HOW TO GENERATE AUTONOMOUS QUESTIONING IN SECONDARY MATHEMATICS TEACHING?

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HOW TO GENERATE AUTONOMOUS QUESTIONING IN SECONDARY MATHEMATICS TEACHING?

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COMMENT GENERER DES QUESTIONNEMENT AUTONOME DANS L'ENSEIGNEMENT DES MATHEMATIQUES AU LYCEE?

Résumé – Dans le domaine de l'enseignement des mathématiques, il demeure un défi majeur : inciter les élèves à soulever des questions, et chercher ensuite des réponses à ces questions, afin d'apprendre les mathématiques. Au cours des trois dernières décennies, la formulation de ce défi a été développée dans le contexte de différentes approches et études empiriques. Cet article décrit les résultats d'une étude empirique où le processus d'enseignement a été élaboré et réalisé dans le cadre de la Théorie Anthropologique du Didactique. Il démontre comment un changement dans le contrat didactique et le fait que les enseignants ont posé des questions ouvertes s'unissent pour soutenir une démarche d'investigation autonome par les élèves. Cette nouvelle approche conduit au développement des connaissances chez les élèves qui vont au-delà des exigences du programme d'étude.

Mots clés: formulation de problèmes, théorie anthropologique du didactique, activités d'étude et de recherche, mathématiques au lycée, fonction exponentielle.

¿CÓMO GENERAR CUESTIONAMIENTO AUTÓNOMA EN LA ENSEÑANZA DE MATEMÁTICAS DE SECUNDARIA?

Resumen – En el campo de la educación matemática sigue siendo un gran desafío hacer que los estudiantes planteen y busquen respuestas a sus propias preguntas con el fin de aprender matemáticas. Durante las últimas tres décadas, la formulación de problemas se ha estudiado a través de diferentes enfoques y en estudios empíricos. En este artículo se presenta el resultado de un estudio empírico donde el proceso de enseñanza se diseñó y llevó a cabo en el marco de la Teoría Antropológica de lo Didáctico. Se muestra cómo un cambio en el contrato didáctico y docente basado en preguntas abiertas puede servir de base para el cuestionamiento autónomo de los estudiantes. Este nuevo planteamiento lleva al desarrollo de conocimientos entre los estudiantes que van más allá de los requisitos curriculares.

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Palabras-claves: formulatión de problemas, teoría antropológica de lo didáctico, matemáticas de secundaria, función exponencial.

HOW TO GENERATE AUTONOMOUS QUESTIONING IN SECONDARY MATHEMATICS TEACHING?

Abstract – In mathematics education it is still a major challenge to find ways to nurture students to pose and pursue their own questions in order to learn mathematics. During the last three decades, problem posing has been explored through different approaches and in empirical studies. This paper presents the result of an empirical study, where teaching was designed and conducted based on The Anthropological Theory of Didactic. It is shown how a changed didactic contract and open generating questions posed by the teacher can support students' autonomous questioning of the taught knowledge. In the study, students developed knowledge that went beyond curriculum requirements through autonomous activities, which were different from more traditional school and pedagogical culture.

Key words: problem-posing, the anthropological theory of didactics, study and research activities, upper secondary mathematics, exponential functions.

INTRODUCTION

A practical and broad characterisation of research is "posing questions and searching for answers". In educational research, efforts have been made to engage students in more research like activities (Artigue & Blomhøj, 2013, p. 797). Since the 1980s mathematics educators have had an interest in the study of how students can be nurtured to pose questions and formulate answers to these. Some of the reasons for this research interest were formulated by Singer, Ellerton & Cai (2013):

> Problem posing improves students' problem-solving skills, attitudes, and confidence in mathematics, and contributes to a broader understanding of mathematical concepts and the development of mathematical thinking. (Singer, Ellerton & Cai, 2013, p. 2)

The quote is from a special issue of *Educational Studies in Mathematics* compiling recent contributions to the research field of problem posing in mathematics education in order to establish a framework for future work. Problem posing and problem solving has been an explicit part of mathematics education ever since Polya published "How to solve it?" (1945), characterising how to conduct mathematical activity. According to Singer (1994) emphasis has so far been on strategies for problem solving, but an element of this process is to keep rephrasing the problem, which essentially is problem posing. Singer further refers to Kilpatrick (1987), and his point of letting problem posing be both a goal and means of instructions in mathematics. Problem solving is an independent research field in mathematics education with linkages to problem posing but not always. An early attempt to gather different ideas on how to teach students problem solving is Kilpatrick (1969) and his literature review on how to deal with problem solving in the teaching of mathematics and how it became part of the agenda for teacher associations and conferences on mathematics teaching in the 1960s. Later, Schoenfeld has considered how students can learn heuristic compentences, inspired by work of Polya, for problem solving (Schoenfeld, 1985). The field of problem solving has kept evolving, as it is shown by Liljedahl and colleagues (2016) who present a state of the art regarding heuristic competences for mathematics problem solving, and discusses the role of creativity and that students pose problems as part of dealing with mathematical problems (Liljedahl et al., 2016). The long term developments in problem solving as well as the potentials of problem posing have led to formulations, in curriculum and educational standards, on requirements for students to pose and treat mathematical problems (e.g. Ministry of Education of Denmark, 2013; NCTM, 2000, p. 335). When inviting students to engage in activities similar to researchers an inquiry approach is often chosen. A common feature in inquiry processes, which stems from the work of Dewey, is the students' formulation of questions (Artigue & Blomhøj, 2013, p. 800). The design tool employed in this study support inquiry based teaching, yet inquiry is not the core interest of this paper. The interest and emphasis will be put on students' autonomous formulation of questions, and on contributing to a new direction in problem posing research: to develop frameworks and structures to guide the problem posing experience (Liljedahl et al., 2016).

Research on students' question formulation range from how students can pose questions based on informations given to them, to how students create problems similar to newly solved problems, on to the description of a phenomena based on which problems should be formulated (Bosch & Winsløw, 2015, p. 371). Despite good intentions and much research, most teaching can still be characterised as transmission of syntheses – that is the cultivated version of knowledge as it is presented in most textbooks. The problems and questions students are supposed to work on, are put forward by teachers, which give students little experience of exploring a piece of knowledge autonomously (Bosch & Winsløw, 2015, p. 362). This is not in alignment with the needs of the students in real life. Referring to Kilpatrick and his argument that in real life most problems must be posed by the person who solves the problem (Kilpatrick, 1987, p. 124), Bosch and Winsløw state that: "Still, the challenge remains: is it feasible and desirable to have students take a more active role in identifying or formulating the questions they work in, and thus make their activity more akin to what Kilpatrick considers the situation "in real life outside school"?" (Bosch & Winsløw, 2015, p. 344). Bosch and Winsløw suggest to develop study and research paths (SRP) based on the Anthropological Theory of Didactics (ATD) as an answer to the above stated question. In ATD, the study processes of students are roughly speaking when students gain knew knowledge from different resources (books, experiments, webpages etc.). The research process is when students combine this new knowledge with, what they already know in order to answer a question or problem – preferably posed by themselves. SRP is a design tool that stages these processes. The motivation of this paper is to analyse to what extend the realisation of students posing and answering their own question can be realised through SRP based teaching. This leads to the following research question:

What didactic notions foster the autonomous questioning from students regarding the mathematical activity carried out? What changes can be introduced to the didactic contract to support students' autonomy?

