Generalized Hardy–Cesaro operators between weighted spaces

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Published in:
Glasgow Mathematical Journal

DOI:
10.1017/S0017089517000398

Publication date:
2019

Document version
Peer reviewed version

Citation for published version (APA):
Generalized Hardy-Cesàro operators between weighted spaces

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December 1, 2017

Abstract

We characterize those non-negative, measurable functions \( \psi \) on \([0, 1]\) and positive, continuous functions \( \omega_1 \) and \( \omega_2 \) on \( \mathbb{R}^+ \) for which the generalized Hardy-Cesàro operator

\[
(U_\psi f)(x) = \int_0^1 f(tx) \psi(t) \, dt
\]

defines a bounded operator \( U_\psi : L^1(\omega_1) \to L^1(\omega_2) \). This generalizes a result of Xiao (\cite{7}) to weighted spaces. Furthermore, we extend \( U_\psi \) to a bounded operator on \( M(\omega_1) \) with range in \( L^1(\omega_2) \oplus C_0 \), where \( M(\omega_1) \) is the weighted space of locally finite, complex Borel measures on \( \mathbb{R}^+ \). Finally, we show that the zero operator is the only weakly compact generalized Hardy-Cesàro operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \).

1 Introduction

A classical result of Hardy (\cite{5}) shows that the Hardy-Cesàro operator

\[
(Uf)(x) = \frac{1}{x} \int_0^x f(s) \, ds
\]

defines a bounded linear operator on \( L^p(\mathbb{R}^+) \) with \( \|U\| = p/(p - 1) \) for \( p > 1 \). Clearly, \( U \) is not bounded on \( L^1(\mathbb{R}^+) \). Hardy’s result has been generalized in various ways, of which we will mention some, which have inspired this paper.

For \( 1 \leq p \leq q \leq \infty \) and non-negative measurable functions \( u \) and \( v \) on \( \mathbb{R}^+ \), Muckenhoupt (\cite{6}) and Bradley (\cite{3}) gave a necessary and sufficient condition for the existence of a constant \( C \) such that

\[
\left( \int_0^\infty \left( u(x) \int_0^x f(t) \, dt \right)^q \, dx \right)^{1/q} \leq C \left( \int_0^\infty (v(x)f(x))^p \, dx \right)^{1/p}
\]

for every positive, measurable function \( f \) on \( \mathbb{R}^+ \). This can be rephrased as a characterization of the weighted \( L^p \) and \( L^q \) spaces on \( \mathbb{R}^+ \) between which the Hardy-Cesàro operator \( U \) is bounded.

\footnote{2010 Mathematics Subject Classification: 44A15, 47B34, 47B38, 47G10}

\footnote{Keywords: Generalized Hardy-Cesàro operators, weighted spaces, weak compactness.}
In a different direction, for a non-negative measurable function $\psi$ on $[0, 1]$, Xiao ([7]) considered the generalized Hardy-Cesàro operators

$$ (U_\psi f)(x) = \int_0^1 f(tx)\psi(t) \, dt $$

for measurable functions $f$ on $\mathbb{R}^n$. We remark that

$$ (U_\psi f)(x) = \frac{1}{x} \int_0^x f(s)\psi(s/x) \, ds $$

for measurable functions $f$ on $\mathbb{R}$. Xiao proved that $U_\psi$ defines a bounded operator on $L^p(\mathbb{R}^n)$ (for $p \geq 1$) if and only if

$$ \int_0^1 \psi(t) \frac{t}{t^{n/p}} \, dt < \infty. $$

Xiao’s result is the main motivation for this paper.

Finally, we mention that Albanese, Bonet and Ricker in a recent series of papers (see, for instance, [1] and [2]) have considered the spectrum, compactness and other properties of the Hardy-Cesàro operator on various spaces of continuous functions and discrete spaces.

