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Published in:
REDIMAT - Journal of Research in Mathematics Education

DOI:
10.17583/redimat.2017.2078

Publication date:
2017

Document Version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Wijayanti, D., & Winsløw, C. (2017). Mathematical Practice in Textbooks Analysis: Praxeological Reference Models. The Case of Proportion. REDIMAT - Journal of Research in Mathematics Education, 6(3), 307-329. https://doi.org/10.17583/redimat.2017.2078

REDIMAT
Journal of Research in Mathematics Education

Hipatia Press
www.hipatiapress.com

Instructions for authors, subscriptions and further details:

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Date of publication: October 24th, 2017
Edition period: October 2017-February 2018

To cite this article: Wijayanti, D., & Winsløw, C. (2017). Mathematical practice in textbooks analysis: Praxeological reference models, the case of proportion. *REDIMAT*, *6*(3), 307-330. doi: 10.1783/redimat.2017.2078

To link this article: http://dx.doi.org/10.17583/redimat.2017.2078

REDIMAT, Vol. 6 No. 3 October 2017 pp. 307-330

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(*Received: 05 May 2016; Accepted: 26 September 2017; Published: 24 October 2017)*

Abstract

We present a new method in textbook analysis, based on so-called praxeological reference models focused on specific content at task level. This method implies that the mathematical contents of a textbook (or textbook part) is analyzed in terms of the tasks and techniques which are exposed to or demanded from readers; this can then be interpreted and complemented by a discussion of the discursive and theoretical level of the text. The praxeological reference model is formed by the analyst to categorize various elements of the text, in particular the tasks and techniques which it explains or requires from readers. We demonstrate the methodological features of this approach by analyzing examples and exercises in three Indonesian textbooks, focusing on the chapters dealing with *arithmetic proportion* (defined theoretically by the model). We also illustrate how this rigorous analysis can be used to provide a quantitative "profile" of textbooks within a topic.

2017 Hipatia Press
ISSN: 2014-3621
DOI: 10.17583redimat.2017.2078

REDIMAT, Vol. 6 No. 3 Octubre 2017 pp. 307-330

Keywords: Textbooks, praxeology, proportion

Prácticas Matemáticas en el Análisis de los Libros de Texto: Modelos Praxeológicos de Referencia, el Caso de la Proporción

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(*Recibido: 05 Mayo 2016; Aceptado: 26 Septiembre 2017; Publicado: 24 Octubre 2017)*

Resumen

Presentamos un nuevo método de análisis de libros de texto, basado en los llamados modelos praxeológicos de referencia. Este método implica que el contenido matemático de un libro de texto (o parte del libro de texto) se analiza en términos de las funciones y técnicas que están expuestas o que se exigen al lector; esto puede interpretarse y complementarse con una discusión del nivel discursivo y teórico del texto. El modelo de referencia práctica sirve para categorizar varios elementos del texto. Mostramos los elementos metodológicos de este enfoque analizando ejemplos y ejercicios en tres libros de texto indonesios. Ilustramos cómo este análisis riguroso puede usarse para proporcionar un "perfil" cuantitativo de los libros de texto dentro de un tema.

Palabras clave: Libro de texto, praxeología, propocionalidad

2017 Hipatia Press
ISSN: 2014-3621
DOI: 10.17583redimat.2017.2078

The importance of "tasks" (exercises, problems and so on) as a main component of students' mathematical activity is increasingly acknowledged by researchers (e.g. Watson & Ohtani, 2015). Indeed, it is a commonly held assumption of both mathematics teachers and researchers that "the detail and content of tasks have a significant effect on learning" (ibid., p. 3). While a school mathematics textbook may at first present itself as a treatise exposing various contents, one of its main functions is in fact be to be a repository of tasks - whether presented together with solutions (often in the "main text"), or proposed as work for students (often in a separate section or volume of "exercises"). Many teachers draw on textbooks as a main source of examples and exercises (Fan, Zhu, & Miao, 2013, p. 643). In choosing a textbook, teachers (or whoever make that decision) will therefore have a significant interest in the contents and quality of the tasks exposed or proposed in the book.

What can teachers (or others) do to examine textbooks from this angle? One can try to assess if the tasks are compatible with any official regulations of mathematics teaching (e.g., the national curriculum). However, such guidelines are not always precise to the point of specifying types of tasks which students should encounter or work on, and so they offer little guideline for analysing examples and exercises in a detailed way. One may also use any relevant national exams to see if the book aligns with types of tasks found there; but in many contexts, such a "measure" will be highly reductive or wholly irrelevant.

In practice, teachers will often depend on others' assessments and opinions about a textbook, such as reviews in magazines or websites for mathematics teachers. Some countries (e.g. Indonesia and Japan) even have a national agency that produces reviews of textbooks and authorizes their use in public schools. But whether such assessments are endorsed by authority or not, one can ask the question: what are they based on? Against what common measure are books evaluated? Could this measure be based on explicit theoretical models, grounded in research? What kinds of theoretical models could enable a systematic and (ideally) reproducible means of analysing and synthesizing the qualities of textbooks, with a special emphasis on tasks?

