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Best Laid Plans of Lions and Men

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Abstract

We answer the following question dating back to J. E. Littlewood (1885–1977): Can two lions catch a man in a bounded area with rectifiable lakes? The lions and the man are all assumed to be points moving with at most unit speed. That the lakes are rectifiable means that their boundaries are finitely long. This requirement is to avoid pathological examples where the man survives forever because any path to the lions is infinitely long. We show that the answer to the question is not always "yes" by giving an example of a region R in the plane where the man has a strategy to survive forever. R is a polygonal region with holes and the exterior and interior boundaries are pairwise disjoint, simple polygons. Our construction is the first truly two-dimensional example where the man can survive.

Next, we consider the following game played on the entire plane instead of a bounded area: There is any finite number of unit speed lions and one fast man who can run with speed $1 + \varepsilon$ for some value $\varepsilon > 0$. Can the man always survive? We answer the question in the affirmative for any constant $\varepsilon > 0$.

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1 Introduction

'A lion and a man in a closed circular arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal?'[1] These words (including the footnote) introduce the now famous lion and man problem, invented by R. Rado in the late thirties, in Littlewood's Miscellany [15]. It was for a long time believed that in order to avoid the lion, it was optimal for the man to run on the boundary of the arena. A simple argument then shows that the

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1 The curve of pursuit (L running always straight at M) takes infinite time, so the wording has its point.

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lion could always catch the man by staying on the radius OM defined by the man while approaching him as much as possible. However, A.S. Besicovitch proved in 1952 that the man has a very simple strategy (following which he will approach but not reach the boundary) that enables him to avoid capture forever no matter what the lion does. See [15] for details.

Throughout this paper, all men, lions, and other animals are assumed to be points. One can prove that two lions are enough to catch the man in a circular arena, and Croft [8] proves that in general a necessary and sufficient number of birds to catch a fly inside an n-dimensional spherical cage is just n (again, we assume that the fly and the birds have equal maximum speeds).

A well-known related discrete game is the *cop and robber game*: Let G be a finite connected undirected graph. Two players called cop C and robber R play a game on G according to the following rules: First C and then R occupy some vertex of G. After that they move alternately along edges of G. The cop C wins if at some point in time C and R are on the same vertex. If the robber R can prevent this situation forever, then R wins. The robber has a winning strategy on many graphs including all cycles of length at least 4. Therefore, the cop player C can be given a better chance by allowing him, say, k cops $C_1, \ldots, C_k$. At every turn C moves any non-empty subset of $\{C_1, \ldots, C_k\}$. Now, the *cop-number of* G is the minimal number of cops needed for C to win. Aigner and Fromme [2] observes that the cop-number of the dodecahedron graph is at least 3, since if there are only 2 cops, the robber can always move to a vertex not occupied by a cop and not in the neighbourhood of any. Furthermore, they prove that the cop-number of any planar graph is at most 3. Thus, the cop-number of the dodecahedron is exactly 3.

Returning to the lion and man game, Bollobás [6] writes that the following open problem was already mentioned by J.E. Littlewood (1885–1977): Can two lions catch a man in a bounded (planar) area with rectifiable lakes? An informal definition of a rectifiable curve is that it has finite length. We require that the boundaries of the lakes and the exterior boundary are all rectifiable curves to avoid pathological examples where the man survives forever because any path to the lions is infinite. Bollobás mentions the same problem in a comment in his edition of Littlewood's Miscellany [15] and in [7]. The problem is also stated by Fokkink et al. [11]. Berarducci and Intrigila [4] prove that the man can survive forever (for some initial positions of the man and lions) if the area is a planar embedding of the dodecahedron graph where each edge is a curve with the same length, say length 1. The proof is essentially the same as the proof by Aigner and Fromme [2] that the cop-number of the dodecahedron is at least 3: When the man is standing at a vertex, there will always be a neighbouring vertex with distance more than 1 to the nearest lion. It is thus safe for the man to run to that vertex. This, however, is a one-dimensional example. Berarducci and Intrigila raise the question whether it is possible to replace the one-dimensional edges by two-dimensional thin lines.

We present a truly two-dimensional region R in the plane where two lions are not enough to ever catch the man. We say that R is truly two-dimensional since R is a polygonal region with holes and the exterior and interior boundaries are all pairwise disjoint, simple polygons – in particular, they are clearly rectifiable. We were likewise inspired by the dodecahedron in the construction of our example. We explain the construction in Section 2.

Rado and Rado [16] and Janković [13] consider the problem where there are many lions and one man, but where the game is played in the entire unbounded plane. They prove that the lions can catch the man if and only if the man starts in the interior of the convex hull of the lions. Inspired by that problem, we ask the following question: What if the lions have maximum speed 1 and the man has maximum speed $1 + \varepsilon$ for some $\varepsilon > 0$? We prove that for any constant ε and any finite number of lions, such a fast man can survive forever provided that he does not start at the same point as one of the lions. We explain a strategy in Section 3.

Other variants of the game with a faster man have been studied previously. Flynn [9, 10] and Lewin [14] study the problem where there is one lion and one fast man in a circular arena. The lion tries to get as close to the man as possible and the man tries to keep the distance as large as possible. Variants of the cop and robber game where the robber is faster than the cops have also been studied. See for instance [3, 12].

