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Published in:
Circulation: mathématiques, historie, enseignement

Publication date:
2018

Circulation: mathématiques, histoire, enseignement,
dir. J. Auvinet, G. Moussard, X. Saint Raymond
Limoges : PULIM, 2018

Creating inquiry-reflective learning environments in mathematics through history and original sources

Tinne Hoff Kjeldsen *

Introduction

During the past decades there has been a lot of emphasis on inquiry-based teaching and learning in science and mathematics education research. The idea is to involve students in inquiry learning experiences that are similar to authentic scientific practices, to how mathematicians and scientists work. The hope is, that this will help students to develop a deep understanding of science and mathematics, the epistemology and the nature of these subjects, and motivate them for further studies in science and mathematics areas. Chinn and Hmelo-Silver interpret authentic inquiry as « activities that scientists engage in while conducting their research[1] ».

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1. Clark A. Chinn et Cindy E. Hmelo-Silver, « Authentic Inquiry : Introduction to the Special Section », *Science Education* 86 (2002), 171-174, p. 171.

This is not an easy task to implement in the teaching of mathematics as a scientific subject in itself. Research processes are seldom explicitly displayed in the teaching of mathematics.

In the present chapter, we will address this issue : How can students be brought in contact with mathematical research ? How can they obtain insights into how mathematics is generated and developed ? How can they come to identify and reflect upon activities that mathematicians engage in while conducting their research ?

One answer is through history of mathematics and teaching with historical sources. As phrased by Évelyne Barbin « Studying the history of mathematics allows one to study the construction of mathematical knowledge and to study mathematical activity – to analyze the role of problems, of proofs, of conjecture, of evidence, of error. *[...]* to see mathematics *[...]* as an activity, a human activity[2] ». Through engagement with sources from past mathematicians' work, students can obtain insights into, identify and come to reflect upon specific research strategies, tools and techniques mathematicians have used to generate mathematical knowledge. Even though our conception of mathematics has changed over time it still makes sense in mathematics education of today, to have students work with mathematical sources from the past as a way to become aware of how mathematicians get ideas, how mathematical objects are introduced into mathematics and take form, which strategies and methods mathematicians use when they create new mathematics, the significance of heuristic arguments in discovering new results, (changing) discussions and opinions about proofs among mathematicians. Students can gain insights into and come to reflect upon the epistemology and the nature of mathematics through engagement with historical episodes and sources displaying these mathematicians' work and thoughts with and about mathematics at a level that is accessible to them. I will use the phrase *inquiry-reflective learning environment in mathematics* to designate a learning environment that provides opportunities for students to gain such kind of insights and to reflect explicitly on inquiries that mathematicians engage in when they do research.

2. Évelyne BARBIN, « The meanings of Mathematical Proofs : On Relations Between History and Mathematical Education », dans *In Eves' Circles*, sous la dir. de Joby Milo ANTHONY, MAA Notes, vol. 34, 1994, p. 41.

It is important to be conscious about the conception of history in one's approach when using and interpreting sources from the past. When we want to use the past as a resource, a « window » through which students can gain insights into and reflect upon what mathematicians do when they create mathematics, we need to take the historical actors' intentions and motivations into account to understand how they had thought and acted in their particular context, i.e., treating mathematics as a cultural and historical product of knowledge. The Danish historian Bernard Eric Jensen[3] advocates for a multiple perspective approach to history, where history is studied from various points of observation, based on concrete projects and activities of the actors. This approach can be adapted into history of mathematics to create an inquiry-reflective learning environment in the sense described above by studying concrete episodes of mathematical research, of former mathematicians' mathematical activities from their practices of mathematics, following the development of their ideas, their arguments, techniques, and views of mathematics.

When we use something from the past in a present context whether it is for historical research in itself or for teaching there is always also a present perspective – the perspective of the historian, or the user of history. In historical research the choice of perspective is determined in a sort of a dialectic process between the historian's project and the historical actors', as they unfold during the research process. When we use history to create an inquiry-reflective learning environment as described above, the perspectives will relate to aspects of mathematical research processes and issues that the teacher wants the students to become aware of[4].