Autonomous questioning refers to situations where students pose questions to an answer they have studied or been presented. In this context, an answer can be a piece of knowledge, a technique or a theorem. Autonomous questioning does not cover questions such as “will you please repeat?” or “please explain?”. The goal is students raising questions, which address a mathematical notion, technique or phenomenon, thus it generates a confirmed mathematical study process for the students posing the question. In this sense, the autonomous questioning relates to Kilpatricks’ claim on who formulate problems in real life.

THEORETICAL BACKGROUND

The study of this paper is based on notions from ATD, which will be presented in this section. SRP was introduced by Yves Chevallard (2006b & 2015) as a tool for designing autonomous transdisciplinary student work in upper secondary education in France (Winsløw, Matheron & Mercier, 2013, p. 269). The reason for transdisciplinary work was in part due to a new work format in French upper secondary school requiring this, but also to put the important questions in forefront of the teaching, that can lead to answers similar to those students are supposed to be taught. A SRP is initiated when a group of students begin the study of a generating question Q_0. In a teaching context the teacher formulates the generating question in advance. The question should be strong enough to guide an exploration of a knowledge domain. Students should understand the question but not be able to answer it, unless they engage in a study and research process. This process is supposed to be driven by initial hypothesis of an answer, which is incomplete and therefore lead to new, derived questions Q_i (Chevallard, 2015, p. 179). In order to answer the derived questions, the students are supposed to study media to gain new knowledge. Media are the works of others, like textbooks, webpages, podcasts and other materials produced in order to disseminate (mathematical) knowledge (Kidron, Artigue, Bosch, Dreyfus & Haspekian, 2014, p. 158). The students are supposed to deconstruct the new knowledge and reconstruct it as answer to a question they work on. In the reconstruction process, the students are supposed to draw on previously acquired knowledge and combine it with the knowledge from studied media. The process of reconstruction of knowledge is characterised as research, which takes place in a milieu. The milieu in ATD is defined as works to study, previous acquired knowledge and the question considered (Kidron et al., 2014). Students’ knowledge construction is then conceived as the result of the dialectics between study and research processes (Winsløw et al., 2013, p. 269). The study and research process leads to a number of paths, detours and dead-ends in the process of developing a coherent answer for the generating Q_0 (see Bosch & Winsløw, 2016, p. 350). The numbering of the derived questions indicates the relation between the derived questions and the paths they belong to.

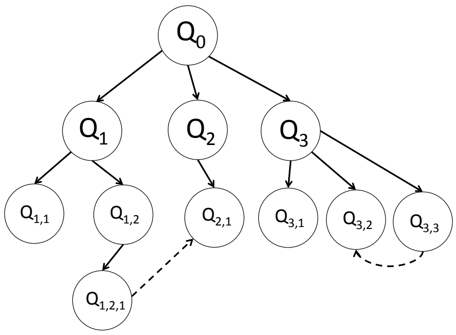

Figure 1. - The tree diagram showing the derived questions and their internal relation

Figure 1 shows a "tree diagram" depicting paths (or branches) of questions. It is possible for questions from different branches to relate, as between $Q_{1,2,1}$ and $Q_{2,1}$ in Figure 1. This means that the answer of $Q_{1,2,1}$ draws on the same technique (e.g. technique for solving equations) as applied in the answer of $Q_{2,1}$. This type of representation has been used for depicting *a priori* analyses of teaching designs and give a sense of the generative power of Q_0 - possible paths and derived questions. The posteriori analysis of realised paths and questions can also be depicted as tree diagrams, showing the realised questions of the process.

The herbartian schema is in this study used to capture the role of media and milieu respectively in the process of students' construction of knowledge. In this study, the herbartian schema is drawn upon in order to explicate how an answer provided by some students made others question the mathematical knowledge presented, which again initiated a further study and research process and hereby the students' learning process was furthered.

The herbartian schema shows the interaction between a didactic system and a milieu, which leads to the development of a personal answer, $A^{\heartsuit}$, to Q_0:

$$[S(X,Y,Q) \curvearrowright M] \hookrightarrow A^{\heartsuit}$$

Here, S represents the didactic system consisting of a group, X, of students, a group, Y, of people assisting the students (this can simply be one teacher), and the question, Q, they study together. The heart is only used for specific answers provided by a group. Generalised versions of answers are not given a heart in the following analysis. The study of Q is conducted as an interaction with a milieu M. A study process is based on students' previously acquired knowledge, which is regarded as previously developed answers, A°, and may appear as elements of the milieu. The available media is denoted O_i. These can be suggested by Y or autonomously be drawn upon by X. Accordingly the milieu can be written as the set: $M=\{A_1{}^{\circ},A_2{}^{\circ},\ldots,A_n{}^{\circ},O_m,\ldots,O_{k+1},Q\}$ (see Bosch & Winsløw, 2016, p. 31; Kidron et al., 2014, p. 157). This analysis can be applied for any kind of teaching, where a group of students work on a question, assisted by a teacher. This way of using the herbartian schema to analyse the dialectic between media and milieu and between questions and answers has previously been explored for the analysis of different ways for teachers to engage in their professional development as teachers (Jessen, 2016).

During the last decade, empirical studies have been conducted showing the potential of SRP's as a design tool for teaching, often as supplement to more common forms of classroom teaching. García and Ruiz-Higueras (2005) used a SRP at lower secondary level for the purpose of teaching proportional relationships and functional relationships, through a generating question on savings plans for an end-of-year trip. Barquero, Bosch and Gascón (2013) studied several iterations of a workshop on modelling attached to a mathematics course in the first year of an engineering programme. The purpose was to support the development of raison d´être for course content and to relate the different elements of the content. Serrano, Bosch and Gascón (2010) studied the tutorials of a mathematics course at first year university studies of economics. The problem to be solved during tutorials concerned a report on sales forecast for a private company and was based on mathematical modelling with one variable calculus.