1.1 Definitions

We follow the conventions of Bollobás et al. [5]. Let $R \subseteq \mathbb{R}^2$ be a region in the plane on which the lion and man game is to be played, and assume that the lion starts at point l_0 and the man at point m_0. We define a *man path* as a function $m\colon [0,\infty) \longrightarrow R$ satisfying $m(0) = m_0$ and the Lipschitz condition $\|m(s) - m(t)\| \leq V \cdot |s - t|$, where V is the speed of the man. In our case, we either have $V = 1$ or, in the case of a fast man, $V = 1 + \varepsilon$ for some small constant $\varepsilon > 0$. Note that it follows from the Lipschitz condition that any man path is continuous. A *lion path* l is defined similarly, but the lions we consider always run with at most unit speed. Let $\mathcal{L}$ be the set of all lion paths and $\mathcal{M}$ be the set of all man paths. Then a *strategy* for the man is a function $M\colon \mathcal{L} \longrightarrow \mathcal{M}$ such that if $l, l' \in \mathcal{L}$ agree on $[0, t]$, then $M(l)$ and $M(l')$ also agree on $[0, t]$. This last condition is a formal way to describe that the man's position $M(l)(t)$, when he follows strategy M, depends only on the position of the lion at points in time before and including time t, i.e., he is not allowed to act based on the lion's future movements. (By the continuity of any man path, the man's position at time t is in fact determined by the lion's position at all times strictly before time t.) A strategy M for the man is *winning* if for any $l \in \mathcal{L}$ and any $t \in [0,\infty)$, it holds that $M(l)(t) \neq l(t)$. Similarly, a strategy for the lion $L\colon \mathcal{M} \longrightarrow \mathcal{L}$ is winning if for any $m \in \mathcal{M}$, it holds that $L(m)(t) = m(t)$ for some $t \in [0,\infty)$. These definitions are extended to games with more than one lion in the natural way.

It might seem unfair that the lion is not allowed to react on the man's movements when we evaluate whether a strategy M for the man is winning. However, we can give the lion full information about M and allow it to choose its path l depending on M *prior* to the start of the game. If M is a winning strategy, the man can also survive the lion running along l.

We call a man strategy M *locally finite* if it satisfies the following property: if l and l' are any two lion paths that agree on $[0, t]$ for some t then the corresponding man paths $M(l)$ and $M(l')$ agree on $[0, t + \delta]$ for some $\delta > 0$ (we allow that δ depends on $l|_{[0,t]}$). Thus, informally, the man commits to doing something for some positive amount of time dependent only on the situation so far. Bollobás et al. [5] prove that if the man has a locally finite winning strategy, then the lion does not have any winning strategy. The argument easily extends to games with multiple lions. At first sight, it might sound absurd to even consider the possibility that the lion has a winning strategy when the man also does. However, it does not follow from the definition that the existence of a winning strategy for the man implies that the lion does not also have a winning strategy. See the paper by Bollobás et al. [5] for a detailed discussion of this (including descriptions of natural variants of the lion and man game where both players have winning strategies). In each of the problems we describe, the winning strategy of the man is locally finite, so it follows that the lions do not have winning strategies. In fact, the strategies we describe satisfy the much stronger condition that they are *equitemporal*, i.e., there is a constant $\Delta > 0$ such that the man at any point in time $i \cdot \Delta$, for $i = 0, 1, \ldots$, decides where he wants to run until time $(i + 1) \cdot \Delta$.

2 The Man Surviving Two Lions in a Bounded Area

In this section, we present a polygonal region R in the plane with 11 lakes. See [1] for an illustration of such a region. The exterior and interior boundaries of R are all pairwise disjoint simple polygons, and a man can survive forever in R against two lions provided that the lions are initially at a sufficient distance.

Consider a planar embedding $\mathcal{D}$ of the dodecahedron where each edge is a polygonal curve. We can obtain that all edges have the same length by prolonging some edges using a zig-zag pattern. This embedding corresponds to an area with 11 lakes and infinitely thin paths between the lakes, and as observed by Berarducci and Intrigila [4], the man can survive forever against two lions on such an embedding by deciding at each vertex which neighbouring vertex to visit next. First, we explain why it is not straight-forward to obtain the region R from $\mathcal{D}$, or, at least, why some natural initial attemps will not work.

We want to "thicken" each edge of $\mathcal{D}$ such that the boundaries of the lakes become disjoint, thus obtaining a truly two-dimensional region $\mathcal{D}'$ containing $\mathcal{D}$ as a subset. However, doing so, the point in $\mathcal{D}'$ corresponding to a vertex of $\mathcal{D}$ does not necessarily lie on the shortest path between its neighbours. We thus cannot simply employ the strategy from $\mathcal{D}$, roughly speaking, because the man must plan in advance which turn to take in the upcoming vertex. Thus, before he reaches the region R_v corresponding to a given vertex, he should already know which neighbouring vertex he will visit afterwards. Then, he can choose a path through R_v that makes the concatenated path shortest possible.

In order to carry out this idea, we first need to describe a winning strategy of the man on the dodecahedron graph with the special property that he does not make his decisions at the vertices. Let $\mathcal{G}$ be a planar embedding of the dodecahedron where all edges have length 4. The distance between two points in $\mathcal{G}$ is the length of a shortest path between the points. Let the *quarters* denote the points on the edges of $\mathcal{G}$ at distance 1 to the closest vertex. Consider a quarter x on the edge ab of $\mathcal{G}$. For a point $p \in \mathcal{G}$, $p \neq x$, let $d_a(x,p)$ be the length of a shortest *simple* path in $\mathcal{G}$ from x to p that initially follows the edge $\{a,b\}$ in direction towards a. Let $d_b(x,p)$ be defined similarly.

When the man is at a quarter x with distance 1 to the vertex a and 3 to the vertex b, we let d_{near} denote the distance from x to the closest lion with respect to d_a, and let d_{far} denote the distance from x to the closest lion with respect to d_b. To avoid confusion, we write them as $d_{\text{near}}(t)$ and $d_{\text{far}}(t)$ when x is the position of the man at the time t.