Of course, analysing all tasks in a textbook could be quite time consuming. If indicators of the overall "quality" of a textbook are aimed for, it is natural to select a few topics which are usually considered problematic or challenging in teaching practice. These problematic topics will typically have attracted considerable attention of mathematics education research, so that the analysis of textbooks focusing on them will have a wide range of research literature to draw on. This could be helpful to set up a sharp theoretical model of the mathematical topic itself, understood as a practice and knowledge with which the text may engage the reader, through its explanations, exercises, etc.

The subject of this paper is the analysis of didactical texts with a focus on one or more mathematical topics - as a case, we consider Indonesian textbooks for grade 7, and the area of mathematics at this level which can loosely be referred to as proportion and ratio in arithmetic. Using this case, we propose a new methodological framework to analyse examples and exercises thoroughly. This framework is based on the anthropological theory of the didactic, and especially the notion of praxeology and praxeological reference model (see Barbé, Bosch, Espinoza, & Gascón, 2005).

The structure of the paper is as follows in Section 2, we present a selection of strongly related background literature for our case study, concerning student and teacher practices related to the proportion in arithmetic, textbook analysis at large, and research into textbook treatments of the proportion topic. In section 3, we introduce our theoretical framework for textbook analysis, based on the notion of praxeology of the anthropological theory of the didactic. In Section 4, we present the main result of the paper, namely a praxeological reference model for the topic of proportion, developed for and from a study of three Indonesian textbooks. As a supplement to the theoretical description of the model, illustrated by textbook excerpts, Section 5 contains a discussion of some methodological challenges and principles for applying the model, illustrated by concrete “limit” cases from textbooks. In Section 6, we show how the model may be used to produce a quantitative “profile” of the three textbooks; similar profiles could be made using the same model on other textbooks, possibly with a slight extension of the model. We discuss, in Section 7, this and wider perspectives of our methodology for producing explicit and systematic accounts of mathematical practices shown or elicited by a textbook within a given area.

Research Background

In this section, we first review some of the main trends and methods available in recent research on mathematics textbooks, focusing on the precision with which topic is analysed. We then consider in more detail two recent studies on the proportion topic.

Textbooks Analysis

In a special issue of *Textbook Research in Mathematics Education* Fan et al. (2013) note the growth of research on mathematics textbooks during the past six decades; it is no longer a "new" field. Fan (2013, p. 773) considers that, in the wider perspective of improving textbooks or mathematics teaching, "it is only the first step to know what the textbooks look like, for example how a specific topic (e.g. algebra or geometry) is treated in a textbook or different textbooks, or how different types of problems are presented in a textbook or in textbooks in different countries". Indeed, (Fan, 2013, p. 774) also mentions that there seems to be a movement from "textbook analysis" towards "textbook research" which encompasses much wider empirical realms than the textbook itself (we could talk of a movement towards "zooming out"). At the same time, the first step may be far from completed - it concerns analytical research, based on solid methodological tools, on the finer details of the mathematical contents of the books. In fact, this paper begins from the premise that theoretical and methodological tools for such a *higher level* of granularity (that is, "zooming in") must be developed. Our analysis of mathematical contents in textbooks must be based on explicit models of such contents, rather than institutional point of view which is implicitly taken for granted.

As an example of research with this higher level of granularity, we refer to a study by Stylianides (2009) who developed an analytical approach to examine tasks (exercises, problems or activities) in American school textbooks for grade sixth, seven, and eight, considering both algebra, geometry and arithmetic. In this framework, Stylianides used 'providing proof' as one of task category and resulted that none of the exercises in the textbooks ask for "generic" (i.e. formal, "general") proofs, but instead asked students to provide various informal explanations, for instance based on a figure or computation. Stylianides' framework on reasoning and proving also played a significant role in a recent special issue of *International Journal of Educational Research* (2014, pp.63-148), focusing on special section: Reasoning and proving in mathematics textbooks: from elementary to the university level).

These categories are certainly specific to certain modalities of work in mathematics (argumentation, reasoning, proof) but they are completely generic with respect to the mathematical contents - the analysis works the same way for tasks on geometry and algebra (for example) and is largely insensitive to specific features of each of these content areas. In fact, concerning research on specific types of mathematical tasks in textbooks, we agree with González-Martín, Giraldo, and Souto (2013, p. 233) that the existing literature is extremely scarce.

In fact, our methodological approach has similarities to the one employed in the study by González-Martín et al. (2013), especially the use of the notion of praxeology to study tasks in textbooks; but the two methodologies also different, as we shall now explain. These authors investigated the case of the introduction of real numbers in Brazilian textbooks, based on a model which has, at its basis, rather broad classes of tasks for the students, such as $\mathcal{T}$*: "*Classifying a given number as rational or irrational". The broadness of this and other task classes considered in that paper stems from the multiplicity of techniques that may be used to solve a given task from this class. For example, for a task like deciding whether $5 + \sqrt{3}$ is irrational or rational, textbooks provide a specific rule: 'the addition of rational and irrational number is irrational' which works here, if the solver knows that 5 is rational and $\sqrt{3}$ is irrational. The scope of this technique is quite limited (one needs only think of the case $\sqrt{0{,}1}$) and corresponds to a much narrower class of tasks than $\mathcal{T}$. The model still suffices to map out "large classes of tasks" which leads to remarkable characteristics of how the textbooks analysed treat the topic; but it does not exhaust the differences in terms of the precise technical knowledge which each of the books could develop among students. By contrast, our approach aims at classifying types of tasks in the precise sense of "tasks which can be solved by a given technique", and to draw up an explicit, precise model of the techniques.