In the present context, where we want to use history to create an inquiry-reflective learning environment where students can come in contact with mathematical research practices, the focus could

3. Bernard Eric Jensen, *Historie-livsverden og fag*, Copenhagen : Gyldendal, 2003, p. 16-17.

4. Tinne Hoff Kjeldsen, « Uses of History for the Learning of and about Mathematics : Towards a theoretical framework for integrating history of mathematics in mathematics education », dans *Proceedings of the International Conference on History and Pedagogy of Mathematics (HPM)*, sous la dir. d'Évelyne Barbin, Sunwook Hwang et Constantinos Tzanakis, Daejeon (Korea), 2012, p. 1-21.

be on capturing features of authentic inquiry, pinpointing driving forces and motivations for research, identifying specific strategies that are effective in mathematical research or key epistemological features of mathematical inquiry which can be found in historical episodes and sources.

In the following we will investigate how history of mathematics and working with original sources can function in the practice of teaching as a means for creating such inquiry-reflective learning environments in mathematics[5].

Experiences from the mathematics program at Roskilde University (RUC) in Denmark with problem-oriented project learning (PPL) will be drawn upon and it will be discussed in what sense an inquiry-reflective learning environment is established. Three student projects will be used as illustration. The question whether such an inquiry-reflective learning environment can be implemented in general mathematics classrooms will be discussed. As an affirmative answer, an analysis of an experimental course that was taught over a couple of weeks in an ordinary Danish high school mathematics classroom will be discussed.

The RUC model of PPL in the mathematics program

The student movement played a significant role in experimenting with new pedagogies when RUC was established in Denmark in 1972. They argued for university programs based on the principles of problem-oriented, interdisciplinary and participant-directed project work (in daily life at RUC referred to as problem-oriented project learning). Together with the principle of exemplarity, these became and still are the key concepts in the educational philosophy and organization of study programs and university pedagogy at Roskilde University. I will not go further into the histori-

5. Parts of earlier versions of these ideas have been published in Tinne Hoff Kjeldsen, « Teaching With and About Nature of Mathematics through History of Mathematics : Enacting Inquiry Learning in Mathematics », *Education Sciences* (2014), sous la dir. de Michael Kourkoulos et Constantinos Tzanakis, 38-55 ; Tinne Hoff Kjeldsen, « Enacting Inquiry Learning in Mathematics through History », dans *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics*, sous la dir. de Luis Radford, Fulvia Furinghetti et Thomas Hausberger, Montpellier : IREM de Montpellier, 2016, p. 453-464.

cal and societal conditions for its establishment, interested readers are referred to part I in Andersen and Heilesen[6].

The notion of exemplarity is an important principle in the theoretical basis for PPL at RUC. In the natural science programs, the understanding and implementation of the principle was inspired by the German physicist Martin Wagenschein who saw exemplarity as an answer to the 20th century increasing amount of content. The idea is that students will obtain insights and knowledge about a scientific subject and its structure through in-depth studies of a few paradigmatic cases that are representative for the science in question.

In the problem-oriented participant-directed project work at RUC it is essential that the students formulate and choose the problem for their project work, that they themselves search and find relevant literature, and decide on and argue for theoretical frameworks and methods. The ideal is that students should study in ways that imitate how researchers work when they do research. The students work in groups of 2-8 students, and each group of students has a professor as their supervisor.

In all programs, each semester begins with a week-long group formation process[7] where all the students (between 20 and 120 depending on the program) at the same level in a particular program participate together with the professors who have been assigned to supervise the projects for this group of students. It has form as a seminar with discussions where the professors present their research and outline areas of interesting problems for the project work. The students also present suggestions for problems which are discussed and qualified in plenary sessions with all students and professors. In close dialogue with all the supervising professors, the students form groups according to interests, and at the end of the week, they hand in a preliminary formulation of a problem to be approved by the study board.

The study board assigns a specific professor to each approved project. The professor supervises the group throughout the semester in weekly meetings, where they together discuss places to look

6. Anders Siig ANDERSEN et Simon Bjørn HEILESEN, *The Roskilde Model : Problem-Oriented Learning and Project Work*, Cham, Heidelberg, New York, Dordrecht, London : Springer, 2015.