We will now show that if the lions are sufficiently far away in the initial situation, there exists a winning strategy for the man where he only takes stock of the situations in the quarters. That is, when he reaches a quarter, he must plan for the next 2 units of time where to run to, and then he has reached a quarter again, and so on.

▶ **Invariant 1.** *In the scenario described above:*

1. *The man is standing on a quarter.*
2. $\min\{d_{\text{near}}, d_{\text{far}}\} \geq 1$.
3. *At least one of the two following statements is true:*
   - $d_{\text{near}} \geq 3$
   - $d_{\text{far}} \geq 7$

▶ **Lemma 2.** *If Invariant 1 is satisfied initially, the man has a winning strategy by which he runs from quarter to quarter at unit speed so that Invariant 1 is true at any quarter. The strategy maintains Invariant 1 Point 2 at all times, that is, that the closest lion is always at least at distance* 1*.*

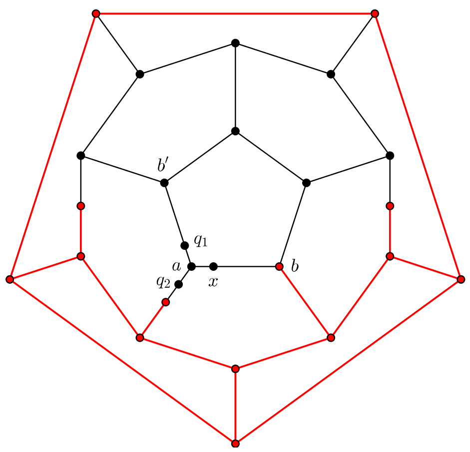


Figure 1 A situation from the proof of Lemma 2. Imagine that all edges have length 4. The lion l_{near} is in the red part.

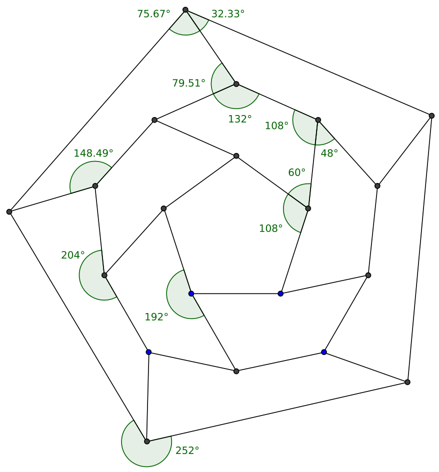


Figure 2 The embedding $\mathcal{D}$ of the dodecahedron. All edges have lengths 1 or 3.

Proof. Let x denote the position of the man at the time t, and assume the invariant holds. We prove that he can run to another quarter x' without getting caught such that the invariant again holds when he reaches x'.

The proof goes by inspecting cases. Let ab be the edge containing x and suppose a be the nearest vertex to x and b the furthest.

Case 1: $d_{\text{far}}(t) \geq 7$. Let y denote the other quarter on the same edge as x. We claim that the man can run to y without violating the invariant. We must thus argue that the invariants are satisfied at time $t+2$ for a man situated at y. First, note that he will not encounter any lion while running towards y because $d_{\text{far}}(t) > 4$. Note also that $d_{\text{far}}(t+2) \geq 1$, since $d_{\text{near}}(t) \geq 1$ and the worst case is that the lion follows the man. Furthermore, $d_{\text{near}}(t+2) \geq 7-4 = 3$, since $d_{\text{far}}(t) \geq 7$ and the worst case is that the man and lion have run towards each other. Thus, the invariant holds at the time $t+2$.

Case 2: $d_{\text{far}} < 7$, and thus $d_{\text{near}}(t) \geq 3$. In this case, we exploit the fact that d_{far} is so small that we can bound $d_{\text{far}}(t+2)$ from below. Let l_{far} denote the lion at distance d_{far} from x, and let l_{near} denote the other lion. Consider the two other quarters at distance 1 from a, call them q_1 and q_2. Assume without loss of generality that q_1 is furthest from l_{near}. The situation is sketched in Figure 1. We now argue that the man can choose to run towards q_1 without getting eaten, and while maintaining the invariant. Let b' denote the vertex at distance 3 to q_1. Note that $d_{b'}(q_1, l_{\text{far}}(t)) \geq 11$ and thus, $d_{b'}(q_1, l_{\text{far}}(t+2)) \geq 9$.

In Figure 1, the points that are both ≥ 3 from x, and (weakly) closer to q_2 than to q_1, are marked with red, and hence by our choice of q_1, l_{near} must be in the subset marked with red at time t. As is easily seen by inspection, $d_{b'}(q_1, l_{\text{near}}(t)) \geq 9$, and thus $d_{b'}(q_1, l_{\text{near}}(t+2)) \geq 7$. But then, $d_{\text{far}}(t+2) \geq \min\{9, 7\} = 7$, and Invariant 1.1 and 3 are maintained.

To see that Invariant 1.2 is still maintained, note that $d_a(q_1, l_{\text{near}}(t)) \geq 3$ and therefore $d_a(q_1, l_{\text{near}}(t+2)) \geq 1$. Similarly, since $d_b(x, l_{\text{far}}(t)) \geq 1$, we have $d_a(q_1, l_{\text{far}}(t)) \geq 3$ so that $d_a(q_1, l_{\text{far}}(t+2)) \geq 1$. Thus, $l_{\text{near}}(t+2) \geq 1$, and we are done. ◀

Our first goal is to find an embedding $\mathcal{G}$ of the dodecahedron in the plane with the properties described below, which will make it easier for us to construct the region R.