Proportion in School Textbooks

Students' and teachers' work with proportion and ratio (or proportional reasoning) is probably one of the most intensively studied topics in mathematics education research. In an early literature study, Tourniaire and Pulos (1985, p. 181) mention that "proportional reasoning has been the object of many research studies in the last 25 years". The authors give an interesting overview of research done during this period, which was largely dominated by cognitive paradigms of research; they also insist on the difficulty of describing explicitly the structure and boundaries of "proportional reasoning".

Research in the cognitive framework was, and is still, often based on test designs. These are of particular relevance to us because such designs sometimes indicate fairly detailed models of the mathematical components of the topic. For instance, to measure student difficulties with the different proportion type of tasks, (Hilton, Hilton, Dole, & Goos, 2013) designed a two-tier diagnostic instrument to measure the degree to which students' master "proportional reasoning". However, the underlying reference model remains implicit in this and many similar studies: it seems that the authors take for granted that readers share the same idea about proportion or proportional reasoning; instead of definitions, the reader is left with the test instrument which, evidently, consists of examples of tasks, rather than explicit types of tasks described theoretically in terms of techniques. It cannot, thus, be used to classify tasks except if they are very similar to the test items, but it can serve as material for validating a given reference model in terms of whether it can classify the items.

In the literature, we find various useful theoretical distinctions of relevance to the theme of proportion, which have supported our model construction (Section 4). For instance, we note the four different kinds of ratio problems defined by van den Heuvel-Panhuizen (1996, p. 238): finding the ratio, comparing ratios, producing equivalents ratio, and finding the fourth proportional.

Considering textbook analysis, there exists a number of studies of proportion in textbooks based on broad models of students work with proportion. Dole and Shield (2008) developed a list of four "specific curricular content goals". Using these goals, the authors examined the extent to which these goals were pursued in two Australian textbooks. The authors later developed their model and extended the analysis to encompass five textbook series (Shield & Dole, 2013). Tasks and examples appear as illustrative cases of the analysis, but the corresponding content requirements (in terms of techniques) are not analysed.

We have also been inspired by a more fine-grained model, developed by Hersant (2005) for the case of "missing number tasks". Hersant developed a completely explicit model for the techniques identified in different programmes and corresponding textbooks. In terms of what we present in this paper, her model corresponds to a fine-grained analysis of possible variations of a specific technique (the one called τ_6 in section 5). Finally, Lundberg (2011) also focused on missing value tasks related to direct proportion. The studies of Hersant and Lundberg are based on the anthropological theory of the didactic, as the present paper, but consider only to illustrative cases while our model is used to characterize the arithmetical proportion topic as it appears in an entire textbook (cf. Section 6).

Theory and Methodology

We now introduce our theoretical framework, based on the Anthropological Theory of the Didactic (ATD), in particular praxeological reference models and the levels of didactic co-determination. On this basis, we introduce the context and methods of the present study.

Praxeologies

The basic idea of this study is to make full use the notion of *praxeology* from ATD, proposed by Chevallard (1999). Praxeology means *praxis* and *logos*, to indicate that a praxeology is a model of some specific amalgam of human practice and knowledge. Concretely a praxeology is a 4-tuple $(T, \tau, \theta, \Theta)$ where the four letters denote different, but closely related, components of the praxeology. While this notion is described in detail by several authors such as (Chevallard, 1999) and (Barbé et al., 2005) , it is so central to our work that we provide our own description here.

At the basis of a praxeology $(T, \tau, \theta, \Theta)$ we have a *type of tasks T* that is a collection of tasks which can be solved by some *technique* τ. Notice that

T and τ are in 1-1 correspondence: T consists of the tasks which can be solved by τ. Notice also that the term "task" in ATD simply means something humans can accomplish with a simple action (the technique); in mathematics, it could be some algorithm or other basic method. Since a praxeology is a model, it depends on the purpose of modelling what kind of human action it will be useful or feasible to distinguish as a technique; the theory does not provide any strict definition of what would count as a technique (and thereby, as a type of task). We note here that the main difference between our approach and the uses of ATD for textbook analysis provided by Lundberg (2011) and González-Martín et al. (2013) is the explicit definition of techniques (presented in Section 4), which enable us to work with types of tasks (in the proper sense of ATD, that is, defined by one technique) rather than the informal use of the term type of task as "a collection of tasks with a similar form and content".

In many contexts (certainly those involving mathematical practice) it is essential to be able to describe and justify techniques. This leads to a "discourse about the technique" which is the element θ in the praxeology. Because θ represents "logos about techniques", it is called a *technology* in ATD (not to be confused with every uses of the term). Finally, the "practical discourse" of how to do task (the technology) is complemented by a discourse about the technology itself, the *theory* Θ. This discourse allows us to challenge, combine and explain the practical discourse independently from specific techniques; for instance, the problem of solving polynomial equations can be discussed at a theoretical level through definitions and existence theorems, and this discourse can then serve to relate, compare, explain and validate concrete techniques for solving more specific kinds of polynomial equations.

