Hardy-Cesàro operators, weighted spaces, weak compactness.
In a different direction, for a non-negative measurable function \( \psi \) on \([0, 1]\), Xiao ([7]) considered the generalized Hardy-Cesàro operators

\[
(U_\psi f)(x) = \int_0^1 f(tx) \psi(t) \, dt
\]

for measurable functions \( f \) on \( \mathbb{R}^n \). We remark that

\[
(U_\psi f)(x) = \frac{1}{x} \int_0^x f(s) \psi(s/x) \, ds
\]

for measurable functions \( f \) on \( \mathbb{R} \). Xiao proved that \( U_\psi \) defines a bounded operator on \( L^p(\mathbb{R}^n) \) (for \( p \geq 1 \)) if and only if

\[
\int_0^1 \frac{\psi(t)}{t^{n/p}} \, dt < \infty.
\]

Xiao’s result is the main motivation for this paper.

Finally, we mention that Albanese, Bonet and Ricker in a recent series of papers (see, for instance, [1] and [2]) have considered the spectrum, compactness and other properties of the Hardy-Cesàro operator on various spaces of continuous functions and discrete spaces.

In this paper we will study the generalized Hardy-Cesàro operators between weighted spaces of integrable functions, and we will obtain a generalization of Xiao’s result in this context. Let \( \omega \) be a positive, continuous function on \( \mathbb{R}^+ \) and let \( L^1(\omega) \) be the Banach space of (equivalence classes of) measurable functions \( f \) on \( \mathbb{R}^+ \) for which

\[
\|f\|_{L^1(\omega)} = \int_0^\infty |f(t)| \omega(t) \, dt < \infty.
\]

In the usual way we identify the dual space of \( L^1(\omega) \) with the space \( L^\infty(1/\omega) \) of measurable functions \( h \) on \( \mathbb{R}^+ \) for which

\[
\|h\|_{L^\infty(1/\omega)} = \text{ess sup}_{t \in \mathbb{R}^+} |h(t)|/\omega(t) < \infty.
\]

We denote by \( C_0(1/\omega) \) the closed subspace of \( L^\infty(1/\omega) \) consisting of the continuous functions \( g \) in \( L^\infty(1/\omega) \) for which \( g/\omega \) vanishes at infinity. Finally, we identify the dual space of \( C_0(1/\omega) \) with the space \( M(\omega) \) of locally finite, complex Borel measures \( \mu \) on \( \mathbb{R}^+ \) for which

\[
\|\mu\|_{M(\omega)} = \int_{\mathbb{R}^+} \omega(t) \, d|\mu|(t) < \infty.
\]

We consider the space \( L^1(\omega) \) as a closed subspace of \( M(\omega) \).

In Section 2 we characterize those functions \( \psi, \omega_1 \) and \( \omega_2 \) for which \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). These operators are extended to bounded operators on \( M(\omega_1) \) in Section 3 where we also obtain results about their ranges. Finally, in Section 4 we show that there are no non-zero weakly compact generalized Hardy-Cesàro operators from \( L^1(\omega_1) \) to \( L^1(\omega_2) \).
2 A characterization of the generalized Hardy-Cesàro operators

For a non-negative, measurable function $\psi$ on $[0, 1]$ and positive, continuous functions $\omega_1$ and $\omega_2$ on $\mathbb{R}^+$, we say that condition (C) is satisfied if there exists a constant $C$ such that

$$
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \leq C \omega_1(s)
$$

for every $s \in \mathbb{R}^+$.

**Theorem 2.1** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Then $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if condition (C) is satisfied.

**Proof** Assume that condition (C) is satisfied and let $f \in L^1(\omega_1)$. Then

$$
\int_0^\infty \int_0^1 |f(t)| \frac{\psi(t)}{t} \omega_2(s/t) \, dt \, ds \leq C \int_0^\infty |f(t)| \omega_1(s) \, ds = C \|f\|_{L^1(\omega_1)} < \infty,
$$

so it follows from Fubini’s theorem that

$$
\int_0^1 \int_0^\infty |f(tx)| \frac{\psi(t)}{t} \omega_2(s/t) \, dx \, dt \leq C \|f\|_{L^1(\omega_1)} < \infty.
$$