In this paper we will study the generalized Hardy-Cesàro operators between weighted spaces of integrable functions, and we will obtain a generalization of Xiao’s result in this context. Let $\omega$ be a positive, continuous function on $\mathbb{R}^+$ and let $L^1(\omega)$ be the Banach space of (equivalence classes of) measurable functions $f$ on $\mathbb{R}^+$ for which

$$ \|f\|_{L^1(\omega)} = \int_0^\infty |f(t)|\omega(t) \, dt < \infty. $$

In the usual way we identify the dual space of $L^1(\omega)$ with the space $L^\infty(1/\omega)$ of measurable functions $h$ on $\mathbb{R}^+$ for which

$$ \|h\|_{L^\infty(1/\omega)} = \text{ess sup}_{t \in \mathbb{R}^+} |h(t)|/\omega(t) < \infty. $$

We denote by $C_0(1/\omega)$ the closed subspace of $L^\infty(1/\omega)$ consisting of the continuous functions $g$ in $L^\infty(1/\omega)$ for which $g/\omega$ vanishes at infinity. Finally, we identify the dual space of $C_0(1/\omega)$ with the space $M(\omega)$ of locally finite, complex Borel measures $\mu$ on $\mathbb{R}^+$ for which

$$ \|\mu\|_{M(\omega)} = \int_{\mathbb{R}^+} \omega(t) \, d|\mu|(t) < \infty. $$

We consider the space $L^1(\omega)$ as a closed subspace of $M(\omega)$.

In Section 2 we characterize those functions $\psi, \omega_1$ and $\omega_2$ for which $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. These operators are extended to bounded operators on $M(\omega_1)$ in Section 3 where we also obtain results about their ranges. Finally, in Section 4 we show that there are no non-zero weakly compact generalized Hardy-Cesàro operators from $L^1(\omega_1)$ to $L^1(\omega_2)$. 

2
2 A characterization of the generalized Hardy-Cesàro operators

For a non-negative, measurable function \( \psi \) on \([0,1]\) and positive, continuous functions \( \omega_1 \) and \( \omega_2 \) on \( \mathbb{R}^+ \), we say that condition (C) is satisfied if there exists a constant \( C \) such that

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \leq C \omega_1(s)
\]

for every \( s \in \mathbb{R}^+ \).

**Theorem 2.1** Let \( \psi \) be a non-negative, measurable function on \([0,1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \) if and only if condition (C) is satisfied.

**Proof** Assume that condition (C) is satisfied and let \( f \in L^1(\omega_1) \). Then

\[
\int_0^1 \int_0^\infty |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) \, dt \, ds \leq C \int_0^\infty |f(s)| \omega_1(s) \, ds = C \|f\|_{L^1(\omega_1)} < \infty,
\]

so it follows from Fubini’s theorem that

\[
\int_0^1 \int_0^\infty f(tx) \psi(t) \omega_2(x) \, dx \, dt = \int_0^1 \int_0^\infty |f(s)| \frac{\psi(t)}{t} \omega_2(s/t) \, dt \, ds \leq C \|f\|_{L^1(\omega_1)} < \infty.
\]

Another application of Fubini’s theorem thus shows that \((U_\psi f)(x)\) is defined for almost all \( x \in \mathbb{R}^+ \) with

\[
\|U_\psi f\|_{L^1(\omega_2)} = \int_0^\infty |(U_\psi f)(x)| \omega_2(x) \, dx \leq \int_0^1 \int_0^\infty f(tx) \psi(t) \omega_2(x) \, dt \, dx
\]

\[
= \int_0^1 \int_0^\infty f(tx) \psi(t) \omega_2(x) \, dx \, dt \leq C \|f\|_{L^1(\omega_1)} < \infty.
\]

Hence \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \).

Conversely, assume that \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). Since \( L^1(\omega_2) \) is a closed subspace of \( M(\omega_2) \) which we identify with the dual space of \( C_0(1/\omega_2) \), it follows from [4, Theorem VI.8.6] that there exists a map \( \rho \) from \( \mathbb{R}^+ \) to \( M(\omega_2) \) for which the map \( s \mapsto \langle g, \rho(s) \rangle = \int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) \) is measurable and essentially bounded on \( \mathbb{R}^+ \) for every \( g \in C_0(1/\omega_2) \) with \( \|U_\psi\| = \text{ess sup}_{x \in \mathbb{R}^+} \|\rho(s)\|_{M(\omega_2)} \) and such that

\[
\langle g, U_\psi f \rangle = \int_0^\infty \langle g, \rho(s) \rangle f(s) \omega_1(s) \, ds = \int_0^\infty \int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) \, f(s) \omega_1(s) \, ds
\]

for every \( g \in C_0(1/\omega_2) \) and \( f \in L^1(\omega_1) \). On the other hand

\[
\langle g, U_\psi f \rangle = \int_0^\infty g(x) (U_\psi f)(x) \, dx
\]