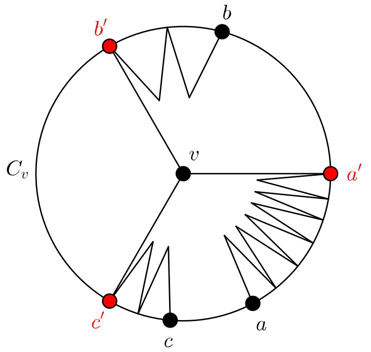


Figure 3 Regardless of angles between a, b, c, we can introduce bends to make the three edges meet at v in angles of size $\frac{3\pi}{2}$ and at the same time extend the lengths suitably.

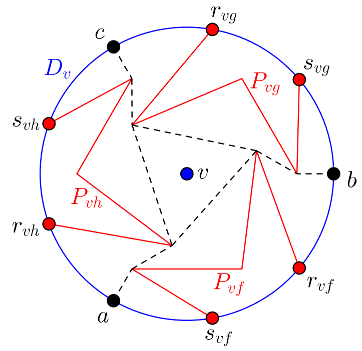


Figure 4 The shortest paths in the circle D_v between any two of a, b, c, that avoid crossing the polygonal curves P_{vf}, P_{vg}, P_{vh} all have length 1/8.

▶ **Lemma 3.** *There exists a planar embedding $\mathcal{G}$ of the dodecahedron such that*
- *all edges have length 4,*
- *all edges consist of line segments with lengths being multiples of $\frac{1}{8}$,*
- *any pair of line segments from different edges that meet at a vertex each have length 1/4 and form an angle of size $\frac{2\pi}{3}$, and*
- *for any vertex v, the circle D_v centered at v with radius 1/16 only intersects the three edges incident to v.*

After proving this lemma, we derive from $\mathcal{G}$ a truly two-dimensional area R in the plane where the man can survive against two lions. Lemma 2 gives a winning strategy for the man in $\mathcal{G}$ where he runs from quarter to quarter. The paths along which he runs in R will be exactly the same as in $\mathcal{G}$ except for inside the circles D_v.

We first need the following elementary geometric observations:

▶ **Observation 4.** *There exists a planar embedding $\mathcal{D}$ of the dodecahedron such that all edges have length 1 or 3. $\mathcal{D}$ furthermore has the property that the circle of radius $\frac{1}{4}$ centered at any vertex v only intersects the three edges incident to v. (See Figure 2.)*

▶ **Lemma 5.** *For any three points a, b, c on a circle C, there exist a equilateral triangle with corners a', b', c' on C where $\{a, b, c\}$ and $\{a', b', c'\}$ are disjoint and such that, when considering the points a, b, c, a', b', c' all together, a is a neighbour of a', and b is a neighbour of b', and c is a neighbour of c'.*

Proof. See Figure 3. The points a, b, c divide C into three arcs. Clearly, we can choose an equilateral triangle with corners on C disjoint from $\{a, b, c\}$ so that not all three corners of the triangle are on the same arc. It is now easy to label the corners of the triangle with a', b', c' to satisfy the lemma. ◀

We are now ready to prove that a planar embedding $\mathcal{G}$ of the dodecahedron exists as stated in Lemma 3.

Proof of Lemma 3. Start with the embedding $\mathcal{D}$ shown in Figure 2, where all edges have length 1 or 3. Consider a vertex v and the circle C_v of radius $r = \frac{1}{4}$ centered at v. Assume the three edges incident to v enter C_v in the points a, b, c, and let u_a, u_b, u_c be the neighbouring

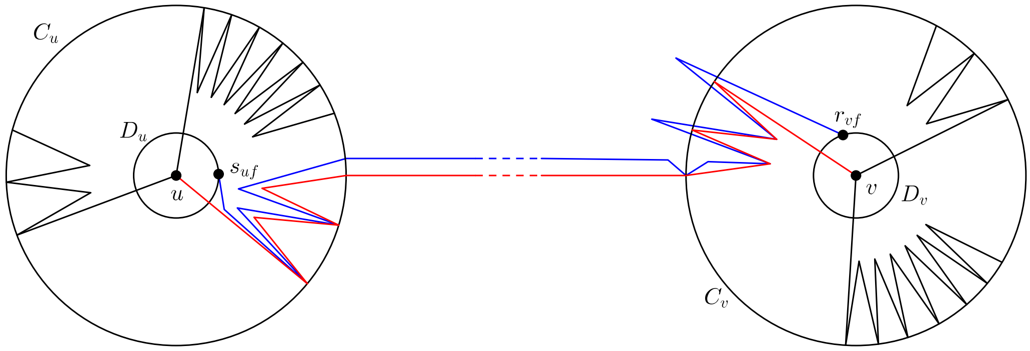


Figure 5 The edge e_{uv} of $\mathcal{G}$ is red and is one of the edges bounding the face f, which is above e_{uv}. The polygonal curve Q_{uv}, which is on the boundary of the lake L_f, is blue.

vertices of v such that a is a point on the edge $\{u_a, v\}$, b is a point on $\{u_b, v\}$, and c is a point on $\{u_c, v\}$. We now delete the segments va, vb, and vc, and therefore need to reconnect a, b, and c to v. We explain how to reconnect a to v; b and c are handled analogously. We find points a', b', c' on C_v as described in Lemma 5. See Figure 3. We first connect a' to v. We now need to connect a to a' using some bends. A *bend* is two segments xy and yz, each of length $r/2 = 1/8$, such that x and z are on C_v and y is in the interior of C_v. If the edge $\{u_a, v\}$ had length 3 in $\mathcal{D}$, we make two bends that together connect a and a'. We thus increase the length of the edge $\{u_a, v\}$ by $1/2$ in each end and the resulting edge has length 4. If the edge $\{u_a, v\}$ had length 1 in $\mathcal{D}$, we connect a and a' by 6 bends, corresponding to extending the length of the edge by 3. The result is a planar embedding $\mathcal{G}$ of the dodecahedron with the properties stated in the lemma. ◀