A *reference praxeological model* for some human activity is then simply an explicit description of praxeological elements $(T, \tau, \theta, \Theta)$ which we use as a reference for analysing the activity. The model can be more more or less detailed according to the purposes of our analysis.

Levels of Didactic Co-Determination

The study of textbooks is full of indications of institutionally stable ways of organising the practice and knowledge which the books aim to engage the students with. First, the textbook will usually indicate the school type and age level it is meant for, as well as the discipline - for instance, one the textbooks analysed in our study has the full title (in English translation): "Mathematics 1: concepts and applications for grade 7, SMP/MTs". Here, SMP/MTs denote two kinds of junior high school in the Indonesian school system, "1" refers to the first year in junior high school, and "7" to the grade while counting also the preceding six years in elementary school. "Mathematics" naturally refers to the school subject which, in turn, can be seen to consist of several levels and elements that are apparent in Chapter and Section headings.

ATD provides a hierarchy of explicit *levels of didactic co-determination* to help explicate and examine these "layers" of organising and structuring the teaching of praxeologies in institutions, usually called *schools*. We do not use the whole hierarchy in this paper, but we will need to use the following levels precisely and coherently:

- The *discipline* is here the school subject mathematics (in Indonesian lower secondary school)
- The *domain* within mathematics is "arithmetic" (cf. Section 4). In general, a domain is a larger part of a discipline which unifies a number of different theories.
- A *sector* is defined by a theory, unifying several praxeologies (sometimes called a *regional organisation*). The one considered in this paper is defined in Section 4 and concerns "proportion" of numbers, which appear also in many social practices outside the school.
- A *theme* is defined by a technology and thus unifies related techniques and types of tasks; it is located within a sector. For instance, "ratio and scale" indicates a discourse with the notions "ratio" and "scale" and central tools to describe and justify specific techniques of calculation within the proportion sector.

We notice that besides textbooks, these levels also appear more or less directly in national curricula of many countries, and while "mathematics" is a discipline in schools almost everywhere, the lower levels may display larger variation.

Our Context

Several factors motivate the special interest of analysing and assessing textbooks for Indonesian schools, for instance:

- The sheer number of students who could, in a given year, be using a textbook (According to Statistic Indonesia (2013) there are 12.125.397 grade 7, 8, 9 students in Indonesia in 2013 ; all are taught in the same language and according to the same national curriculum)
- The fact that only 37% of the teachers who have the required education level (The World Bank, 2011) results in a dependency on textbooks by many Indonesian teachers.

Indonesia has nine years of general, compulsory education (6 years of elementary school for students aged 7-13, and 3 years of lower secondary school for age levels 13-16). All authorised textbooks are made available electronically and can be downloaded at www.bse.go.id.

In the Indonesian curriculum, students are supposed to learn proportion within the arithmetic domain during the first grade of lower secondary school. However, the curriculum does not specify the detailed contents of the sector “proportion”. Thus, one might expect a large variation in how textbooks treat the theme. In this paper we analysed the proportion sector as it appears in the following three textbooks, which are the only textbooks which are both authorized for grade 7 in the year 2014 and digitally available : Nuharini and Wahyuni (2008). The digital (online) access of the three books means that they are widely used. These textbooks were all produced in 2008 at the occasion of a major curriculum reform.

Methodology

The way to construct and use a praxeological reference model needs further explanation. First, the model is not constructed independently from the material to be analysed, but it is constructed *along with the analysis* and serves, in the end, to make that analysis completely explicit. It should then also be *reproducible* in the sense that the same analysis would be made by other researchers who have familiarized themselves with the model.

Next, to analyse a sector we need to identify what part of the textbook it corresponds to. As Indonesian textbooks follow the national curriculum quite closely, it is easy to identify the parts of the textbooks which correspond to proportion. Then, within these parts of the books, we begin to analyse all *examples* to identify the *techniques* they present students with, and the corresponding types of tasks. The examples give us explicit information about the techniques which students may use when solving exercises. The exercises are then solved and analysed in terms of the types of tasks found in examples; if needed, new types of tasks are added to the model, in order to be able to classify and to describe all exercises precisely and objectively. We emphasize that the reference praxeological model is as much a result as it is a tool of our analysis.

Praxeological Reference Model for The Sector of Proportion

In accordance with the literature reviewed in Section 2, we consider *proportion* as concerned with numbers and quantities, thus belonging to arithmetic in the broad sense of “calculation with positive real numbers” (possibly with units and occasionally including also zero) in school and other social contexts. We note here that a *quantity* can be seen, abstractly, as a positive real number together with a unit, such as 0, 75 litres or 5 apples. Here, the unit (litre or years) corresponds to some measure that the number “counts”. In the domain of *algebra, one* can consider magnitudes as products of numbers and unit symbols, but with the domain of arithmetic, units have to treat with more semantic than syntactic means of control; in particular, operations are done only with numbers, and the questions of units must be handled separately, with reference to the context of measurement.