7. *Ibid.*

for literature, the problem formulation, methodology etc. However, it is the students' responsibility to provide input to these discussions. If they do not provide anything before a meeting with the supervisor, he or she will ask the group for input on the relevant issues before the next meeting. During the semester, the first preliminary formulation of the problem is often modified, changed and sometimes discarded and replaced by a new problem along with the students' learning process in discussions with their supervisor. In the final project report written by the students, they are also required to explain the epistemology, and to argue for the theoretical foundation of their method. The students have discussed these issues in meetings with the supervisor, in a midterm seminar with peer-review and feedback from another group of students and their supervisor and at a similar seminar towards the end of the semester before they hand in their project report for the exam. The project work takes up half of the students' workload in each semester. The other half constitute regular course work in more traditional sense.

There is no specific pre-determined content matter attached to the projects. The projects are defined by themes that guide the students' work. There are three projects in the mathematics program : 1) a *modeling* project (on the bachelor level), 2) a *mathematics as a scientific discipline* project, and 3) a *master thesis*. It is in the mathematics as a scientific discipline project that the students are supposed to get insights into how mathematics functions as a (growing) body of knowledge. It is the students' first project on the master level, and their 7$^{\text{th}}$ PPL project. Before the students enter into the master's program they have completed a three year interdisciplinary Bachelors of Science program where they have specialized in mathematics and another subject, one semester of study in each specialization. In the study regulation, the mathematics as a scientific discipline project is described as follows :

> *The project should deal with the nature of mathematics and its « architecture » as a scientific subject such as its concepts, methods, theories, foundation etc., in such a way that the nature of mathematics, its epistemological status, its historical development and/or its place in society get illuminated.*

There is a wide variety among the projects that students at

RUC have worked with under the mathematics as a scientific discipline over the years, in which history is used, e.g. *The contribution of Galois to the Development of Abstract Algebra* ; *Euler and Bolzano : A mathematical analysis in an epistemological perspective* ; *Fourier and the Function Concept : From Euler's to Dirichlet's concept of a function* ; *The Real Numbers – Constructions in the 1870s.*

Together, the PPL and the requirements for the mathematics as a scientific discipline project create an inquiry-reflective learning environment in mathematics in the sense outlined above. This will be illustrated in the next section through analyses of three projects.

Analysis of three PPL student projects

The three project reports represent different aspects of what influences and drives research in mathematics. The first project illustrates ways in which new concepts are introduced into mathematics and generalizations, symmetry and simplicity as motivations for research ; the second deals with discussions of proofs, i.e. the validation of mathematical knowledge ; and the third one deals with discussions and research motivated by foundational issues in mathematics.

The title of the first project is *Generalizations in the Theory of Integration : An Investigation of the Lebesgue Integral, the Radon Integral and the Perron Integral* – a 72 pages long project report[8] written by two students. The background for the project was their curiosity about the Lebesgue integral which was mentioned in passing, but not treated, in the textbook for their analysis course. They wanted to know the reason behind the introduction of the concept of the Lebesgue integral into mathematics, and to learn something about the theory. They began to investigate the integral concept, and, as they write in the introduction to their project report[9] :

8. http ://milne.ruc.dk/imfufatekster/pdf/403.pdf.
9. All translations of titles and quotes from the project reports are mine.

> *Very soon more questions than answers showed up, for example we discovered that there is a whole variety of different integrals ; the Denjoy-, Perron-, Henstock-, Radon-Stieltjes- and Burkill-integral – to name just a few. All these integrals are most often described in the literature as* generalizations*, and sometimes as* extensions*, of either the Riemann or the Lebesgue integral. This gave rise to questions such as : What do these integrals do ? Why have so many types of integrals been developed ? Why is it always the Lebesgue integral we hear about ? What is meant by* generalizations *in this respect ? In what sense are the various integrals generalizations of former definitions of integrals ? Are the generalizations of the same character*[10] *?*

Regarding the « invitation » into the practice of mathematical research, the students wrote :

> *The development of mathematics is both guided by the people who create the new mathematics and the significance of the mathematics that is being developed. In order for some mathematics to be created within an area, there have to be some mathematicians who are actually interested in investigating and uncovering the area. By studying the motivation of the people who have helped develop the integral concept, we can gain insight into why a mathematical field is being studied*[11]*.*

– and « how this went on in practice », they might have added.