Another application of Fubini’s theorem thus shows that $(U_\psi f)(x)$ is defined for almost all $x \in \mathbb{R}^+$ with

$$
\|U_\psi f\|_{L^1(\omega_2)} = \int_0^\infty \| (U_\psi f)(x) \|_{L^1(\omega_2)} \, dx \leq \int_0^\infty \int_0^1 |f(tx)| \frac{\psi(t)}{t} \omega_2(s/t) \, dx \, dt
$$

$$
= \int_0^1 \int_0^\infty |f(tx)| \frac{\psi(t)}{t} \omega_2(s/t) \, dx \, dt \leq C \|f\|_{L^1(\omega_1)} < \infty.
$$

Hence $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

Conversely, assume that $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Since $L^1(\omega_2)$ is a closed subspace of $M(\omega_2)$ which we identify with the dual space of $C_0(1/\omega_2)$, it follows from [4, Theorem VI.8.6] that there exists a map $\rho$ from $\mathbb{R}^+$ to $M(\omega_2)$ for which the map $s \mapsto (g, \rho(s)) = \int_{\mathbb{R}^+} g(x) \, d\rho(s)(x)$ is measurable and essentially bounded on $\mathbb{R}^+$ for every $g \in C_0(1/\omega_2)$ with $\|U_\psi \| = \sup_{s \in \mathbb{R}^+} \|\rho(s)\|_{M(\omega_2)}$ and such that

$$
(g, U_\psi f) = \int_0^\infty (g, \rho(s)) f(s) \omega_1(s) \, ds = \int_0^\infty \int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) f(s) \omega_1(s) \, ds
$$

for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$. On the other hand

$$
(g, U_\psi f) = \int_0^\infty g(x) (U_\psi f)(x) \, dx
$$

$$
= \int_0^\infty \int_0^x \frac{g(x)}{x} f(s) \psi(s/x) \, ds \, dx
$$

$$
= \int_0^\infty \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx \, f(s) \omega_1(s) \, ds
$$
for every \( g \in C_0(1/\omega_2) \) and \( f \in L^1(\omega_1) \), so it follows that

\[
\int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) = \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx
\]

for almost all \( s \in \mathbb{R}^+ \) and every \( g \in C_0(1/\omega_2) \) (considering both sides as elements of \( L^\infty(\mathbb{R}^+) \)). Considered as elements of \( M(\omega_2) \) we thus have

\[
d\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s} \, dx
\]

for almost all \( s,x \in \mathbb{R}^+ \). Hence \( \rho(s) \in L^1(\omega_2) \) with

\[
\|\rho(s)\|_{L^1(\omega_2)} = \int_0^\infty \omega_2(x) \, d\rho(s)(x)
= \frac{1}{\omega_1(s)} \int_0^\infty \frac{1}{x} \psi(s/x) 1_{x \geq s} \omega_2(x) \, dx
= \frac{1}{\omega_1(s)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx
= \frac{1}{\omega_1(s)} \int_0^1 \frac{\psi(t)}{t} \omega_2(s/t) \, dt
\]

for almost all \( s \in \mathbb{R}^+ \). Therefore

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \|\rho(s)\|_{L^1(\omega_2)} \omega_1(s) \leq \|U_\psi\| \omega_1(1)
\]

for almost all \( s \in \mathbb{R}^+ \). Since both sides of the inequality are continuous functions of \( s \), the inequality holds for every \( s \in \mathbb{R}^+ \), so condition (C) holds.

Letting \( s = 0 \) in condition (C) we see that Xiao’s condition is necessary in our situation.

**Corollary 2.2** Let \( \psi \) be a non-negative, measurable function on \([0, 1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). If \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \), then

\[
\int_0^1 \psi(t) \frac{t}{t} \, dt < \infty.
\]

The following straightforward consequences can be deduced from Theorem 2.1.

**Corollary 2.3** Let \( \psi \) be a non-negative, measurable function on \([0, 1]\)

(a) Let \( \omega \) be a decreasing, positive, continuous function on \( \mathbb{R}^+ \), and assume that

\[
\int_0^1 \psi(t)/t \, dt < \infty.\]

Then \( U_\psi \) defines a bounded operator from \( L^1(\omega) \) to \( L^1(\omega) \).