\[
= \int_0^\infty \int_0^x \frac{g(x)}{x} f(s) \psi(s/x) \, ds \, dx
\]

\[
= \int_0^\infty \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx \, f(s) \omega_1(s) \, ds
\]

for almost all \( x \in \mathbb{R}^+ \).
for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$, so it follows that
\[
\int_{\mathbb{R}^+} g(x) \, d\rho(s)(x) = \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx
\]
for almost all $s \in \mathbb{R}^+$ and every $g \in C_0(1/\omega_2)$ (considering both sides as elements of $L^\infty(\mathbb{R}^+)$). Considered as elements of $M(\omega_2)$ we thus have
\[
d\rho(s)(x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s} \, dx
\]
for almost all $s, x \in \mathbb{R}^+$. Hence $\rho(s) \in L^1(\omega_2)$ with
\[
\|\rho(s)\|_{L^1(\omega_2)} = \int_0^\infty \omega_2(x) \, d\rho(s)(x)
\]
\[
= \frac{1}{\omega_1(s)} \int_0^\infty \frac{1}{x} \psi(s/x) 1_{x \geq s} \omega_2(x) \, dx
\]
\[
= \frac{1}{\omega_1(s)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx
\]
\[
= \frac{1}{\omega_1(s)} \int_0^s \frac{1}{t} \psi(t) \omega_2(s/t) \, dt
\]
for almost all $s \in \mathbb{R}^+$. Therefore
\[
\int_0^s \frac{1}{t} \psi(t) \omega_2(s/t) \, dt = \|\rho(s)\|_{L^1(\omega_2)} \omega_1(s) \leq \|U_\psi\| \omega_1(s)
\]
for almost all $s \in \mathbb{R}^+$. Since both sides of the inequality are continuous functions of $s$, the inequality holds for every $s \in \mathbb{R}^+$, so condition (C) holds. □

Letting $s = 0$ in condition (C) we see that Xiao’s condition is necessary in our situation.

**Corollary 2.2** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. If $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then
\[
\int_0^1 \frac{\psi(t)}{t} \, dt < \infty.
\]

The following straightforward consequences can be deduced from Theorem 2.1.

**Corollary 2.3** Let $\psi$ be a non-negative, measurable function on $[0, 1]$.

(a) Let $\omega$ be a decreasing, positive, continuous function on $\mathbb{R}^+$, and assume that $\int_0^1 \psi(t)/t \, dt < \infty$. Then $U_\psi$ defines a bounded operator from $L^1(\omega)$ to $L^1(\omega)$.

(b) Let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$, and assume that $\omega_2$ is increasing. If $U_\psi$ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then there exists a constant $C$ such that $\omega_2(s) \leq C \omega_1(s)$ for every $s \in \mathbb{R}^+$. 

(c) Let \( \omega \) be an increasing, positive, continuous function on \( \mathbb{R}^+ \), and assume that there exists \( a < 1 \) and \( K > 0 \) such that \( \psi(t) \geq K \) almost everywhere on \([a,1]\). If \( U_\psi \) defines a bounded operator from \( L^1(\omega) \) to \( L^1(\omega) \), then there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \omega(s) \leq \int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \leq C_2 \omega(s)
\]

for every \( s \in \mathbb{R}^+ \).

**Proof**  
(a): We have

\[
\int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \leq \int_0^1 \frac{\psi(t)}{t} \, dt \omega(s)
\]

for every \( s \in \mathbb{R}^+ \), so condition (C) is satisfied with \( \omega_1 = \omega_2 = \omega \) and the result follows.

(b): We have

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \geq \int_0^1 \frac{\psi(t)}{t} \, dt \omega_2(s)
\]

for every \( s \in \mathbb{R}^+ \). Since condition (C) is satisfied, the result follows.

(c): We have

\[
\int_0^1 \omega(s/t) \frac{\psi(t)}{t} \, dt \geq K \int_a^1 \omega(s/t) \, dt \geq K(1-a)\omega(s)
\]

for every \( s \in \mathbb{R}^+ \). The other inequality is just condition (C) with \( \omega_1 = \omega_2 = \omega \). \( \square \)

We finish the section with some examples of functions \( \psi, \omega_1 \) and \( \omega_2 \) for which \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \).