Recent research seeks to explore the potential of designing course activities not anchored in traditional lectures and other common classroom activities. Jessen (2014) designed a bidisciplinary project for upper secondary education, where students were supposed to write individual papers combining mathematics and biology. In order to guide the study process, a few derived questions were posed together with the generating question. Students were guided through email correspondence based on questions posed by the students. Rasmussen designed a course element of an interdisciplinary course combining mathematics and science called "Health – risk or chance". Rasmussen focused his design on supporting the autonomy of the students by methods called 'selective picking', side questions and student diaries (Rasmussen, 2016). Most recently Florensa, Bosch and Gascón have designed an engineering course on elasticity of materials, where questions for weekly status reports handed in by the students guide the study and research process (Florensa et al., 2016, p. 7). Otaki, Miyakawa and Hamanaka (2016) designed three lessons of proving activities as SRP drawing on Internet search as media for further study. We will not go further into the conditions of the settings where SRP based teaching has been experimented, but only notice that they have been realised in very different contexts of secondary, tertiary education and pre-service teacher training. However, as the single papers show the conditions and institutional constraints affect what is possible to realise in each case.

All of these studies share the aim of students' pursuing a multitude of paths and posing what can be characterised as "implicit questions". In their account for students' work, researchers interpret elements of reports or diaries as signs of interest for further study or research but not necessarily as explicit questions (Rasmussen, 2016, p. 167), (Jessen, 2014, p. 205). In the study of Rasmussen, students were asked to pose questions for the teachers of the course, which they then would like teachers to give a presentation of at the beginning of the following session (Rasmussen, 2016, p. 166). These studies substantiate the advantages in students' construction of knowledge through questioning and relate to the first of three issues raised by Bosch and Winsløw (2015) with respect to the practical realisation of sustainable questioning:

> What are the didactic and mathematical infrastructures (and resources), as well as the associated knowledge, required for the design, monitoring and evaluation of sustainable study and research processes? (Bosch & Winsløw, 2015, p. 33).

Rasmussen addresses this point explicitly by employing "selective picking" and the use of "side questions" (Rasmussen, 2016). Further, he discusses SRP-based teaching in a course frame with a "monumentalistic curriculum" and a changed didactic contract. The SRP took up 20% of the course activities in a pre-service teacher education course (Rasmussen, 2016, p. 161). In the case study by Author, no content-based curriculum existed for the projects but the knowledge acquired by the students should relate to the content of the curriculum for mathematics and biology separately (Jessen, 2014, p. 203). In order to explore the potentials mentioned above in a classroom where specific curriculum requirements must be met, we propose to consider Study and Research Activities (SRA) as a design tool.

1. The notion of SRA

It is a challenge to design teaching based on generative questions when current curriculum structures are "monumentalistic" and to some extent list a number of works to visit (Rasmussen, 2016; Chevallard, 2006). In ATD, SRA are described as a practical organisation of the study of works described in curriculum. In the case of teaching based on SRA, Chevallard points to the risk of atomising the mathematical knowledge, which could easily lack the rationale of the developed techniques and the motivation for the questions posed to the students (Chevallard, 2006a, p. 18). Barquero and Bosch regard SRA as a special branch of a SRP focusing on a certain answer $A_k^{\diamond}$. Whenever the teaching is based on a generative question, Barquero & Bosch argue that: "what then appears is a sequence of linked study and research activities called study and research paths (SRP)." (Barquero & Bosch, 2015, p. 261). There is no clear line between SRP and SRA but it can be said that together, SRP and SRA provide tools for describing teaching and learning processes ranging from transmission to inquiry based approaches (Barquero & Bosch, 2015, p. 262). Barquero, Serano and Ruiz-Munzón identify three types of SRA's starting with SRA disseminating a pre-established answer to the type of SRA, which engage students in "search, de- and re-construction of external answers and objects according to the new SRP needs" (Barquero, Serrano & Ruiz-Munzón, 2016, p. 3). The common feature of these descriptions of SRA's is that they support students' development of answers being similar to the answer intended by the teacher: $A^{\heartsuit} \sim A_{Y^{\diamond}}$. In the study reported on in this paper SRA is used as a model for ordinary classroom teaching aiming at students' development of certain "monuments" or techniques mentioned in curriculum but also part of the rationale behind the technique.

The tree diagrams have been used for illustrating sequences of SRA's and their interrelation. It can be depicted as in Figure 2, which Barquero, Serrano and Ruiz-Munzón used to illustrate a sequence of SRA's on modelling different functions and sequences.

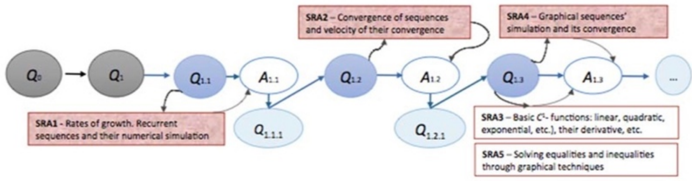

Figure 2. - The picture of a sequence of SRA indicating how these are interrelated and the derived questions they generate (Barquero et al., 2016, p. 6).

2. Expectations and responsibilities in study and research processes

When engaging students in more inquiry based teaching, expecting students to autonomously study resources and pose questions the mutual expectations and responsibilities between teacher and students changes. Previously Schoenfeld (1988, p. 161) has characterised traditional mathematics teaching as: rules presented, explained and rehearsed. The kind of teaching where students expect the teacher to present a rule, provide an example and then ask them to rehearse the use of the rule on a number of examples is still dominant. Chevallard (2015) has characterised this prevailing teaching paradigm as visiting monuments, meaning that students are presented some rule, which they are supposed to appreciate through its use in textbook examples and exercises. The didactic contract of this teaching leaves the responsibility of presenting rules and reasons to the teacher. Brousseau characterises the didactic contract as:

> These (specific) habits of the teacher are expected by the student and the behaviour of the student is expected by the teacher; this is the didactical contract. (Brousseau, 1997, p. 225).

Mutual but different expectations and interpretations of teachers' and students' activities in classrooms will always be in play. At upper secondary level the students have gained experience with the school system and therefore they have an expectation regarding their own and their teacher's roles in the classroom. In the sequence of SRA's in the present study, the roles, the expectations and responsibilities of teacher and students were changed according to the dynamics of study and research processes as described above.