In our reference model, *proportion* will actually be a *sector* within the *domain* of arithmetic. It is unified by a *theory* that keeps together the two *themes* which the sector consists of; each of the themes has their explicit technology, which in turn unifies and relates the *subjects* within the theme. We first describe the theory level of our model which, in fact, is quite distant from the texts we have analysed, but which is indispensable for the describing and applying the rest of the model (the themes and subjects) with precision. Then, we present two themes that provide types of tasks and techniques in English translation.

Basics of a Theory of Proportion

A systematic reference model requires precise notation and terminology. As researchers, we establish this from the basis, in our own terms (while it is be inspired from the literature reviewed above, especially (Miyakawa & Winsløw, 2009), the "principle of detachment" (Barbé et al., 2005) is a main point of ATD, to avoid whole sale assumption of established institutional jargon, or even of ideas and terms that are often taken for granted by scholars.

In the following we designate numbers or quantities by letters (x, y_j etc.) to describe a theory which involves *only numbers and quantities* (and only occasional "letters" in their place, in the case of "unknowns" to be determined). In the rest of this section, letters are understood to represent positive real numbers or quantities.

Definition. Two pairs (x_1, x_2) and (y_1, y_2) are said to be *proportional* if $x_1 \cdot y_2 = x_2 \cdot y_1$; we write this in short as $(x_1, x_2) \sim (y_1, y_2)$. More generally, two *n*-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are said to be proportional if $(x_i, x_j) \sim (y_i, y_j)$$(\mathrm{x_i}, \mathrm{x_j}) \sim (\mathrm{y_{i,}} \mathrm{y_j})$ for all $i, j = 1, \ldots, n$; we then write $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$.

It is easy to prove that ~ is an equivalence relation on $\mathbb{R}_+^n$ for all *n* = 2,3, ... (one can make use of P1 below). A number of other useful properties of this relation are listed below, where, for the sake of brevity, we just formulate the results for 2-tuples:

> P1. If we define the internal ratio of a pair $(\mathrm{x}_1, \mathrm{x}_2)$ as $\frac{\mathrm{x}_1}{\mathrm{x}_2}$, then $(\mathrm{x}_1, \mathrm{x}_2) \sim (\mathrm{y}_1, \mathrm{y}_2)$ is logically equivalent to equality of the internal ratios $\frac{\mathrm{x}_1}{\mathrm{x}_2}$ and $\frac{\mathrm{y}_1}{\mathrm{y}_2}$.
> P2. Similarly, $(\mathrm{x}_1, \mathrm{x}_2) \sim (\mathrm{y}_1, \mathrm{y}_2)$ holds if and only if the external ratios $\frac{\mathrm{x}_1}{\mathrm{y}_1}$ and $\frac{\mathrm{x}_2}{\mathrm{y}_2}$ are equal (notice that external ratio concerns two tuples, while internal ratio depends only on one).
> P3. If $(\mathrm{x}_1, \mathrm{x}_2) \sim (\mathrm{y}_1, \mathrm{y}_2)$ and $\mathrm{x}_1 < \mathrm{x}_2$, then $\mathrm{y}_1 < \mathrm{y}_2$.
> P4. If $(\mathrm{x}_1, \mathrm{x}_2) \sim (1, \mathrm{r})$ if and only if $\frac{\mathrm{x}_2}{\mathrm{x}_1} = \mathrm{r}$, that is, if and only if $\mathrm{x}_2 = \mathrm{r} \cdot \mathrm{x}_1$.

We finally note that for 2-tuples (x_1, x_2) and (y_1, y_2), the property $(x_1, x_2) \sim (y_2, y_1)$ is sometimes called *inverse proportion*; when it holds, we say that (x_1, x_2) and (y_1, y_2) are *inverse proportional*. This corresponds to a relation on 2-tuples which, however, is not an equivalence relation (it lacks transitivity); also, it does not have natural generalisation to *n*-tuples. It is, nevertheless, as the definition also shows, closely related to proportion (sometimes called “direct proportion” to distinguish it from inverse proportion).

With this theoretical basis of the sector we can now describe the rest of our reference model, consisting of two themes, each constituted by several types of tasks.

Theme 1: Ratio and Scale

Property P4 above deals with the special case of proportion where one of the tuples is of the form $(1, r)$. This case is closely linked to a technology involving *ratio* and *scale*. Both terms refer to the number *r* in P4 (and thus a property of a single pair of numbers); we use *scale* for the special cases where r or 1/*r* is an integer, and ratio for the general case. In any cases, when *r* is a fraction of integers *m*/*n*, the notation *m*:*n* is often used, as in the alternative formulation $x_2 : x_1 = m : n$ of the characteristic property in P4. In Table 1, we present three tasks $(\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3)$ that exemplify the three types of tasks in the theme ratio and scale.

In table 1, t_1 and t_3 are tasks that appear in an example in the textbook quoted. Thus, the technique can be read off from the quote.