**Example 2.4**

(a) For \( \alpha > 0 \), let \( \psi(t) = t^\alpha \) for \( t \in [0,1] \). Also, for \( \beta_1, \beta_2 \in \mathbb{R} \), let \( \omega_i(x) = (1 + x)^{\beta_i} \) for \( x \in \mathbb{R}^+ \) and \( i = 1,2 \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \) if and only if \( \beta_2 \leq \beta_1 \) and \( \beta_2 < \alpha \).

(b) For \( \alpha > 0 \), let \( \psi(t) = t^\alpha \) for \( t \in [0,1] \). Also, let \( \omega_1(x) = e^{-x/(1+x)} \) and \( \omega_2(x) = e^{-x} \) for \( x \in \mathbb{R}^+ \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). Moreover, it is not possible to replace \( \omega_1(x) \) by a function tending faster to zero at infinity.

(c) Let \( \psi(t) = e^{-t^2} \) for \( t \in [0,1] \). Also, let \( \omega_1(x) = e^{x^2/4}/x \) and \( \omega_2(x) = e^{x} \) for \( x \in \mathbb{R}^+ \). Then \( U_\psi \) defines a bounded operator from \( L^1(\omega_1) \) to \( L^1(\omega_2) \). Moreover, it is not possible to replace \( \omega_1(x) \) by a function tending slower to infinity at infinity.

**Proof**  
(a): For \( s \geq 1 \) and \( t \in [0,1] \) we have \( s/t < 1 + s/t \leq 2s/t \), so

\[
\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1} \, dt
\]

\[
\simeq s^{\beta_2} \int_0^1 t^{\alpha-\beta_2-1} \, dt
\]

\[
\simeq s^{\beta_2}
\]
for $s \geq 1$ if $\beta_2 < \alpha$ (where $F(s) \simeq G(s)$ for positive functions $F$ and $G$ on $[1, \infty)$ indicates the existence of positive constants $C_1$ and $C_2$ such that $C_1 F(s) \leq G(s) \leq C_2 F(s)$ for all $s \in [1, \infty]$), whereas the integrals diverge if $\beta_2 \geq \alpha$. Moreover, the expression

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_0^1 \left(1 + \frac{s}{t}\right) \beta_2 t^{\alpha-1} dt$$

defines a positive, continuous function of $s$ on $\mathbb{R}^+$, so it follows that condition (C) is satisfied if and only if $\beta_2 \leq \beta_1$ and $\beta_2 < \alpha$.

(b): For $s \geq 1$ we have

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_s^\infty \omega_2(x) x^{-\alpha} dx \leq \int_s^\infty \frac{e^{-x}}{x} dx \leq \frac{e^{-s}}{s}.$$

Moreover,

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \leq \int_0^1 \psi(t) dt < \infty$$

for all $s \in \mathbb{R}^+$, so condition (C) is satisfied and $U_\psi$ thus defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. On the other hand, since

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt \geq \int_s^\infty \omega_2(x) x^{-\alpha} dx \geq \frac{1}{2^{\alpha+1}s} \int_s^\infty x^{-\alpha} dx \geq \frac{1}{2^{\alpha+2}} \frac{e^{-s}}{s}$$

for $s \geq 1$, it is not possible to replace $\omega_1(x)$ by a function tending faster to zero at infinity.

(c): For $s \in \mathbb{R}^+$ we have

$$\int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} dt = \int_s^\infty \omega_2(x) x^{-\alpha} dx \geq \int_s^\infty \frac{e^{-x}}{x^{\alpha/2}} dx = \int_1^\infty \frac{e^{s\alpha-y^2}}{y} dy.$$
3 Extensions to weighted spaces of measures

Identifying the dual space of $L^1(\omega)$ with $L^\infty(1/\omega)$ as in the introduction, we have the following result about the adjoint of $U_\psi$.

**Proposition 3.1** Let $\psi$ be a non-negative, measurable function on $[0,1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \rightarrow L^1(\omega_2)$ is a bounded operator, and consider the adjoint operator $U_\psi^* : L^\infty(1/\omega_2) \rightarrow L^\infty(1/\omega_1)$.

(a) For $h \in L^\infty(1/\omega_2)$ we have

$$(U_\psi^*h)(x) = \int_0^1 h(x/t) \frac{\psi(t)}{t} dt$$

for almost all $x \in \mathbb{R}^+$.

(b) $U_\psi^*$ maps $C_0(1/\omega_2)$ into $C_0(1/\omega_1)$.