In the SRP and SRA as we have seen above, much initiative lies with the students since they are the ones who must engage in study and research processes, pose derived questions, and select and study media. Bosch and Winsløw report that in experiments with SRP, it is frequently observed that the students are resistant to accept the new didactic contract, but also teachers have had a tendency to revert to the prevailing didactic contract (Bosch & Winsløw, 2015, p. 369). Rasmussen identified a group of derived questions as a "residual group". This group of questions was interpreted as a possible "metadidactic resistance to the changed didactic contract" (Rasmussen, 2016, p. 169). In the experiment of this paper, the change of the didactic contract was made explicit through requirements for the students. In particular, students were expected to formulate questions and answers and share those with the class. Below the context of the study is presented together with the management of the SRA in the classroom. Before going into further detail with the realised sequence of SRA the theoretical background of this paper enables us to rephrase the research question in terms of ATD and didactic contract as the following:

How can SRA nurture or promote students' autonomous questioning regarding the mathematical activity carried out? What mathematical and didactic infrastructures in terms of suggested media, organisation of time and activities can be introduced to change the didactic contract in order to support students' autonomy?

CONTEXT OF THE STUDY

The teaching experiment was conducted at upper secondary level in Denmark. The class had their main elective focus in humanities (languages) and only took one year of mathematics. Hence they were expected to be less keen on mathematics. The class had 24 students aged 15-16. Mathematics at this level is evaluated in a high stake oral exam where students present written thematic projects on various mathematical topics. A thematic project is a synopsis covering a number of non-standard mathematical questions. It is prepared for the oral exam and commented upon by the teacher (not corrected) before the exam. At the oral exam, the students present the improved version of the synopsis (see further description by Grønbæk, Misfeldt & Winsløw (2010)). The sequence of SRA's prepared the students to write a synopsis on what characterises exponential functions and its applications.

The author was both the ordinary mathematics teacher of the class as well as a didactic researcher. The experiment was conducted with the author as the teacher and another mathematics teacher who observed the lessons. The observing teacher taught the class in another discipline. Hence he knew the students and they knew him. The field notes, taken by the observing teacher, include the dialogue from students' presentation of preliminary answer at the whiteboard, what they wrote, and pointing gestures. In addition, pictures were taken of the boards to support the notes.

It is worth to notice that the class had already completed a sequence of SRA's on linear functions. This means that students were familiar with the changed didactic contract: they were familiar with the requirement of providing an answer for the posed question and to present preliminary versions at the whiteboard.

The class was divided into eight groups where the members of one group performed equally in mathematics.

What was supposed to be taught in the class was as stated in the curriculum:

> Students should be able to […] use relations between variables for the purpose of modelling data, predict how the modelled system evolves, and be able to discuss how well the model fits the system. (Danish Ministry of Education, 2013).

Furthermore, they should work with "equations describing […] exponential relations between variables […]". The curriculum is supported by guidelines, which suggest that the teaching of exponential functions should build on arithmetic calculations with exponents and how to interpret expressions such as $\sqrt{2}$ or 10^{100} (Danish Ministry of Education, 2010, p. 5). However, official documents leave out what rationale could or should be developed for the notion of exponents and exponential functions. Textbooks at this level do not explain the expression a^x, where $x \in \mathbb{R}$, except from pointing out that it can easily be found using a calculator as if the exponent is a natural number (Jessen, 2015, p. 72). Textbooks disregard that the expression cannot be interpreted as the product of finite number of a's (Winsløw, 2013, p. 5). This example indicates the lack of rationales in the teaching of exponential functions at this level. Which rationale the students might develop through the SRA is not fixed but potential elements are described in the *a priori* analysis below.

METHODOLOGY

The methodology employed in this study draws on tree diagrams and question-answer maps as depicted in Figure 1. Concretely, we follow the ideas of didactic engineering as a research methodology (Barquero & Bosch, 2015) realising the four phases: preliminary analysis; prepare the *a priori* design of the teaching proposal; implement, observe and collect date from the realisation and finally we carry out the *a posteriori* analysis. Above we presented a part of the preliminary (epistemological) analysis of what should be taught and under what constraints and condition this should be done. This is further elaborated below. From this the generating questions of the design are formulated.

This means that they are formulated as questions, which are supposed to lead students to develop the knowledge listed in curriculum by creating a need for students to study e.g. the notion of doubling time and use this knowledge in developing an answer. Whether this goal can be fulfilled is argued in the *a priori* analysis of the generating question. In the *a priori* analysis possible derived questions are formulated, related and depicted in a tree diagram as the one shown in Figure 4. What questions the teacher (and here researcher) can imagine the students to pose and answer are based on the mathematical knowledge, the experience as teachers and a study of possible media, which students might consult. This means that from a pure mathematical point of view the generating question might lead to a certain number of derived questions. Studying available media, the designer of the SRA might add further derived questions to the map. Furthermore, the experiences of teachers with students' misconceptions or preferred strategies might even add more paths to the tree diagram of the *a priori* analysis. Some of these paths and strategies might be wrong or dead-ends or just less direct strategies compared to those of a mathematician. Still they play an important role in preparing the managing and foreseeing the realisation of the SRA and therefor are included in the *a priori* analysis. The *a priori* analysis is depicted in Figure 4.

The implementation and data collection was done as described above with the author as the teacher and a colleague, also mathematics teacher, observing. The *a posteriori* analysis is based on field notes and pictures of the whiteboards. These data were analysed with respect to what questions and answers they provide with respect to the generating question. Some derived questions were explicitly, and others implicitly posed. An example of an implicit question was when a group started to isolate x in the equation $10\,000 = 5\,000 \cdot 1.025^x$. They wanted to answer how long it took before the balance was doubled, counted from the starting point where $x = 0$. It means they had asked the question: "How can we solve this equation with respect to x, which represents the time it takes to double the balance?". The actual formulation of the question was never made explicit, but is implicitly given through the answer the students provided.

An explicitly formulated question was: "Why does that happen?" pointing to another group's answer, where the group used a specific mathematical rule on logarithms. In this example the question was rephrased as: "Why does this specific rule solve the problem?".

Diagram 5 shows the results of the posteriori analysis. Grey circles mark the autonomous questions posed by students. The sense in which these represent autonomous questioning will be explained below. Further it is worth noticing that there is differences between Diagram 4 and 5 meaning, that not all expected paths were realised and some realised paths were not foreseen. This will be further discussed later.

