
Theory including future not excluded: Formulation of complex action theory II

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In Ref. [1] we have found errata. They are composed of two parts: one part is for the body, which is also explained in our recently published book [2], while the other part is for the appendix, which is mainly a result of the corrections to Ref. [3]. They do not influence the result of the manuscript. Rather, the latter part provides us a new additional result: the Schrödinger equation described with the Hamiltonian \(\hat{H}_B\) has been derived for the future state \(|B(t)\rangle\) via the Feynman path integral in the complex action theory.

In the fifth line below Eq. (5.8), where \(f(D)f(D)^\dagger\mathpzc{=}F\mathpzc{=}(\sum_{\lambda}\lambda|\lambda\rangle\langle\lambda|)^{-1}\), we have chosen \(f(D)\) such that \((P\mathpzc{=})^{-1}(f(D)f(D)^\dagger)^{-1}P\mathpzc{=}F(\hat{H}^\dagger)\), which is rewritten as \((f(D)f(D)^\dagger)^{-1}\mathpzc{=}F(D)\). However, this relation does not stand, because the left-hand side is Hermitian, while the right-hand side is not Hermitian. Accordingly, the expression \(Q\mathpzc{=}F(\hat{H}^\dagger)Q\) below Eq. (5.8), which was introduced based on the above relation, has to be corrected. In addition, the next statement, “\(F(\hat{H}^\dagger)Q\mathpzc{=}F(\hat{H}_B^\dagger)Q\) for the restricted subspace,” is not right. This is because, for any reasonable function \(h\) and any state \(|A(t)\rangle\mathpzc{=}\sum_i a_i(t)|\lambda_i\rangle\) that obeys the Schrödinger equation \(i\hbar\frac{d}{dt}\langle A(t)\rangle\mathpzc{=}\hat{H}|A(t)\rangle\), the following relation holds for large \(t\mathpzc{=}T_A\): \(h(\hat{H})|A(t)\rangle\mathpzc{=}h(\hat{H}_{\text{eff}}+iB\Lambda_A)|A(t)\rangle\mathpzc{=}h(\hat{H}_{\text{eff}})|\tilde{A}(t)\rangle\), where we have used the automatic Hermiticity mechanism and introduced \(\tilde{A}(t)\mathpzc{=}\sum_{i\in A}a_i(t)|\lambda_i\rangle\), \(\Lambda_A\mathpzc{=}\sum_{i\in A}(|\lambda_i\rangle\langle\lambda_i|Q\), and another function \(\tilde{h}\) such that \(\tilde{h}(\text{Re}\lambda_i)\mathpzc{=}h(\text{Re}\lambda_i+iB)\). Similarly, the statement “\(Q\mathpzc{=}F(\hat{H}_{\text{eff}})Q\) for the restricted subspace” given in Eq. (5.6) has to be corrected.

To correct the above points, on behalf of \(F(Re\lambda_i)\mathpzc{=}|b_i|^2\) and Eq. (5.6), we introduce functions \(G\) and \(\tilde{G}\) such that \(G(\text{Re}\lambda_i+iB)\mathpzc{=}\tilde{G}(\text{Re}\lambda_i)\mathpzc{=}b_i\), and express \(Q\) as follows:

\[
Q_2 = \sum_{i\in A} |b_i|^2 |\lambda_i\rangle_B \langle \lambda_i|B
\]

\[
= \sum_{i\in A} G(\hat{H}_{\text{eff}}+iB\Lambda_A)^\dagger|\lambda_i\rangle_B \langle \lambda_i|G(\hat{H}_{\text{eff}}+iB\Lambda_A)
\]

\[
= \tilde{G}(\hat{H}_{\text{eff}})^\dagger Q\Lambda_A \tilde{G}(\hat{H}_{\text{eff}}),
\]

where, in the second and third equalities, supposing that \(\text{Re}\lambda_i\)'s are not degenerate, we have used \(|\lambda_i\rangle_B\mathpzc{=}Q|\lambda_i\rangle\), and \(b(\lambda_i)G(\text{Re}\lambda_i+iB)\mathpzc{=}b(\lambda_i)G(\hat{H}_{\text{eff}}+iB\Lambda_A)\) for \(i\in A\). We note that
\[ Q \Lambda_A = Q \sum_{i \in A} |\lambda_i\rangle \langle \lambda_i|_Q \] is Hermitian, and so is \( Q_2 \). Next we define \( Q' \) by \( Q' = G(\hat{H})^\dagger QG(\hat{H}) = (PG^{-1})^{-1}PG^{-1} \), where \( PG^{-1} = G(\hat{H})^{-1}P \) diagonalizes \( \hat{H} \). (\( PG^{-1})^{-1}P^{-1}PG^{-1} = P^{-1}\hat{H}P = D \).

