Generalized Partial Benders Decomposition of Two Stage Stochastic Programs

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Abstract

This paper introduces the concept of Generalized Partial Benders Decomposition (GPBD), a solution method for two-stage stochastic programs, possibly mixed-integer at both stages, which revisits the classical Benders decomposition algorithm. Richer master problems are obtained by retaining selected second-stage variables and constraints. The scope is that of providing the master problem additional structure to produce better solutions in shorter times, and thus to reduce the reliance on the cuts generated. A number of simple and general-purpose strategies for retaining second-stage variables and constraints in the master problem are proposed. GPBD is compared to classical Benders decomposition in an extensive computational study based on instances of the Capacitated Facility Location Problem with Stochastic Demand. The study illustrates that GPBD improves Benders decomposition on the great majority of the instances and that the improvements are stable across a set of heterogeneous instances. In addition, it illustrates that even simple retaining strategies may grant significant improvements and that the best improvements are obtained retaining a relatively small number of variables and constraints. Finally, results show that efficient retaining strategies can be discovered by training the method on small-scale instances.
Keywords: Stochastic Programming, Benders Decomposition, Facility Location.

1 Introduction

Two-stage Stochastic Programs (SPs) have, for long time, interested the operations research community. The formal theoretical foundation was laid by G.B. Dantzig in 1955 [Dantzig, 1955]. Essentially, some decisions are made here-and-now (so called *first-stage* decisions), then the value of some random parameters of the problem is realized and, finally, other decisions are made (so called, *recourse* or *second-stage* decisions). Their intuitive structure, and the potential benefits for decision makers dealing with uncertainty, attracted significant attention in areas such as vehicle routing [Gendreau et al., 2014, 2016], power production [Shiina and Birge, 2004, Papavasiliou and Oren, 2013], fleet planning [Pantuso et al., 2016, Mørch et al., 2017], and network design [Crainic et al., 2011, Bai et al., 2014]. Today SPs represent a well-established framework and research avenue.

Benders decomposition [Benders, 1962, BD], adapted to SPs by Van Slyke and Wets [1969], has been among the first and most used exact methods for solving SPs. In a nutshell, the second-stage components of the problem are dualized and projected onto the subspace generated by the first-stage variables only, generating the so called *master problem* [Geoffrion, 1970b]. The dualized components, initially relaxed, are iteratively reconstructed through the addition of optimality and feasibility cuts obtained by solving second-stage problems as subproblems.

Several improvements have been proposed to the original algorithm, see Rahmaniani et al. [2016]. Particularly, the authors report that less than 5% of the contributions focused on the decomposition strategy, that is on the partitioning of the problem in order to obtain a master and subproblems. Nevertheless, the formulation of the problem (and thus of the master and subproblems) has a crucial impact on the performances of BD [see, e.g., Geoffrion and Graves, 1974, Magnanti and Wong, 1981]. As Crainic et al. [2014] point out, the initial relaxation typically generates a rather weak formulation of the master problem, without any information about recourse decisions, and with a number of consequent computational challenges such as erratic bounds and slow convergence. In this paper we revisit the classical decomposition strategy applied in BD and thus propose improvements along the above-mentioned decomposition strategy.

As far as stochastic programs are concerned, alternative decomposition strategies have recently been proposed by Crainic et al. [2014] and Crainic et al. [2016]. The former introduce the idea of a *partial Benders decomposition* (PBD) obtained by retaining selected second-stage subproblems in the master problem, thus relaxing the original problem only partially, compared to BD. The intuition behind the method is to provide the master problem second-stage information from the beginning as to enable it to make better decisions in shorter times. Fewer cuts and iterations are consequently expected. The authors propose several strategies for selecting subproblems to retain in the master problem. Tests on instances of the Stochastic Fixed Charge Multicommodity Network Design Problem illustrated that the new decomposition approach improves plain BD. Crainic et al. [2016] extend the idea of Crainic et al. [2014] including *scenario creation* strategies. That is, artificial scenarios are generated with the scope of improving the bounds provided by the master, should the corresponding subproblems be retained. Also in this case, tests show a significant improvement compared to the original BD.

Motivated by the need of studies of alternative decomposition strategies [see Rahmaniani et al., 2016] and by the positive results of Crainic et al. [2014] and Crainic et al. [2016], we also pursue a partial decomposition of SPs. Particularly, we propose the idea of retaining in the master problem selected second-stage variables and constraints, thus projecting the original problem onto the subspace made of the first-stage variables and selected second-stage variables. We refer to this decomposition strategy as *Generalized Partial Benders Decomposition* (GPBD). With respect to BD, GPBD relaxes the second-stage components only partially, thus leaving the master problem additional structure to generate higher quality solutions already in the initial iterations of the algorithm. With respect to PBD, GBPD retains in the master problem only selected variables and constraints of the (selected) subproblems, as opposed to retaining entire subproblems. The aim is that of providing the master problem useful information without excessively increasing its size by adding entire subproblems, and thus combining the benefits of an improved but light master problem formulation. GPBD can be seen as a generalization of PBD which allows the user to tailor the master problem based on the specific SP at hand without necessarily retaining entire subproblems. Thus, PBD represents an instance of GPBD where all the variables of selected subproblems are retained. The expected advantages of an augmented MP are essentially two: tighter bounds and early production of better first-stage candidate solutions, in turn leading to fewer iterations and required cuts. However, a trade-off between the amount of information kept in the master problem and its complexity is to be found. Therefore, an important exercise in GPBD is that of identifying the most relevant second-stage information (i.e., variables and constraints). While knowledge of the specific SP is a key element, in this paper we propose a number of simple general-purpose retaining strategies. Furthermore, we illustrate that effective retaining strategies can be identified by training the method on small instances of the problem.

The remainder of this paper is organized as follows. In Section 2 we briefly summarize the BD algorithms, in Section 3 we provide a more thorough description of GPBD, and in Section 4 we introduce general-purpose retaining strategies. In Section 5 we report our experience from an extensive computational study based on instances of the Capacitated Facility Location Problem with Stochastic Demand and, finally, we draw conclusions in Section 6.

2 Benders Decomposition

Consider the following two-stage stochastic program where we assume the uncertain parameters follow a discrete distribution with a set $\mathcal{S}$ of possible outcomes:

$$\min_{x \in X} \{c^T x + Q(x) | Ax = b\} \tag{1a}$$

where $Q(x) = \sum_{s \in \mathcal{S}} p_s Q(x, \xi_s)$ is referred to as the *recourse function*, p_s is the probability of scenario $s \in \mathcal{S}$, and

$$Q(x, \xi_s) = \min_y \{q_s^T y | W_s y = h_s - T_s x, y_s \geq 0\}. \tag{1b}$$

Here $x \in X \subseteq \mathbb{R}^{n1}$ are first-stage decision variables and $y \in \mathbb{R}^{n2}$ are second-stage (recourse) decision variables. Parameters $c \in R^{n1}$,$b \in R^{m1}$, and $A \in R^{m1 \times n1}$ are known and deterministic. However, parameters $q_s \in R^{n2}$, $h_s \in R^{m2}$, $W_s \in R^{m2 \times n2}$ and $T_s \in R^{m2 \times n1}$ are stochastic and depend on the scenario s that eventually materializes. The collection of random parameters is referred to as $\xi_s^T = (q_s^T, W_s^1, \ldots, W_s^{m2}, T_s^1, \ldots, T_s^{m2}, h_s^T)$, where W_s^i, and T_s^i are the i-th row of W_s and T_s, respectively.

Benders decomposition first projects problem (1) onto the space of the x variables only, then performs an *outer-linearization* of the recourse function, and finally relaxes and dynamically reconstructs the constraints generated by the projection and outer-linearization phase [see Geoffrion, 1970a,b]. More specifically, BD generates the following *master problem* (MP):

$$\min \ c^T x + \theta \tag{2a}$$

$$\text{s.t. } Ax = b \tag{2b}$$

$$\theta \geq \sum_{s\in\mathcal{S}} p_s(\pi_s^i)^T(h_s - T_s x) \qquad i = 1,\dots,I \tag{2c}$$

$$(\sigma_s^j)^T(h_s - T_s x) \leq 0 \qquad j = 1,\dots,J_s, s \in \mathcal{S} \tag{2d}$$

$$x \in X \tag{2e}$$

where π_s^i, $i = 1,\dots,I$ are extreme points and σ_s^j, $j = 1,\dots,J_s$ extreme rays of the dual to second-stage problem (1b). Constraints (2c) are referred to as *optimality cuts* and represent the outer-linearization of the recourse function, while (2d) are referred to as *feasibility cuts* and represent second-stage feasibility conditions. Constraints (2c)-(2d) are initially relaxed and iteratively reconstructed. At a each iteration, given a candidate solution $(\hat{x}, \hat{\theta})$ to MP, second-stage problems are solved as subproblems for all $s \in \mathcal{S}$. If all second-stage problems are feasible and $\hat{\theta} \geq Q(\hat{x})$, then $\hat{x}$ is optimal to (1). Otherwise, a violated constraint (2c) or (2d) is found and added to MP which is the re-solved. In the rest of this treatise, we assume that second-stage variables are continuous as this allows us a convenient representation of duality-based cuts. However, this is without loss of generality as alternative cuts for problems with integer second-stage variables can be found for example in the seminal works of Laporte and Louveaux [1993] and Carøe and Tind [1998]. Finally, note that when X imposes integrality restrictions on x the algorithm sketched in this section is embedded in a branch-and-cut framework [see Laporte and Louveaux, 1993].

3 Generalized Partial Benders Decomposition

Generalized Partial Benders Decomposition (GPBD) amounts to dualize and relax only partially the second-stage components of problem (1). Consider, for each $s \in \mathcal{S}$, an integer r_s such that $0 \leq r_s \leq n2$ – we remind the reader that $n2$ is number of second-stage variables, see Section 2. Let $y_s^R = (y_{is})_{i=1,\dots,r_s}$ be a selection of r_s second-stage variables from y_s we wish to retain in the master problem, and $y_s^{NR} = (y_{is})_{i=r_s+1,\dots,n2}$ the remaining variable in y_s. The s-th second-stage problem can be reformulated, without loss of generality, as follows:

$$\min q_s^R y_s^R + q_s^{NR} y_s^{NR} \tag{3a}$$

$$\begin{bmatrix} W_s^R & \mathbf{0} \\ W_s^{NR1} & W_s^{NR2} \end{bmatrix} \begin{bmatrix} y_s^R \\ y_s^{NR} \end{bmatrix} = \begin{bmatrix} h_s^R \\ h_s^{NR} \end{bmatrix} - \begin{bmatrix} T_s^R \\ T_s^{NR} \end{bmatrix} x \tag{3b}$$

$$y_s^R, y_s^{NR} \geq 0 \tag{3c}$$

where q_s^R and q_s^{NR} are an r_s-dimensional and an $(n2 - r_s)$-dimensional sub-vector of q_s, respectively, W_s^R is a $v \times r_s$ sub-matrix of W_s, $\mathbf{0}$ is a $v \times (n2 - r_s)$ zero-matrix, W_s^{NR1} is an $(m2 - v) \times r_s$, possibly null, sub-matrix of W_s, and W_s^{NR2} is an $(m2 - v) \times (n2 - r_s)$ non-null sub-matrix of W_s. Furthermore, h_s^R and h_s^{NR} are a v-dimensional and an $(m2 - v)$-dimensional sub-vector of h_s, respectively, and finally T_s^R and T_s^{NR} are a $v \times n1$ and an $(m2 - v) \times n1$ sub-matrix of T_s, respectively. Thus, $0 \leq v \leq m2$ is the number of second-stage constraints involving only y_s^R variables (i.e., the second-stage constraints where the coefficients of y_s^{NR} are null) and depends on the specific selection of y_s^R variables which is discussed in Section 4. For now we assume that a selection of second-stage variables y_s^R exists for each $s \in \mathcal{S}$. GPBD consists of projecting the original problem onto the subspace defined by the x and $(y_s^R)_{s\in\mathcal{S}}$, decision variables, thus we can formulate the master problem (MP) as follows:

$$\min\ c^T x + \sum_{s\in\mathcal{S}} p_s q_s^R y_s^R + \theta \tag{4a}$$

$$\text{s.t. } Ax = b, \tag{4b}$$

$$W_s^R y_s^R = h_s^R - T_s^R x, \qquad s \in \mathcal{S}, \tag{4c}$$

$$\theta \geq \sum_{s\in\mathcal{S}} p_s (\pi_s^i)^T (h_s^{NR} - T_s^{NR} x - W_s^{NR1} y_s^R), \qquad i = 1, \ldots, I, \tag{4d}$$

$$(\sigma^j)^T (h_s^{NR} - T_s^{NR} x - W_s^{NR1} y_s^R) \leq 0, \qquad j = 1, \ldots, J_s, s \in \mathcal{S}, \tag{4e}$$

$$x \in X, \tag{4f}$$

$$y_s^R \geq 0, \qquad s \in \mathcal{S} \tag{4g}$$

Notice the following differences with respect to the master problem in BD (2). The objective function (4a) includes a term for the cost of the variables retained. Constraints (4c) are second-stage constraints which are added to MP as a consequence of retaining variables $(y_s^R)_{s\in\mathcal{S}}$. The optimality and feasibility cuts, (4d) and (4d), now account for the fact that variables y_s^R belong to the MP and are thus right-hand-side parameters in the subproblems. Finally, constraints (4g) set the range for the variables y_s^R retained in MP. Thus, given a solution $(\hat{x}, \hat{y}_1^R, \ldots, \hat{y}_{|\mathcal{S}|}^R, \hat{\theta})$ to MP, for each scenario s we solve the following subproblem:

$$\min\ q_s^{NR} y_s^{NR} \tag{5a}$$

$$W_s^{NR2} y_s^{NR} = h_s^{NR} - T_s^{NR} \hat{x} - W_s^{NR1} \hat{y}_s^R \tag{5b}$$

$$y_s^{NR} \geq 0 \tag{5c}$$

Notice that the constraints which involve y_s^{NR} (and possibly also y_s^R) variables belong exclusively to the subproblems, while the constraints involving only the y_s^R variables are in MP, see (4c). The algorithm is thus exactly the same as for BD (see Section 2) except for a preprocessing phase which consists of selecting the y_s^R variables to keep in MP for each $s \in \mathcal{S}$.

In general, GPBD amounts to solving slightly bigger MPs and slightly smaller subproblem compared with BD as the MP formulation is strengthened due to the addition of constraints (4c). The rationale behind GPBD is that the second-stage variables and, particularly, the constraints (4c) retained, might help MP provide higher quality solutions and better bounds already in the initial iterations of the algorithm. This in turn may correspond to shorter computation times and fewer cuts required. However, while the search space of MP is restricted, MP is heavier and in principle more difficult to solve. Thus, it is necessary to trade off the benefits of additional structure and the burden of a more difficult MP. Compared with PBD, the MP in GPBD is, in general, not augmented by full subproblems. Rather, GPBD provides the option of retaining (for some or all of the subproblems) only the structure which is thought more relevant for MP. This might in turn yield the advantages of an augmented formulation with a relatively little increase in the size and complexity of the MP. Finally, notice that BD is an instance of GPBD where $r_s = 0$ for all $s \in \mathcal{S}$, while PBD is an instance of GPBD where $r_s = n2$ for selected $s \in \mathcal{S}$. Thus, GPBD can be thought of as a generalization of PBD.

4 Retaining Strategies

A crucial step in GPBD is that of selecting, for each scenario $s \in S$, variables y_s^R to be retained in MP. In Section 3 we assumed, without loss of generality, that the first r_s variables of subproblem s were retained. More generally, this step amounts to selecting r_s out of $n2$ variables, thus a subset $\mathcal{R}_s \subseteq \{1, \ldots, n2\}$, such that $|\mathcal{R}_s| = r_s$. Let, thus, a *Retaining Strategy* (RS) be a criteria for selecting subsets $\mathcal{R}_s$ of $\{1, \ldots, n2\}$. While the selection of $\mathcal{R}_s$ can be arbitrary, a successful application of GPBD requires some intelligence. In fact, selecting variables y_s^R which are not tied by second-stage constraints (i.e., a W_s^R matrix – see (3b)) is arguably of little practical use. This would amount to adding unconstrained variables to MP (similar to θ) whose value would have to be adjusted through the addition of cuts, and providing as such very little information. Instead, we advocate retaining second-stage variables generating (4c) constraints which are "representative" of selected groups of second-stage constraints. In what follows, we propose a number of general-purpose RSs following this line of reasoning. While any type of sophisticated rule can be applied, the RSs we propose are chosen following the criteria of "simplicity", meaning that they are based on simple mnemonic rules. We proceed as follows: first, given a set of representative constraints, we illustrate how the variables to retain in the MP (i.e., the subset $\mathcal{R}_s$) are found and, second, we clarify what a group of constraints is in our context and propose simple rules for identifying representative constraints for a given group.