Table 1.
Tasks that exemplify the types of tasks in the theme “ration and scale”

Task	Text
t_1	The price of eggs was Rp. 10.000,00 per kg. But the price of egg increased 6:5 from the original price. What is the current price of egg per kg? Answer: Current price: original price = 6:5 Current price = $\frac{6}{5}$ × Rp. 10.000,00 =Rp. 12.000, - (Nuharini & Wahyuni, 2008, p. 148)
t_2	A mother gives R.p 5.000,00 for pocket money to a kid. $\frac{2}{5}$ of pocket money is used to buy stationery. How much pocket money is left? (Nuharini & Wahyuni, 2008, p. 147)

Task	Text
t_3	Ali saves Rp. 300.000,00 in the bank and Budi saves Rp. 450.000,00. Determine the ratio of Alis' saving and Budi's saving? Answer: $\text{ratio} = \frac{\text{Rp.}300.000{,}00}{\text{Rp.}450.000{,}00} = \frac{2}{3}$ (Wagiyo et al., 2008, p. 115)

We found that although t_1 and t_2 look quite similar at first, they are different because the given ratio should be applied differently (multiply or divide) and it is a real point that students should distinguish and choose among those options. The task t_3 is clearly different as the students are asked to compute the ratio. We note that tasks involving scale (as defined above) are not included in the table, and could look as follow (this is an example of a task of the same type as t_1).

> A map has a scale 1: 2.000.000 and distance between A and B on the map is 3,5 cm. Determine the real distance from A to B (Wagiyo et al., 2008, p. 113).

Table 2.

Types of Task Related to Ratio and Scale

Type of task	Technique
T_1: Given x_1 and r, find x_2 so that $(x_1, x_2) \sim (1, r)$.	τ_1: $x_2 = r.x_1$ (multiplying by the given ratio)
T_2: Given x_2 and r, find x_1 so that $(x_1, x_2) \sim (1, r)$.	τ_2: $x_1 = x_2/r$ (dividing by the given ratio)
T_3: Given x_1 and x_2, find r so that $(x_1, x_2) \sim (1, r)$.	τ_3: $r = x_2 / x_1$ (finding the ratio)

Based on analysing these and many other tasks occurring in the textbooks, we defined three types of tasks $(\mathrm{T}_1 - \mathrm{T}_3)$, and the corresponding techniques$(\tau_1 - \tau_3)$, as shown in Table 2; the connection between tables 1 and 2 is, naturally, that t_i is of type T_i (i=1,2,3).

Theme 2: Direct and Inverse Proportion

In theme 1, the tasks really involve only one tuple; the implicit tuple (1, *r*) is either completely identified with one number (the ratio). We now proceed to a theme which is unified by a technology on certain relations between *two* tuples (most often, but not always, 2-tuples); these can either be *directly* or *inversely* proportional; both relations have important and common examples in real life (e.g. s. Here, we identified four types of tasks. As before, we first give characteristic examples for each of them (Table 3).

Table 3.
Tasks that exemplify the types of tasks in the theme "direct and inverse proportion"

Task	Text
t_4	The order of numbers in a proportion must be correct. Indicate for each statement if it is false: a. My age: father's age = 4:1 b. Population of Jakarta: population of Bandar c. Lampung = 1:10 Toni's age: Toni younger brother's age = 3:2 (Wagiyo et al., 2008, p. 116)
t_5	In Bu Ina's grocery, the price of a package containing 2 kg of sugar is Rp. 9.400,00 and the price of a package containing 5 kg of sugar is Rp. 22.750,00. Which package is cheaper? What would you do to solve that problem? (Wintarti et al., 2008, p. 194)
t_6	The price of 2 m fabric is Rp. 45.000,00. How much does 10 m fabric cost? Answer: The price of 2 m fabric is Rp. 45.000,00. So, the price of 1 m fabric is $= \frac{Rp.\ 45.000,00}{2} = Rp. 22.500,00.$ Thus, the price of 10 m fabric is: $10 \times \text{Rp.} 22.500,00 = \text{Rp.} 225.000,00$ (Wagiyo et al., 2008, p. 120)

Task	Text
t_7	A package of candies was distributed to 20 children, so that each child receives 10 candies. How many candies would each child receive if the same package of candies were distributed to 50 children? Answer: 20 children= 10 candies 50 children= n candies Based on inverse proportion, one gets $\frac{20}{50} = \frac{n}{10}$ $\Leftrightarrow 50 \times n = 20 \times 10 \Leftrightarrow n = 4$ (Wagiyo et al., 2008, p. 124)

The corresponding types of tasks are shown in Table 4. Notice that the technique τ_4 can be justified by property P3 (proportional tuples have the same order relations).

About direct and inverse proportion, these four types of tasks exhaust almost all exercises and examples in the three textbooks; the exceptions and limit cases are discussed in the next section.

Table 4.
Types of task related to "direct and inverse proportion"

Type of task	Technique
t_4: Given numbers a, b and given that $x > y$ are relations with a, b. Can it be true that $(x, y) \sim (a, b)$?	τ_4: The answer is yes only if $a > b$.
t_5: Given (x_1, x_2) and (y_1, y_2), compare internal ratios	τ_5: Calculate $\frac{x_1}{x_2}$ and $\frac{y_2}{y_2}$, and compare.
t_6: Given (x_1, x_2) and y_1 find y_2 so that $(x_1, x_2) \sim (y_1, y_2)$	τ_6: Calculate $y_2 = \frac{x_2 . y_1}{x_1}$.
t_7: Given x_1, x_2, y_1 find y_2 such that (x_1, x_2) and (y_1, y_2) are in inverse proportion	τ_7: Calculate $y_2 = \frac{x_1 . y_1}{x_2}$.