In addition, we introduce \( |\lambda_i\rangle^{G^{-1}} = G(\hat{H})^{-1}|\lambda_i\rangle \), so that \( |\lambda_i\rangle^{G^{-1}} \) is \( Q' \)-orthogonal, i.e., orthogonal with regard to the proper inner product \( I_Q' : I_Q'(|\lambda_i\rangle^{G^{-1}}, |\lambda_j\rangle^{G^{-1}}) \equiv G^{-1} \langle \lambda_i|Q'| \lambda_j\rangle^{G^{-1}} = \delta_{ij} \).

We use the automatic Hermiticity mechanism for large \( t - T_A \). Then, since \( |A(t)\rangle \) behaves as \( |\tilde{A}(t)\rangle = \sum_{i \in A} a_i(t)|\lambda_i\rangle \), \( Q' \) used in the normalized matrix element \( \langle O^{\dagger}_A |Q_2 - t \rangle \) is estimated in the subspace restricted by \( A \) as follows:

\[
Q' \approx G(\hat{H}_{\text{eff}} + iB\Lambda_A)^\dagger Q \Lambda_A G(\hat{H}_{\text{eff}} + iB\Lambda_A) \quad \text{for the restricted subspace}
\]

\[
= \tilde{G}(\hat{H}_{\text{eff}})^\dagger Q \Lambda_A \tilde{G}(\hat{H}_{\text{eff}})
\]

\[
= Q_2,
\]

where in the last equality we have used Eq. (1). The three sentences “We first point out … replaced with \( |\tilde{A}(t)\rangle \)” below Eq. (5.8) should be replaced with the above argument.

A \( dt \)-dependent normalization factor, say \( \frac{1}{\alpha(dt)} \), should be inserted on the right-hand sides of Eq. (A.2) and of the first line of Eq. (A.4). The following sentence should be inserted after the sentence “\( C \) is an arbitrary … complex plane” below Eq. (A.2): “In addition, \( \alpha(dt) \) is a \( dt \)-dependent normalization factor, which is properly fixed later.” The factor \( \sqrt{2\pi\hbar dt/m} \) in the second line of Eq. (A.4) should be deleted. The following sentences should be inserted after the phrase “where … Eq. (3.7)” below Eq. (A.4): “Here we have taken \( \alpha(dt) = \sqrt{2\pi\hbar dt/m} \) so that both sides of Eq. (A.4) correspond to each other in the vanishing limit of \( dt \). Then Eq. (A.4) is reduced to \( |\psi(t + dt)\rangle = e^{-\frac{i}{\hbar}\hat{H} dt} |\psi(t)\rangle \).” The next sentence, “Thus we have found that … Eq. (A.2),” below Eq. (A.4) should be replaced with “Thus we have derived the Schrödinger equation and found that … Eq.(A.2).” The following sentence should be added after the above replaced sentence: “Such a derivation of the Schrödinger equation is well known in the real action theory [4].” Factors \( \frac{1}{\alpha(dt)^2}, \frac{1}{\alpha(-dt)^2} \), and \( \frac{1}{\alpha(-dt)} \) should be inserted on the right-hand side of the equation in the second sentence of the last paragraph of the appendix, on the right-hand sides of Eqs (A.5) and (A.6), respectively. The second sentence below Eq. (A.6), “Indeed, \( \hat{H}_B \) is given … \( \hat{H}^\dagger \),” should be replaced with “Indeed, we obtain the Schrödinger equation \( |B(t - dt)\rangle = e^{\frac{\pi}{\hbar}\hat{H}_B dt} |B(t)\rangle \), where \( \hat{H}_B \) is given … \( \hat{H}^\dagger \).”

References