Assume a set $\mathcal{N}_s \subseteq \{1, \ldots, m2\}$ of representative constraints to retain for subproblem s has been identified. We define the set of variables retained as $\mathcal{R}_s = \{i : \exists\, W_s^{ij} \neq 0, 1 \leq i \leq n2, j \in \mathcal{N}_s\}$, where W_s^{ij} is the i,j-th element of W_s. That is, we retain the variables y_s which appear with a non-null W_s^{ij} coefficient at least once in the representative constraints. This corresponds to include in MP all the variables which are strictly necessary for generating the desired constraints (4c). As an example, consider the W_s matrix (6), and assume we want to represent the first block of constraints $\{1, \ldots, 4\}$, by means of the first and third constraint (shown in boldface), i.e., $\mathcal{N}_s = \{1, 3\}$. We can then retain variables $\mathcal{R}_s = \{1, 2, 3\}$ as there exist at least one $W_s^{ij} \neq 0$, for $i = 1, \ldots, 3$ and $j \in \mathcal{N}_s$.

$$W_s = \begin{bmatrix} \mathbf{4} & \mathbf{5} & \mathbf{5} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ 3 & 0 & 5 & 3 & 0 & \ldots & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ 1 & 0 & 1 & 0 & 0 & \ldots & 0 \\ & & \ldots\ldots\ldots & & & & \end{bmatrix} \tag{6}$$

Generally, blocks of constraints can be easily identified in the structure $W_s y_s = h_s - T_s x$ in (1b). A block of constraints typically represents the same requirements applied to a number of similar entities. As an example, in the Capacitated Facility Location Problem at least two blocks of constraints can be identified, one ensuring that demand is satisfied for each customer, the other ensuring that the capacity of each facility is respected [see, e.g., Fischetti et al., 2016a,b].

In the basic version of the Stochastic Unit Commitment Problem it is possible to identify one block of constraints ensuring the satisfaction of power demand, and another ensuring that each generating unit produces within the respective operating range [see, e.g., Shiina and Birge, 2004]. However, several additional blocks of restrictions, such as ramping limits, are typically included [see, e.g., Papavasiliou and Oren, 2013]. Finally, in the Stochastic Network Design Problem it is possible to find at least a block of flow-conservation constraint and a block of constraints which ensures that the capacity of individual network links is respected [see, e.g., Crainic et al., 2011, 2016]. Thus, it is generally straightforward to identify different blocks of constraints, each representing a different physical-world requirement.

Let m be the number of blocks of constraints in the second-stage problem and $\mathcal{M}_1, \ldots, \mathcal{M}_m$ a partition of $\{1, \ldots, m2\}$, i.e., disjoint groups of second-stage constraints. We refer to the rows of W_s corresponding to the i-th block of constraints $\mathcal{M}_i$ as $W_s(i) = (W_s^j)_{j\in\mathcal{M}_i}$, where W_s^j is the j-th row of W_s. Similarly, let $T_s(i) = (T_s^j)_{j\in\mathcal{M}_i}$, and $h_s(i) = (h_s^j)_{j\in\mathcal{M}_i}$, where h_s^j is the j-th element of vector h_s. Thus, $W_s(i)$, $T_s(i)$ and $h_s(i)$ is the data describing the i-th block of constraints. Such data is used by the RSs we propose.

We propose a number of general-purpose RSs based on simple but widely applicable rules for selecting representative constraints and consequently variables to retain. Clearly, more sophisticated method could be employed. However, for the scope of this paper we prefer to test GPBD only with simple selection rules in order to show its potential. The RS we propose are summarized in Appendix A. Most of the RSs we propose are based on finding highest and lowest values, or on finding random samples. The most sophisticated RS we propose is based on a clustering algorithm which is easy implementable with standard programming languages. The rationale behind the RSs proposed is that once MP is aware of "extreme" constraints, such as bearing highest or lowest values, or of representative constraints to satisfy, it can more easily tailor its solutions to the second-stage information provided, and thus produce better first-stage solutions already in the initial phases of the algorithm, in turn improving convergence.

In RSs $H^W(i,n)$, $H^T(i,n)$ and $H^h(i,n)$ the letter H stands for "highest". These RSs have the scope of representing constraints block i by the n constraints of the block with the "highest" W_s^j, $j \in \mathcal{M}_i$, values (respectively, T_s^j and h_s^j). More precisely, let $W_s^{(j)}$ be the j-th order statistic where $\sum_{k=1}^{n2} W_s^{k,(1)} \geq \sum_{k=1}^{n2} W_s^{k,(2)} \geq \cdots \geq \sum_{k=1}^{n2} W_s^{k,(|\mathcal{M}_i|)}$. Similarly, let $T_s^{(j)}$ and $h_s^{(j)}$ be order statistics such that $\sum_{k=1}^{n1} T_s^{k,(1)} \geq \sum_{k=1}^{n1} T_s^{k,(2)} \geq \cdots \geq \sum_{k=1}^{n1} T_s^{k,(|\mathcal{M}_i|)}$ and $h_s^{(1)} \geq h_s^{(2)} \geq \cdots \geq h_s^{(|\mathcal{M}_i|)}$. For constraints block i let the representative constraints be $\mathcal{N}_s = \{j : W_s^j \in \{W_s^{(1)}, \ldots, W_s^{(n)}\}\}$ (respectively $\mathcal{N}_s = \{j : T_s^j \in \{T_s^{(1)}, \ldots, T_s^{(n)}\}\}$ and $\mathcal{N}_s = \{j : h_s^j \in \{h_s^{(1)}, \ldots, h_s^{(n)}\}\}$). Thus, RSs $H^W(i,n)$, $H^T(i,n)$ and $H^h(i,n)$ amount to representing the i-th block by means of the constraints with the n highest sums of W_s coefficients, with the n highest sums of T_s coefficients, and with the n highest h_s coefficients, respectively.

In RSs $L^W(i,n)$, $L^T(i,n)$ and $L^h(i,n)$ the letter L stands for "lowest". These RSs have the scope of representing constraints block i by the n constraints of the block with the "lowest" W_s^j, $j \in \mathcal{M}_i$, values (respectively, T_s^j and h_s^j). For constraints block i let the representative constraints be $\mathcal{N}_s = \{j : W_s^j \in \{W_s^{(|\mathcal{M}_i|-n+1)}, \ldots, W_s^{(|\mathcal{M}_i|)}\}\}$ (respectively $\mathcal{N}_s = \{j : T_s^j \in \{T_s^{(|\mathcal{M}_i|-n+1)}, \ldots, T_s^{(|\mathcal{M}_i|)}\}\}$ and $\mathcal{N}_s = \{j : h_s^j \in \{h_s^{(|\mathcal{M}_i|-n+1)}, \ldots, h_s^{(|\mathcal{M}_i|)}\}\}$). Thus, RSs

$L^W(i,n)$, $L^T(i,n)$ and $L^h(i,n)$ amount to representing the i-th block by means of the constraints with the n lowest sums of W_s coefficients, with the n lowest sums of T_s coefficients, and with the n lowest h_s coefficients, respectively.

In RSs $EX^W(i,n)$, $EX^T(i,n)$ and $EX^h(i,n)$ the letters EX stand for "extremes". These RSs represent a combination between the H and the L-type strategies. That is, they have the scope of representing constraints block i by means of the n constraints of the block with the "lowest" and "highest" W_s^j values (respectively, T_s^j and h_s^j) $j \in \mathcal{M}_i$. Thus RSs $EX^W(i,n)$, $EX^T(i,n)$ and $EX^h(i,n)$ retain twice as many constraints as the corresponding H and L-type strategies.

In RS $S(i,n)$ the letter S stands for "sampling". This RS amounts to representing constraints $\mathcal{M}_i$ by sampling n of its constraints. Any sampling technique can be adopted. In Section 5 we use random sampling.

Finally, in RSs $C^W(i,n)$, $C^T(i,n)$ and $C^h(i,n)$ the letter C stands for "clustering". These RSs cluster the constraints in block $i \in \{1,\dots,m\}$ into n subsets of "similar" constraints, and then, for each cluster, choose one constraint as representative of the cluster. In RS $C^W(i,n)$ the population of constraints is clustered based on the similarity between the rows of $W_s(i)$. Similarly, in $C^T(i,n)$ and $C^h(i,n)$, constraints $\mathcal{M}_i$ are clustered based on the similarity between the rows of $T_s(i)$ and the elements of $h_s(i)$, respectively. For each cluster $c = 1,\dots,n$, the point $j \in \mathcal{M}_i$ closest to the centroid of the cluster is selected as the representative for the cluster. Constraint j is added to $\mathcal{N}_s$ and the variables forming constraint j will be retained (as well as constraint j itself). Any measure of similarity and clustering algorithm can be applied (in Section 5 we use the Euclidean distance and the `k-means++` algorithm of Arthur and Vassilvitskii [2007]).

Additional RSs can be thought of by combining the ones proposed in this section in such a way to represent different groups of constraints, possibly using a different strategy for each of them.

5 Computational Study

In this section we report on our computational experience based on several difficult instances of the *Capacitated Facility Location Problem with Stochastic Demand* (CFLPSD). The computational study comprises a *training phase* and *test phase*. In the training phase the GPBD algorithm is used to solve a set of small *training instances* with the scope of identifying the most promising RSs. In the test phase GPBD, with the selected RSs, is used to solve a set of *test instances* which comprise larger and numerically different problems.

GPBD and BD are implemented in Java using the Cplex 12.6.2 callable libraries. Cplex 12.6.2 is also used to implement and solve (when possible) the extensive problem. The implementation includes default solver settings (except for the tolerances necessary to ensure cuts are correctly enforced) and no additional algorithmic enhancement that could theoretically improve the results obtained. This allows us to isolate the effect of a revisited decomposition strategy. Possible algorithmic enhancements, such as stabilization, are discussed in Section 5.5. All tests are performed on a cluster of machines equipped with 12 × 2.39 GHz CPU and 23.59 GB RAM. All training and test instances are solved to a target relative optimality gap of 10^{-6}.

5.1 The Capacitated Facility Location Problem with Stochastic Demand

The Capacitated Facility Location Problem with Stochastic Demand (CFLPSD) consists of choosing which facilities to open, given a set of candidates, and how to assign uncertain customer demand to the open facilities. Facility location decisions must be made before the demand become known and are thus first-stage decisions, while the allocation of demand to facilities can be decided once the demand become known. The CFLPSD can be thus modeled as a two-stage stochastic program with recourse. The CFLPSD is NP-Hard as the deterministic Capacitated Facility Location Problem, which is a special case of the CFLPSD , is known to be NP-hard [Cornuejols et al., 1991].

An instance of the CFLPSD is made of a set of facilities $\mathcal{I}$, a set of customers $\mathcal{J}$, and a set of demand scenarios $\mathcal{S}$, the probability π_s for each $s \in \mathcal{S}$, the opening cost F_i and the capacity Q_i for each facility $i \in \mathcal{I}$, the cost C_{ij} of allocating one unit of the demand of customer $j \in \mathcal{J}$ to facility $i \in \mathcal{I}$ and, finally, D_{js} the demand of customer $j \in \mathcal{J}$ under scenario $s \in \mathcal{S}$. Let binary variable x_i be equal 1 if facility $i \in \mathcal{I}$ is open, 0 otherwise, and let continuous variable y_{ijs} represent the amount of demand of customer $j \in \mathcal{J}$ allocated to facility $i \in \mathcal{I}$ under scenario $s \in \mathcal{S}$. The CFLPSD is thus:

$$\min \left\{ \sum_{i \in \mathcal{I}} F_i x_i + Q(x) \;\middle|\; x_i \in \{0,1\}, i \in \mathcal{I} \right\}. \tag{7a}$$

where $x = (x_i)_{i \in \mathcal{I}}$, $Q(x) = \sum_{s \in \mathcal{S}} \pi_s Q(x, s)$ and

$$Q(x,s) = \min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} C_{ij} y_{ijs} \tag{7b}$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{I}} y_{ijs} = D_{js}, \qquad j \in \mathcal{J}, \tag{7c}$$

$$y_{ijs} \leq Q_i x_i, \qquad i \in \mathcal{I}, j \in \mathcal{J}, \tag{7d}$$

$$\sum_{j \in \mathcal{J}} y_{ijs} \leq Q_i x_i, \qquad i \in \mathcal{I}, \tag{7e}$$

$$y_{ijs} \geq 0 \qquad i \in \mathcal{I}, j \in \mathcal{J}. \tag{7f}$$

The objective function in (7a) consists of the sum of opening costs and expected cost of fulfilling customer demands. First-stage decision are of a binary type, thus problem (7) is a two-stage stochastic program with integer first-stage. In the second-stage problem, objective function (7b) represents the cost of allocating demand to facilities given a scenario s and a first-stage decision x. Constraints (7c) ensure that the demand of each customer is completely satisfied, constraints (7d) ensure that demand is allocated to a facility only if the facility is open and, finally, constraints (7e) ensure that the capacity of each facility is respected. Constraints (7f) set the range for the second-stage variables. Furthermore, we add constaint $\sum_{i \in \mathcal{I}} Q_i x_i \geq \max_{s \in \mathcal{S}} \sum_{j \in \mathcal{J}} D_{js}$ which ensures second-stage feasibility and is known to be useful in a BD approach [see, e.g., Fischetti et al., 2016a]. Finally, note that constraints (7d) are redundant in this formulation, but they are know to yield tighter linear programming relaxation bounds [Wentges, 1996].

To the best of our knowledge, no available set of benchmark instances exists for the CFLPSD . For this reason we use the well-known instances of the Capacitated Facility Location Problem generated by Beasley [1988], also available in the OR-library, and extend them by including stochastic demand. The instances in Beasley [1988] are randomly generated and have been used by several authors including, e.g., Wentges [1996], Guastaroba and Speranza [2012] and Fischetti et al. [2016a]. Consistently with Fischetti et al. [2016a], we use the instances sets A, B and C as they generate large and difficult problems. Each set contains four instances, and all instances have 100 facilities and 1000 customers. In order to obtain stochastic demand we follow the procedure adopted by Laporte et al. [1994] for a facility location problem. Particularly, customer demands are assumed normally distributed with mean equal to the deterministic value of the demand in Beasley [1988]. Different instances are obtained by setting different numbers of scenarios, values of demand standard deviation (SD – indicated as a percentage of the mean demand) and percentages of strongly correlated customer demands (C – a 0.8 correlation is used), as summarized in Table 1. Particularly, the set of *test instances* is made as follows: for each instance in Beasley [1988] and for each $|\mathcal{S}| \in \{30, 60, 90\}$, there exists one instance for each $SD \in \{5, 10, 20\%\}$ and $C = 0\%$, and one instance for each $C \in \{20, 50, 80\%\}$ and $SD = 3\%$, for a total of 216 instances. The set of *training instances* contains, for each instance in Beasley [1988], one instances $|\mathcal{S}| = 5$, $SD = 3\%$ and $C = 0\%$, for a total of 12 instances. Scenarios are obtained by sampling realizations from the underlying normal distribution.

Table 1: Instances of the CFLPSD .

Instances	Type	$\lvert\mathcal{S}\rvert$	St. Deviation	Correlated Demands	# Instances
Training	Beasley [1988]'s A,B,C	5	3%	0	12
Test	Beasley [1988]'s A,B,C	30,60,90	3,5,10,20%	0, 20, 50, 80 %	216

5.2 Retaining Strategies for the CFLPSD

We use several of the general-purpose RSs introduced in Section 4 as well as additional problem-specific RSs. In problem (7) it is possible to identify two groups of constraints. The first group, (7c) - which we refer to as A - ensures satisfaction of the demand. The second group, (7e) - which we refer to as B - ensures the respect of the capacity of the facilities. The constraints in block A can be distinguished by their right-hand-side only, $h_s(A) = (D_{js})_{j \in \mathcal{J}}$, while the constraints in block B can be distinguished by the rows of the $T_s(B)$ matrix only, which corresponds to an $|\mathcal{I}| \times |\mathcal{I}|$ diagonal matrix where the elements on the diagonal are Q_i for $i \in \mathcal{I}$. For this reason, we adopt only RSs $C^h(A, n)$, $H^h(A, n)$, $L^h(A, n)$, $EX^h(A, n)$, $C^T(B, n)$, $H^T(B, n)$, $L^T(B, n)$, $EX^T(B, n)$, in addition to $S(A, n)$ and $S(B, n)$.