(b) Let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \), and assume that \( \omega_2 \) is increasing. If \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \), then there exists a constant \( C \) such that \( \omega_2(s) \leq C \omega_1(s) \) for every \( s \in \mathbb{R}^+ \).
(c) Let \( \omega \) be an increasing, positive, continuous function on \( \mathbb{R}^+ \), and assume that there exists \( a < 1 \) and \( K > 0 \) such that \( \psi(t) \geq K \) almost everywhere on \([a, 1]\). If \( U_\psi \) defines a bounded operator from \( L^1(\omega) \) to \( L^1(\omega) \), then there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1\omega(s) \leq \int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \leq C_2\omega(s)
\]
for every \( s \in \mathbb{R}^+ \).

Proof (a): We have
\[
\int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \leq \int_0^1 \frac{\psi(t)}{t} \, dt \omega(s)
\]
for every \( s \in \mathbb{R}^+ \), so condition (C) is satisfied with \( \omega_1 = \omega_2 = \omega \) and the result follows.
(b): We have
\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \geq \int_0^1 \frac{\psi(t)}{t} \, dt \omega_2(s)
\]
for every \( s \in \mathbb{R}^+ \). Since condition (C) is satisfied, the result follows.
(c): We have
\[
\int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \geq K \int_0^1 \omega(s/t) \, dt \geq K(1-a)\omega(s)
\]
for every \( s \in \mathbb{R}^+ \). The other inequality is just condition (C) with \( \omega_1 = \omega_2 = \omega \). \( \square \)

We finish the section with some examples of functions \( \psi, \omega_1 \) and \( \omega_2 \) for which \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \).

Example 2.4

(a) For \( \alpha > 0 \), let \( \psi(t) = t^\alpha \) for \( t \in [0, 1] \). Also, for \( \beta_1, \beta_2 \in \mathbb{R} \), let \( \omega_i(x) = (1 + x)^{\beta_i} \) for \( x \in \mathbb{R}^+ \) and \( i = 1, 2 \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \) if and only if \( \beta_2 \leq \beta_1 \) and \( \beta_2 < \alpha \).

(b) For \( \alpha > 0 \), let \( \psi(t) = t^\alpha \) for \( t \in [0, 1] \). Also, let \( \omega_1(x) = e^{-x}/(1 + x) \) and \( \omega_2(x) = e^{-x} \) for \( x \in \mathbb{R}^+ \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). Moreover, it is not possible to replace \( \omega_1(x) \) by a function tending faster to zero at infinity.

(c) Let \( \psi(t) = e^{-1/t^2} \) for \( t \in [0, 1] \). Also, let \( \omega_1(x) = e^{x^2/4}/x \) and \( \omega_2(x) = e^x \) for \( x \in \mathbb{R}^+ \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). Moreover, it is not possible to replace \( \omega_1(x) \) by a function tending slower to infinity at infinity.

Proof (a): For \( s \geq 1 \) and \( t \in [0, 1] \) we have \( s/t < 1 + s/t \leq 2s/t \), so
\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1} \, dt
\]
\[
\simeq s^{\beta_2} \int_0^1 t^{\alpha-\beta_2-1} \, dt
\]
\[
\simeq s^{\beta_2}
\]
for \( s \geq 1 \) if \( \beta_2 < \alpha \) (where \( F(s) \simeq G(s) \) for positive functions \( F \) and \( G \) on \([1, \infty)\) indicates the existence of positive constants \( C_1 \) and \( C_2 \) such that \( C_1 F(s) \leq G(s) \leq C_2 F(s) \) for all \( s \in [1, \infty) \)), whereas the integrals diverge if \( \beta_2 \geq \alpha \). Moreover, the expression

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} \frac{\alpha}{t^{\alpha-1}} \, dt
\]

defines a positive, continuous function of \( s \) on \( \mathbb{R}^+ \), so it follows that condition (C) is satisfied if and only if \( \beta_2 \leq \beta_1 \) and \( \beta_2 < \alpha \).