The students decided to focus on Lebesgue, Perron and Radon. They justified their choice by referring to the difference between the character of these mathematicians' work, which they found to be so different from one another that an analysis of their work could span a « space » of possible answers to the students' original questions. With this restriction, the problem formulation that guided the students' project work read as follows :

> *What were Lebesgue, Perron and Radon motivated by in their respective generalizations of the integral concept ? What character and scope do the generalizations introduced by Lebesgue, Perron and Radon have, and what are the differences between them*[12] *?*

10. Stine Timmermann et Eva Uhre, *Generalizations in the theory of integration : an investigation of the Lebesgue integral, the Radon integral and the Perron integral*, in Danish, IMFUFA, text 403, Roskilde University, 2001, p. 1, italic in the original.
11. *Ibid.*, p. 2.
12. *Ibid.*, p. 3.

The students read literature from historians of mathematics, and they traced and read mathematical papers and books of Lebesgue, Perron and Radon. They looked for and discussed with their supervisor how these mathematicians worked with the mathematical ideas that motivated them to develop their respective integrals. In their work with the original sources, the students analyzed and compared the content of the sources with respect to the motivation of the mathematicians, why they created these generalized integral concepts, and the differences and similarities between the characteristics and scope of the generalizations. The students placed the work of these mathematicians in historical context, discussing the development from Riemann to Lebesgue of the function concept, Fourier series, and the measure concept with focus on the work by Jordan and Borel.

To be more specific, the students studied five of Lebesgue's notes in *Comptes Rendus de l'Académie des sciences de Paris* published in the period 1899-1901, and parts of his thesis *Intégrale, Longueur, Aire* which was published in 1902. They presented, discussed and analyzed the publications with respect to the mathematical problems, concepts, techniques, proofs and approaches that Lebesgue used in order to give some answers to their problem formulation. They found that Lebesgue was motivated by two factors : his interests in determining areas of surfaces and lengths of curves, and the (lacking) symmetry in Riemann's integral concept of what Lebesgue called « the fundamental problem of integral calculus », i.e. to find a function when its derivative is known. Lebesgue addressed the second aspect explicitly in his thesis, stating that :

> *... integration, as defined by Riemann, does not allow us to solve all cases of the fundamental problem of the integral calculus : Find a function when the derivative is known. It may therefore seem natural to look for another definition of the integral, such that, in more general cases, integration is the inverse operation of differentiation*[13].

From the secondary literature they learned that Lebesgue had difficulties getting his thesis published and that it was received with some reservation, because of his treatment of discontinuous functions and functions without continuous derivatives. They conclu-

13. *Ibid.*, p. 20.

ded that the Lebesgue integral is an extension of the Riemann integral for bounded functions, that it is not the same as the Cauchy-Riemann integral (improper integral) for unbounded functions, and that Lebesgue's integral is based on generalization of the measure concept.

Perron published his integral concept a decade later in 1914 in the paper « Über den Integralbegriff ». In the paper he gave the following motivation :

> *In what follows I propose a definition of the definite integral which, as I shall show, is at least as far-reaching as Lebesgue's. But it is much more elementary, and the proof of the fundamental theorems appears much easier. Nothing at all is assumed from the theory of point sets*[14].

The students found that Perron based his definition on the concepts of upper and lower adjoint functions for a bounded function f defined on an interval $[a, b]$. These adjoint functions can be interpreted as approximating anti-derivatives of f. The integral is based on the supremum and infimum of the values in b of the upper and lower adjoint functions respectively.

The students concluded that Perron was motivated by trying to find, what he considered a more elementary integral concept than Lebesgue's, and that he with « elementary » meant the avoidance of measure theory. The students considered Perron's integral to be more abstract than Lebesgue's because the intuitive conception of the integral as an area or a measure is lost, but that it has, what they called the « didactical advantage » that the definition is based on the anti-derivative.

The last integral concept, the students looked at, was the one introduced by Radon in his paper « Theorie und Anwendungen der absolut additive Megenfunktionen » from 1913. The students concluded that Radon first generalized the Stieltjes integral so it became possible to integrate any additive set function, where after he generalized this integral concept in the same manner as Lebesgue generalized the Riemann integral. Radon's motivation to generalize the integral concept came out of a need in his work with integral equations.

14. *Ibid.*, p. 37.