PRESENTATION OF THE TEACHING DESIGN

The aim of the sequence of SRA's was for the students to know and be able to use the notion of exponential relation between variables y and x as in: $y = b \cdot a^x$, a and b being constants of real values. Further students were supposed to be able to find the expression of the exponential function passing through two points on its curve, (x_1, y_1) and (x_2, y_2), using the formula $a = \sqrt[x_2 - x_1]{\frac{y_2}{y_1}}$ and $b = \frac{y_1}{a^{x_1}}$. Finally, students must be able to find the doubling time of the growth using the formula $T_2 = \frac{\log(2)}{\log(a)}$. These formulas represent the "monuments" of the curriculum for exponential growth at this level (Danish Ministry of Education, 2010, p. 5) and the intended answers of the teaching. Students were supposed to acquire the use of them but need not to know the rationale of these answers, or what questions initially led to these answers. In contrast to curriculum, the sequence of SRA's aimed at students developing answers through work on questions, which cover these techniques along with some reasoning behind the techniques. The analysis of this paper will focus on the students work with doubling time.

The SRA's explored are linked together and have the potentials for students to develop the above monuments as coherent knowledge. An overarching Q_0 capturing the sequence of SRA's is the following:

Q_0: What characterises an exponential function and where can it be applied?

This question was answered in the students' thematic projects. To guide the study of Q_0, students were posed the following questions, which were studied jointly by the groups in class:

Q_1: Grandparents starts a saving account for their newborn grandchild by putting 5 000 dkr into an account at an annual rate of interest of 2.5%. Bank regulations say that the balance may not exceed 50 000 dkr. Will that be a problem?

Q_2: The neighbours have a similar savings account for their child and initially they put 5000 dkr into his account. After 10 years

the balance has increased to 5947.22dkr. How much money can the neighbours' kid withdraw?

Q_3: If the children only are allowed to withdraw their money when the amount is doubled, how long should they wait?

Through their work with these questions, students will gradually expand their knowledge on exponential growth, and they need to use knowledge from former SRA's in the construction of answers for the next question. The last question, Q_4, concerned regression carried out using spreadsheets.

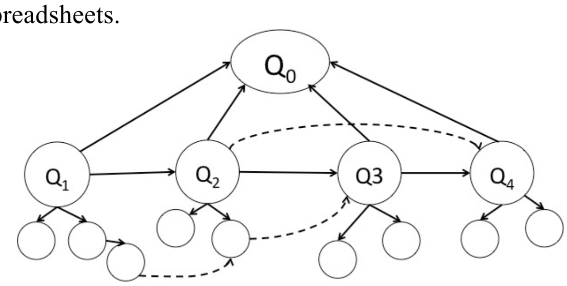

Figure 3. - The relation between questions in the sequence of SRA's, which constitute the teaching of exponential function.

The reason for putting Q_0 above the others is that each SRA individually should generate a subanswer to Q_0. At the same time the questions are related as indicated by the horizontal arrows. The blank circles indicate potential derived questions and how they can be related to other derived questions. In the presentation below we focus on Q_3, since this was the question, which most clearly led students to pose questions autonomously and beyond what was expected.

For each SRA, students were guided in their work by media explicitly proposed by the teacher. The media included certain pages in a textbook, online pages and video clips. The division of the class based on their previous achievements was done to secure that students existing answers were similar. This is important because of the intention of students' construction of answers based on their existing knowledge and the de- and reconstruction of studied knowledge. If one student performed better than the remaining group, there is a risk he presents the others with monuments to visit and hinder their learning potential. After 5-7 minutes, the groups were asked to present their work in fields occupying 1/8 of the whiteboard. The groups were not allowed to erase anything, as they wrote and drew their preliminary ideas.

The *a priori* analysis of the SRA

The *a priori* analysis of Q_3 is shown in Figure 4. The numbers in the circles correspond to those of the questions listed below the figure.

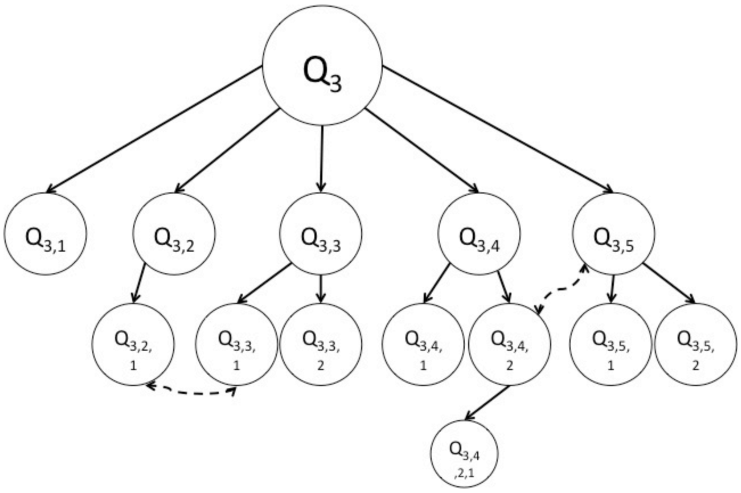

Figure 4. - The *a priori* analysis showing the possible paths to an answer to Q3.

The questions below are formulated by the author as *a priori* analysis of Q_3.

$Q_{3,1}$: How can we solve the problem by "trial and error" meaning, calculate the value of y for different values of x in the expression: $y = 5{,}000 \cdot 1.025^x$ until we get $y = 10{,}000$?

$Q_{3,2}$: How many times must the initial amount of money be multiplied by the factor 1.025 to exceed 10,000?

$Q_{3,2,1}$: How can we answer the question if the calculations from the above question are plotted in a coordinate system, the graph drawn and the x-value corresponding to $y = 10{,}000$ found graphically?

$Q_{3,3}$: How can we find the solution by drawing the graph, which show the relation $y = b \cdot a^x$ in a coordinate system?

$Q_{3,3,1}$: How can the above strategy be done with pen and paper?

$Q_{3,3,2}$: How can it be done with a computer program? (CAS or spreadsheet)

$Q_{3,4}$: How can we solve the equation $10{,}000 = 5{,}000 \cdot 1.025^x$ with respect to x?

$Q_{3,4,1}$: How can the equation be solved by a CAS-tool (Maple, Geogebra, TI Nspire, etc.)?

$Q_{3,4,2}$: How can we solve the equation$2 = 1.025^x$ with respect to x?

$Q_{3,4,2,1}$: How can we solve the equation by "trial and error" with different values of x?

$Q_{3,4,2,2}$: How can the identity $\log(a^x) = x \cdot \log(a)$ solve the equation?

$Q_{3,5}$: How can the formula $T_2 = \frac{\log(2)}{\log(a)}$ solve the problem?

$Q_{3,5,1}$: How can it simply be used as an algorithm?

$Q_{3,5,2}$: How can the formula be justified?