**Proof** (a): Let $h \in L^\infty(1/\omega_2)$. Since $|h(x/t)| \leq \|h\|_{L^\infty(1/\omega_2)} \omega_2(x/t)$ for almost all $x,t \in \mathbb{R}^+$, it follows from condition (C) that $\int_0^1 h(x/t) \psi(t)/t \, dt$ is defined and satisfies

$$\left| \int_0^1 h(x/t) \frac{\psi(t)}{t} dt \right| \leq \|h\|_{L^\infty(1/\omega_2)} \int_0^1 \omega_2(x/t) \frac{\psi(t)}{t} dt \leq C \|h\|_{L^\infty(1/\omega_2)} \omega_1(x)$$

for almost all $x \in \mathbb{R}^+$. Hence the function $x \mapsto \int_0^1 h(x/t) \psi(t)/t \, dt$ belongs to $L^\infty(1/\omega_1)$. Also, for $f \in L^1(\omega_1)$ we have

$$\langle f, U_\psi^* h \rangle = \langle U_\psi f, h \rangle = \int_0^\infty (U_\psi f)(s) h(s) \, ds$$

$$= \int_0^\infty \int_0^s \frac{1}{s} f(x) \psi(x/s) h(s) \, dx \, ds$$

$$= \int_0^\infty \int_x^\infty \frac{h(s)}{s} \psi(x/s) \, ds f(x) \, dx$$

from which it follows that

$$(U_\psi^*h)(x) = \int_x^\infty \frac{h(s)}{s} \psi(x/s) \, ds = \int_0^1 h(x/t) \frac{\psi(t)}{t} dt$$

for almost all $x \in \mathbb{R}^+$.

(b): It suffices to show that $U_\psi^*$ maps $C_c(\mathbb{R}^+)$ (the continuous functions on $\mathbb{R}^+$ with compact support) into $C_0(1/\omega_1)$. Let $g \in C_c(\mathbb{R}^+)$, let $x_0 \in \mathbb{R}^+$ and let $(x_n)$ be a sequence in $\mathbb{R}^+$ with $x_n \rightarrow x_0$ as $n \rightarrow \infty$. Then

$$(U_\psi^*g)(x_n) - (U_\psi^*g)(x_0) = \int_0^1 (g(x_n/t) - g(x_0/t)) \frac{\psi(t)}{t} dt$$

for $n \in \mathbb{N}$. Since $g$ is bounded on $\mathbb{R}^+$ and since $\int_0^1 \psi(t)/t \, dt < \infty$ by Corollary 2.2, it follows from Lebesgue’s dominated convergence theorem that $(U_\psi^*g)(x_n) \rightarrow (U_\psi^*g)(x_0)$ as $n \rightarrow \infty$. Hence $U_\psi^*g$ is continuous on $\mathbb{R}^+$. Finally, from the expression

$$(U_\psi^*g)(x) = \int_x^\infty \frac{g(s)}{s} \psi(x/s) \, ds$$
Hence $\psi \in \mu$ for Corollary 3.2 then immediately have the following result.

Let $\psi$ be the restriction of $U_\psi^*$ to $C_0(1/\omega_2)$ considered as a map into $C_0(1/\omega_1)$. We then immediately have the following result.

**Corollary 3.2** Let $\psi$ be a non-negative, measurable function on $[0,1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. The bounded operator $\overline{U}_\psi = U_\psi^*$ from $M(\omega_1)$ to $M(\omega_2)$ is an extension of $U_\psi$.

Let $\psi$ be a non-negative, continuous function on $[0,1]$ with $\psi(0) = 0$. For $\mu \in M(\omega_1)$ and $x > 0$ let

$$(W_\psi \mu)(x) = \frac{1}{x} \int_{(0,x)} \psi(s/x) \, d\mu(s).$$

**Proposition 3.3** Let $\psi$ be a non-negative, continuous function on $[0,1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. Then $W_\psi \mu \in L^1(\omega_2)$ and

$$\overline{U}_\psi \mu = W_\psi \mu + \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \mu(\{0\}) \delta_0$$

for $\mu \in M(\omega_1)$. In particular $\operatorname{ran} \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$ and $\overline{U}_\psi$ maps $M((0,\infty),\omega_1)$ into $L^1(\omega_2)$.