(b): For \( s \geq 1 \) we have

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_s^\infty \frac{\omega_2(x)}{x} \psi(s/x) \, dx = \int_s^\infty \frac{e^{-x} s^\alpha}{x^{\alpha+1}} \, dx \leq \int_s^\infty \frac{e^{-x}}{x} \, dx \leq \frac{e^{-s}}{s}.
\]

Moreover,

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \leq \int_0^1 \frac{\psi(t)}{t} \, dt < \infty
\]

for all \( s \in \mathbb{R}^+ \), so condition (C) is satisfied and \( U_\psi \) thus defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). On the other hand, since

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \geq \int_s^1 \frac{e^{-x} s^\alpha}{x^{\alpha+1}} \, dx \geq \frac{1}{2^\alpha s} \int_s^1 e^{-x} \, dx \geq \frac{1}{2^{\alpha+2}} \frac{e^{-s}}{s}
\]

for \( s \geq 1 \), it is not possible to replace \( \omega_1(x) \) by a function tending faster to zero at infinity.

(c): For \( s \in \mathbb{R}^+ \) we have

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_s^\infty \frac{\omega_2(x)}{x} \psi(s/x) \, dx = \int_s^\infty \frac{e^{-x} s^2}{x^{\alpha+2}} \, dx = \int_1^\infty \frac{e^{sy-y^2}}{y} \, dy.
\]

Moreover, for \( s \geq 4 \)

\[
\int_{s/4}^\infty \frac{e^{sy-y^2}}{y} \, dy \leq \frac{4}{s} \int_{s/4}^\infty e^{-(y-s/2)^2+s^2/4} \, dy = 4 \int_{-s/4}^\infty e^{-u^2} \, du \frac{e^{s^2/4}}{s}
\]

and

\[
\int_{1/4}^{s/4} \frac{e^{sy-y^2}}{y} \, dy \leq \int_{1/4}^{s/4} e^{sy} \, dy \leq \frac{e^{s^2/4}}{s},
\]

so condition (C) is satisfied and \( U_\psi \) thus defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). On the other hand, the estimate

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_1^\infty \frac{e^{sy-y^2}}{y} \, dy \geq \frac{1}{s} \int_{s/2}^{s+1} e^{-(y-s/2)^2+s^2/4} \, dy = \int_0^1 \frac{e^{-u^2}}{u} \frac{e^{s^2/4}}{s}
\]

for \( s \geq 2 \) shows that it is not possible to replace \( \omega_1(x) \) by a function tending slower to infinity at infinity.

\[\square\]

In Example 2.24(b) we have \( \omega_2(x)/\omega_1(x) \to \infty \) as \( x \to \infty \), which should be compared to the conclusion in Corollary 2.33(b). Conversely, Example 2.21(c) shows an example where we need \( \omega_2(x)/\omega_1(x) \to 0 \) rapidly as \( x \to \infty \) in order for \( U_\psi \) to be defined.
3 Extensions to weighted spaces of measures

Identifying the dual space of $L^1(\omega)$ with $L^\infty(1/\omega)$ as in the introduction, we have the following result about the adjoint of $U_\psi$.

**Proposition 3.1** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$ is a bounded operator, and consider the adjoint operator $U_\psi^* : L^\infty(1/\omega_2) \rightarrow L^\infty(1/\omega_1)$.

(a) For $h \in L^\infty(1/\omega_2)$ we have

$$(U_\psi^* h)(x) = \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt$$

for almost all $x \in \mathbb{R}^+$.

(b) $U_\psi^*$ maps $C_0(1/\omega_2)$ into $C_0(1/\omega_1)$.

**Proof** (a): Let $h \in L^\infty(1/\omega_2)$. Since $|h(x/t)| \leq \|h\|_{L^\infty(1/\omega_2)} \omega_2(x/t)$ for almost all $x, t \in \mathbb{R}^+$, it follows from condition (C) that $\int_0^1 h(x/t) \psi(t)/t \, dt$ is defined and satisfies

$$\left| \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt \right| \leq \|h\|_{L^\infty(1/\omega_2)} \int_0^1 \omega_2(x/t) \frac{\psi(t)}{t} \, dt \leq C \|h\|_{L^\infty(1/\omega_2)} \omega_1(x)$$