There is a clear relation between $Q_{3,4,2,2}$ and $Q_{3,5,2}$ in the sense that the technique of $Q_{3,4,2,2}$ is part of the reasoning needed to answer $Q_{3,5,2}$. The questions could reasonably be pursued further, however to pose these two questions is already beyond the scope of the curriculum and the ministerial guidelines. Other of these potential questions represent strategies to solve the problem at hand in more or less precise ways, which was assumed to be possible paths for the students to follow with their former achievements in mind.

RESULTS

Two cases will be presented where students pursued their own questions in the study and research processes, as examples of autonomous questioning. In the first case students study the nature of logarithms. In the second case students investigate certain "time constants" based on that knowledge regarding logarithms.

The class raised a number of initial questions from the top of their heads before they studied the proposed media and conducted any research:

$Q_{3,0,1}$:"What figures do we have from earlier? 5,000 and 2.5%?"

$Q_{3,0,2}$:"Can we use the $K_n = K_0 \cdot (1 + r)^n$ formula and isolate n?"

$Q_{3,0,3}$:"How do you take the n^{th} root using nSpire [the CAS-tool]"? And "How can you solve this problem [pointing towards the equation $1.025^x \cdot 5000 = 10{,}000$]?"

The numbering indicates that it was preliminary questions. All groups got the same reply form the teacher: the advice to study the proposed media, denoted by O_1, O_2, O_3. The media is a classic textbook for this level of mathematics (Clausen et al, 2010, p. 72-74), and two YouTube videos from a Danish website similar to Khan Academy run by high school teachers (Clausen & Clausen, 2014). The questions indicate that students start to pose questions immediately. Moreover, the questions were all addressing the mathematical content of the question and therefore they constitute autonomous questioning with respect to Q_3.

Below the answers of the groups are numbered as in the *a priori* analysis above, showing paths realised by the students. The tree diagram of the *a posteriori* analysis is presented in Figure 5.

$Q_{3,4}$: How can we solve the equation $10{,}000 = 1.025^x \cdot 5000$ with respect to x?

$Q_{3,4,2}$: How can we solve the equation $2 = 1.025^x$ with respect to x?

$Q_{3,4,2,2}$: How can the identity $\log(a^x) = x \cdot \log(a)$ help to solve the equation?

$Q_{3,4,2,2,1}$: How can we prove the identity $\log(a^x) = x \cdot \log(a)$?

$Q_{3,4,2,2,1,1}$: What defines the logarithm?

$Q_{3,5}$: How can the formula $T_2 = \frac{\log(2)}{\log(a)}$ solve the problem?

$Q_{3,5,1}$: How can it simply be used as an algorithm?

$Q_{3,5,1,1}$: How can we use the formula if the doubling happens twice?

$Q_{3,5,1,1,1}$: Why is it 28 years every time, when the interests are increasing?

$Q_{3,5,2}$: How can the formula be justified?

$Q_{3,6}$: Can the two doubling time constants be described as one, T_4?

$Q_{3,6,1}$: How can we "prove" a formula of T_4?

$Q_{3,6,2}$: Can we deduce a T_8, which describe the time required for all the savings to be increased by a factor 8?

Group 1 followed the branch of $Q_{3,4}$. The answer they presented ended by posing $Q_{3,4,2}$. They used previously acquired knowledge and they translated the concrete problem into an equation, which led them to the expression $2 = 1.025^x$. One can argue that this branch stems from the immediate plenum question $Q_{3,0,3}$. This is shown in Figure 5 below.

The next four groups did not answer the derived question, $Q_{3,4,2}$, but presented their synthesis. They gave various versions of the following answer:

$A_{3,5}$: The doubling time is given by the formula:

$$T_2 = \frac{\log(2)}{\log(a)}$$

In our case $a = 1.025$, therefore $T_2 = 28.07$.

These answers clearly drew on the study of suggested media since the formula was presented in them. At this level of mathematics, the answer represents what the students were expected to do. In the written media students were encouraged not to worry about the meaning of "log(a)" apart from considering it as a button on their calculators. Group number 2-5 followed this path.

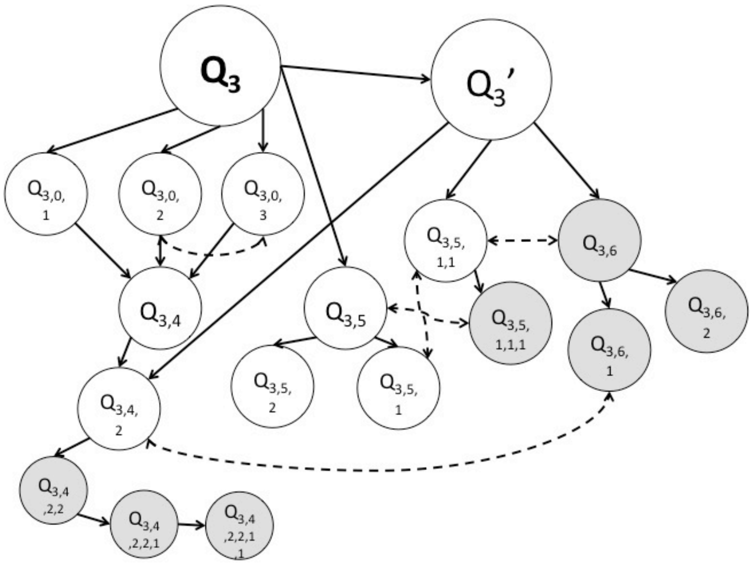

Figure 5. - The result of the analysis of the third SRA. The grey circles indicate students autonomous questioning.

1. First example of autonomous questioning

Group 6 started where Group 1 posed their question on how to solve $1.025^x = 2$ with respect to x. The group continued by writing:

$$x \cdot \log(1.025) = \log(2)$$

$$(x \cdot \log(1.025)) / (\log(1.025)) = (\log(2))/(\log(1.025))$$

$$x = 28.7$$

Hence the group gave an answer, $A_{3,4,2,2}$♥, based on using a rule on logarithms. Group 7 yet another version of $A_{3,5}$. The last group claimed they had used the same technique as in $A_{3,5}$, but instead of formulating their answer right away they ask: "[…] but why does that happen?" [the group pointed towards Group 6's use of logarithms]. Hence, they questioned $A_{3,4,2,2}$, by posing $Q_{3,4,2,2,1}$.

Group 6 replied: "We used the rule: $\log(a^x) = x \cdot \log(a)$. Isn't it like if $f(x) = 10^x$ then the opposite is $f(x) = \log(x)$?".