We also consider a number of problem-specific RSs. RS $C^F(B, n)$ consists of partitioning locations i into n clusters based on their opening costs. For each cluster, the point closest to the centroid is chosen as representative, and the constraint B for the corresponding locations are retained. RSs $H^F(B, n)$, $L^F(B, n)$, $EX^F(B, n)$ consist of selecting the constraints

B corresponding to the location with the highest, lowest, and extreme (highest and lowest) opening costs, respectively. Similarly, RS $C^C(A, n)$ consists of partitioning customers j into n clusters based on their allocation costs. For each cluster, the point closest to the centroid is chosen as representative, and the constraint A for the corresponding customers are retained. RSs $H^C(A, n)$, $L^C(A, n)$, $EX^C(A, n)$ consist of selecting the constraints A corresponding to the customers with the highest, lowest, and extreme (highest and lowest) allocation costs, respectively. Particularly, for each customer $j \in \mathcal{J}$ the allocation costs considered is the maximum among all locations, that is, $\max_{i\in\mathcal{I}}\{C_{ij}\}$.

Finally, we adopt a number of RSs which allow us to evaluate whether it is profitable to represent more than one group of constraints at a time. For example, RSs $EX^{h,F}(n)$ consists of the simultaneous adoption of RSs $EX^h(A, n)$ and $EX^F(B, n)$. A summary of all RSs can be found in Appendix A.

5.3 Training Phase

The scope of the training phase is that of identifying, by means of a set of small training instances, the RSs to use in the test phase. For the sake of our computational study, we perform an extensive training phase. All the training instances are solved with BD and GPBD (using all applicable RSs – see Section 5.2 – on all subproblems), with a 10 minute time limit. The RSs (if any) which provide better solutions than BD are then selected for the test phase. For the RSs used, values $n = 1$ and 2 are used. In fact, preliminary tests show that the performance of GPBD worsens with more constraints retained, providing a preliminary confirmation of the soundness of retaining only portions of subproblems.

Table 2 reports the results of the training phase. The statistics reported in Table 2 are the average gap obtained after reaching the time limit (Gap [%]), the percentage of instances solved to optimality (Sol %) and the average solution time in second (Time [s]). Consistently with Cplex statistics [IBM, 2015], the optimality gap is calculated as $100 * (\texttt{BestInteger} - \texttt{BestBound})/\texttt{BestBound}$, where `BestInteger` is the best objective value found with the specific RS and for the specific instance while `BestBound` is the best available bound for the corresponding instance. This calculation is consistent with recent work such as Guastaroba and Speranza [2012] and Fischetti et al. [2016a]. The results in Table 2 are shown with increasing total average gap.

A number of elements can be pointed out. First, there exist several RSs for which GPBD outperforms BD but also a number of RSs for which GPBD does not improve BD and a number of RSs for which GPBD performs worse than solving the extensive problem. Second, the RSs based on representing constraints block A are in general successful. All RSs applied to constraints block A solve the same number of instances solved by BD, and all except $H^h(A, 2)$ reduce (in some cases significantly) the average optimality gap. On the contrary, the RSs based on representing constraints block B lead to worse results than BD. It is worth noticing that constraints A are affected by uncertainty in the right-hand-side h, while constraints B are not affected by uncertainty. Third, when using RSs $EX^{h,T}(n)$, $EX^{h,F}(n)$, $EX^{C,T}(n)$ and $EX^{C,F}(n)$, GPBD is always outperformed by BD and in some cases also by Cplex. This is due to the fact that such RSs retain more constraints ($4n$) than simple RSs. This finding illustrates that in this case generating a heavy MP is detrimental to a partial decomposition strategy, and confirms the need of methods to calibrate the size of the MP in the spirit of GPBD. As a result of the training phase, all the RSs applied to constraints block A will be used in the test phase.

Table 2: Results for the training phase for the A, B and C testbed instances and 600 seconds time limit. The average optimality gap (Gap) over all instances, the percentage of problems solved to optimality (Sol) and the average solution time (Time) are reported. The results are sorted with increasing global average gap across all instances. The results for BD and Cplex are shaded.

	A			*B*			*C*			Total	
RS	Gap[%]	Sol[%]	Time[s]	Gap[%]	Sol[%]	Time[s]	Gap[%]	Sol[%]	Time[s]	Gap[%]	Time[s]
$C^h(A,2)$	2.344	25.0	546.78	2.485	0.0	602.80	3.030	0.0	603.11	2.620	584.23
$S(A,2)$	2.404	25.0	533.13	2.046	0.0	601.38	4.530	0.0	602.66	2.994	579.06
$C^h(A,1)$	3.201	25.0	563.30	3.387	0.0	603.06	3.113	0.0	602.42	3.234	589.59
$EX^C(A,2)$	3.000	25.0	563.22	2.822	0.0	603.01	3.940	0.0	603.66	3.254	589.96
$H^C(A,1)$	2.417	25.0	588.97	3.813	0.0	602.50	3.891	0.0	603.99	3.374	598.49
$EX^h(A,1)$	2.702	25.0	539.95	2.546	0.0	601.19	5.566	0.0	603.52	3.605	581.55
$C^C(A,2)$	3.172	25.0	601.74	3.875	0.0	602.87	4.106	0.0	602.17	3.718	602.26
$EX^h(A,2)$	2.841	25.0	553.05	4.841	0.0	601.87	3.601	0.0	600.64	3.761	585.19
$H^C(A,2)$	3.004	25.0	565.55	4.096	0.0	603.26	4.557	0.0	602.85	3.886	590.55
$C^C(A,1)$	3.013	25.0	593.36	3.871	0.0	602.48	4.975	0.0	601.81	3.953	599.22
$L^h(A,1)$	2.916	25.0	577.84	3.167	0.0	602.73	5.883	0.0	602.49	3.989	594.35
$L^h(A,2)$	2.779	25.0	544.11	3.462	0.0	602.76	5.755	0.0	601.19	3.999	582.68
$S(A,1)$	2.793	25.0	558.59	3.859	0.0	603.05	5.360	0.0	603.79	4.004	588.47
$H^h(A,1)$	3.146	25.0	557.45	3.465	0.0	602.23	5.494	0.0	604.32	4.035	588.00
$EX^C(A,1)$	2.740	25.0	576.17	5.418	0.0	601.32	4.228	0.0	601.80	4.129	593.10
$L^C(A,2)$	3.368	25.0	590.04	3.264	0.0	602.29	5.908	0.0	603.15	4.180	598.49
$L^C(A,1)$	2.883	25.0	571.22	4.373	0.0	604.04	5.343	0.0	601.60	4.200	592.29
BD	2.945	25.0	567.12	5.192	0.0	603.23	4.780	0.0	602.76	4.306	591.03
$H^h(A,2)$	2.831	25.0	535.06	3.269	0.0	602.38	7.332	0.0	603.58	4.477	580.34
$EX^{h,T}(1)$	3.263	0.0	602.63	4.134	0.0	602.22	6.656	0.0	605.50	4.685	603.45
$EX^{C,T}(1)$	3.097	25.0	600.95	5.124	0.0	601.16	7.565	0.0	602.24	5.262	601.45
$H^F(B,1)$	2.649	25.0	600.47	7.499	0.0	601.64	7.353	0.0	601.87	5.833	601.33
$C^F(B,1)$	3.429	0.0	600.59	7.158	0.0	601.63	7.036	0.0	602.52	5.874	601.58
$H^F(B,2)$	4.977	0.0	601.90	8.158	0.0	601.96	7.729	0.0	602.78	6.955	602.21
$EX^T(B,1)$	3.673	0.0	601.16	8.510	0.0	601.57	9.597	0.0	601.39	7.260	601.37
$H^T(B,1)$	3.673	0.0	600.75	8.510	0.0	601.62	9.597	0.0	602.02	7.260	601.46
$L^T(B,1)$	3.673	0.0	601.29	8.510	0.0	602.96	9.597	0.0	601.45	7.260	601.90
$C^F(B,2)$	3.411	25.0	602.31	9.179	0.0	601.26	9.290	0.0	602.68	7.293	602.08
$C^T(B,2)$	6.161	0.0	602.36	8.295	0.0	602.00	10.304	0.0	600.88	8.253	601.75
$L^T(B,2)$	12.677	0.0	601.84	10.411	0.0	602.66	10.850	0.0	601.64	11.313	602.05
$EX^T(B,2)$	12.677	0.0	601.08	10.411	0.0	604.39	10.850	0.0	601.83	11.313	602.44
$H^T(B,2)$	12.677	0.0	603.18	10.411	0.0	603.33	10.850	0.0	601.81	11.313	602.77
$S(B,1)$	22.591	0.0	602.77	8.508	0.0	600.87	10.949	0.0	602.55	14.016	602.06
Cplex	28.412	25.0	515.86	35.423	0.0	600.46	30.110	25.0	557.35	31.315	557.89
$EX^{C,T}(2)$	Inf	0.0	601.41	13.101	0.0	600.32	21.975	0.0	601.75	Inf	601.16
$EX^{h,T}(2)$	6.657	0.0	600.20	Inf	0.0	602.17	Inf	0.0	601.24	Inf	601.20
$C^T(B,1)$	3.709	0.0	601.21	Inf	0.0	605.23	5.722	0.0	602.09	Inf	602.84
$S(B,2)$	Inf	0.0	604.31	12.161	0.0	601.56	13.250	0.0	603.49	Inf	603.12
$EX^F(B,1)$	Inf	0.0	690.95	15.833	0.0	617.12	Inf	0.0	607.85	Inf	638.64
$EX^{h,F}(1)$	Inf	0.0	723.30	Inf	0.0	604.19	Inf	0.0	602.98	Inf	643.49
$EX^{C,F}(1)$	Inf	0.0	721.16	Inf	0.0	603.75	Inf	0.0	608.22	Inf	644.38
$L^F(B,1)$	Inf	0.0	754.83	Inf	0.0	605.01	Inf	0.0	609.41	Inf	656.42
$EX^{C,F}(2)$	Inf	0.0	781.99	Inf	0.0	651.59	Inf	0.0	610.76	Inf	681.45
$EX^F(B,2)$	Inf	0.0	676.85	Inf	0.0	717.47	Inf	0.0	708.52	Inf	700.95
$EX^{h,F}(2)$	Inf	0.0	703.79	Inf	0.0	708.49	Inf	0.0	707.81	Inf	706.70
$L^F(B,2)$	Inf	0.0	729.67	Inf	0.0	756.15	Inf	0.0	731.45	Inf	739.09

5.4 Test Phase

The scope the test phase is to asses whether i) GPBD yields any advantage compared to BD on large-scale problems, ii) the performance of GPBD is stable across heterogeneous instances, and iii) it is possible to rely on the training phase for selecting RSs. The test phase consists of solving the test instances by means of BD and GPBD using the RSs selected in the training phase, namely $H^h(A,n)$, $L^h(A,n)$, $EX^h(A,n)$, $C^h(A,n)$, $H^C(A,n)$, $L^C(A,n)$, $EX^C(A,n)$, $C^C(A,n)$ and $S(A,n)$, with $n = 1, 2$. The RSs are applied to an increasing percentage (REP) of representative scenario subproblems, namely 20, 50, 80 and 100%. The subproblems are selected using the *representation strategy* described by Crainic et al. [2014]. All instances are solved with a 2400 second time limit.

Table 3, Table 4 and Table 5 report the results with different correlations settings for the test instances with $|\mathcal{S}| = 30$, 60 and 90, respectively. Similarly, Table 6, Table 7 and Table 8 report the results with different standard deviation settings for the test instances with $|\mathcal{S}| = 30$, 60 and 90 scenarios, respectively. The statistics reported are the average gap (Gap [%]) with respect to the best known bound (see Section 5.3), the smallest and highest optimality gaps (Min[%] and Max[%], respectively), and the percentage of instances solved to optimality (Sol[%]).

Tables 3 to 8 illustrate that GPBD has significant potential to improve BD and that, when the least number of subproblems are represented (i.e., REP=20%), GPBD outperforms BD in the great majority of the cases. In several cases a the optimality gap reduction is significantly high. Furthermore, the gap reduction appears even more substantial as the size of the instances increases. As an example, in Table 5 RS $S(A,1)$ with REP $= 20\%$, reduces the average BD optimality gap by almost ten percentage points for all correlations settings. RS $S(A,1)$ is highly effective also on the instances with $|\mathcal{S}| = 90$ and different standard deviations (see Table 8). Remarkably, RS $S(A,1)$ simply amounts to adding a random constraint from block A to MP, illustrating that also very simple RSs can generate significant improvements. In Table 8 it is also possible to notice that RSs $L^h(A,n)$ nearly halves the optimality gap of BD for the instances with $SD = 5\%$. Its performances are confirmed also on the largest instances with different correlations settings (Table 5). Finally, the highest optimality gap (Max[%]) is often much smaller than for BD, particularly on the largest instances. For the instances with $|\mathcal{S}| = 30$ and 60, the optimality gap is also consistently smaller than for BD.

The results show also that, in general, the performances of GPBD worsen as the number of scenarios represented (REP) increases, particularly on the largest instances. On the one hand, this illustrates that increasing the size of MP excessively is detrimental, on the other hand, it certifies the need for tailoring the decomposition strategy to the specific problem in the spirit of GPBD. Finally, we can address the first scope of the computational study by concluding that GPBD significantly improves BD with all RSs considered, and particularly when a few scenario subproblems are represented. Remarkably, a very simple RS consisting of retaining a random constraint, provides a substantial reduction of the optimality gap, particularly on the largest instances.

To address the second question of the computational study, Tables 3 to 8 illustrate that the performance of GPBD is stable across all instances tested when the number of subproblems is represented is small. Consider the cases with REP $\leq 50\%$. It can be noticed that GPBD outperforms BD on the great majority of the instances independently on the correlations and standard deviation settings.

Finally, it emerges that training the GPBD on small instances is a suitable way for identifying the most effective RSs. All the RSs selected in the training phase outperform BD across a large set of instances with different number of scenarios, standard deviations and correlations. Furthermore, the training instances were solved with a time limit of 600 seconds. This shows that a suitable way for identifying efficient RSs consists of solving a small-scale instance (possibly not to optimality) with a few well-thought RSs and selecting the RSs which yield the best results. Our computational experience shows that the RSs which perform well on small-scale instances maintain their performances also on large-scale instances.

Table 3: Minimum (Min), average (Gap) and maximum (Max) optimality gaps and percentage of instances solved (Sol) across the A, B and C instances with $|\mathcal{S}| = 30$ and 3% standard deviation of the demand, for different percentages C of customer demands strongly correlated (0.8) and subproblems represented REP. Shaded cells correspond to average optimality gaps lower than for BD.