**Proof** By Corollary 2.2 we have $\int_0^1 \psi(t)/t \, dt < \infty$, so it follows that $\psi(0) = 0$. Let $\mu \in M(\omega_1)$ with $\mu(\{0\}) = 0$. By condition (C) we have

$$\int_{(0,\infty)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x) \, dx \, d|\mu|(s) = \int_{(0,\infty)} \int_0^1 \omega_2(s/t) \frac{\psi(t)}{t} \, dt \, d|\mu|(s)$$

$$\leq C \int_{(0,\infty)} \omega_1(s) \, d|\mu|(s) = C \|\mu\|_{M(\omega_1)} < \infty,$$

so it follows from Fubini’s theorem that

$$\int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x) \, d|\mu|(s) \omega_2(x) \, dx < \infty.$$

Hence $W_\psi \mu \in L^1(\omega_2)$. Moreover, for $g \in C_0(1/\omega_2)$ we have

$$\langle g, \overline{U}_\psi \mu \rangle = \langle V_\psi g, \mu \rangle = \int_{(0,\infty)} \int_0^1 g(s/t) \frac{\psi(t)}{t} \, dt \, d\mu(s)$$

$$= \int_{(0,\infty)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) \, dx \, d\mu(s)$$

$$= \int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x) \, d\mu(s) \, g(x) \, dx$$

$$= \int_0^\infty (W_\psi \mu)(x) g(x) \, dx = \langle g, W_\psi \mu \rangle,$$

8
so we conclude that $\overline{U}_\psi \mu = W_\psi \mu$. Finally, for $g \in C_0(1/\omega_2)$ we have

$$\langle g, \overline{U}_\psi \delta_0 \rangle = \langle V_\psi g, \delta_0 \rangle = (V_\psi g)(0) = g(0) \int_0^1 \frac{\psi(t)}{t} \, dt = \langle g, \int_0^1 \frac{\psi(t)}{t} \, dt \cdot \delta_0 \rangle.$$ 

Since $W_\psi \delta_0 = 0$ this finishes the proof. \hfill \Box

The conclusion about the range of $\overline{U}_\psi$ can be generalized to the case, where $\psi$ is not assumed to be continuous.

**Proposition 3.4** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. Then $\text{ran} \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$.

**Proof** Choose a sequence of non-negative, continuous functions $(\psi_n)$ on $[0, 1]$ with $\psi_n \leq \psi$ and

$$\int_0^1 \frac{\psi(t) - \psi_n(t)}{t} \, dt \to 0 \quad \text{as} \ n \to \infty.$$ 

For $\mu \in M(\omega_1)$ and $g \in C_0(1/\omega_2)$ we have

$$|\langle g, (\overline{U}_\psi - \overline{U}_{\psi_n}) \mu \rangle| = |\langle (V_\psi - V_{\psi_n}) g, \mu \rangle| = \left| \int_{\mathbb{R}^+} \int_0^1 g(x/t) \frac{\psi(t) - \psi_n(t)}{t} \, dt \, d\mu(x) \right|$$

$$\leq \|g\|_{C_0(1/\omega_2)} \int_{\mathbb{R}^+} \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} \, dt \, d|\mu|(x).$$

Let

$$p_n(x) = \int_0^1 \omega_2(x/t) \frac{\psi(t) - \psi_n(t)}{t} \, dt$$

for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. By condition (C) there exists a constant $C$ such that $p_n(x) \leq C\omega_1(x)$ for every $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Moreover, for every $x \in \mathbb{R}^+$ we have $p_n(x) \to 0$ as $n \to \infty$ by Lebesgue's dominated convergence theorem. Hence

$$\| (\overline{U}_\psi - \overline{U}_{\psi_n}) \mu \|_{M(\omega_2)} = \sup_{\|g\|_{C_0(1/\omega_2)} \leq 1} |\langle g, (\overline{U}_\psi - \overline{U}_{\psi_n}) \mu \rangle| \leq \int_{\mathbb{R}^+} p_n(x) \, d|\mu|(x) \to 0$$

as $n \to \infty$ again by Lebesgue’s dominated convergence theorem. Consequently, $\overline{U}_{\psi_n} \to \overline{U}_\psi$ strongly as $n \to \infty$. Since $\text{ran} \overline{U}_{\psi_n} \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$ for $n \in \mathbb{N}$ by Proposition 3.3 the same thus holds for $\text{ran} \overline{U}_\psi$. \hfill \Box

**Corollary 3.5** Let $\psi$ be a non-negative, measurable function on $[0, 1]$ and let $\omega_1$ and $\omega_2$ be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. For $s > 0$ we then have $(\overline{U}_{\psi_\delta_s})(x) = \psi(s/x)/x$ for almost all $x \geq s$ and $(\overline{U}_{\psi_\delta_s})(x) = 0$ for almost all $x < s$. 