The class was unfamiliar with the concept of inverse functions but still some students dared to share a vague idea about it, being like "the opposite operation of addition is subtraction". Moreover, Group 6's answer to Group 8 can be interpreted as a new question, $Q_{3,4,2,2,1}$: Is the rule $\log(a^x) = x \cdot \log(a)$ based on the fact that, $y = \log(x) \Leftrightarrow \log(x) = 10^y$?

To give a full answer to $Q_{3,4,2,2,1}$, one needs to question how $\log(a^x) = x \cdot \log(a)$ can be justified, which represents $Q_{3,4,2,2,1,1}$. The textbook's answer to this is based on rules for calculating with exponentials as $(a^p)^q = a^{pq}$ (Clausen et al., 2010). This last technique was never explicitly mentioned and no explicit answer for $Q_{3,4,2,2,1}$ was provided. We note that, the textbook disregard to differ between cases where a and b are natural numbers and when they are real numbers.

Simultaneously with the teacher, Group 6 looked up logarithms in the textbook. Both group and teacher suggested this as a media to study by the other groups. These pages could lead to an expansion of the milieu. The episode represents a branch in the SRA, which exemplifies autonomous questioning. The teacher did not expect students at this level to study logarithms as more than a calculator button. This branch is marked in Figure 5 by grey circles, and starts with $Q_{3,4,2,2}$.

2. Further study and research on doubling time

Since most of the class gave versions of $A_{3,5}$ as their answer, the teacher wanted to know what idea the students had about the doubling time. Therefore, she asked a derived question to the whole class, Q_3': "How long will it take if the money stays in the account until another doubling of the balance occur. When will that be?"

Most groups answered that it takes another doubling time, hence the total time can be calculated as: $2 \cdot T_2$. This represents $A_{3,5,1,1}$. But others were in doubt, asking $Q_{3,5,1,1,1}$: "Why is it 28 years every time, when the accrued interest is increasing?" Another group answers, $A_{3,5,1,1}^{\blacktriangledown}$: "When the rate of growth (a) is the same, the time for doubling is the same". This relates back to $Q_{3,5}$. It is unclear how the group came up with this answer. The textbook (Clausen et al., 2010) has a graphic representation of the doubling time, which indicates that the doubling time is the same regardless where you look at the graph. This could have inspired the answering.

The first group argued that finding the time it takes for the balance to increase by a factor 4, is equivalent to solving an equation they reduced to: $1.025^x = 4$. This is similar to their work with Q_3 and is indicated by the arrow from Q_3' unto $Q_{3,4,2}$ in Figure 5. This group did not study the answers of other groups. The last two groups continued the work of Group 1. They argued for a "double doubling time" constant: $T_4 = \frac{\log(4)}{\log(a)}$. Their argument was built on the equation: $1.025^x = 4$. It can be interpreted as they raise the questions:

$Q_{3,6}$: Can the "two doubling times" be described as one, T_4?

$Q_{3,6,1}$: How can we "prove" a formula of T_4?
The groups deconstructed and reconstructed the answer of Group 6 using the rule: $\log(a^x) = x \cdot \log(a)$. It was an example of autonomous questioning since the students raised derived questions based on their study of answer $A_{3,4,2,2}$. The two groups used Group 6's answers in the sense of de- and reconstructing the knowledge into an argument, which led to $A_{3,6,1}{}^{\blacktriangledown}$. Group 8 made further hypothesis about an "8-time constant", T_8. Yet, the claim was never investigated further or formalised by the students. It is depicted as $Q_{3,6,2}$ in Figure 5.

The investigation of constants T_4 and T_8 was certainly beyond the scope of the ministerial guidelines and does not represent core mathematical content for upper secondary mathematics. Nevertheless, in this study it spurs students' mathematical curiosity to do so. Students formulated and solved problems without being directly required. The questions on logarithms and time constants were surprising outcomes of the study.

3. Autonomous questioning analysed with herbartian schema

Using the herbartian schema, to describe Group 6's answer to Q_3 looks as below:

$$[S(X_6, y, Q_{3,4,2,2}) \curvearrowright M] \hookrightarrow A_{3,4,2,2}{}^{\blacktriangledown}$$

Group 6 developed their answer by exploring the milieu: $M = \{A_1^\circ, A_2^\circ, \ldots, A_n^\circ, O_1, O_2, O_3, Q_{3,4,2,2}\}$. The group must have studied more media than those suggested by the teacher, since they knew the relation between logarithms and exponential functions – or a member had picked it up earlier, in another context. The Internet is flooded by webpages offering tutorials and guidance of varying quality of these topics.

Group 8 adressed the answer of Group 6 explicitly by raising $Q_{3,4,2,2,1}$. This gave rise to a joint study of the two groups, X_6 and X_8. Group 8 provided the question based on their shallow study of $A_{3,4,2,2}{}^{\blacktriangledown}$. Group 6 sought to answer $Q_{3,4,2,2,1}$ based on media familiar to them and their acquired knowledge. This can be described as:

$$[S(X_8, X_6, y, Q_{3,4,2,2,1}) \curvearrowright M] \hookrightarrow A_{3,4,2,2,1}{}^{\blacktriangledown}$$

Group 6 initiated the formulation of an incomplete answer and the group suggested further media for the class to study. The definition of logarithms as the inverse function of $y = 10^x$, was a known answer of group 6, but a work to study for Group 8 and the rest of the class.

Through the answer Group 8 provided for Q_3, it was clear that they had studied the suggested media and applied the formula given, $T_2 = \log(2) / \log(a)$. In the textbook, no justification of the formula was provided. Whether Group 8 linked the answer of Group 6 to the answer they had studied, is unclear from data. But when the group was introduced to $A_{3,4,2,2,1}$ they started to formulate questions. And Group 6 was eager to help construct or develop answers.

Further, when the teacher posed the generating question Q_3' group 7 and 8 drew on Group 6's answer to Q_3. This means that their study and research process can be described as the following:

$$[S(X_8,X_6y,Q_{3,6,1}) \curvearrowright M] \hookrightarrow A_{3,6,1}^{\heartsuit}$$

Here $M = \{A_{3,5}^{\diamond}, A_{3,4,2,2}^{\diamond}, A_{3,4,2,2,1}^{\diamond}, \ldots, O_1, O_2, O_3, Q_{3,6,1}\}$. Both Group 7 and 8 used the answer provided by Group 6 to study how they can reason for the existing of a T_4. In this way each question of Figure 5 relates to a herbartian schema that describes the dynamics of the construction of its answer.