for almost all $x \in \mathbb{R}^+$. Hence the function $x \mapsto \int_0^1 h(x/t) \psi(t)/t \, dt$ belongs to $L^\infty(1/\omega_1)$. Also, for $f \in L^1(\omega_1)$ we have

$$\langle f, U_\psi^* h \rangle = \langle U_\psi f, h \rangle = \int_0^\infty (U_\psi f)(s) h(s) \, ds$$

$$= \int_0^\infty \int_0^s \frac{1}{s} f(x) \psi(x/s) h(s) \, dx \, ds$$

$$= \int_0^\infty \int_{\infty}^x \frac{h(s)}{s} \psi(x/s) \, ds \, f(x) \, dx$$

from which it follows that

$$(U_\psi^* h)(x) = \int_0^\infty \frac{h(s)}{s} \psi(x/s) \, ds = \int_0^1 h(x/t) \frac{\psi(t)}{t} \, dt$$

for almost all $x \in \mathbb{R}^+$.

(b): It suffices to show that $U_\psi^*$ maps $C_c(\mathbb{R}^+)$ (the continuous functions on $\mathbb{R}^+$ with compact support) into $C_0(1/\omega_1)$. Let $g \in C_c(\mathbb{R}^+)$, let $x_0 \in \mathbb{R}^+$ and let $(x_n)$ be a sequence in $\mathbb{R}^+$ with $x_n \rightarrow x_0$ as $n \rightarrow \infty$. Then

$$(U_\psi^* g)(x_n) - (U_\psi^* g)(x_0) = \int_0^1 (g(x_n/t) - g(x_0/t)) \frac{\psi(t)}{t} \, dt$$

for $n \in \mathbb{N}$. Since $g$ is bounded on $\mathbb{R}^+$ and since $\int_0^1 \psi(t)/t \, dt < \infty$ by Corollary 2.2, it follows from Lebesgue’s dominated convergence theorem that $(U_\psi^* g)(x_n) \rightarrow (U_\psi^* g)(x_0)$ as $n \rightarrow \infty$. Hence $U_\psi^* g$ is continuous on $\mathbb{R}^+$. Finally, from the expression

$$(U_\psi^* g)(x) = \int_{x_0}^\infty \frac{g(s)}{s} \psi(x/s) \, ds$$

7
it follows that \( \text{supp} U^*_\psi g \subseteq \text{supp} g \), so we conclude that \( U^*_\psi g \in C_1(\mathbb{R}^+) \subseteq C_0(1/\omega_1) \). \( \square \)

Let \( V_\psi \) be the restriction of \( U^*_\psi \) to \( C_0(1/\omega_2) \) considered as a map into \( C_0(1/\omega_1) \). We then immediately have the following result.

**Corollary 3.2** Let \( \psi \) be a non-negative, measurable function on \([0, 1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( U_\psi : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. The bounded operator \( \overline{U}_{\psi} = V^*_\psi \) from \( M(\omega_1) \) to \( M(\omega_2) \) is an extension of \( U_\psi \).

Let \( \psi \) be a non-negative, continuous function on \([0, 1] \) with \( \psi(0) = 0 \). For \( \mu \in M(\omega_1) \) and \( x > 0 \) let

\[
(W_\psi\mu)(x) = \frac{1}{x} \int_{(0, x)} \psi(s/x) \, d\mu(s).
\]

**Proposition 3.3** Let \( \psi \) be a non-negative, continuous function on \([0, 1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( U_\psi : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. Then \( W_\psi \mu \in L^1(\omega_2) \) and

\[
\overline{U}_{\psi}\mu = W_\psi\mu + \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \mu(\{0\})\delta_0
\]

for \( \mu \in M(\omega_1) \). In particular \( \text{ran} \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0 \) and \( \overline{U}_\psi \) maps \( M((0, \infty), \omega_1) \) into \( L^1(\omega_2) \).