Through this project, the students gained insights into mathematicians' motivations for defining a new or extending an existing integral concept. They wrote in their report that :

> *There have been different motivations associated with the development of integral concepts* [...]*. This illustrates that* [...] *there may well be several reasons involved in the creation of new mathematics. This also illustrates that mathematics does not necessarily evolve along a beaten path and that perhaps it is only in hindsight that a certain approach appears to be the most natural*[15].

They also came to reflect upon and identify « generalization » as a way in which mathematics works, so to speak, as a strategy that guides some research in mathematics. They discussed what they called « the status » of the three generalizations with respect to their significance for the development of mathematics. They found that a mathematical result's importance in the history of mathematics can be due to applications that it led to, and/or its role as a driving force for initiating new developments in mathematics, and/or as a kind of « cleaning up » result, and/or its importance for clarifying the theoretical foundation.

In the two next examples I will only draw out the essence with respect to inquiry in mathematics. The first example is the project *D'Alembert and the Fundamental Theorem of Algebra*[16]. This project was developed by two students in 2003. They were curious about the reception of mathematical proofs. They took point of departure in D'Alembert's proof from 1748 of the fundamental theorem of algebra and the discussion about its « validity ». They primarily focused on Gauss, but they also took interpretations from historians of mathematics into account. I will not go into the technical details of the students' work but focus on the insights they gained into mathematical research. They realized that mathematicians' understanding of what constitute a proof is subject to changes over time, and that new standards emerge. They discovered that even though Gauss' proof for the theorem became accepted as complete, mathematicians continued to find new proofs for the theorem. They realized that an important component of

15. *Ibid.*, p. 57.

16. Signe Høg Andersen et Trine Fich Pedersen, *D'Alembert and the Fundamental Theorem of Algebra*, in Danish, Project Report, Imfufa, Roskilde University, 2003.

mathematical research is to find new and other kind of proofs for mathematical theorems that already belong to the body of mathematical knowledge – and that such work is an important part of inquiry in mathematics.

The last example deals with foundational issues in mathematics. The project is written by a student who was interested in the role played by the paradoxes in the development of set theory[17]. He focused on the discussion of Zermelo's notion of « definit », which Zermelo introduced in his axiom of separation in his 1908 paper « Untersuchungen über die Grundlagen der Mengenlehre, 1 ». Zermelo's definition of « definit » was found to be too vague by Fraenkel in a paper from 1922 « The notion of 'definit' and the independence of the axiom of choice » where Fraenkel investigated the independence of the axiom of choice in Zermelo's system. He discussed the shortcomings of Zermelo's notion of « definit », and introduced another formulation of the notion. The Norwegian mathematician and logician Thoralf Skolem gave a talk at the 5th Scandinavian Congress of Mathematicians in Helsinki in 1922, where he criticized mathematicians' attempts to build a foundation for mathematics on the axiomatization in terms of sets, which he found to be unsatisfactory as an ultimate foundation. He presented yet another formulation of « definit ». Zermelo continued the discussion of the notion of 'definit' in a paper from 1929, in which he disagreed with Fraenkel's approach. Skolem published a reaction to Zermelo's paper a year later. The student followed the mathematicians' discussion by studying these five papers. He found that the most important theme in the discussion of the notion of « definit »was :

> [...] *the discussion between Zermelo and Skolem about the status of the natural numbers with respect to set theory, and in connection with this, the difference between an axiomatic formulation (Zermelo) and a constructive formulation (Skolem) of the notion of « definit*[18] *».*

If we compare the analyzes of the three projects in relation to reflections on inquiry in mathematics as a research discipline,

17. Steffen Holm, *Paradoxes in Set Theory and Zermelo's 3th. Axiom*, Project Report, Imfufa, Roskilde University, 1996.
18. *Ibid.*, p. 42.

we can say that with respect to these three projects, an inquiry-reflective learning environment in mathematics was created in the sense that the students obtained in-depth insights into discussions among mathematicians in the development of mathematics – discussions about which kind of functions should be treated, about what constitute a proper proof, about how to formulate a notion. They became aware of, through history, that mathematicians sometimes introduce new definitions of concepts for the sake of simplicity or because they find another definition to be more « natural », and that such work can motivate further research, spur new developments and make new connections possible. They realized that new ways of proving and arguing for statements give rise to new insights and discussions and disagreements about approaches in mathematical research. It is characteristic for all three projects that the students, through the historical episodes and their own inquiries about mathematics in their work with the original sources, became engaged in understanding some of the problems mathematicians face and some of the resources and strategies they use when they do mathematical research.