Hence, it became explicit that responsibility had changed regarding who supported the study and research process, who posed and answered questions and who delivered media for further study. The changed didactic contract led students to present and study each other's answers, which induced an autonomous questioning and the development of reasoning related to the rule of calculating the doubling time. It is worth noticing that the responsibility taken by the students does not mean the teacher is not needed. The teacher's role is to set a scene with potentials for autonomous study and research processes for the students. Further the students use the teacher to validate their vague ideas on inverse functions as well as this choice of media on logarithms.

DISCUSSION

The two episodes described above suggest the feasibility of students take an active role in identifying and formulating the questions they work with, as discussed by Bosch and Winsløw (2015). Though autonomous questioning was realised, the sustainability can be questioned. The study realised the potential of continued formulation of questions and answers from the students. Although, from a pure mathematical perspective, one might wish for more. Why did the students not question none-integer exponents? What are they? How does the calculator find the decimal number representing 3^{π}? As argued earlier, these questions are beyond the curriculum and the media treat exponents as something natural, not to be questioned. This indicates, if the studied works treat notions as something not to be questioned, it might limit the students' initiatives regarding problem posing. When students studied other groups' answers it seemed "legal" to question it. It was natural to question the use of logarithms and pursue this further when a vague answer using inverse functions did not satisfy the other groups. This aligns with recent results of Otaki, Miyakawa and

Hamanaka who reports that SRPs makes it easier for students to formulate "why-questions" and pursue these (Otaki et al., 2016, p. 17). This indicate that the management of SRA or SRP based teaching should facilitate students immediate study of other students (initial) answers through some kind of sharing settings. In this study the board was used but other ways might be useful as well.

Hence the answer to the research question of this paper is that explicit requirements to share preliminary answers for an open question supports students' autonomous questioning. Moreover, the changed didactic contract seemed to reinforce the milieu in order to promote students' formulation of questions and pursuing them.

The strict time frame performed a constraint securing no group presented a perfect answer. Other students could always question elements of the other groups' answers. And no group would waste a lot of time on questions they could not overcome. Group 1 kept encountering equations with exponential notions, which they were not able to solve. They did not use the rule: $\log(a^x) = x \cdot \log(x)$, as other groups did. For this group the time frame might have been too strict. However, their final thematic project employed the rule. Whether they studied the works of the others based on their notes is impossible to determine but in the end, they were able to present a coherent answer to Q_0.

In order to address the question raised by Bosch and Winsløw (2015) on the mathematical and didactic infrastructures needed to realise sustainable study and research processes, this study has realised some key potentials of SRP's regarding students' problem posing and development of answers. With respect to didactical infrastructures, the planning of the lesson – including the *a priori* analysis and choice of appropriate media – it takes more time than preparing common classroom activities. Similarly, the four SRA's took three lessons of 95 minutes to complete. It is worth noticing that the class did not need the teacher to institutionalise the intended knowledge. After the SRA's, students solved standard exercises and performed better than similar classes at the oral exam, measured on the grades given. Here the thematic projects were coherent and reflected the questions and answers, which were presented at the whiteboards, including the use of logarithms.

In order to monitor the work of students, the requirement to use the whiteboard functioned well. In the beginning (the SRA's on Q_1 and Q_2) the teacher initiated the students' study of the other groups' answers. The teacher explicitly asked what similarities and differences the class could find between the presented answers. This led to discussions on such topics as notation but also to studying the relation between multiplying by the factor 1.025 ten times versus calculating $f(10)$ when $f(x) = 5000 \cdot 1.025^x$. This might not be a ground shaking mathematical discussion but it gradually expanded their techniques and autonomy for solving problems about exponential expressions.

The disadvantage of the rigid requirement of all groups presenting sometimes similar answers were the time consumed and that it became boring to attend. For a longer study alternative, configurations of use of boards might be needed.

While designing the SRA's, the mathematical infrastructures were taken into account and the interrelation between the SRA's as made explicit through the story on grandparents. Through the *a priori* analysis the possibility of reinvestment of techniques and strategies was successfully aimed for. This might strengthen the students' inclination to use a developed answer in their subsequent study and research process. For Q_1 it should be possible to answer the question based on previously acquired knowledge. Hence the question opens for the possibility of answering the question based on research activity. However, the problem is much more directly approached if the students study the suggested media. This idea was continued in all the questions of the SRA.

CONCLUDING REMARKS AND FURTHER PERSPECTIVES

Whether the SRA designed and studied in this paper is akin to what Kilpatrick described as the situation in real life outside school, is hard to determine. But it seems evident that it suggests new approaches to problem posing in teaching. When the students answer Q_3' by employing rules on logarithms which have never been presented by the teacher, it indicates a significant development of problem solving skills. This is not just solving problems similar to presented examples by imitating a procedure. Students in this study reused procedures of other students, but after questioning the nature of the rule. This supports the development of "a broader understanding of mathematical concepts and the development of mathematical thinking" (Singer, Cai & Ellerton, 2013, p. 2). Based on the study of this paper, it seems crucial for the problem posing that students develop questions and answers in a genuine study and research process – where students point to gaps or inconsistencies, which they need to mend. When this happens, the students have the possibility of developing coherent mathematical knowledge within a prescribed area. This result of the study supports that problem posing should be considered a part of inquiry approaches to teaching (as in Artigue & Blomhøj, 2013) rather than some special activity conducted apart from standard classroom activities as listed by Bosch & Winsløw (2015, p. 371).

Based on the thematic projects, we confirm that no group simply adopted the rules of the textbooks. The groups' answers were interrelated and they gave reasons for their solution methods. In that sense the result of the taught sequence of SRA's avoided a common tendency to atomise the mathematical knowledge. In that light, the SRA taught under the condition and restrictions of strict time frame, being group based, with required sharing of answers (spoken and written), teacher proposed media and finalised in a thematic project, seemed fruitful for curriculum bound teaching.

The external constraints of this experiment might be stronger than in previous experimental studies on SRP's. A full SRP certainly has advantages regarding the potential of students' learning e.g. regarding the dialectics of media and milieu. But as the study of Jessen (2014) and Otaki, Miyakawa and Hamanake (2016) shows, the students might realise (also qualitative) very different paths. Rasmussen (2015) planned only 20% of the course activities as SRP because of the challenges in securing the students' acquaintance with the "monuments of curriculum". The other studies were not core course activities. In light of this more empirical work exploring potentials of generating questions and how to manage these in the ordinary classroom must be conducted. Furthermore, setting up longitudinal studies where students can adapt to the changed contract through sequences of SRA's combined with full SRP's, would be interesting for the study of students' problem posing. In this sense, some elements of the teaching could be close to a given curriculum, while other parts could be in depth studies of pieces of mathematical knowledge with students' problem posing as a core element.

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