**Proof** By Corollary 3.2 we have \( \int_0^1 \psi(t)/t \, dt < \infty \), so it follows that \( \psi(0) = 0 \). Let \( \mu \in M(\omega_1) \) with \( \mu(\{0\}) = 0 \). By condition (C) we have

\[
\int_{(0, \infty)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx \, d|\mu|(s) = \int_{(0, \infty)} \int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \, d|\mu|(s)
\]

\[
\leq C \int_{(0, \infty)} \omega_1(s) \, d|\mu|(s) = C \|\mu\|_{M(\omega_1)} < \infty,
\]

so it follows from Fubini’s theorem that

\[
\int_0^\infty \frac{1}{x} \int_{(0, x)} \psi(s/x) \, d|\mu|(s) \omega_2(x) \, dx < \infty.
\]

Hence \( W_\psi \mu \in L^1(\omega_2) \). Moreover, for \( g \in C_0(1/\omega_2) \) we have

\[
\langle g, \overline{U}_\psi \mu \rangle = \langle V_\psi g, \mu \rangle = \int_{(0, \infty)} \int_0^1 g(s/t) \frac{\psi(t)}{t} \, dt \, d\mu(s)
\]

\[
= \int_{(0, \infty)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx \, d\mu(s)
\]

\[
= \int_0^\infty \frac{1}{x} \int_{(0, x)} \psi(s/x) \, d\mu(s) \, g(x) \, dx
\]

\[
= \int_0^\infty (W_\psi \mu)(x)g(x) \, dx = \langle g, \overline{U}_\psi \mu \rangle,
\]

8
so we conclude that $U_\psi \mu = W_\psi \mu$. Finally, for $g \in C_0(1/\omega)$ we have

$$
\langle g, U_\psi \delta_0 \rangle = \langle V_\psi g, \delta_0 \rangle = (V_\psi g)(0) = g(0) \int_0^1 \frac{\psi(t)}{t} dt = \langle g, \int_0^1 \frac{\psi(t)}{t} dt \cdot \delta_0 \rangle.
$$

Since $W_\psi \delta_0 = 0$ this finishes the proof. \[\square\]

The conclusion about the range of $U_\psi$ can be generalized to the case, where $\psi$ is not assumed to be continuous.

**Proposition 3.4** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. Then $\text{ran}\, U_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C} \delta_0$.

**Proof** Choose a sequence of non-negative, continuous functions $(\psi_n)$ on $[0, 1]$ with $\psi_n \leq \psi$ and

$$
\int_0^1 \frac{\psi(t) - \psi_n(t)}{t} dt \to 0 \quad \text{as} \quad n \to \infty.
$$

For $\mu \in M(\omega_1)$ and $g \in C_0(1/\omega_2)$ we have

$$
|\langle g, (U_\psi - U_{\psi_n})\mu \rangle| = |\langle (V_\psi - V_{\psi_n})g, \mu \rangle| = \left| \int_{\mathbb{R}^+} \int_0^1 g(x/t) \frac{\psi(t) - \psi_n(t)}{t} dt \, d\mu(x) \right|
$$

$$
\leq \|g\|_{C_0(1/\omega_2)} \int_{\mathbb{R}^+} \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} dt \, d|\mu|(x).
$$

Let

$$
p_n(x) = \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} dt
$$

for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. By condition (C) there exists a constant $C$ such that $p_n(x) \leq C\omega_1(x)$ for every $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Moreover, for every $x \in \mathbb{R}^+$ we have $p_n(x) \to 0$ as $n \to \infty$ by Lebesgue’s dominated convergence theorem. Hence

$$
\| (U_\psi - U_{\psi_n})\mu \|_{M(\omega_2)} = \sup_{\|g\|_{C_0(1/\omega_2)} \leq 1} |\langle g, (U_\psi - U_{\psi_n})\mu \rangle| \leq \int_{\mathbb{R}^+} p_n(x) \, d|\mu|(x) \to 0
$$

as $n \to \infty$ again by Lebesgue’s dominated convergence theorem. Consequently, $U_{\psi_n} \to U_\psi$ strongly as $n \to \infty$. Since $\text{ran}\, U_{\psi_n} \subseteq L^1(\omega_2) \oplus \mathbb{C} \delta_0$ for $n \in \mathbb{N}$ by Proposition 3.3 the same thus holds for $\text{ran}\, U_\psi$. \[\square\]

**Corollary 3.5** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. For $s > 0$ we then have $(U_\psi \delta_s)(x) = \psi(s/x)/x$ for almost all $x \geq s$ and $(U_\psi \delta_s)(x) = 0$ for almost all $x < s$. 