The mathematics program and the problem-oriented project learning at RUC supply the institutional and pedagogical structure for such inquiry-reflective learning environments where it is possible to implement a multiple perspective approach to history as illustrated by the analysis of the three projects. This is a rather unique study program, so the question is, how can these ideas of establishing inquiry-reflective learning environments in mathematics through history be implemented in mathematics education in a broader sense, in mathematics classrooms in general, where such a structure is not provided by the institution and/or study regulations ? It can be done on a small scale with specific, carefully designed student activities – in courses and/or projects that can run for a week or two along with the ordinary mathematics teaching. Even though history of mathematics might not be in the curricula, mathematical thinking strategies, argumentation and proofs are core elements in any mathematics education, elements that students will be engaged in if using history to create an inquiry-reflective learning environment in the sense discussed above.

In the next section an example of such a course from a teaching experiment in a Danish high school will be presented as an

« existence proof ». The experiment was carried out by a graduate student from Roskilde University.

Analysis of a teaching experiment in high school

The teaching experiment « The concept of a function viewed through historical and contemporary glasses » was carried out in a Danish high school class (11$^{\text{th}}$ graders)[19]. The purpose of the experiment was to investigate possible benefits of integrating history in mathematics teaching. The experiment was also part of a research project, namely to test the hypothesis that history can function at the core of teaching mathematics as a means to reveal meta-discursive rules of mathematics and make them explicit objects of students' reflections and (maybe) change[20]. The high school students' discussions in group work sessions were audio-taped. The teaching experiment will be analyzed with respect to its potential to create an inquiry-reflective learning environment in mathematics through history[21].

The historical foci, that were used in the experiment, were divided into four parts : 1) The development of Leonard Euler's (1707-1783) concept of a function, 2) the debate of the vibrating string in the 18$^{\text{th}}$ century, 3) changes in the organization of mathematics education in connection with the French revolution in 1789, and 4) Dirichlet's (1805-1859) concept of a function from the 19$^{\text{th}}$ century. The students were introduced to two meta-rules : the so-called « general validity of analysis » which refers to the norm that results, techniques and statements of analysis should be generally valid ; and « the generality of the variable » which refers to the norm that a variable in a function could take on all values – it could not be restricted to e.g. an interval. These rules were

19. Pernille Hviid Petersen, *Potentielle vindinger ved inddragelse af matematikhistorie i matematikundervisningen*, Master Thesis, Imfufa, Roskilde University, 2011.

20. Tinne Hoff Kjeldsen et Morten Blomhøj, « Beyond Motivation - History as a method for the learning of meta-discursive rules in mathematics », *Educational Studies in Mathematics* 80 (2012), 327-349.

21. For results about the role of history for revealing students' meta-discursive rules, see Tinne Hoff Kjeldsen et Pernille Hviid Petersen, « Bridging History of the Concept of Function with Learning of Mathematics : Students' Meta-Discursive Rules, Concept Formation and Historical Awareness », *Science & Education* 23 (2014), 29-45.

accepted in the mathematical discourse of Euler but they were not part of Dirichlet's meta-discursive rules. The intentions were on the one hand, that students should become aware that there are meta-rules in mathematics and that they are historically given, and on the other hand that the students should come to reflect upon the role of proofs and the domain of a function in contemporary mathematics.

The students worked with the historical episodes and analyzed excerpt of original sources (translated into Danish), e.g. Euler's definition of a variable and a function in *Introductio in Analysin Infinitorum* from 1748 ; D'Alembert's solution of the wave equations from 1747 and the discussion of the vibrating string, which led Euler to extend his concept of function into what he called discontinuous functions, because the motion of a plucked string was excluded from D'Alembert's solution and Euler was of the opinion that mathematics should be able to account for all situations in physics[22] ; and an excerpt of Dirichlet's concept of a function from 1837.

The students' work was also guided by a list of different kinds of resources in history of mathematics, and some history of mathematics books and articles. Marks were placed in the material to assist the students' use of the materials. The main guidance of the students' work was done by worksheets that were explicitly designed to lead the students into discussions of meta-discursive rules of the past and to compare them with how these issues are conceived of today, and to have them reflect upon the concept of a domain of a function in contemporary mathematics and the role of proofs.