		C=20%				C=50%				C=80%			
RS	REP	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]
BD	0	0.000	10.434	20.149	8.3	3.524	11.892	18.229	0.0	0.000	11.142	16.946	8.3
$H^h(A,1)$	20	0.000	10.648	21.055	8.3	0.000	9.514	15.587	8.3	0.000	9.457	17.270	8.3
$H^h(A,2)$	20	0.000	9.596	21.047	8.3	0.000	9.071	18.258	8.3	0.000	8.291	13.294	8.3
$L^h(A,1)$	20	0.000	10.140	21.886	8.3	0.000	9.041	16.999	8.3	0.000	9.667	16.942	8.3
$L^h(A,2)$	20	0.000	9.837	22.174	8.3	0.418	9.091	17.087	0.0	0.000	9.567	17.370	8.3
$H^C(A,1)$	20	0.000	10.772	21.911	8.3	1.906	10.840	15.686	0.0	0.000	10.272	17.098	8.3
$H^C(A,2)$	20	3.310	11.093	17.128	0.0	3.771	9.705	15.885	0.0	0.000	9.927	15.984	8.3
$S(A,1)$	20	0.000	9.556	19.137	8.3	0.000	10.039	15.154	8.3	0.000	8.952	14.171	8.3
$S(A,2)$	20	0.000	9.739	23.438	8.3	0.000	8.116	13.147	8.3	0.000	7.990	13.051	8.3
$EX^h(A,1)$	20	0.000	10.185	22.455	8.3	0.000	9.680	17.235	8.3	0.000	9.371	15.157	8.3
$EX^h(A,2)$	20	0.000	8.769	16.996	8.3	0.000	7.790	13.080	8.3	0.000	9.260	15.920	8.3
$C^h(A,1)$	20	0.000	8.948	17.795	8.3	0.000	9.059	15.079	8.3	1.753	9.060	14.417	0.0
$C^h(A,2)$	20	0.000	8.874	15.851	8.3	0.000	9.597	16.663	8.3	0.000	8.799	15.291	8.3
$EX^C(A,1)$	20	0.681	10.631	18.229	0.0	0.000	10.768	22.259	8.3	4.159	10.551	18.690	0.0
$EX^C(A,2)$	20	0.681	9.942	16.277	0.0	2.592	9.526	13.840	0.0	0.000	9.024	14.756	8.3
$C^C(A,1)$	20	0.000	11.648	20.869	8.3	3.862	10.799	15.843	0.0	0.683	10.583	16.702	0.0
$C^C(A,2)$	20	2.791	10.705	16.341	0.0	2.072	10.290	14.598	0.0	3.513	10.917	15.956	0.0
$L^C(A,1)$	20	0.000	10.591	20.064	8.3	0.682	9.387	15.357	0.0	1.603	10.090	14.699	0.0
$L^C(A,2)$	20	2.187	10.320	20.739	0.0	0.000	9.341	16.499	8.3	0.000	9.388	15.122	8.3
$H^h(A,1)$	50	0.000	10.187	20.991	8.3	0.000	8.379	12.892	8.3	0.000	8.492	14.170	8.3
$H^h(A,2)$	50	0.000	10.151	19.484	8.3	0.000	8.403	13.451	8.3	0.000	8.188	13.523	8.3
$L^h(A,1)$	50	0.000	9.496	20.297	8.3	0.000	9.491	14.775	8.3	0.000	8.896	15.101	8.3
$L^h(A,2)$	50	0.000	9.448	17.962	8.3	0.000	8.719	15.714	8.3	0.000	9.085	14.687	8.3
$H^C(A,1)$	50	4.508	10.418	16.644	0.0	0.000	10.572	16.884	8.3	0.000	10.480	18.707	8.3
$H^C(A,2)$	50	5.179	11.822	17.638	0.0	0.000	10.844	20.080	8.3	0.000	9.947	18.056	8.3
$S(A,1)$	50	0.000	8.644	16.014	8.3	0.000	9.779	18.581	8.3	0.000	9.491	15.235	8.3
$S(A,2)$	50	0.000	9.217	15.129	8.3	1.079	8.698	14.628	0.0	0.000	8.682	18.419	8.3
$EX^h(A,1)$	50	0.000	10.111	19.474	8.3	0.000	8.907	14.570	8.3	0.000	9.106	14.597	8.3
$EX^h(A,2)$	50	0.000	10.098	19.250	8.3	0.000	10.010	16.160	8.3	0.421	9.456	16.648	0.0
$C^h(A,1)$	50	0.000	9.492	17.997	8.3	0.000	9.615	17.817	8.3	0.000	8.244	14.393	8.3
$C^h(A,2)$	50	0.000	9.358	18.230	8.3	0.000	9.207	15.396	8.3	0.000	8.835	13.943	8.3
$EX^C(A,1)$	50	4.199	11.166	21.222	0.0	0.000	9.343	13.926	8.3	6.375	11.328	20.806	0.0
$EX^C(A,2)$	50	0.000	9.174	15.631	8.3	0.000	8.735	13.010	8.3	0.421	8.937	14.423	0.0
$C^C(A,1)$	50	0.000	10.865	21.627	8.3	0.418	10.511	15.346	0.0	0.000	9.898	15.757	8.3
$C^C(A,2)$	50	1.963	10.142	16.008	0.0	3.148	10.383	16.266	0.0	4.134	9.941	15.452	0.0
$L^C(A,1)$	50	2.791	10.727	17.921	0.0	1.748	9.747	16.430	0.0	0.000	8.969	15.097	8.3
$L^C(A,2)$	50	0.412	10.755	22.665	0.0	0.000	9.255	15.918	8.3	0.000	9.316	14.449	8.3
$H^h(A,1)$	80	0.000	10.435	21.078	8.3	0.000	8.217	13.819	8.3	0.000	9.418	14.349	8.3
$H^h(A,2)$	80	0.000	9.546	17.157	8.3	0.418	10.013	20.512	0.0	0.000	8.902	14.776	8.3
$L^h(A,1)$	80	0.000	10.689	22.346	8.3	0.000	9.292	14.501	8.3	0.000	9.913	17.418	8.3
$L^h(A,2)$	80	0.000	10.033	18.252	8.3	0.000	8.763	14.290	8.3	0.000	9.407	16.034	8.3
$H^C(A,1)$	80	0.681	10.222	19.489	0.0	3.353	10.859	17.898	0.0	0.000	10.761	16.364	8.3
$H^C(A,2)$	80	0.000	10.462	20.816	8.3	3.385	10.404	16.807	0.0	0.421	10.837	21.177	0.0
$S(A,1)$	80	0.000	9.096	16.570	8.3	0.000	7.998	12.933	8.3	0.000	7.848	12.422	8.3
$S(A,2)$	80	0.000	9.865	15.197	8.3	0.000	8.834	13.286	8.3	0.000	9.684	20.530	8.3
$EX^h(A,1)$	80	0.000	9.175	15.572	8.3	0.000	9.109	16.441	8.3	0.000	9.090	15.583	8.3
$EX^h(A,2)$	80	1.084	12.392	29.358	0.0	1.079	11.165	20.833	0.0	0.000	13.100	21.809	8.3
$C^h(A,1)$	80	0.000	8.992	15.856	8.3	0.000	8.312	13.998	8.3	0.000	8.756	13.819	8.3
$C^h(A,2)$	80	0.000	9.886	19.002	8.3	0.000	9.634	15.822	8.3	0.000	9.960	17.697	8.3
$EX^C(A,1)$	80	1.971	11.463	19.239	0.0	0.000	9.748	17.382	8.3	1.856	10.852	17.080	0.0
$EX^C(A,2)$	80	1.864	11.392	18.769	0.0	0.000	8.834	16.254	8.3	0.000	9.484	18.116	8.3
$C^C(A,1)$	80	0.000	10.671	20.869	8.3	0.000	10.539	15.826	8.3	0.683	10.339	16.034	0.0
$C^C(A,2)$	80	2.763	10.645	17.548	0.0	0.000	11.026	17.577	8.3	0.000	10.594	18.043	8.3
$L^C(A,1)$	80	0.000	10.528	20.869	8.3	0.000	10.628	17.073	8.3	0.000	10.506	16.313	8.3
$L^C(A,2)$	80	0.000	10.429	20.149	8.3	0.682	9.599	14.465	0.0	0.421	9.183	13.963	0.0
$H^h(A,1)$	100	0.000	9.574	20.488	8.3	0.000	9.150	13.697	8.3	0.000	9.054	14.905	8.3
$H^h(A,2)$	100	3.981	11.296	18.056	0.0	0.000	9.915	17.882	8.3	0.000	9.781	16.357	8.3
$L^h(A,1)$	100	0.000	10.854	20.589	8.3	0.000	9.720	16.080	8.3	0.000	9.823	16.492	8.3
$L^h(A,2)$	100	0.000	9.536	15.271	8.3	0.000	8.451	12.834	8.3	1.075	9.580	15.777	0.0
$H^C(A,1)$	100	0.000	11.007	20.334	8.3	2.642	10.879	19.195	0.0	0.421	11.098	20.641	0.0

$H^C(A,2)$	100	0.000	9.301	14.800	8.3	0.418	10.575	20.497	0.0	0.000	9.315	16.218	8.3
$S(A,1)$	100	0.000	8.901	13.420	8.3	0.000	8.875	14.239	8.3	0.000	8.547	13.662	8.3
$S(A,2)$	100	0.000	9.433	16.482	8.3	0.418	10.129	17.072	0.0	0.000	10.080	17.471	8.3
$EX^h(A,1)$	100	0.000	9.931	18.786	8.3	0.000	10.200	19.086	8.3	0.000	10.288	16.664	8.3
$EX^h(A,2)$	100	0.681	13.359	23.808	0.0	0.418	11.946	24.966	0.0	0.000	13.480	27.485	8.3
$C^h(A,1)$	100	0.000	9.047	16.227	8.3	0.000	8.966	15.417	8.3	0.000	10.311	18.899	8.3
$C^h(A,2)$	100	0.000	11.269	18.408	8.3	0.000	10.082	17.846	8.3	0.000	10.060	15.662	8.3
$EX^C(A,1)$	100	0.412	9.806	18.105	0.0	0.000	8.785	19.135	8.3	0.000	9.316	17.755	8.3
$EX^C(A,2)$	100	1.666	11.171	21.604	0.0	1.667	12.171	27.151	0.0	0.000	11.197	19.744	8.3
$C^C(A,1)$	100	0.000	10.285	17.969	8.3	1.972	9.554	16.060	0.0	3.413	10.098	15.596	0.0
$C^C(A,2)$	100	0.412	9.739	16.730	0.0	0.418	9.706	16.153	0.0	0.000	9.626	14.077	8.3
$L^C(A,1)$	100	2.412	11.286	18.147	0.0	0.418	10.620	16.627	0.0	0.000	10.542	16.609	8.3
$L^C(A,2)$	100	0.000	9.486	15.324	8.3	0.000	8.675	14.969	8.3	1.075	9.140	13.931	0.0

Table 4: Minimum (Min), average (Gap) and maximum (Max) optimality gaps and percentage of instances solved (Sol) across the A, B and C instances with $|\mathcal{S}| = 60$ and 3% standard deviation of the demand for different percentages C of customer demands strongly correlated (0.8) and subproblems represented REP. Shaded cells correspond to average optimality gaps lower than for BD.

		C=20%				C=50%				C=80%			
RS	REP	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]
BD	0	7.748	18.896	33.068	0.0	5.040	18.739	30.347	0.0	9.735	17.756	28.054	0.0
$H^h(A,1)$	20	3.047	13.695	24.538	0.0	4.128	12.738	18.963	0.0	3.260	12.744	19.532	0.0
$H^h(A,2)$	20	5.532	12.631	21.992	0.0	7.489	12.471	19.905	0.0	3.260	12.777	22.693	0.0
$L^h(A,1)$	20	3.472	13.726	20.167	0.0	4.128	13.722	22.725	0.0	6.561	13.578	21.296	0.0
$L^h(A,2)$	20	5.829	13.096	20.656	0.0	4.128	12.780	21.939	0.0	3.520	12.278	22.297	0.0
$H^C(A,1)$	20	13.973	17.090	21.351	0.0	9.457	16.707	22.126	0.0	8.512	15.874	21.944	0.0
$H^C(A,2)$	20	10.912	16.448	22.676	0.0	13.304	17.741	28.063	0.0	11.025	15.124	19.247	0.0
$S(A,1)$	20	3.047	12.844	21.626	0.0	6.637	12.705	19.390	0.0	2.818	11.926	18.494	0.0
$S(A,2)$	20	3.472	13.568	23.104	0.0	5.253	13.089	22.790	0.0	5.902	12.063	18.876	0.0
$EX^h(A,1)$	20	5.532	13.006	19.531	0.0	4.562	12.606	19.816	0.0	3.920	13.569	30.042	0.0
$EX^h(A,2)$	20	6.712	15.924	25.271	0.0	5.802	14.784	23.662	0.0	4.787	15.327	24.827	0.0
$C^h(A,1)$	20	3.047	14.023	25.526	0.0	4.128	12.843	20.580	0.0	2.818	13.760	21.253	0.0
$C^h(A,2)$	20	4.969	12.660	20.382	0.0	4.128	12.868	21.581	0.0	3.260	11.975	18.170	0.0
$EX^C(A,1)$	20	3.749	15.690	22.614	0.0	11.128	16.994	24.401	0.0	8.070	14.348	21.944	0.0
$EX^C(A,2)$	20	7.003	15.300	23.720	0.0	7.396	13.948	22.431	0.0	7.681	13.268	20.694	0.0
$C^C(A,1)$	20	10.156	15.927	20.976	0.0	5.864	15.485	21.504	0.0	8.000	15.048	20.964	0.0
$C^C(A,2)$	20	9.307	16.925	25.873	0.0	9.077	16.600	21.700	0.0	12.054	16.910	21.432	0.0
$L^C(A,1)$	20	9.747	17.392	22.739	0.0	11.150	17.587	22.505	0.0	7.976	15.329	21.944	0.0
$L^C(A,2)$	20	9.247	15.749	22.961	0.0	10.890	14.697	19.569	0.0	10.749	15.211	21.905	0.0
$H^h(A,1)$	50	4.966	12.502	17.344	0.0	6.706	11.830	16.759	0.0	3.260	12.746	20.772	0.0
$H^h(A,2)$	50	5.925	18.217	38.249	0.0	4.562	15.529	27.014	0.0	3.920	17.911	44.901	0.0
$L^h(A,1)$	50	3.047	12.896	22.283	0.0	4.562	12.678	21.812	0.0	6.092	13.480	22.135	0.0
$L^h(A,2)$	50	4.969	13.576	19.778	0.0	5.253	12.470	20.369	0.0	3.260	13.796	23.158	0.0
$H^C(A,1)$	50	5.182	16.406	26.158	0.0	4.562	16.097	22.126	0.0	10.119	16.201	21.944	0.0
$H^C(A,2)$	50	8.946	14.180	19.607	0.0	7.136	13.321	19.895	0.0	7.657	13.967	18.964	0.0
$S(A,1)$	50	5.301	13.004	20.695	0.0	5.802	11.906	17.443	0.0	2.818	12.292	18.869	0.0
$S(A,2)$	50	6.782	17.862	37.112	0.0	8.915	16.911	26.595	0.0	8.599	16.634	27.375	0.0
$EX^h(A,1)$	50	3.047	16.097	30.677	0.0	9.235	17.610	30.755	0.0	3.260	14.099	24.386	0.0
$EX^h(A,2)$	50	6.762	23.818	43.031	0.0	7.978	19.654	31.826	0.0	7.375	20.003	33.476	0.0
$C^h(A,1)$	50	6.782	12.873	18.487	0.0	5.802	13.103	20.801	0.0	3.260	13.237	19.262	0.0
$C^h(A,2)$	50	9.381	18.335	30.254	0.0	8.318	16.886	26.944	0.0	7.775	16.108	32.294	0.0
$EX^C(A,1)$	50	5.182	15.467	27.877	0.0	4.562	13.639	21.703	0.0	7.502	13.727	21.364	0.0
$EX^C(A,2)$	50	5.925	21.206	53.398	0.0	6.064	24.302	44.563	0.0	9.898	23.458	43.730	0.0
$C^C(A,1)$	50	11.735	17.398	22.532	0.0	8.724	15.773	23.606	0.0	6.688	15.420	19.995	0.0
$C^C(A,2)$	50	12.169	16.137	21.662	0.0	11.308	16.099	23.476	0.0	3.260	13.988	19.861	0.0
$L^C(A,1)$	50	3.047	14.731	22.779	0.0	9.754	16.861	24.699	0.0	6.690	15.418	21.441	0.0
$L^C(A,2)$	50	5.079	13.639	19.053	0.0	6.827	13.355	18.558	0.0	3.260	12.886	18.425	0.0
$H^h(A,1)$	80	3.047	14.840	25.071	0.0	6.827	15.282	24.093	0.0	3.920	15.300	26.703	0.0
$H^h(A,2)$	80	5.182	24.300	50.379	0.0	9.189	21.586	36.492	0.0	2.818	18.874	32.649	0.0
$L^h(A,1)$	80	7.261	14.452	23.138	0.0	8.130	13.731	21.330	0.0	4.948	12.947	20.794	0.0
$L^h(A,2)$	80	7.785	15.156	22.676	0.0	5.253	16.551	32.188	0.0	4.463	15.771	28.095	0.0
$H^C(A,1)$	80	7.958	15.624	26.158	0.0	5.253	18.441	45.408	0.0	4.751	18.103	44.636	0.0
$H^C(A,2)$	80	3.047	14.326	21.622	0.0	8.348	14.967	22.126	0.0	8.006	15.510	25.151	0.0
$S(A,1)$	80	3.472	17.139	28.586	0.0	4.562	16.201	27.020	0.0	7.087	15.494	24.013	0.0
$S(A,2)$	80	9.963	21.953	38.281	0.0	6.113	21.755	38.416	0.0	6.417	20.941	35.777	0.0
$EX^h(A,1)$	80	8.948	20.622	33.539	0.0	4.562	18.658	37.287	0.0	7.243	21.868	35.801	0.0
$EX^h(A,2)$	80	5.730	22.017	37.511	0.0	8.939	22.082	31.300	0.0	9.736	26.382	45.439	0.0
$C^h(A,1)$	80	3.472	14.770	31.090	0.0	6.074	16.697	29.021	0.0	3.520	15.780	32.253	0.0
$C^h(A,2)$	80	13.688	22.774	32.664	0.0	7.879	21.579	30.935	0.0	9.147	22.140	41.845	0.0
$EX^C(A,1)$	80	6.867	16.488	26.548	0.0	4.128	15.595	23.926	0.0	7.994	16.100	27.242	0.0
$EX^C(A,2)$	80	5.532	23.519	50.621	0.0	8.573	23.769	43.507	0.0	9.504	23.611	38.979	0.0
$C^C(A,1)$	80	7.860	17.014	23.523	0.0	13.832	16.981	20.734	0.0	8.830	16.363	22.602	0.0
$C^C(A,2)$	80	8.579	19.289	58.169	0.0	6.074	17.442	35.432	0.0	2.818	16.092	30.647	0.0