We now describe how to make the region R. We want each quarter of $\mathcal{G}$ to be a point in R and we want all pairs of quarters to have the same distances in $\mathcal{G}$ and R. It will then follow from Lemma 2 that the man has a winning strategy by running from quarter to quarter in R. We make one lake L_f corresponding to each face f of $\mathcal{G}$. Here, we also consider the outer boundary of R to be the boundary of an unbounded lake corresponding to the exterior face of $\mathcal{G}$. The shortest paths in R will be polygonal paths with corners at convex corners of the lakes. Outside the circles D_v, the paths along which the man will run are exactly the paths in $\mathcal{G}$. Inside a circle D_v, we need to take special care to ensure that the man can always run along an optimal path.

We now explain the construction of the lakes L_f corresponding to faces f of $\mathcal{G}$. Consider a vertex v of $\mathcal{G}$ and the faces f, g, h on which v is a vertex. We first describe how the boundaries of L_f, L_g, L_h look in the circle D_v of radius $1/16$ centered at v. See Figure 4. Let a, b, c be the points where the edges incident to v enter D_v. Suppose that the arc on D_v from a to b is in the face f, the arc from b to c is in g, and the arc from c to a is in h. We now create three polygonal curves P_{vf}, P_{vg}, P_{vh} inside D_v so that the shortest path between any two of a, b, c contained in D_v and not crossing any of P_{vf}, P_{vg}, P_{vh} has length $1/8$. The curve P_{vf} starts at a point r_{vf} on D_v and ends at a point s_{vf} on D_v, and the endpoints r_{vf}, s_{vf} are inside f, and similarly for the faces g, h. These properties are easy to obtain by a construction as shown in Figure 4. The curves P_{vf}, P_{vg}, P_{vh} will be part of the boundary of the lakes L_f, L_g, L_h, respectively.

We now explain how to construct the rest of the boundary of each lake L_f. Consider a face f of $\mathcal{G}$ and assume that the vertices on f are $uvxyz$ in that order on f. The curves $P_{uf}, P_{vf}, P_{xf}, P_{yf}, P_{zf}$ appear on the boundary of L_f in that order. In the following, we describe how to connect the end s_{uf} of P_{uf} with the start r_{vf} of P_{vf} – the other curves are connected in a completely analogous way. See Figure 5. Let e_{uv} be the edge of $\mathcal{G}$ between u and v, thus, e_{uv} is a polygonal curve. Let a *corner* of e_{uv} be a common point of two neighbouring segments of e_{uv}. We make a polygonal curve Q_{uv} corresponding to e_{uv}. Q_{uv} starts at s_{uf} and ends at r_{vf} so that it connects P_{uf} and P_{vf}. Q_{uv} stays near e_{uv} inside f and touches e_{uv} at the corners of e_{uv} which are convex corners of f. To summarize, Q_{uv} has the following properties:

1. Q_{uv} starts at s_{uf} and ends at r_{vf},
2. Q_{uv} is completely contained in f,
3. Q_{uv} is, except for the endpoints s_{uf}, r_{vf}, outside the circles D_u and D_v,
4. Q_{uv} and $Q_{u'v'}$ are completely disjoint for any ordered pair $(u'v') \neq (u,v)$ so that $\{u', v'\}$ is an edge of $\mathcal{G}$, and
5. Q_{uv} touches e_{uv} at a point p if and only if p is a corner of e_{uv} which is a convex corner of f.

Observe that Q_{vu} (note: not Q_{uv}!) touches e_{uv} at the corners which are concave corners of f, since those are convex corners of the neighbouring face on the other side of e_{uv}.

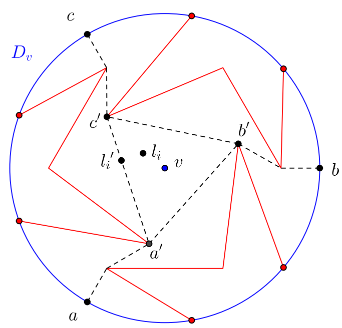
▶ **Theorem 6.** *There exists a polygonal region R in the plane with holes where the exterior and interior boundaries are all pairwise disjoint and such that the man has a winning strategy against two lions.*

Proof. R is the region that we get by removing from $\mathbb{R}^2$ the interior of each of the lakes L_f. Thus, the boundary of each lake is included in R, so that R is a closed set. R is also bounded because we remove the interior of the unbounded lake corresponding to the exterior face of $\mathcal{G}$. Note that any point on an edge e_{uv} of $\mathcal{G}$ which is outside the circles D_u and D_v is a point in R. Since the quarters of e_{uv} are outside the circles D_u and D_v, it follows that they are also points in R. Furthermore, our construction ensures that the distance in R between any two quarters is the same as in $\mathcal{G}$. Let $\mathcal{G}'$ be the points in R which are on some shortest path between two quarters in R. Thus, $\mathcal{G}'$ are the points that the man can possibly visit when running along shortest paths in R from quarter to quarter.