The course was implemented in two steps using a matrix structure : in step 1, the students were divided into four groups, called « basis » groups. They worked in the basis groups for 5 lessons of 50 minutes with homework between the lessons. Each basis group worked with their own topic (see below). In step 2, four so-called « expert » groups were formed consisting of at least one student from each of the basis groups. In this way, all the knowledge that had been produced in the basis groups was in principle available in

22. Jesper Lützen, « Funktionsbegrebets udvikling fra Euler til Dirichlet », *Nordisk Matematisk Tidskrift* 25 (1978), 5-32.

the expert groups. The students worked in the expert groups for 4 lessons with homework between the lessons. The topics for the four basis groups were : 1) Historical definitions of a function ; 2) The debate of the vibrating string ; 3) Euler, Dirichlet and the society in which they lived ; and 4) The modern concept of a function. Each basis group wrote a report completing the tasks formulated in their worksheet. The 4 expert groups of step 2 all worked with the same task. They were asked to write a paper to be « published » in the journal *Nordisk Matematisk Tidsskrift*. They received a made up invitation from the editor, who told them that there was a heated debate among two groups of mathematicians : one group was of the opinion that mathematical concepts are static, timeless entities, whereas the second group believed that mathematical concepts are the result of a process of development. Each group was asked to contribute to the discussion, forming their own opinion about this issue. They were requested to argue for their opinions based on the collected work that had been done in the basis groups.

The experiment showed that the high school students found that the present concept of a function was the result of a historical development, and that they became acquainted with inquiry processes in mathematics, as illustrated by the following quotes from two groups of students' paper for the journal :

> *The reason why Euler began to work with* [Euler-] *discontinuous functions was because of a debate between contemporary mathematicians. The debate concerned the fact that the functions the mathematicians worked with could not describe a vibrating string.*
>
> [...] *the development of the concept of a function was among other things due to human attitudes and interpretations, which were important factors. For example, some of Euler's contemporary mathematics colleagues were of the opinion that Euler's extended function concept should not be used because it went against the principle of mathematics. They thought it was cheating. This meant that Euler's extended function concept never came to be used as intended, and a new function concept was developed by Dirichlet*[23].

These students became aware of at least one source for mathematical research questions, and of discussions among mathema-

23. Petersen, *Potentielle vindinger ved inddragelse af matematikhistorie i matematikundervisningen*, *op. cit.*, App. C.

ticians that relate to the validation of mathematical knowledge. In this sense, this experimental teaching course confirms that it is possible to use historical episodes and sources in mathematics teaching in high school to create an inquiry-reflective learning environment in mathematics.

Conclusion

The analyses of the students' projects at RUC and the high school teaching experiment show that it is possible to create inquiry-reflective learning environments in mathematics (in the sense explained in the introduction) through history and original sources. In both settings the students were brought into contact with aspects of mathematical research. They gained insights into and came to reflect explicitly upon inquiries that mathematicians engage in when they do research – and, to quote Évelyne Barbin again, they studied aspects of « the construction of mathematical knowledge », and they were presented with mathematics « as an activity, a human activity[24] ».

If we compare the PPL in the mathematics program at Roskilde University and the teaching experiment from the Danish high school, a key issue in the creation of the inquiry-reflective learning environments in mathematics is the use of history and original sources, that allow the students to get glimpses into aspects of authentic research processes in mathematics and related debates and discussions, together with requirements in the study regulation or carefully designed instructions for reflections. In Johansen and Kjeldsen (2018)[25] we have presented a first version of our development of a methodology for using history and original sources to create and implement inquiry-reflective learning environments in mathematics more generally in mathematics classrooms.

24. Barbin, « The meanings of Mathematical Proofs : On Relations Between History and Mathematical Education », *op. cit.*, p. 41.

25. Mikkel Willum Johansen et Tinne Hoff Kjeldsen, « Inquiry-Reflective Learning Environments and the Use of the History of Artifacts as a Resource in Mathematics Education », dans *Mathematics, Education and History : Towards a Harmonious Partnership*, sous la dir. de Kathleen Clark, Tinne Hoff Kjeldsen, Sebastian Schorcht et Constantinos Tzanakis, ICME-13 Monographs, Cham : Springer, 2018, p. 27-42.

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