$L^C(A,1)$	80	10.426	18.444	30.063	0.0	8.186	16.281	22.505	0.0	7.994	15.824	21.253	0.0
$L^C(A,2)$	80	6.882	13.990	22.298	0.0	4.128	15.064	38.283	0.0	2.818	13.595	21.052	0.0
$H^h(A,1)$	100	6.143	19.209	34.886	0.0	4.434	17.034	36.748	0.0	4.110	17.491	30.328	0.0
$H^h(A,2)$	100	11.869	27.279	41.839	0.0	5.124	23.590	40.407	0.0	10.857	23.987	49.855	0.0
$L^h(A,1)$	100	3.989	14.556	23.726	0.0	3.318	13.557	23.959	0.0	4.373	13.742	19.832	0.0
$L^h(A,2)$	100	9.038	24.144	38.328	0.0	11.629	24.500	44.297	0.0	5.324	21.157	35.832	0.0
$H^C(A,1)$	100	8.431	16.950	25.922	0.0	8.893	16.532	24.890	0.0	7.809	16.129	24.007	0.0
$H^C(A,2)$	100	11.135	25.046	43.655	0.0	5.040	19.936	47.245	0.0	10.683	Inf	Inf	0.0
$S(A,1)$	100	5.925	18.597	40.516	0.0	6.030	19.695	34.556	0.0	7.234	20.731	38.669	0.0
$S(A,2)$	100	7.641	24.375	47.608	0.0	12.776	25.242	42.090	0.0	7.696	21.337	36.038	0.0
$EX^h(A,1)$	100	5.664	23.881	43.041	0.0	3.318	23.961	42.520	0.0	5.396	21.593	35.597	0.0
$EX^h(A,2)$	100	8.603	23.308	43.733	0.0	10.684	30.213	70.015	0.0	4.373	24.427	37.704	0.0
$C^h(A,1)$	100	4.418	17.845	31.634	0.0	3.749	16.673	28.829	0.0	3.665	17.213	32.765	0.0
$C^h(A,2)$	100	4.697	23.892	44.926	0.0	6.174	24.672	55.305	0.0	12.909	22.897	32.947	0.0
$EX^C(A,1)$	100	5.009	18.761	28.883	0.0	6.074	20.386	36.262	0.0	2.818	21.520	39.308	0.0
$EX^C(A,2)$	100	7.871	20.713	36.012	0.0	8.820	22.217	42.827	0.0	9.257	21.988	34.954	0.0
$C^C(A,1)$	100	11.628	16.311	24.318	0.0	4.562	14.248	20.734	0.0	6.833	13.948	19.332	0.0
$C^C(A,2)$	100	8.986	22.806	48.982	0.0	5.948	22.710	66.024	0.0	6.309	23.554	47.268	0.0
$L^C(A,1)$	100	5.072	14.903	23.179	0.0	10.213	15.065	21.435	0.0	8.862	14.706	21.961	0.0
$L^C(A,2)$	100	10.918	25.734	55.465	0.0	8.558	21.813	37.054	0.0	6.290	20.426	40.511	0.0

Table 5: Minimum (Min), average (Gap) and maximum (Max) optimality gaps and percentage of instances solved (Sol) across the A, B and C instances with $|\mathcal{S}| = 90$ and 3% standard deviation of the demand for different percentages C of customer demands strongly correlated (0.8) and subproblems represented REP. Shaded cells correspond to average optimality gaps lower than for BD.

		C=20%				C=50%				C=80%			
RS	REP	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]
BD	0	14.767	24.616	43.577	0.0	13.571	24.306	39.373	0.0	17.706	25.068	34.859	0.0
$H^h(A,1)$	20	7.472	17.037	27.909	0.0	8.448	16.405	22.849	0.0	7.729	15.284	22.875	0.0
$H^h(A,2)$	20	6.648	16.324	24.640	0.0	6.815	16.135	28.325	0.0	8.573	17.142	30.626	0.0
$L^h(A,1)$	20	6.223	15.755	26.115	0.0	5.656	16.375	27.111	0.0	7.729	15.308	24.799	0.0
$L^h(A,2)$	20	9.011	16.232	27.202	0.0	5.656	15.619	25.403	0.0	6.487	15.084	22.583	0.0
$H^C(A,1)$	20	13.438	19.768	31.713	0.0	15.246	22.007	30.604	0.0	17.689	20.715	26.958	0.0
$H^C(A,2)$	20	8.933	18.841	26.003	0.0	18.066	21.431	29.483	0.0	15.707	20.213	34.573	0.0
$S(A,1)$	20	7.449	15.982	25.063	0.0	6.478	14.994	24.678	0.0	6.032	15.457	25.587	0.0
$S(A,2)$	20	7.812	17.031	30.924	0.0	4.961	18.167	28.884	0.0	6.487	16.859	29.664	0.0
$EX^h(A,1)$	20	7.513	17.730	26.291	0.0	4.961	15.672	25.949	0.0	9.545	16.366	22.875	0.0
$EX^h(A,2)$	20	5.939	24.613	36.128	0.0	9.511	21.117	32.327	0.0	7.997	19.488	36.733	0.0
$C^h(A,1)$	20	7.472	17.031	27.308	0.0	8.723	15.568	25.224	0.0	8.229	15.036	21.625	0.0
$C^h(A,2)$	20	7.449	17.463	27.116	0.0	4.527	16.188	25.004	0.0	7.170	15.027	22.309	0.0
$EX^C(A,1)$	20	16.190	20.391	24.869	0.0	13.843	19.510	25.390	0.0	15.023	20.371	29.464	0.0
$EX^C(A,2)$	20	7.983	16.635	34.351	0.0	13.185	17.858	23.705	0.0	11.257	15.813	19.009	0.0
$C^C(A,1)$	20	14.981	22.046	35.213	0.0	17.614	21.266	31.068	0.0	15.865	20.489	25.203	0.0
$C^C(A,2)$	20	14.981	21.228	30.613	0.0	15.428	20.556	25.525	0.0	13.526	18.969	24.456	0.0
$L^C(A,1)$	20	15.304	20.494	27.308	0.0	11.692	19.106	24.497	0.0	15.366	21.428	27.023	0.0
$L^C(A,2)$	20	14.475	19.941	25.583	0.0	15.428	18.882	22.738	0.0	14.211	17.833	24.089	0.0
$H^h(A,1)$	50	5.504	17.566	29.169	0.0	4.961	16.642	36.377	0.0	8.521	16.803	27.710	0.0
$H^h(A,2)$	50	14.069	27.550	47.737	0.0	13.283	31.588	59.303	0.0	10.923	30.670	59.736	0.0
$L^h(A,1)$	50	5.939	15.974	26.115	0.0	9.336	16.466	27.301	0.0	9.007	15.930	24.227	0.0
$L^h(A,2)$	50	9.623	18.432	31.292	0.0	6.207	20.265	38.515	0.0	13.112	22.026	38.398	0.0
$H^C(A,1)$	50	14.624	21.001	31.713	0.0	11.022	20.831	45.580	0.0	13.444	21.496	31.641	0.0
$H^C(A,2)$	50	12.234	17.782	23.516	0.0	8.912	17.790	23.982	0.0	12.704	19.284	28.722	0.0
$S(A,1)$	50	5.504	16.791	27.651	0.0	9.004	18.485	29.747	0.0	9.173	18.573	36.416	0.0
$S(A,2)$	50	13.497	30.923	63.339	0.0	5.656	30.496	57.717	0.0	10.895	26.467	70.841	0.0
$EX^h(A,1)$	50	11.411	30.135	52.344	0.0	12.264	33.116	105.081	0.0	10.791	29.955	57.253	0.0
$EX^h(A,2)$	50	8.420	29.590	46.670	0.0	8.829	Inf	Inf	0.0	13.642	28.895	52.523	0.0
$C^h(A,1)$	50	5.504	19.627	30.083	0.0	4.527	17.468	26.320	0.0	8.263	16.152	27.806	0.0
$C^h(A,2)$	50	14.805	31.207	61.663	0.0	13.163	32.650	64.024	0.0	11.371	29.309	53.032	0.0
$EX^C(A,1)$	50	9.228	18.179	26.527	0.0	6.470	17.569	25.911	0.0	12.418	19.180	28.237	0.0
$EX^C(A,2)$	50	14.752	29.107	48.619	0.0	15.024	30.804	52.546	0.0	15.306	26.640	43.935	0.0
$C^C(A,1)$	50	18.203	22.390	32.451	0.0	13.683	20.424	29.722	0.0	11.493	18.773	25.445	0.0
$C^C(A,2)$	50	10.911	22.569	54.078	0.0	10.049	21.470	54.497	0.0	8.917	21.115	49.962	0.0
$L^C(A,1)$	50	15.274	21.024	27.308	0.0	9.765	20.115	28.318	0.0	14.238	20.319	29.464	0.0
$L^C(A,2)$	50	11.968	18.611	27.691	0.0	11.219	17.940	24.036	0.0	12.418	16.938	21.537	0.0
$H^h(A,1)$	80	9.880	26.434	60.787	0.0	8.561	25.096	56.141	0.0	9.893	21.065	43.112	0.0
$H^h(A,2)$	80	15.714	Inf	Inf	0.0	13.604	29.784	53.266	0.0	11.279	32.971	54.605	0.0
$L^h(A,1)$	80	10.194	16.520	24.102	0.0	8.385	17.675	26.440	0.0	11.767	17.369	23.609	0.0
$L^h(A,2)$	80	15.003	29.115	52.273	0.0	4.527	26.585	48.235	0.0	13.400	26.973	49.322	0.0
$H^C(A,1)$	80	12.025	20.228	29.450	0.0	9.274	18.476	25.280	0.0	9.773	18.061	31.079	0.0
$H^C(A,2)$	80	14.308	28.569	42.781	0.0	13.068	33.645	59.705	0.0	13.093	27.662	60.886	0.0
$S(A,1)$	80	11.644	27.444	55.898	0.0	10.203	25.255	50.808	0.0	6.032	26.414	53.776	0.0
$S(A,2)$	80	13.247	30.176	55.455	0.0	4.961	27.392	51.874	0.0	19.406	31.013	49.952	0.0

$EX^h(A,1)$	80	11.275	30.311	61.988	0.0	6.518	24.443	38.979	0.0	13.486	27.634	45.755	0.0
$EX^h(A,2)$	80	10.318	Inf	Inf	0.0	18.345	32.807	59.472	0.0	15.234	30.432	54.523	0.0
$C^h(A,1)$	80	8.450	22.489	38.146	0.0	6.478	22.135	40.432	0.0	9.744	22.084	37.540	0.0
$C^h(A,2)$	80	13.636	28.943	42.937	0.0	10.504	27.109	45.028	0.0	10.441	25.144	40.436	0.0
$EX^C(A,1)$	80	13.311	28.361	49.728	0.0	8.393	28.334	61.913	0.0	13.296	26.232	44.366	0.0
$EX^C(A,2)$	80	12.952	30.618	48.076	0.0	15.127	Inf	Inf	0.0	14.871	29.194	55.411	0.0
$C^C(A,1)$	80	14.040	19.646	31.415	0.0	12.086	18.753	31.966	0.0	11.647	17.931	25.111	0.0
$C^C(A,2)$	80	11.229	30.597	50.226	0.0	15.778	32.665	67.409	0.0	6.487	27.913	47.159	0.0
$L^C(A,1)$	80	12.037	18.773	26.555	0.0	13.077	18.985	32.003	0.0	13.759	18.642	26.941	0.0
$L^C(A,2)$	80	10.829	30.945	55.186	0.0	11.236	27.994	45.904	0.0	15.213	28.703	46.303	0.0
$H^h(A,1)$	100	14.944	30.940	63.047	0.0	12.987	Inf	Inf	0.0	12.042	30.346	57.150	0.0
$H^h(A,2)$	100	10.135	Inf	Inf	0.0	23.207	37.373	56.777	0.0	21.532	34.523	72.901	0.0
$L^h(A,1)$	100	11.682	24.225	56.306	0.0	6.730	23.970	42.837	0.0	13.786	25.660	49.791	0.0
$L^h(A,2)$	100	19.366	34.860	58.955	0.0	13.260	31.885	49.650	0.0	13.587	29.295	44.584	0.0
$H^C(A,1)$	100	15.542	23.490	38.039	0.0	8.509	19.649	37.007	0.0	14.287	20.974	31.415	0.0
$H^C(A,2)$	100	16.294	34.280	57.545	0.0	12.119	33.637	58.508	0.0	13.970	Inf	Inf	0.0
$S(A,1)$	100	18.995	32.171	71.352	0.0	14.808	28.014	46.728	0.0	11.064	30.821	55.997	0.0
$S(A,2)$	100	11.818	33.119	61.381	0.0	14.248	31.673	45.953	0.0	15.698	33.423	54.776	0.0
$EX^h(A,1)$	100	15.387	34.813	59.737	0.0	16.326	Inf	Inf	0.0	16.861	30.904	45.971	0.0
$EX^h(A,2)$	100	26.029	Inf	Inf	0.0	18.009	Inf	Inf	0.0	18.666	Inf	Inf	0.0
$C^h(A,1)$	100	16.630	33.148	54.841	0.0	11.175	31.376	57.751	0.0	16.070	35.908	58.035	0.0
$C^h(A,2)$	100	14.620	35.220	56.503	0.0	7.458	34.342	56.313	0.0	10.862	39.646	86.732	0.0
$EX^C(A,1)$	100	14.813	31.790	50.368	0.0	10.569	30.086	60.310	0.0	14.992	30.961	66.364	0.0
$EX^C(A,2)$	100	11.062	Inf	Inf	0.0	15.534	Inf	Inf	0.0	21.400	Inf	Inf	0.0
$C^C(A,1)$	100	13.283	19.522	28.283	0.0	9.034	18.123	30.618	0.0	12.751	20.598	38.324	0.0
$C^C(A,2)$	100	20.322	33.779	63.656	0.0	8.763	29.222	57.070	0.0	7.997	Inf	Inf	0.0
$L^C(A,1)$	100	12.015	19.238	26.068	0.0	12.395	21.133	47.855	0.0	9.211	18.753	28.237	0.0
$L^C(A,2)$	100	13.627	31.533	48.559	0.0	9.561	28.852	45.530	0.0	6.032	26.556	44.611	0.0

Table 6: Minimum (Min), average (Gap) and maximum (Max) optimality gaps and percentage of instances solved (Sol) across the A, B and C instances with $|\mathcal{S}| = 30$ and uncorrelated customer demands, for different values of standard deviation SD of the demand and percentages of subproblems represented REP. Shaded cells correspond to average optimality gaps lower than for BD.