9
Proof For \( \psi \) continuous, this follows from Proposition 3.3. For general \( \psi \) it follows from the approach in the proof of Proposition 3.4 using \( \mathcal{U}_\psi \rightarrow \mathcal{U}_\psi \) strongly as \( n \rightarrow \infty \).

It follows from Corollary 3.5 that

\[
\| \mathcal{U}_\psi \delta_s \|_{M(\omega_2)} = \int_{s}^{\infty} \frac{\omega_2(x)}{x} \psi(s/x) \, dx = \int_{0}^{1} \frac{\omega_2(s/t)}{t} \psi(t) \, dt,
\]

whereas \( \| \delta_s \|_{M(\omega_1)} = \omega_1(s) \). Since \( \mathcal{U}_\psi \) is bounded we thus recover condition (C). If we without using Theorem 2.1 could show that if \( \mathcal{U}_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2) \) is a bounded operator, then is has a bounded extension \( \mathcal{U}_\psi : M(\omega_1) \rightarrow M(\omega_2) \) for which Corollary 3.5 holds, then we would in this way obtain an alternative proof of condition (C).

4 Weakly compact operators

We finish the paper by showing that there are no non-zero, weakly compact generalized Hardy-Cesàro operators between \( L^1(\omega_1) \) and \( L^1(\omega_2) \).

Proposition 4.1 Let \( \psi \) be a non-negative, measurable function on \([0, 1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( \mathcal{U}_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2) \) is a bounded operator. If \( \psi \neq 0 \), then \( \mathcal{U}_\psi \) is not weakly compact.

Proof For \( f \in L^1(\omega_1) \) and \( x \in \mathbb{R}^+ \) we have

\[
(\mathcal{U}_\psi f)(x) = \frac{1}{x} \int_{0}^{x} f(s) \psi(s/x) \, ds = \int_{0}^{\infty} f(s) \rho(s)(x) \omega_1(s) \, ds,
\]

where (with a slight change of notation compared to the proof of Theorem 2.1)

\[
\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s}
\]

for \( x, s \in \mathbb{R}^+ \). In the proof of Theorem 2.1 we saw that \( \rho(s) \in L^1(\omega_2) \) with \( \| \rho(s) \|_{L^1(\omega_2)} \leq C \) for a constant \( C \) for almost all \( s \in \mathbb{R}^+ \). It thus follows from [4, Theorem VI.8.10] that \( \mathcal{U}_\psi \) is weakly compact if and only if \( \{ \rho(s) : s \in \mathbb{R}^+ \} \) is contained in a weakly compact set of \( L^1(\omega_2) \) (except possibly for \( s \) belonging to a null-set). Consider \( \rho(s) \) as an element of \( C_0(1/\omega_2)^* \) for \( s \in \mathbb{R}^+ \) and let \( g \in C_0(1/\omega_2) \). Then

\[
\langle g, \rho(s) \rangle = \int_{0}^{\infty} g(x) \rho(s)(x) \, dx
\]

\[
= \frac{1}{\omega_1(s)} \int_{s}^{\infty} \frac{g(x)}{x} \psi(s/x) \, dx
\]

\[
= \frac{1}{\omega_1(s)} \int_{0}^{1} g(s/t) \frac{\psi(t)}{t} \, dt.
\]

Since \( g(s/t) \rightarrow g(0) \) as \( s \rightarrow 0^+ \) for all \( t > 0 \), it follows from Lebesgue’s dominated convergence theorem that

\[
\langle g, \rho(s) \rangle \rightarrow \frac{1}{\omega_1(0)} g(0) \int_{0}^{1} \frac{\psi(t)}{t} \, dt
\]

10
as \( s \to 0^+ \). We therefore conclude that

\[
\rho(s) \to \frac{1}{\omega_1(0)} \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \delta_0
\]

weak-star in \( M(\omega_2) \) as \( s \to 0^+ \). Since \( \delta_0 \notin L^1(\omega_2) \), it follows that \( \{\rho(s) : s \in \mathbb{R}^+\} \) is not contained in a weakly compact set of \( L^1(\omega_2) \) (even excepting null sets), and the result follows.

\[\Box\]

References


Thomas Vils Pedersen
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark
vils@math.ku.dk