
Theory including future not excluded: Formulation of complex action theory II

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In Ref. [1] we have found errata. They are composed of two parts: one part is for the body, which is also explained in our recently published book [2], while the other part is for the appendix, which is mainly a result of the corrections to Ref. [3]. They do not influence the result of the manuscript. Rather, the latter part provides us a new additional result: the Schrödinger equation described with the Hamiltonian $\hat{H}_B$ has been derived for the future state $|B(t)\rangle$ via the Feynman path integral in the complex action theory.

In the fifth line below Eq. (5.8), where $f(D)^{-1}$ should have been replaced with $(f(D)f(D)^{\dagger})^{-1}$, we have chosen $f(D)$ such that $(P^f)^{-1}(f(D)f(D)^{\dagger})^{-1}P^f = F(\hat{H}^f)$, which is rewritten as $(f(D)f(D)^{\dagger})^{-1} = F(D)^{\dagger}$. However, this relation does not stand, because the left-hand side is Hermitian, while the right-hand side is not Hermitian. Accordingly, the expression $Q' = F(\hat{H}^f)Q$ below Eq. (5.8), which was introduced based on the above relation, has to be corrected. In addition, the next statement, “$F(\hat{H}^f)Q \simeq F(\hat{H}_B^f)Q$ for the restricted subspace,” is not right. This is because, for any reasonable function $\lambda$ and any state $|A(t)\rangle = \sum_i a_i(t)|\lambda_i\rangle$ that obeys the Schrödinger equation $ih\frac{d}{dt}|A(t)\rangle = \hat{H}|A(t)\rangle$, the following relation holds for large $t - T_A$: $h(\hat{H})|A(t)\rangle \simeq h(\hat{H}_B + iA\Lambda_A)|A(t)\rangle \equiv \hat{h}(\hat{H}_B)|A(t)\rangle$, where we have used the automatic Hermiticity mechanism and introduced $|\tilde{A}(t)\rangle \equiv \sum_{i \in A} |a_i(t)|\lambda_i\rangle$, $\Lambda_A \equiv \sum_{i \in A} |\lambda_i\rangle\langle \lambda_i|$, and another function $\tilde{h}$ such that $\tilde{h}(\text{Re} \lambda_i) = h(\text{Re} \lambda_i + iB)$. Similarly, the statement “$Q_2 = F(\hat{H}_B^f)Q$ for the restricted subspace” given in Eq. (5.6) has to be corrected.

To correct the above points, on behalf of $F(\text{Re}\lambda_i) = |b_i|^2$ and Eq. (5.6), we introduce functions $G$ and $\tilde{G}$ such that $G(\text{Re} \lambda_i + iB) = \tilde{G}(\text{Re} \lambda_i) = b_i$, and express $Q_2$ as follows:

$$Q_2 = \sum_{i \in A} |b_i|^2|\lambda_i\rangle_B \langle B|\lambda_i|$$

$$= \sum_{i \in A} G(\hat{H}_B + iB\Lambda_A)^{\dagger}|\lambda_i\rangle_B \langle B|\lambda_i|G(\hat{H}_B + iB\Lambda_A)$$

$$= \tilde{G}(\hat{H}_B)^{\dagger}Q\Lambda_A\tilde{G}(\hat{H}_B),$$

where, in the second and third equalities, supposing that $\text{Re} \lambda_i$’s are not degenerate, we have used $|\lambda_i\rangle_B = Q|\lambda_i\rangle$, and $b(\lambda_i|G(\text{Re} \lambda_i + iB) = b(\lambda_i|G(\hat{H}_B + iB\Lambda_A)$ for $i \in A$. We note that
\[ Q\Lambda_A = Q\sum_{i\in A} |\lambda_i\rangle\langle\lambda_i|_Q \] is Hermitian, and so is \( Q_2 \). Next we define \( Q' \) by \( Q' \equiv G(\hat{H})^\dagger QG(\hat{H}) = (P_{G-1})^{-1}P_{G-1}, \) where \( P_{G-1} \equiv G(\hat{H})^{-1}P \) diagonalizes \( \hat{H} : (P_{G-1})^{-1}\hat{H}P_{G-1} = P^{-1}\hat{H}P = D. \) In addition, we introduce \( |\lambda_i|^{G^{-1}} \equiv G(\hat{H})^{-1}|\lambda_i|, \) so that \( \varepsilon_{ij} \equiv G(\hat{H})^{-1}|\lambda_i|Q'| |\lambda_j|^{G^{-1}} = \delta_{ij} \).

We use the automatic Hermiticity mechanism for large \( t - T_A \). Then, since \( |A(t)| \) behaves as \( |A(t)| \equiv \sum_{i\in A} a_i(t)|\lambda_i|, \) \( Q' \) used in the normalized matrix element \( \langle O^{A_A}_{Q} \rangle \) is estimated in the subspace restricted by \( A \) as follows:

\[
\begin{align*}
Q' &\simeq G(\hat{H}_{\text{eff}} + iB\Lambda_A)^\dagger Q\Lambda_A G(\hat{H}_{\text{eff}} + iB\Lambda_A) \quad \text{for the restricted subspace} \\
&= \tilde{G}(\hat{H}_{\text{eff}})^\dagger Q\Lambda_A \tilde{G}(\hat{H}_{\text{eff}}) \\
&= Q_2,
\end{align*}
\]

where in the last equality we have used Eq. (1). The three sentences “We first point out … replaced with \( |A(t)| \)” below Eq. (5.8) should be replaced with the above argument.

A \( dt \)-dependent normalization factor, say \( \frac{1}{\alpha(dt)} \), should be inserted on the right-hand sides of Eq. (A.2) and of the first line of Eq. (A.4). The following sentence should be inserted after the sentence “\( C \) is an arbitrary … complex plane” below Eq. (A.2): “In addition, \( \alpha(dt) \) is a \( dt \)-dependent normalization factor, which is properly fixed later.” The factor \( \sqrt{\frac{2\pi i\hbar dt}{m}} \) in the second line of Eq. (A.4) should be deleted. The following sentences should be inserted after the phrase “where … Eq. (3.7)” below Eq. (A.4): “Here we have taken \( \alpha(dt) = \sqrt{\frac{2\pi i\hbar dt}{m}} \) so that both sides of Eq. (A.4) correspond to each other in the vanishing limit of \( dt \). Then Eq. (A.4) is reduced to \( |\psi(t + dt)| = e^{-i\frac{\hbar}{m}dt}|\psi(t)| \)” The next sentence, “Thus we have found that … Eq. (A.2),” below Eq. (A.4) should be replaced with “Thus we have derived the Schrödinger equation and found that … Eq. (A.2).” The following sentence should be added after the above replaced sentence: “Such a derivation of the Schrödinger equation is well known in the real action theory [4].” Factors \( \frac{1}{\alpha(dt)^2}, \frac{1}{\alpha(-dt)^2} \), and \( \frac{1}{\alpha(-dt)} \) should be inserted on the right-hand-side of the equation in the second sentence of the last paragraph of the appendix, on the right-hand sides of Eqs (A.5) and (A.6), respectively. The second sentence below Eq. (A.6), “Indeed, \( \hat{H}_B \) is given … \hat{H}^\dagger \)” should be replaced with “Indeed, we obtain the Schrödinger equation \( |B(t - dt)| = e^{-i\frac{\hbar}{m}dt}|B(t)| \), where \( \hat{H}_B \) is given … \hat{H}^\dagger.”

References