		SD=5%				SD=10%				SD=20%			
RS	REP	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]
BD	0	0.406	11.007	19.039	0.0	0.000	10.818	18.974	8.3	0.681	10.451	17.708	0.0
$H^h(A,1)$	20	0.000	9.398	20.987	8.3	0.000	9.505	16.124	8.3	0.000	9.600	17.019	8.3
$H^h(A,2)$	20	0.000	9.318	20.879	8.3	0.000	9.280	18.343	8.3	0.000	9.278	19.188	8.3
$L^h(A,1)$	20	0.000	10.897	18.825	8.3	0.000	9.922	18.242	8.3	0.000	10.263	18.734	8.3
$L^h(A,2)$	20	0.000	9.031	17.522	8.3	0.000	10.193	19.422	8.3	0.000	9.179	19.871	8.3
$H^C(A,1)$	20	2.470	10.606	16.584	0.0	1.740	10.880	18.575	0.0	1.844	10.208	16.380	0.0
$H^C(A,2)$	20	4.441	9.328	13.493	0.0	0.000	9.593	14.324	8.3	3.614	10.156	15.335	0.0
$S(A,1)$	20	0.000	9.605	17.240	8.3	1.894	9.521	14.469	0.0	0.000	9.066	14.246	8.3
$S(A,2)$	20	0.000	8.546	15.521	8.3	0.000	8.681	15.941	8.3	0.000	8.373	14.501	8.3
$EX^h(A,1)$	20	0.000	9.323	17.289	8.3	0.000	8.616	15.858	8.3	0.000	10.067	21.892	8.3
$EX^h(A,2)$	20	0.000	8.194	13.348	8.3	0.000	9.306	17.931	8.3	0.000	8.739	15.535	8.3
$C^h(A,1)$	20	0.000	10.636	22.932	8.3	0.000	10.290	18.857	8.3	1.082	10.068	18.469	0.0
$C^h(A,2)$	20	0.000	8.356	17.333	8.3	0.000	10.066	20.340	8.3	0.000	10.066	21.279	8.3
$EX^C(A,1)$	20	3.530	9.834	14.218	0.0	2.774	10.526	19.167	0.0	0.000	10.558	16.730	8.3
$EX^C(A,2)$	20	2.981	10.462	20.462	0.0	1.863	9.432	15.705	0.0	0.000	9.258	14.653	8.3
$C^C(A,1)$	20	1.085	10.381	21.281	0.0	4.683	10.738	18.849	0.0	0.000	9.697	19.278	8.3
$C^C(A,2)$	20	0.000	9.766	16.939	8.3	0.000	9.809	16.816	8.3	3.533	9.841	15.986	0.0
$L^C(A,1)$	20	0.000	9.994	17.607	8.3	0.000	10.192	18.866	8.3	3.334	11.174	19.278	0.0
$L^C(A,2)$	20	0.000	9.919	20.187	8.3	1.611	9.327	14.419	0.0	0.000	9.282	15.366	8.3
$H^h(A,1)$	50	0.000	7.873	14.238	8.3	0.000	8.787	14.889	8.3	0.393	8.577	16.137	0.0
$H^h(A,2)$	50	0.000	8.449	14.279	8.3	0.000	8.716	14.787	8.3	0.000	8.827	14.834	8.3
$L^h(A,1)$	50	0.000	8.502	15.038	8.3	0.000	8.752	16.072	8.3	0.000	9.187	22.028	8.3
$L^h(A,2)$	50	0.000	9.535	18.779	8.3	0.000	9.634	19.044	8.3	0.000	10.084	20.550	8.3
$H^C(A,1)$	50	3.693	10.540	18.675	0.0	1.084	10.734	20.590	0.0	0.000	10.595	19.058	8.3
$H^C(A,2)$	50	0.681	9.203	14.752	0.0	0.000	10.558	18.093	8.3	3.854	11.258	17.009	0.0
$S(A,1)$	50	0.000	8.971	18.206	8.3	0.000	9.925	20.113	8.3	0.000	8.330	13.642	8.3
$S(A,2)$	50	0.000	8.371	14.654	8.3	0.000	8.591	14.188	8.3	0.000	8.291	12.907	8.3
$EX^h(A,1)$	50	0.000	9.014	13.770	8.3	0.000	8.639	13.815	8.3	0.000	9.190	15.496	8.3
$EX^h(A,2)$	50	0.000	9.477	15.849	8.3	0.402	9.103	14.268	0.0	0.000	10.778	23.303	8.3
$C^h(A,1)$	50	0.000	7.687	14.777	8.3	0.000	9.310	14.413	8.3	0.000	9.111	17.927	8.3
$C^h(A,2)$	50	0.000	9.123	17.097	8.3	0.000	7.980	12.223	8.3	0.000	8.651	14.847	8.3
$EX^C(A,1)$	50	0.000	9.959	16.744	8.3	0.000	9.152	15.550	8.3	0.393	11.362	20.064	0.0
$EX^C(A,2)$	50	0.000	8.807	13.937	8.3	0.000	9.694	15.572	8.3	0.000	9.764	16.839	8.3
$C^C(A,1)$	50	0.681	9.315	17.807	0.0	0.000	10.204	16.468	8.3	0.000	9.948	19.278	8.3
$C^C(A,2)$	50	0.000	9.503	17.860	8.3	0.402	10.722	20.111	0.0	0.000	10.158	16.504	8.3
$L^C(A,1)$	50	0.681	9.891	17.923	0.0	3.700	10.506	18.301	0.0	0.000	10.522	16.175	8.3
$L^C(A,2)$	50	2.072	9.907	19.236	0.0	0.402	10.049	18.747	0.0	0.000	10.226	16.763	8.3

$H^h(A,1)$	80	0.000	8.500	14.435	8.3	0.000	9.500	15.424	8.3	0.000	8.722	16.513	8.3
$H^h(A,2)$	80	0.406	10.468	17.428	0.0	0.000	9.559	15.569	8.3	0.000	11.294	26.890	8.3
$L^h(A,1)$	80	0.000	8.661	17.563	8.3	0.000	8.825	15.310	8.3	0.000	9.564	18.493	8.3
$L^h(A,2)$	80	0.000	9.153	16.785	8.3	0.000	9.372	16.595	8.3	0.000	8.757	14.730	8.3
$H^C(A,1)$	80	0.000	10.157	18.944	8.3	0.000	11.159	18.195	8.3	0.000	10.579	18.937	8.3
$H^C(A,2)$	80	0.000	10.120	17.509	8.3	2.471	10.577	17.249	0.0	0.000	9.734	17.468	8.3
$S(A,1)$	80	0.000	7.833	12.628	8.3	0.000	8.470	15.091	8.3	0.000	8.288	13.095	8.3
$S(A,2)$	80	0.000	9.664	15.576	8.3	0.000	8.802	15.954	8.3	0.393	9.758	17.299	0.0
$EX^h(A,1)$	80	0.000	9.369	16.291	8.3	0.000	10.002	18.812	8.3	0.000	8.717	14.552	8.3
$EX^h(A,2)$	80	0.000	11.769	20.979	8.3	0.402	12.081	19.396	0.0	0.000	12.789	22.804	8.3
$C^h(A,1)$	80	0.000	9.020	15.914	8.3	0.000	8.148	13.226	8.3	0.000	8.701	16.772	8.3
$C^h(A,2)$	80	0.000	8.461	15.771	8.3	0.000	10.090	18.655	8.3	0.000	9.998	20.923	8.3
$EX^C(A,1)$	80	3.635	10.215	16.699	0.0	0.000	9.304	15.224	8.3	4.303	10.747	15.963	0.0
$EX^C(A,2)$	80	1.085	10.084	17.450	0.0	0.000	10.426	17.434	8.3	0.393	9.662	15.909	0.0
$C^C(A,1)$	80	1.864	10.553	17.803	0.0	0.000	10.198	20.732	8.3	0.393	9.887	19.278	0.0
$C^C(A,2)$	80	3.970	10.807	21.392	0.0	0.000	10.441	17.962	8.3	0.000	10.767	17.763	8.3
$L^C(A,1)$	80	1.612	10.676	19.292	0.0	6.764	11.181	19.843	0.0	0.000	11.458	21.168	8.3
$L^C(A,2)$	80	3.514	10.324	18.490	0.0	1.084	9.891	18.175	0.0	0.681	10.426	15.919	0.0
$H^h(A,1)$	100	0.000	9.578	17.175	8.3	0.000	9.250	16.752	8.3	0.000	9.508	15.660	8.3
$H^h(A,2)$	100	0.000	11.197	18.849	8.3	0.402	10.286	20.418	0.0	0.000	12.346	21.463	8.3
$L^h(A,1)$	100	0.000	9.196	18.366	8.3	0.000	9.376	18.406	8.3	0.000	9.571	16.746	8.3
$L^h(A,2)$	100	0.000	10.001	17.090	8.3	0.000	9.282	15.375	8.3	0.000	9.678	17.547	8.3
$H^C(A,1)$	100	0.000	11.342	22.800	8.3	0.402	11.936	22.463	0.0	3.657	10.434	17.971	0.0
$H^C(A,2)$	100	1.085	10.471	20.045	0.0	0.000	9.843	17.693	8.3	0.000	9.938	17.017	8.3
$S(A,1)$	100	0.000	9.159	18.412	8.3	0.000	8.867	17.896	8.3	0.000	8.545	12.359	8.3
$S(A,2)$	100	0.000	10.356	21.743	8.3	0.000	9.196	15.749	8.3	1.082	10.425	19.140	0.0
$EX^h(A,1)$	100	0.000	10.807	20.022	8.3	0.000	10.859	20.529	8.3	0.393	10.814	23.240	0.0
$EX^h(A,2)$	100	1.612	12.482	21.301	0.0	0.402	11.967	23.694	0.0	0.000	12.712	25.871	8.3
$C^h(A,1)$	100	0.000	9.335	21.043	8.3	0.000	9.051	16.521	8.3	0.000	9.238	15.231	8.3
$C^h(A,2)$	100	0.000	10.902	23.106	8.3	0.000	10.290	21.148	8.3	0.000	11.206	22.548	8.3
$EX^C(A,1)$	100	0.406	9.106	14.529	0.0	0.000	9.235	15.431	8.3	0.681	9.306	14.800	0.0
$EX^C(A,2)$	100	0.000	10.349	18.022	8.3	1.665	11.573	19.875	0.0	0.393	10.560	17.268	0.0
$C^C(A,1)$	100	0.000	10.407	17.387	8.3	1.611	10.147	17.071	0.0	0.000	9.899	14.914	8.3
$C^C(A,2)$	100	0.681	9.482	16.815	0.0	3.999	10.371	16.362	0.0	0.000	10.541	18.364	8.3
$L^C(A,1)$	100	0.000	10.531	18.904	8.3	0.000	10.085	17.816	8.3	1.082	10.422	16.027	0.0
$L^C(A,2)$	100	0.681	9.479	16.295	0.0	2.188	9.466	16.100	0.0	0.393	8.919	14.016	0.0

Table 7: Minimum (Min), average (Gap) and maximum (Max) optimality gaps and percentage of instances solved (Sol) across the A, B and C instances with $|\mathcal{S}| = 60$ and uncorrelated customer demands, for different values of standard deviation SD of the demand and percentages of subproblems represented REP. Shaded cells correspond to average optimality gaps lower than for BD.

		SD=5%				SD=10%				SD=20%			
RS	REP	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]
BD	0	7.256	16.205	25.658	0.0	5.732	18.427	32.039	0.0	9.624	18.609	26.336	0.0
$H^h(A,1)$	20	2.868	14.055	25.256	0.0	5.793	15.461	27.659	0.0	3.145	12.686	22.817	0.0
$H^h(A,2)$	20	3.288	12.679	22.328	0.0	3.087	13.838	25.932	0.0	3.145	13.866	23.872	0.0
$L^h(A,1)$	20	4.526	13.762	24.079	0.0	7.217	14.441	23.599	0.0	7.259	14.598	25.044	0.0
$L^h(A,2)$	20	3.288	13.119	21.284	0.0	5.044	12.695	21.700	0.0	4.261	13.088	22.218	0.0
$H^C(A,1)$	20	5.409	15.748	27.327	0.0	12.329	17.199	23.476	0.0	11.298	17.185	23.625	0.0
$H^C(A,2)$	20	9.631	16.801	23.953	0.0	9.600	15.615	22.830	0.0	10.279	16.174	23.900	0.0
$S(A,1)$	20	2.868	13.128	27.327	0.0	3.506	12.929	23.427	0.0	3.145	12.674	26.310	0.0
$S(A,2)$	20	3.288	12.686	18.267	0.0	3.087	13.227	23.094	0.0	7.006	13.157	19.081	0.0
$EX^h(A,1)$	20	2.868	11.833	16.945	0.0	3.087	12.769	20.031	0.0	3.145	13.686	24.371	0.0
$EX^h(A,2)$	20	2.868	15.890	26.353	0.0	3.087	15.047	32.448	0.0	3.145	17.493	47.216	0.0
$C^h(A,1)$	20	4.785	12.670	17.694	0.0	3.506	13.973	23.931	0.0	5.057	13.514	26.665	0.0
$C^h(A,2)$	20	2.868	13.077	21.937	0.0	3.789	13.821	21.713	0.0	5.064	15.342	24.545	0.0
$EX^C(A,1)$	20	12.526	16.614	27.777	0.0	7.786	16.466	26.681	0.0	6.961	15.895	22.125	0.0
$EX^C(A,2)$	20	2.868	14.964	24.153	0.0	9.873	14.920	20.859	0.0	8.699	14.927	26.795	0.0
$C^C(A,1)$	20	7.042	16.414	25.116	0.0	3.087	15.311	22.836	0.0	4.804	15.343	22.376	0.0
$C^C(A,2)$	20	9.438	16.620	26.543	0.0	6.382	15.722	26.446	0.0	8.600	17.067	27.683	0.0
$L^C(A,1)$	20	7.441	16.114	24.754	0.0	6.708	16.044	22.836	0.0	13.280	17.859	26.315	0.0
$L^C(A,2)$	20	7.059	15.267	23.067	0.0	8.855	15.887	20.726	0.0	8.326	15.792	26.511	0.0
$H^h(A,1)$	50	4.581	13.373	22.057	0.0	4.204	12.851	22.024	0.0	3.560	13.926	25.945	0.0
$H^h(A,2)$	50	3.984	16.584	27.778	0.0	4.804	19.986	38.222	0.0	4.261	20.939	41.998	0.0
$L^h(A,1)$	50	3.288	12.855	21.284	0.0	3.087	13.196	21.073	0.0	5.057	13.351	22.468	0.0
$L^h(A,2)$	50	3.288	13.185	20.446	0.0	4.748	13.777	19.757	0.0	3.847	13.890	21.816	0.0
$H^C(A,1)$	50	7.621	17.034	25.967	0.0	12.191	17.588	27.085	0.0	11.114	16.737	26.568	0.0
$H^C(A,2)$	50	4.660	13.548	20.586	0.0	3.087	13.775	25.709	0.0	4.862	13.394	21.814	0.0
$S(A,1)$	50	4.581	13.120	22.762	0.0	3.087	12.213	23.445	0.0	3.145	12.838	23.457	0.0
$S(A,2)$	50	2.868	16.523	38.551	0.0	6.549	17.471	30.273	0.0	3.560	14.796	25.337	0.0
$EX^h(A,1)$	50	3.569	15.112	29.408	0.0	5.099	16.031	42.018	0.0	3.145	22.170	60.607	0.0
$EX^h(A,2)$	50	8.242	22.188	41.583	0.0	8.195	22.942	42.939	0.0	10.577	28.742	77.801	0.0
$C^h(A,1)$	50	3.288	13.276	21.180	0.0	3.087	13.222	22.300	0.0	3.847	13.560	23.789	0.0