Let l_1 and l_2 be two lions in R. We define projections l_1' and l_2' of the lions l_1 and l_2 to be the closest points in $\mathcal{G}'$ (with respect to distances in R). We now define l_1'' and l_2'' to be projections of l_1' and l_2' in $\mathcal{G}$ in the following way. Outside the circles D_v, $\mathcal{G}$ and $\mathcal{G}'$ coincide, and here we simply define $l_i'' := l_i'$. Suppose now that l_i' is inside a circle D_v for some vertex v of $\mathcal{G}$. See Figure 6. Suppose that the three edges incident to v enter D_v at the points a, b, c. The projection l_i' is a point on one of the shortest paths between a pair of the points a, b, c. Recall that these shortest paths all have length $1/8$. Assume without loss of generality that l_i' is on the path from a to c. Let d be the distance from a to l_i' in R, so that $0 \leq d \leq 1/8$. If $d = 1/16$, we define $l_i'' := v$. Otherwise, if $d < 1/16$, we let l_i'' be the point on the segment av in $\mathcal{G}$ with distance d to a, i.e., $l_i'' \in av$ so that $\|al_i''\| = d$. Similarly, if $d > 1/16$, we let l_i'' be the point on bv with distance $1/8 - d$ to b.

We now prove that l_i'' moves with at most unit speed in $\mathcal{G}$. It will then follow from Lemma 2 that the man has a winning strategy.

$\mathcal{G}'$ subdivides R into some regions $R_1', \dots, R_k'$, which are the connected components of $R \setminus \mathcal{G}'$. Let $R_i = \overline{R_i'}$ be the closure of R_i'. Now, $R = \bigcup_{i=1}^{k} R_i$. Inside each circle D_v, there is a *triangular* region bounded by three segments from $\mathcal{G}'$. All other regions are bounded by



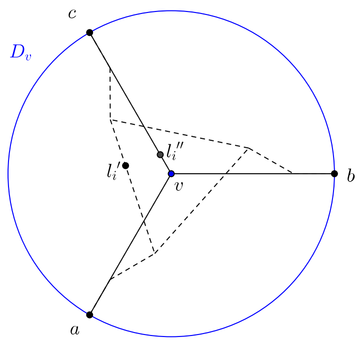


Figure 6 The projection of the lion's position l_i onto the point l_i' of $\mathcal{G}'$ (left), and the projection of l_i' onto the point l_i'' of $\mathcal{G}$ (right). The dashed lines illustrate $\mathcal{G}'$, and the solid lines illustrate $\mathcal{G}$. In the left figure, l_i' is the closest point on $\mathcal{G}'$ to l_i. In the right figure, the length of the segment cl_i'' equals the length of the dashed path from c to l_i'.

a polygonal curve $C \subset \partial L_f$ on the boundary of some lake L_f and a concave chain $\mathcal{H} \subset \mathcal{G}'$. Call such a region *normal*. If the lion l_i is in a normal region R_j with boundary $\partial R_j = C \cup \mathcal{H}$ as described before, the projection l_i' is on $\mathcal{H}$. It then follows from the concavity of $\mathcal{H}$ that l_i', and thus also l_i'', moves continuously and with at most unit speed.

However, when l_i is inside a triangular region in D_v, the projection l_i' might jump from one segment of the triangle to another. Suppose that the three edges incident to v enter D_v at the points a, b, c as in Figure 6. Let a' be the point where the shortest paths from a to b and c separate and define b' and c' similarly. Thus, the points $a'b'c'$ are the corners of the triangular region. Suppose that l_i' jumps from $a'b'$ to $a'c'$. Then, the distance from l_i to $a'b'$ and $a'c'$ is the same and the distance from a to l_i' before and after the jump is at most $1/16$, since otherwise, l_i would be closer to the segment $b'c'$ than to $a'b'$ and $a'c'$. It follows that l_i' jumps from one point to another which have the same projection l_i''. Thus, l_i'' moves continuously and with at most unit speed.

The man now employs the strategy from Lemma 2 in the following way. He imagines that he is playing in the dodecahedron $\mathcal{G}$ against the lions l_1'' and l_2''. Assume therefore that Invariant 1 holds initially. The strategy tells the man to which neighbouring quarter to run. That quarter also exists in $\mathcal{G}'$, and has the same distance, so the man runs to that quarter in $\mathcal{G}'$. Since l_1'' and l_2'' run with at most unit speed, the man can escape them forever. When the man is outside the circles D_v, it is a necessary condition for the lions to catch the man that l_1'' or l_2'' coincide with the man, so we conclude that they cannot catch him outside the circles. When the man is inside a circle D_v, we know from Lemma 2 that l_1'' and l_2'' are at least 1 away from the man. Therefore, l_1 and l_2 must be outside D_v, and hence they cannot catch him in that case either. Thus, the man survives forever in R. ◀

3 The Fast Man Surviving any Number of Lions in the Plane

Finally, we consider the case where the man is just slightly faster than the lions in the unbounded plane without obstacles. In this case, the man is able to escape arbitrarily many lions. The full proofs of some of the claims below can be found in [1].

▶ **Theorem 7.** *In the plane $\mathbb{R}^2$, for any $\varepsilon > 0$, a man able to run at speed $1+\varepsilon$ has a locally finite strategy to escape the convex hull of any number $n \in \mathbb{N}$ of unit-speed lions, provided* *that the man does not start at the same point as a lion. Thus, the man has a locally finite winning strategy.*

In fact, we prove that the man is able to keep some minimum distance $d_{\varepsilon,n}$ to any lion, where $d_{\varepsilon,n}$ only depends on ε, n, and the initial distances to the lions. Thus, if the n lions and man were disks with radius $< \frac{1}{2}d_{\varepsilon,n}$, the man is still able to escape.

We proceed by induction on the number n of lions. We define strategies M_j for the man to keep distance c_j to the first j lions. The j'th strategy yields a curve consisting of line segments all of the same length.

Inductively, the man can keep a safety distance c_{n-1} to the $n-1$ first lions by running at speed $1+\varepsilon_{n-1}$, where $\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n < \varepsilon$. The bends of the curve defined by strategy M_{n-1} are milestones that he runs towards when avoiding n lions. If the n'th lion ℓ_n is in the way, the man makes an *avoidance move*, keeping a much smaller safety distance c_n to ℓ_n and running slightly faster at speed ε_n (see Figure 9). Intuitively, when performing avoidance moves, the man runs counter-clockwise around a fixed-diameter circle centered at the lion.