9
Proof. For \( \psi \) continuous, this follows from Proposition 3.3. For general \( \psi \) it follows from the approach in the proof of Proposition 3.4 using \( U_{\psi_n} \to U_{\psi} \) strongly as \( n \to \infty \).

It follows from Corollary 3.5 that

\[
\| U_{\psi} \delta_s \|_{M(\omega_2)} = \int_0^1 \omega_2(s/t) \psi(t) \frac{t}{s} dt,
\]

whereas \( \| \delta_s \|_{M(\omega_1)} = \omega_1(s) \). Since \( U_{\psi} \) is bounded we thus recover condition (C). If we without using Theorem 2.1 could show that if \( U_{\psi} : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator, then is has a bounded extension \( \overline{U}_{\psi} : M(\omega_1) \to M(\omega_2) \) for which Corollary 3.5 holds, then we would in this way obtain an alternative proof of condition (C).

4 Weakly compact operators

We finish the paper by showing that there are no non-zero, weakly compact generalized Hardy-Cesàro operators between \( L^1(\omega_1) \) and \( L^1(\omega_2) \).

Proposition 4.1 Let \( \psi \) be a non-negative, measurable function on \([0,1]\) and let \( \omega_1 \) and \( \omega_2 \) be positive, continuous functions on \( \mathbb{R}^+ \). Assume that condition (C) is satisfied so that \( U_{\psi} : L^1(\omega_1) \to L^1(\omega_2) \) is a bounded operator. If \( \psi \neq 0 \), then \( U_{\psi} \) is not weakly compact.

Proof. For \( f \in L^1(\omega_1) \) and \( x \in \mathbb{R}^+ \) we have

\[
(U_{\psi} f)(x) = \frac{1}{x} \int_0^x f(s) \psi(s/x) ds = \int_0^\infty f(s) \rho(s/x) \omega_1(s) ds,
\]

where (with a slight change of notation compared to the proof of Theorem 2.1)

\[
\rho(s/x) = \frac{1}{\omega_1(s)} \frac{1}{x} \psi(s/x) 1_{x \geq s}
\]

for \( x, s \in \mathbb{R}^+ \). In the proof of Theorem 2.1 we saw that \( \rho(s) \in L^1(\omega_2) \) with \( \| \rho(s) \|_{L^1(\omega_2)} \leq C \) for a constant \( C \) for almost all \( s \in \mathbb{R}^+ \). It thus follows from [4, Theorem VI.8.10] that \( U_{\psi} \) is weakly compact if and only if \( \{ \rho(s) : s \in \mathbb{R}^+ \} \) is contained in a weakly compact set of \( L^1(\omega_2) \) (except possibly for \( s \) belonging to a null-set). Consider \( \rho(s) \) as an element of \( C_0(1/\omega_2)^* \) for \( s \in \mathbb{R}^+ \) and let \( g \in C_0(1/\omega_2) \). Then

\[
\langle g, \rho(s) \rangle = \int_0^\infty g(x) \rho(s/x) dx = \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x} \psi(s/x) dx = \frac{1}{\omega_1(s)} \int_0^1 g(s/t) \frac{\psi(t)}{t} dt.
\]

Since \( g(s/t) \to g(0) \) as \( s \to 0^+ \) for all \( t > 0 \), it follows from Lebesgue’s dominated convergence theorem that

\[
\langle g, \rho(s) \rangle \to \frac{1}{\omega_1(0)} g(0) \int_0^1 \frac{\psi(t)}{t} dt
\]
as $s \to 0_+$. We therefore conclude that

$$\rho(s) \to \frac{1}{\omega_1(0)} \int_0^1 \frac{\psi(t)}{t} dt \cdot \delta_0$$

weak-star in $M(\omega_2)$ as $s \to 0_+$. Since $\delta_0 \notin L^1(\omega_2)$, it follows that $\{\rho(s) : s \in \mathbb{R}^+\}$ is not contained in a weakly compact set of $L^1(\omega_2)$ (even excepting null sets), and the result follows. □

References