$C^h(A,2)$	50	4.526	16.195	29.193	0.0	3.087	17.442	35.471	0.0	5.057	18.446	31.283	0.0
$EX^C(A,1)$	50	6.501	14.145	22.173	0.0	4.204	14.044	21.832	0.0	7.299	14.408	23.335	0.0
$EX^C(A,2)$	50	10.845	22.131	33.368	0.0	3.789	22.079	38.893	0.0	11.886	22.383	39.283	0.0
$C^C(A,1)$	50	11.766	17.351	23.204	0.0	11.700	16.961	24.895	0.0	4.862	16.601	26.109	0.0
$C^C(A,2)$	50	2.868	14.896	21.493	0.0	4.204	14.308	22.533	0.0	3.145	15.308	22.524	0.0
$L^C(A,1)$	50	9.915	17.335	23.067	0.0	8.335	15.810	22.176	0.0	6.862	17.084	25.067	0.0
$L^C(A,2)$	50	8.089	14.302	22.840	0.0	6.419	13.715	19.757	0.0	3.145	13.320	21.474	0.0
$H^h(A,1)$	80	5.734	15.425	29.871	0.0	3.506	15.114	27.700	0.0	5.285	17.110	29.880	0.0
$H^h(A,2)$	80	5.119	23.785	53.909	0.0	6.725	21.589	43.305	0.0	7.850	25.636	46.582	0.0
$L^h(A,1)$	80	7.180	13.574	20.002	0.0	4.204	13.649	22.075	0.0	6.967	14.572	22.468	0.0
$L^h(A,2)$	80	8.179	17.301	29.455	0.0	3.087	17.915	42.237	0.0	7.166	16.832	25.359	0.0
$H^C(A,1)$	80	10.291	17.195	26.692	0.0	6.902	16.189	25.764	0.0	6.479	16.498	28.790	0.0
$H^C(A,2)$	80	7.592	16.583	27.937	0.0	7.623	16.369	27.994	0.0	9.543	16.718	27.006	0.0
$S(A,1)$	80	2.868	14.858	25.145	0.0	3.087	15.102	29.430	0.0	4.261	15.710	27.247	0.0
$S(A,2)$	80	7.065	20.901	37.594	0.0	8.752	18.934	33.221	0.0	8.801	20.826	34.183	0.0
$EX^h(A,1)$	80	2.868	21.364	35.500	0.0	7.746	24.332	67.316	0.0	5.097	25.592	53.075	0.0
$EX^h(A,2)$	80	5.000	22.241	55.260	0.0	10.671	26.435	39.710	0.0	11.220	26.544	47.081	0.0
$C^h(A,1)$	80	3.288	14.885	23.370	0.0	7.746	14.451	22.235	0.0	6.001	15.433	26.285	0.0
$C^h(A,2)$	80	6.327	21.612	36.749	0.0	3.789	21.687	32.524	0.0	7.564	23.379	45.858	0.0
$EX^C(A,1)$	80	8.826	18.372	27.945	0.0	7.609	17.086	28.931	0.0	7.120	16.534	27.320	0.0
$EX^C(A,2)$	80	9.600	Inf	Inf	0.0	10.693	24.677	46.782	0.0	5.817	22.271	59.650	0.0
$C^C(A,1)$	80	6.588	17.036	29.344	0.0	8.070	16.697	31.910	0.0	4.862	17.496	25.342	0.0
$C^C(A,2)$	80	5.850	16.544	36.756	0.0	7.063	15.192	23.513	0.0	10.674	17.130	30.820	0.0
$L^C(A,1)$	80	9.451	16.482	23.067	0.0	9.241	16.487	24.159	0.0	13.032	18.285	27.680	0.0
$L^C(A,2)$	80	7.540	14.578	23.457	0.0	9.705	15.696	23.605	0.0	5.064	15.007	24.636	0.0
$EX^C(A,1)$	100	7.535	20.576	35.959	0.0	4.204	19.976	42.548	0.0	7.937	21.177	36.880	0.0
$EX^C(A,2)$	100	9.893	22.718	55.234	0.0	8.823	24.444	49.800	0.0	5.057	21.905	45.189	0.0
$C^C(A,1)$	100	3.984	14.643	22.346	0.0	8.827	15.630	23.199	0.0	11.538	16.147	22.524	0.0
$C^C(A,2)$	100	9.470	Inf	Inf	0.0	9.975	Inf	Inf	0.0	7.442	19.879	32.545	0.0
$L^C(A,1)$	100	9.149	15.993	22.314	0.0	4.804	14.822	20.941	0.0	7.565	14.641	22.417	0.0
$L^C(A,2)$	100	4.660	21.965	33.869	0.0	3.087	20.226	37.951	0.0	8.011	17.589	29.210	0.0
$H^h(A,1)$	100	3.253	19.100	37.540	0.0	3.708	19.569	34.268	0.0	5.910	23.973	52.265	0.0
$H^h(A,2)$	100	7.465	27.568	53.449	0.0	8.325	26.271	48.469	0.0	5.714	26.764	42.695	0.0
$L^h(A,1)$	100	3.253	14.554	22.345	0.0	3.708	13.924	23.111	0.0	7.896	15.592	23.805	0.0
$L^h(A,2)$	100	5.177	22.467	33.569	0.0	4.832	22.731	46.918	0.0	10.176	24.422	37.436	0.0
$H^C(A,1)$	100	12.501	18.160	24.056	0.0	6.574	16.567	26.515	0.0	3.982	16.696	26.291	0.0
$H^C(A,2)$	100	9.162	22.723	37.335	0.0	10.980	Inf	Inf	0.0	14.147	25.069	61.033	0.0
$S(A,1)$	100	5.392	18.804	32.344	0.0	3.708	20.255	41.147	0.0	4.690	17.953	36.254	0.0
$S(A,2)$	100	7.465	24.158	38.513	0.0	9.076	23.052	40.195	0.0	3.982	24.383	46.842	0.0
$EX^h(A,1)$	100	4.917	23.799	57.050	0.0	5.978	24.527	50.264	0.0	8.036	25.537	39.086	0.0
$EX^h(A,2)$	100	15.972	Inf	Inf	0.0	5.857	24.023	42.506	0.0	15.365	32.382	57.464	0.0
$C^h(A,1)$	100	3.253	18.684	32.438	0.0	9.622	18.159	35.384	0.0	3.982	17.148	25.633	0.0
$C^h(A,2)$	100	8.764	23.226	44.106	0.0	3.708	19.303	34.831	0.0	8.440	24.291	55.023	0.0

Table 8: Minimum (Min), average (Gap) and maximum (Max) optimality gaps and percentage of instances solved (Sol) across the A, B and C instances with $|\mathcal{S}| = 90$ and uncorrelated customer demands, for different values of standard deviation SD of the demand and percentages of subproblems represented REP. Shaded cells correspond to average optimality gaps lower than for BD.

		SD=5%				SD=10%				SD=20%			
RS	REP	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]	Min [%]	Gap [%]	Max [%]	Sol [%]
BD	0	19.821	30.055	73.460	0.0	13.231	24.630	44.014	0.0	11.038	24.471	36.704	0.0
$H^h(A,1)$	20	6.745	17.779	28.551	0.0	7.123	17.675	27.704	0.0	8.181	17.645	26.408	0.0
$H^h(A,2)$	20	6.309	19.369	31.356	0.0	6.407	18.264	31.232	0.0	6.129	18.695	27.769	0.0
$L^h(A,1)$	20	7.463	16.141	25.661	0.0	7.123	16.490	25.438	0.0	8.619	15.435	25.564	0.0
$L^h(A,2)$	20	7.463	16.213	24.313	0.0	9.814	17.206	25.438	0.0	9.302	15.755	25.826	0.0
$H^C(A,1)$	20	14.830	22.943	31.511	0.0	15.573	20.792	30.857	0.0	11.954	19.103	25.455	0.0
$H^C(A,2)$	20	14.465	21.192	33.288	0.0	15.952	21.620	32.857	0.0	17.255	20.961	33.176	0.0
$S(A,1)$	20	8.291	17.496	25.961	0.0	7.123	16.994	28.036	0.0	8.455	18.237	32.341	0.0
$S(A,2)$	20	6.745	18.770	34.787	0.0	8.171	17.634	31.922	0.0	6.560	17.755	31.085	0.0
$EX^h(A,1)$	20	9.767	18.767	30.290	0.0	6.407	17.650	32.833	0.0	6.129	17.857	28.819	0.0
$EX^h(A,2)$	20	12.512	23.720	38.762	0.0	5.974	24.348	47.427	0.0	6.560	25.428	51.650	0.0
$C^h(A,1)$	20	6.309	16.950	29.742	0.0	8.555	16.452	26.760	0.0	8.119	17.653	26.559	0.0
$C^h(A,2)$	20	8.388	17.314	29.314	0.0	8.171	17.735	34.776	0.0	6.560	16.744	27.386	0.0
$EX^C(A,1)$	20	15.770	21.195	29.042	0.0	12.036	20.203	25.993	0.0	11.360	19.988	30.553	0.0
$EX^C(A,2)$	20	10.059	18.475	37.852	0.0	12.814	18.106	32.510	0.0	12.950	18.710	39.221	0.0
$C^C(A,1)$	20	18.773	22.741	31.019	0.0	17.064	22.891	34.286	0.0	14.855	19.765	31.126	0.0
$C^C(A,2)$	20	15.077	21.672	30.723	0.0	15.362	22.245	33.972	0.0	15.885	19.780	27.033	0.0
$L^C(A,1)$	20	14.952	21.838	27.504	0.0	18.315	23.445	29.829	0.0	11.320	19.635	30.891	0.0
$L^C(A,2)$	20	15.165	20.504	27.111	0.0	14.923	20.792	27.726	0.0	15.119	20.055	28.627	0.0
$H^h(A,1)$	50	6.309	20.177	32.634	0.0	8.171	20.095	46.682	0.0	6.129	19.218	30.838	0.0
$H^h(A,2)$	50	12.395	30.276	52.517	0.0	10.023	34.324	64.525	0.0	13.153	Inf	Inf	0.0
$L^h(A,1)$	50	10.332	16.870	24.285	0.0	9.814	17.459	25.438	0.0	9.980	17.112	25.826	0.0

$L^h(A,2)$	50	8.023	23.315	44.301	0.0	11.515	20.691	37.227	0.0	11.397	20.893	37.349	0.0
$H^C(A,1)$	50	16.281	24.116	45.341	0.0	15.252	21.465	31.162	0.0	17.042	21.114	30.827	0.0
$H^C(A,2)$	50	12.571	20.602	33.549	0.0	11.481	19.507	35.142	0.0	13.355	19.207	27.505	0.0
$S(A,1)$	50	6.309	19.256	31.961	0.0	5.974	19.232	31.903	0.0	8.081	18.550	35.986	0.0
$S(A,2)$	50	10.647	28.136	46.225	0.0	14.131	31.360	58.624	0.0	15.518	28.462	57.019	0.0
$EX^h(A,1)$	50	9.154	28.825	67.138	0.0	11.970	30.760	57.063	0.0	12.360	30.298	65.799	0.0
$EX^h(A,2)$	50	10.647	Inf	Inf	0.0	15.905	32.201	48.715	0.0	17.642	34.044	62.259	0.0
$C^h(A,1)$	50	6.309	18.944	30.694	0.0	5.974	18.621	38.496	0.0	8.202	18.147	33.051	0.0
$C^h(A,2)$	50	16.099	29.867	57.647	0.0	14.595	31.495	64.314	0.0	14.076	30.636	63.503	0.0
$EX^C(A,1)$	50	10.458	20.740	40.892	0.0	11.282	20.307	39.733	0.0	9.117	18.640	34.591	0.0
$EX^C(A,2)$	50	12.349	27.739	48.167	0.0	13.630	29.692	44.981	0.0	17.627	29.873	53.949	0.0
$C^C(A,1)$	50	16.633	22.572	29.673	0.0	16.549	21.441	29.689	0.0	7.278	20.036	30.653	0.0
$C^C(A,2)$	50	10.249	23.268	47.932	0.0	12.854	22.724	43.977	0.0	12.047	20.385	42.103	0.0
$L^C(A,1)$	50	18.192	22.833	28.502	0.0	16.674	22.477	30.063	0.0	9.634	20.280	30.891	0.0
$L^C(A,2)$	50	9.847	17.753	31.075	0.0	12.490	18.628	27.678	0.0	13.139	17.912	27.033	0.0
$H^h(A,1)$	80	12.182	23.970	44.640	0.0	7.681	Inf	Inf	0.0	10.062	Inf	Inf	0.0
$H^h(A,2)$	80	9.878	30.006	47.550	0.0	12.501	Inf	Inf	0.0	9.861	30.682	52.295	0.0
$L^h(A,1)$	80	10.161	16.533	23.353	0.0	8.024	17.089	24.855	0.0	10.444	17.072	24.354	0.0
$L^h(A,2)$	80	13.235	29.777	49.472	0.0	8.294	28.943	53.652	0.0	8.754	26.514	55.432	0.0
$H^C(A,1)$	80	9.266	19.149	31.563	0.0	12.487	20.393	32.058	0.0	11.855	18.572	26.045	0.0
$H^C(A,2)$	80	14.529	Inf	Inf	0.0	13.972	Inf	Inf	0.0	15.628	29.116	44.282	0.0
$S(A,1)$	80	12.896	24.723	38.273	0.0	13.949	26.755	41.989	0.0	10.704	22.032	38.271	0.0
$S(A,2)$	80	12.176	27.606	36.677	0.0	11.060	30.463	57.361	0.0	10.721	28.984	42.262	0.0
$EX^h(A,1)$	80	11.301	27.770	50.103	0.0	14.163	31.732	53.479	0.0	18.492	27.736	42.262	0.0
$EX^h(A,2)$	80	15.373	Inf	Inf	0.0	22.197	Inf	Inf	0.0	12.508	Inf	Inf	0.0
$C^h(A,1)$	80	9.856	24.542	42.730	0.0	7.681	23.775	44.175	0.0	10.046	27.530	60.446	0.0
$C^h(A,2)$	80	10.394	25.976	39.475	0.0	10.639	29.597	59.883	0.0	17.212	32.444	60.372	0.0
$EX^C(A,1)$	80	17.529	Inf	Inf	0.0	14.513	Inf	Inf	0.0	11.553	33.845	56.936	0.0
$EX^C(A,2)$	80	23.059	29.445	39.766	0.0	12.367	31.688	49.634	0.0	22.499	33.498	52.288	0.0
$C^C(A,1)$	80	13.125	18.771	25.562	0.0	15.253	20.258	30.843	0.0	14.539	20.303	29.334	0.0
$C^C(A,2)$	80	10.668	30.347	53.163	0.0	16.673	29.428	48.886	0.0	20.360	30.350	57.200	0.0
$L^C(A,1)$	80	10.868	19.453	31.019	0.0	11.237	20.251	32.604	0.0	13.867	18.893	28.614	0.0
$L^C(A,2)$	80	15.261	34.175	64.655	0.0	15.168	29.433	44.637	0.0	10.432	29.576	43.456	0.0
$H^h(A,1)$	100	18.521	Inf	Inf	0.0	10.664	36.740	108.918	0.0	17.315	Inf	Inf	0.0
$H^h(A,2)$	100	21.571	Inf	Inf	0.0	17.207	37.797	67.248	0.0	10.855	Inf	Inf	0.0
$L^h(A,1)$	100	13.330	29.073	93.777	0.0	15.562	25.157	43.535	0.0	11.127	24.559	46.628	0.0
$L^h(A,2)$	100	13.102	33.375	96.274	0.0	17.744	33.330	68.726	0.0	19.518	35.189	62.849	0.0
$H^C(A,1)$	100	13.676	27.266	65.461	0.0	12.760	23.048	33.459	0.0	12.256	21.952	37.595	0.0
$H^C(A,2)$	100	13.511	37.217	82.331	0.0	17.208	35.624	82.405	0.0	17.095	Inf	Inf	0.0
$S(A,1)$	100	17.586	41.605	97.384	0.0	18.368	33.034	61.389	0.0	13.402	33.424	78.831	0.0
$S(A,2)$	100	14.121	41.425	89.703	0.0	17.182	35.128	65.797	0.0	13.370	33.268	65.417	0.0
$EX^h(A,1)$	100	14.641	43.033	122.665	0.0	13.390	Inf	Inf	0.0	13.820	38.336	68.556	0.0
$EX^h(A,2)$	100	12.677	Inf	Inf	0.0	20.065	Inf	Inf	0.0	17.599	Inf	Inf	0.0
$C^h(A,1)$	100	16.499	37.823	76.760	0.0	21.260	35.429	58.067	0.0	16.199	36.575	72.566	0.0
$C^h(A,2)$	100	16.303	44.528	112.173	0.0	14.690	36.308	62.765	0.0	21.135	35.524	73.046	0.0
$EX^C(A,1)$	100	16.033	31.527	47.020	0.0	12.508	32.079	66.370	0.0	17.510	Inf	Inf	0.0
$EX^C(A,2)$	100	14.628	Inf	Inf	0.0	18.412	Inf	Inf	0.0	16.843	32.142	51.695	0.0
$C^C(A,1)$	100	12.256	18.339	22.816	0.0	14.081	20.272	29.314	0.0	12.016	20.122	28.119	0.0
$C^C(A,2)$	100	17.818	31.098	52.086	0.0	12.147	Inf	Inf	0.0	17.251	36.000	74.102	0.0
$L^C(A,1)$	100	12.402	19.992	29.389	0.0	14.225	22.471	41.697	0.0	12.201	19.486	27.344	0.0
$L^C(A,2)$	100	15.709	30.934	57.832	0.0	13.558	29.477	65.187	0.0	18.536	31.943	66.534	0.0

5.5 Other Observations

In this section we provide some additional insights drawing from our computational experience. We begin by examining the progression of the lower bound and of the incumbent for both BD and GPBD (with the RSs used in Section 5.4). For the sake of brevity, the progression is exemplarily illustrated for the instances with highest number of customer demands positively correlated ($C = 80\%$), Figure 1, and highest standard deviation of the demand ($SD = 20\%$), Figure 2. In both cases the instances with the highest number of scenarios ($|\mathcal{S}| = 90$) are considered. Both Figure 1 and Figure 2 show that, for all instances, the curves corresponding to BD almost "envelope" the curves corresponding to the GPBD RSs. This pattern illustrates that, during the entire solution process, the incumbent provided by GPBD is typically smaller (better in our case) than that provided by BD, and that the lower bound provided by GPBD is typically higher (tighter in our) that that provided by BD. In turn, this shows that GPBD is