Methodological Remarks

In this section, we discuss some methodological challenges we encountered with the above model, above all tasks which we found hard or impossible to classify with it. These occur in four main groups.

Combination with Techniques from Other Sectors

Many exercises contain more than one question and each of these can be a task, or a combination of tasks, in the sense of ATD. In order to relate "old knowledge" with the knowledge taught in a given chapter, exercises may draw on other sectors besides that of the chapter. Specifically, when analysing exercises from a chapter on proportion, some of the techniques required to solve the exercise may come from other sectors and even domains; we then simply disregard this part in our analysis. However, sometimes the two techniques (one from the sector we study, one from without) may be rather difficult to separate, or we need to make strong assumptions to classify the tasks. We found two such cases in the three textbooks: one exercise (Nuharini & Wahyuni, 2008) in which students need to use knowledge about similar triangles (and then solve a task of type T_1), another one in which substantial modelling needs to be done from a described situation before one gets to an inverse proportion problem (of type T_7). Our model can only be used to account for the proportion part of these exercises.

Combinations of Two Techniques Can Replace a Third

In some cases, a technique is equivalent to the combination of two other techniques. Here is a typical case of a problem for which both the simple technique and the combination appear quite naturally, taken from an example in a textbook (Table 5). There are three known numbers (3, 24, and 45) and students are asked to find one unknown number. The textbook demonstrates two solutions to the problem above (see Table 6).

Table 5.

A task with combination of two techniques can replace a third (Nuharini & Wahyuni, 2008, p. 152)

A car needs 3 litres of gasoline to go 24 km. How many kilometres can the car reach with 45 litres of gasoline?

In the first solution, the authors are using τ_3 to find the ratio. Then, the answer can be found by multiplying the result with the ratio, following τ_1. In the second solution, the technique τ_6 is used. Thus, the simple technique τ_6 is in fact shown to be equivalent to a combination of two techniques $(\tau_3 + \tau_1)$. Both approaches result in the same answer, however, in the first solution, some extra information is produced, namely the distance which the car can run on one litre of gasoline, while this is not asked for in the problem itself. In view of the form of the question (three given numbers, one to be found), we decided to count this task only as belonging to T_6 and to treat similar exercises in the same way. Even though students can develop their reasoning by using $\tau_3 + \tau_1$, we have classified this task in T_6, based on the simplicity of τ_6 that would make it a more likely choice for students, in comparison to the more complicated one $(\tau_3 + \tau_1)$.

Table 6.

Two solutions to the same problem (Nuharini & Wahyuni, 2008, p. 152)

Solution:

1st approach:

With 3 liters of gasoline, a car can go 24 km, thus 1 liter gasoline can reach = $\frac{24}{3}$ km = 8 km

The distance that can be reached with 45 liters of gasoline is 45 x 8 km= 360 km

2nd approach:

gasoline	Distance
3 liters	24 km
45 liters	x

$x = \frac{45}{3} \times 24\ km = 36\ km$

Thus, the distance that can be reached with 45 liters of gasoline is 360 km

Combinations of Two Techniques

For other - much rarer - problems, it is necessary to combine two techniques. Table 7 shows an instance, which is based on the same notation as the case considered in Section 5.1.

Table 7.
A task with combination of two techniques (Wagiyo et al, 2008, p. 123)

Determine : y: z .   a. $x: y = 1: 2$ and $y: z = 3: 4$ $x: y = 2: 3$ and $y: z = 4: 5$

This problem requires that one combine the ratios of two couples which are related to each other because the second element of the first couple is identical to the first element of the second couple. For instance, to solve task 'b', one can first use τ_3 and then τ_6, as follows:

$$(2, 3) \sim (1, \mathrm{r}) \text{ gives } \mathrm{r} = \frac{3}{2} \ (\tau_3) \ ;$$

$$\left(\frac{3}{2}, \ \mathrm{y}_2\right) \sim (4,5) \text{ gives } \mathrm{y}_2 = \frac{\frac{3}{2} \cdot 5}{4} = \frac{15}{8} \ (\tau_6). \text{ So } \mathrm{x}: \mathrm{y}: \mathrm{z} = 1: \frac{3}{2}: \frac{15}{8}.$$

Unlike the case considered in section 5.2, the task cannot be solved directly by one of the simple techniques of the model, so the use of two techniques is actually needed. We classify this problem as containing a task of type T_3 and a task of type T_6.

Non-Classified Problems

We now present the result of applying the reference model to the three textbooks. In the parts of the three textbooks that were identified with the sector "Proportion", we found in total 30 tasks located in examples, and 276 tasks located within exercises. For each book, we first classified the tasks that occurred within examples (Table 8) and then the tasks within exercises (Table 9). Most tasks located in exercises are of a type already located in examples; in this case, we classified the task as belonging to that type (with exception of the case mentioned in Section 5.2, where both a simple technique and a combination of techniques were demonstrated in an example).