After a limited number of avoidance moves, the man can make an *escape move*, where he simply runs towards the milestone defined by the strategy M_{n-1}.

By choosing c_n sufficiently small, we can make sure that the detour caused by the n'th lion is so small that it can only annoy the man once for each of the segments of the strategy M_{n-1}, and thus that he is ensured to have distance at least $c_{i-1}/2$ to the position defined by M_{n-1} and hence not in danger of the $(n-1)$'st lions.

▶ **Theorem 8.** *A man able to run at speed $1+\varepsilon$ for any $\varepsilon > 0$ has a locally finite strategy to escape the convex hull of any number $n \in \mathbb{N}$ of unit-speed lions, provided that the man does not start at the same point as a lion. Thus, the man has a locally finite winning strategy.*

Proof. We assume without loss of generality that $\varepsilon < 1$. Let $l_1, \ldots, l_n$ be n arbitrary lion paths and let the man start at position m_0 such that $m_0 \neq l_i(0)$ for all i. We show that the man has a strategy M_n with the following properties:

1. The man is always running at speed $1+\varepsilon_n$, where $\varepsilon_n := (1-2^{-n}) \cdot \varepsilon$.
2. The path defined by $M_n(l_1, \ldots, l_n)$ is a polygonal path with corners $m_0 m_1 \ldots$ and each segment $m_i m_{i+1}$ has the same length $\Delta_n \cdot (1+\varepsilon_n)$. Thus, the time it takes the man to run from m_i to m_{i+1} is Δ_n.
3. Let $t_i := i \cdot \Delta_n$ be the time where the man leaves m_i in order to run to m_{i+1}. The point m_{i+1} can be determined from the positions of the lions at time t_i.
4. There exists a *safety distance* $c_n > 0$ such that for any $i = 1, \ldots$, any $t \in [t_i, t_{i+1}]$, and any point $x \in m_i m_{i+1}$, it holds that $\operatorname{dist}(x, \{l_1(t), \ldots, l_n(t)\}) \geq c_n$.
5. There is a corner $m_i = M_n(t_i)$ such that for all $t \geq t_i$,

$$M_n(l_1, \ldots, l_n)(t) \notin \mathcal{CH}\{l_1(t), \ldots, l_n(t)\}.$$

Clearly, it follows from the properties that M_n is a winning strategy for the man fulfilling the requirements in the theorem. We prove the statement by induction on n. If there is only one lion, the man will run on the same ray all the time with constant speed $1+\varepsilon_1 = 1+\varepsilon/2$. The man chooses the direction of the ray to be $m_0 - l_1(0)$. This strategy obviously satisfies the stated properties. Assume now that a strategy M_{n-1} with the stated properties exists for $n-1 \geq 1$ lions and consider a situation with n lions running along paths $l_1, \ldots, l_n$.

Assume without loss of generality that the lions are numbered according to their (increasing) distance to the man at time 0, i.e., $\|m_0 l_1(0)\| \leq \|m_0 l_2(0)\| \leq \cdots \leq \|m_0 l_n(0)\|$. For any $i \in \{1, \ldots, n\}$, let M_i be shorthand for $M_i(l_1, \ldots, l_i)$ and m shorthand for M_n.

At any time t, let the succeeding corner on the strategy M_{n-1} be

$$g(t) := M_{n-1}(\lfloor t/\Delta_{n-1} + 1 \rfloor \cdot \Delta_{n-1}).$$

By property 3, the man can always compute the point $g(t)$.

We first describe the intuition behind the man's strategy without specifying all details, and later give a precise description. In the situation with n lions, the man attempts to run according to the strategy for the $n-1$ first lions, i.e., the strategy M_{n-1}. Thus, at any time t, the man's goal is to run towards the point $g(t)$. However, the lion l_n might prevent him from doing so. Compared to the case with $n-1$ lions, the man has increased his speed by $1+\varepsilon_n-(1+\varepsilon_{n-1}) = 2^{-n}\varepsilon$, so he has time to take detours while still following the strategy M_{n-1} approximately.

Assume that we have defined the man's strategy up to time t. If he is close to the n'th lion, i.e., the distance $\|m(t)l_n(t)\|$ is close to r, for some small constant $r > 0$ to be specified later, he runs counterclockwise around the lion, maintaining approximately distance r to the lion. He does so until he gets to a point where running directly towards $g(t)$ will not decrease his distance to the lion. He then escapes from the lion, running directly towards $g(t)$. Doing so, he can be sure that the lion cannot disturb him anymore until he reaches $g(t)$ or $g(t)$ has changed.

We choose r so small that when the man is running around the lion, we are in one of the following cases:

- The lion is so close to $g(t)$ that the man is within the safety distance c_{n-1} from $g(t)$, and thus in no danger of the lions $l_1, \dots, l_{n-1}$.
- After running around the lion in a period of time no longer than $12\pi r/\varepsilon_n$, the man escapes by running directly towards $g(t)$ without decreasing the distance to the lion. By choosing r sufficiently small, we can therefore limit the duration, and hence the length, of the detour that the lion can force the man to run, so that the man is ensured to be within the safety distance from the lions $l_1, \dots, l_{n-1}$ during the detour.