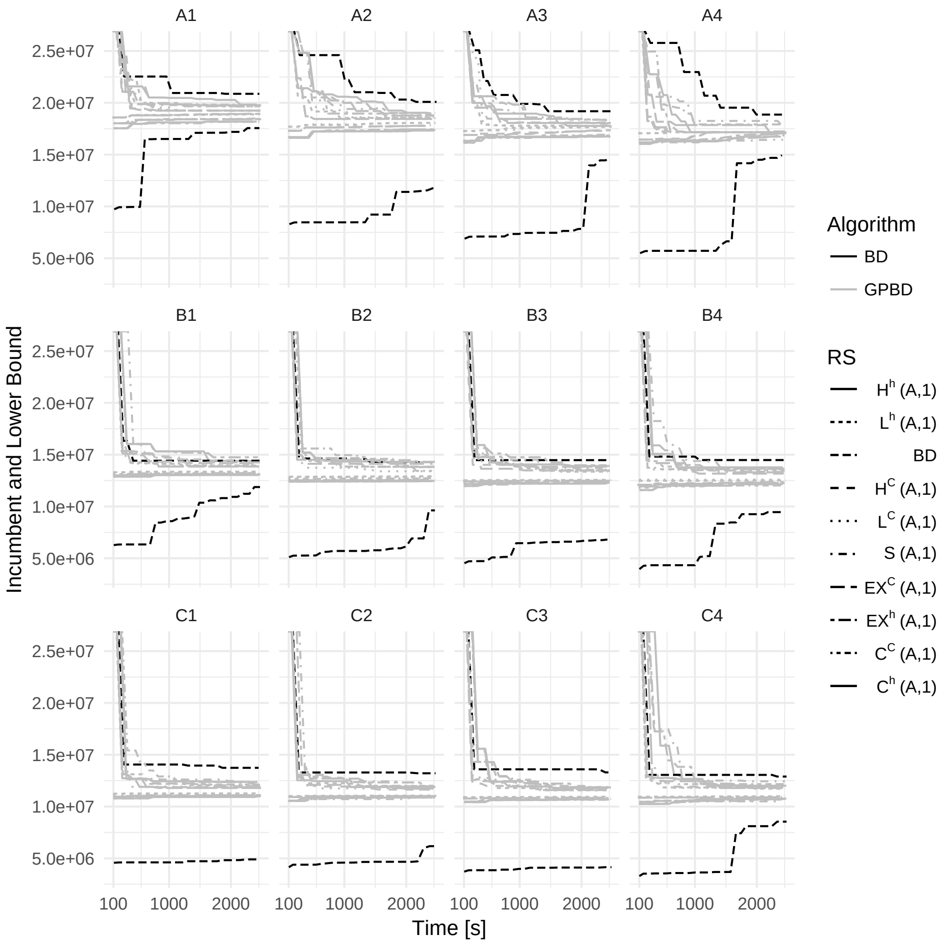

Figure 1: Progression of lower bound and incumbent in BD and GPBD (with all selected RSs, $n = 1$ and REP$= 20\%$) for all instance with $|\mathcal{S}| = 90$ and $C = 80\%$.

able to provide better candidate solutions and better bounds already in the initial phases of the algorithm. Notice particularly the steep descent of the incumbent during the first 500 seconds and the remarkably higher bound already from the beginning of the solution process.

The reasons of the improved bounds and candidate solutions are to be ascribed mostly to the improved formulation of MP. Figure 3 reports the Branch and Cut nodes explored using BD and GPBD for the instances with $|\mathcal{S}| = 90$ and REP=20%. It can be noticed that, in general, GPBD explores fewer nodes than BD and, in some cases, significantly fewer. Thus, the improved incumbent and bound are not due to a wider exploration of the Branch and Cut tree, but rather to a better formulation of MP. The fact that GPBD explores fewer nodes can be justified by a heavier and more time-consuming LP relaxation with consequent tighter bounds leading to slimmer trees. In addition, we observed that both GPBD and BD generated approximately the same number of optimality cuts, confirming that the improved results are mainly due to the fact that a stronger MP formulation can provide better solutions already in the initial phases of the algorithm. Finally, we observed that when using GPBD, in general, the solver (i.e., Cplex 12.6.2 in this case) was able to add more valid inequalities to improve the LP relaxation.

Finally, in addition to the benefits provided by the revisited decomposition strategy illus-

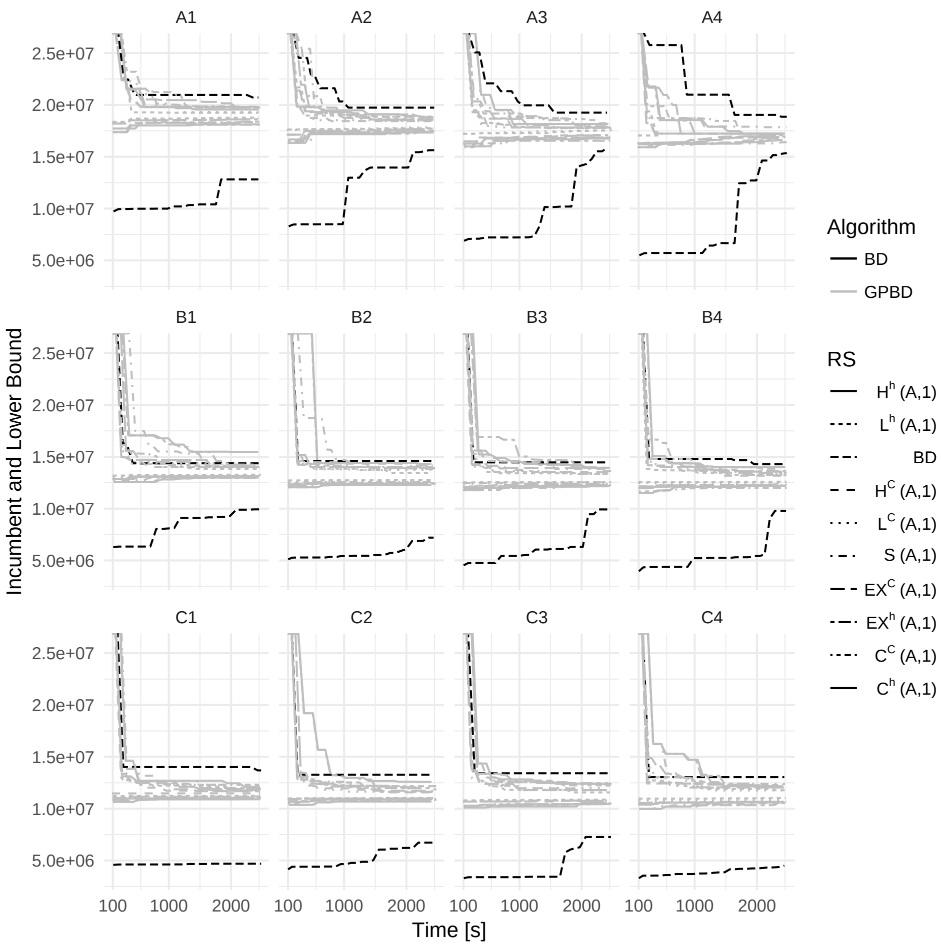

Figure 2: Progression of lower bound and incumbent in BD and GPBD (with all selected RSs, $n = 1$ and REP$= 20\%$) for all instance with $|\mathcal{S}| = 90$ and $SD = 20\%$.

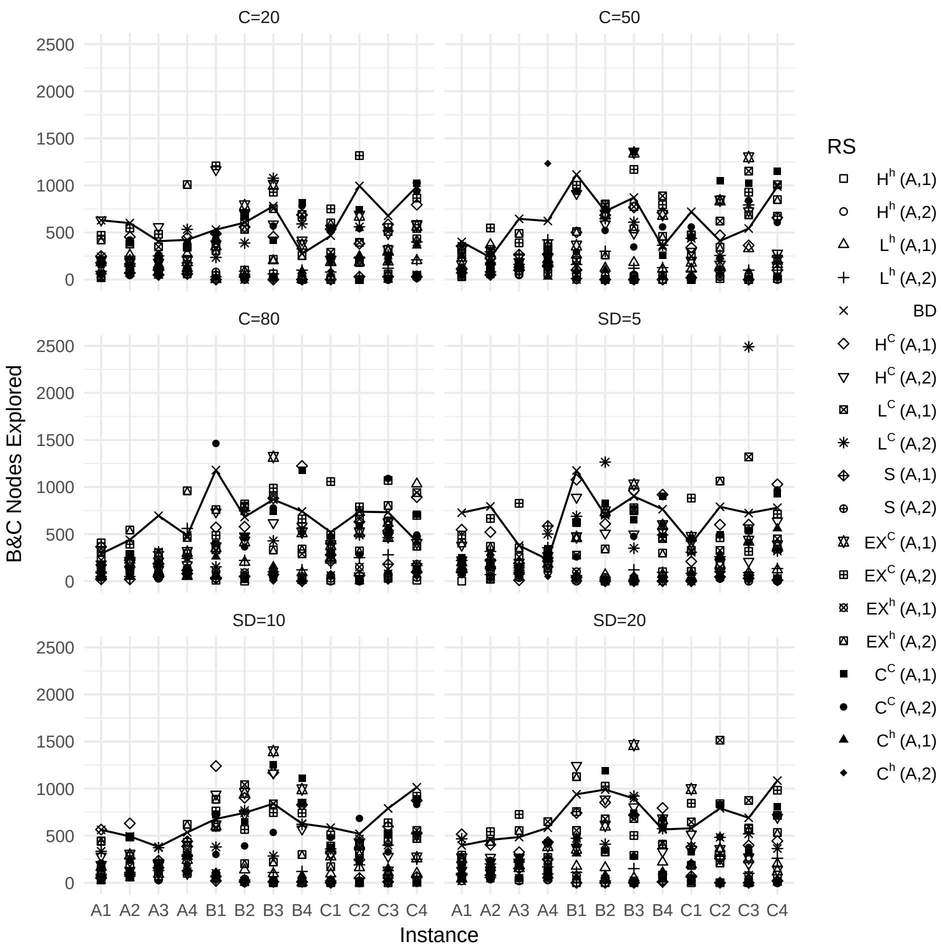

Figure 3: Number of B&C nodes explored by BD and GPBD (with the selected RSs and REP = 20%) for all instances with $|\mathcal{S}| = 90$.

trated in this article, GPBD is amenable to several of the enhancements suggested for the classical BD. The reader can find a survey of the possible enhancements in Rahmaniani et al. [2016]. Among these, we report the *stabilization* procedure which was recently pointed out by Fischetti et al. [2016b] as the most important ingredient in their BD approach for solving large-scale instances of the uncapacitated facility location problem. In a nutshell, stabilization implies generating cuts on a "stabilized" versions of the solution proposed by MP. The stabilized point is obtained by choosing a point in between the solution to MP and a stabilizer decided a priori, in this case $(1, \ldots, 1)^T$. The scope is that of avoiding the production of erratic solutions especially in the initial iterations. We tested the stabilization technique described by [Fischetti et al., 2016b] and observed that, in general, it grants a further noticeable reduction of the optimality gap (for some instances the optimality gap was remarkably reduced by approximately 80%). This offers significant opportunities to further improve the results in Section 5.3 which are obtained without additional efficiency measures in order to isolate the effect of a mere alternative decomposition strategy.

6 Conclusions

The proposed Generalized Partial Benders Decomposition of two-stage stochastic programs revisits the classical Benders decomposition by granting the possibility to generate more "informed" master problems. Selected portions of second-stage subproblems (variables and constraints) are retained in the master problem with the scope of providing additional information for generating better solutions in the early phases of the algorithm. The method provides enough flexibility for adapting the amount of second-stage information in MP on the specific case. A number of simple and general purpose strategies have been proposed for selecting second-stage variables and constraints to retain in the master problem.

The method has been applied to computationally difficult large-scale instances of the Capacitated Facility Location Problem with Stochastic Demand. An extensive computational study shows that GPBD improves BD on almost all instances tested and that the improvements are stable across a set of heterogeneous instances. The optimality gap reduction is particularly high on the largest instances (with 90 scenarios). In some cases the optimality gap is nearly halved. In addition, it emerges that GPBD performs better when the number of variables and constraints retained is small, meaning for example that they are selected from only a small number of scenario-subproblems (20% of the subproblems in our case). Conversely, the performances of GPBD worsen dramatically when the master problem becomes too heavy. The analysis of the solution process shows that GPBD provides high quality bounds and candidate solutions already in the early stages of the algorithm. Particularly, the initial bounds are significantly higher than the bounds provided by BD.

Our study also illustrates that efficient retaining strategies can be discovered simply by testing a number of candidate retaining strategies on small-scale instances. In fact, the corresponding retaining strategies remain effective also on larger and numerically different instances. Remarkable results on the largest instances were provides by retaining in the master-problem only one, randomly selected, second-stage constraint (for each of the represented subproblems). This illustrates that even simple retaining strategies may grant satisfactory results and that retaining a small amount of second-stage information in the master problem may be extremely beneficial.

A Summary of the retaining strategies

Table 9: Retaining Strategies

RS	Description
	General Purpose
$C^W(i,n)$	Represents the i-th block of constraints by partitioning the constraints in block i in n clusters based on the rows of $W_s(i)$ and selecting, for each cluster, the constraint closest to its centroid
$H^W(i,n)$	Represents the i-th block of constraints by the n constraints of block i with highest values of $W_s(i)$
$L^W(i,n)$	Represents the i-th block of constraints by the n constraints of block i with lowest values of $W_s(i)$
$EX^W(i,n)$	Applies simultaneously RSs $L^T(i,n)$ and $H^T(i,n)$
$C^T(i,n)$	Represents the i-th block of constraints by partitioning the constraints in block i in n clusters based on the rows of $T_s(i)$ and selecting, for each cluster, the constraint closest to its centroid
$H^T(i,n)$	Represents the i-th block of constraints by the n constraints of block i with highest values of $T_s(i)$
$L^T(i,n)$	Represents the i-th block of constraints by the n constraints of block i with lowest values of $T_s(i)$
$EX^T(i,n)$	Applies simultaneously RSs $L^T(i,n)$ and $H^T(i,n)$
$C^h(i,n)$	Represents the i-th block of constraints by partitioning the constraints in block i in n clusters based on the coefficients $h_s(i)$ and selecting, for each cluster, the constraint closest to its centroid
$H^h(i,n)$	Represents the i-th block of constraints by the n constraints of block i with highest values of $h_s(i)$
$L^h(i,n)$	Represents the i-th block of constraints by the n constraints of block i with lowest values of $h_s(i)$
$EX^h(i,n)$	Applies simultaneously RSs $L^h(i,n)$ and $H^h(i,n)$
$S(i,n)$	Represents the i-th block of constraints by sampling n constraints from the subset $\mathcal{M}_i$
	Capacitated Facility Location Problem with Stochastic Demand
$C^F(B,n)$	Represents constraints block B i.e., (7e), by partitioning facilities in n clusters based on their opening costs, and selecting, for each cluster, the constraint in B corresponding to the location closest to the centroid
$H^F(B,n)$	Represents constraints block B i.e., (7e), by means of the n constraint corresponding to the n facilities with the highest opening cost
$L^F(B,n)$	Represents constraints block B i.e., (7e), by means of the n constraint corresponding to the n facilities with the lowest opening cost
$EX^F(B,n)$	Applies simultaneously RSs $L^F(B,n)$ and $H^F(B,n)$
$C^C(A,n)$	Represents constraints block A i.e., (7c), by partitioning customers in n clusters based on their allocation costs, and selecting, for each cluster, the constraint in A corresponding to the customer closest to the centroid
$H^C(A,n)$	Represents constraints block A i.e., (7c), by means of the n constraint corresponding to the n customers with the highest allocation cost
$L^C(A,n)$	Represents constraints block A i.e., (7c), by means of the n constraint corresponding to the n customers with the lowest allocation cost
$EX^C(A,n)$	Applies simultaneously RSs $L^C(A,n)$ and $H^C(A,n)$
$EX^{h,F}(n)$	Applies simultaneously RSs $EX^h(A,n)$ and $EX^F(B,n)$
$EX^{C,F}(n)$	Applies simultaneously RSs $EX^C(A,n)$ and $EX^F(B,n)$
$EX^{h,T}(n)$	Applies simultaneously RSs $EX^h(A,n)$ and $EX^T(B,n)$
$EX^{C,T}(n)$	Applies simultaneously RSs $EX^C(A,n)$ and $EX^T(B,n)$

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