Table 8.

Types of tasks in the examples, number of occurrence

	T_1	T_2	T_3	T_4	T_5	T_6	T_7
Wagiyo et al (2008)	1	1	8	0	0	5	1
Wintarti et al (2008)	1	1	2	0	0	2	0
Nuharini and Wahyuni (2008)	2	0	2	0	0	3	1
	4	2	12	0	0	10	2

All three textbooks have exercises with tasks that cannot be solved by techniques demonstrated in a worked example within the book (and hence appear in Table 12 but not in Table 11). These tasks tend to be exceptional and some of them gave rise to specific (new) types of tasks in the reference model (T_4 and T_5).

The most eye catch thing in these two tables is the similar pattern we find in the two textbooks: the sector "proportion" is, essentially, constituted by five types of tasks (T_1, T_2, T_3, T_6, and T_7) which account for 90% of the examples and 81% of the exercises. These dominant tasks have numerous occurrences in exercises and appear also as examples.

Table 9.

Types of tasks in the exercises, number of occurrence

	T_1	T_2	T_3	T_4	T_5	T_6	T_7
Wagiyo et al (2008)	14	2	62	4	4	26	18
Wintarti et al (2008)	11	6	40	0	1	18	9
Nuharini and Wahyuni (2008)	7	8	24	0	0	10	12
	32	16	126	4	5	54	39

Many Indonesian teachers follow the textbooks closely when structuring and carrying out their teaching. One could therefore expect that these five dominant types of tasks capture most of the “realised” curriculum in Indonesian schools, as far as proportion is concerned. However, in the national curriculum for lower secondary school, there is no detailed discussion on how proportion should be taught and certainly nothing as precise as these types of tasks is even mentioned. Nevertheless, our analysis of textbooks (in this case, three state authorized textbooks used in almost every school) reveals these five types of tasks as a national “profile” of the proportion sector within arithmetic. While this profile cannot be traced to the curriculum, it seems to be well rooted in the didactic tradition of the school institution, which is especially carried and continued by textbooks.

Discussion

As illustrated by the short quantitative overview of the three textbooks, the praxeological reference model presented in Section 4 can be used to identify five dominant tasks which, together, form the core of the proportion sector in Indonesian school. At the same time, the model allowed us to single out a few exceptional types of tasks which complete the two themes of the sector and adds some autonomy to the student activity which the books can generate. We conjecture that different “exceptional” types of tasks may be found in other Indonesian textbooks (non-authorized, or older) while the five dominant types would probably also dominate there. At any rate, both the similarity and differences in the mathematical core of the textbooks’ treatment of the sector appears from a presentation such as given in Table 8 and 9.

In this paper, we have focused on types of tasks and techniques. In other words, we have not analysed corresponding technology or theory presented in the textbooks, which we will consider elsewhere; this will be of particular importance for analyzing the connections with other domains, such as algebra and geometry. Similarly, we have not considered the ecological aspect of proportion, i.e. institutional conditions and constraints of Indonesian school, which are necessary to *explain* (rather than to analyze) the shape of the themes in the present textbooks, or to discuss alternative designs, *raison-d’être* of the themes, etc. Thus, this paper is far from exhausting the potential of textbook analysis based on ATD. However, we claim that such an analysis will have to include, at its basis, an analysis of the granularity and precision demonstrated in this paper, and that our approach shows more generally that such a granularity with respect to the mathematical content of examples and exercises is indeed possible and useful in textbook analysis.

Our main point in this paper was, thus, to give a first demonstration of how the notion of praxeological reference model enables us to analyse the mathematical core of textbooks in a quite objective and detailed way, which could contribute to "common measures" for both comparative and historical studies of how a sector or theme appears in mathematics textbooks. About the practical level of exercises and examples, which is crucial to the mathematical activity it can support among students, teachers can use such a reference model to examine a textbook. For example, a teacher may compare the type tasks found in a textbook to those appearing in national examinations. Also for textbook authors, comprehensive analyses of themes as given in Section 6 may be useful to consider, to develop a more deliberate profile than what can be done by personal experience and more or less arbitrary variation of single types of tasks.

We acknowledge that the methodology proposed here only attends to certain specific aspects of textbooks, while leaving others untouched. It mainly focuses on mathematical themes, not - for instance - on the use of daily life contexts, style of presentation, or connections with other themes. It also does not question the ecology of the textbooks, for instance, the coherence or genesis of the national curriculum, or the conditions under which the textbooks are used in Indonesian schools.

For further research, it is also important to strengthen the reference model by applying it on different textbooks from different contexts (e.g. private textbooks or foreign textbooks). Including a wider array of empirical data, the reference model will not only have to be extended, but will also gain in solidity and use, for instance for comparative purposes. Finally, we currently work on extending the reference model to include themes from other related domains, such as similarity in plane geometry and linearity in algebra. This will enable us to identify actual or potential relations between the three domains, which are naturally important qualities in textbooks - in the absence of establishing explicit links between themes, they will tend to support the "thematic autism" (Barbé et al., 2005) which can be identified as one of the main challenges of school mathematics.

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