We now describe the details that make this idea work. We define

$$\begin{aligned}
r &:= \min\left\{\frac{\Delta_{n-1}\varepsilon_n(\varepsilon_n-\varepsilon_{n-1})}{2+2\varepsilon_n+18\pi(1+\varepsilon_n)}, \frac{\varepsilon_n c_{n-1}}{4+4\varepsilon_n+24\pi(1+\varepsilon_n)}\right\},\\
\rho &:= 2r/\varepsilon_n,\\
\theta &:= \arccos\frac{1}{1+\varepsilon_n},\\
\varphi &\in (0,\pi/2] \quad \text{so that} \quad \tan\theta = \frac{\rho\sin\varphi}{\rho\cos\varphi - 2r}, \quad \text{and}
\end{aligned}$$

$$\Delta_n > 0 \text{ so that } 2\arcsin\frac{(1+\varepsilon_n)\Delta_n}{2(r-\Delta_n)} + \frac{\Delta_n}{\rho} \le \varphi,\ \Delta_n < \frac{r}{3+\varepsilon_n},\ \text{and } \Delta_{n-1}/\Delta_n \in \mathbb{N}.$$

We note that φ can be chosen since the function $x \longmapsto \frac{\rho\sin x}{\rho\cos x - 2r}$ is 0 for $x=0$ and tends to $+\infty$ as $\rho\cos x$ decreases to $2r$. As for Δ_n, the function $x \longmapsto 2\arcsin\frac{(1+\varepsilon_n)x}{2(r-x)} + \frac{x}{\rho}$ is 0 for $x=0$ and increases continuously, and hence Δ_n can be chosen.

Define a point in time t to be a *time of choice* if t has the form $t_i := i\Delta_n$ for $i \in \mathbb{N}_0$. At any time of choice t_i, the man chooses the point $m(t_{i+1})$ at distance $(1+\varepsilon_n)\Delta_n$ from his current position $m(t_i)$ by the following strategy (see Figures 7–9):

A. Suppose first that $\|m(t_i)l_n(t_i)\| \ge r + \Delta_n(1+\varepsilon_n)$. Then the man chooses the direction directly towards $g(t_i)$. In the exceptional case that $m(t_i) = g(t_i)$, he chooses an arbitrary direction.

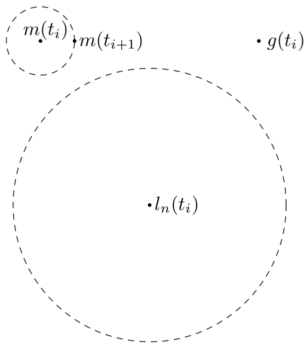


Figure 7 A free move. The circles with centers $m(t_i)$ and $l_n(t_n)$ have radii $(1+\varepsilon_n)\Delta_n$ and r, respectively.

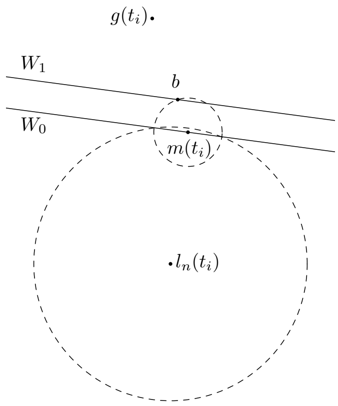


Figure 8 An escape move. The man runs to b.

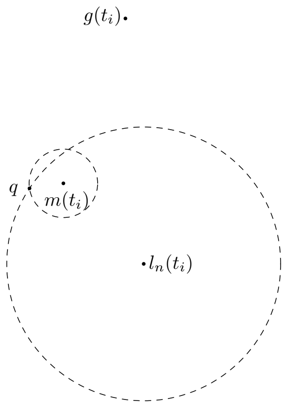


Figure 9 An avoidance move. The man runs to q.

B. Suppose now that $\|m(t_i)l_n(t_i)\| < r+\Delta_n(1+\varepsilon_n)$ and consider the case $m(t_i) \neq g(t_i)$. Let b be the point at distance $(1+\varepsilon_n)\Delta_n$ from $m(t_i)$ in the direction towards $g(t_i)$. If there exist two parallel lines W_0 and W_1 such that $m(t_i) \in W_0$, $b \in W_1$, $\text{dist}(l_n(t_i), W_0) \geq r - \Delta_n$, and $\text{dist}(l_n(t_i), W_1) \geq \text{dist}(l_n(t_i), W_0) + \Delta_n$, then the man runs to b.

C. In the remaining cases, the circles $C(m(t_i), \Delta_n(1+\varepsilon_n))$ and $C(l_n(t_i), r)$ intersect at two points p and q such that the arc on $C(l_n(t_i), r)$ from p counterclockwise to q is in the interior of $C(m(t_i), \Delta_n(1+\varepsilon_n))$. The man then runs towards the point q.

A move defined by case A, B, or C is called a *free move*, an *escape move*, or an *avoidance move*, respectively. Let *move* i be the move that the man does during the interval $[t_i, t_{i+1})$.

▶ **Claim 9.** *At any time of choice t_i, it holds that*

$$\|m(t_i)l_n(t_i)\| \geq r - \Delta_n$$

and if the preceding move was an avoidance move, it also holds that

$$\|m(t_i)l_n(t_i)\| \leq r + \Delta_n.$$

Furthermore, at an arbitrary point in time $t \in [t_{i-1}, t_i]$ and any point $m' \in m([t_{i-1}, t_i])$ it holds that

$$0 < r - (3+\varepsilon_n)\Delta_n \leq \|m'l_n(t)\|$$

and if move $i-1$ is an avoidance move then additionally

$$\|m'l_n(t)\| \leq r + (3+\varepsilon_n)\Delta_n.$$

Proof. (Sketch) The proof is by induction on i. The induction step for the first inequality follows easily from the speed of the man and the speed of the lion l_n and (in case of an escape or avoidance move) from considering the distance $\|m(t_i)l_n(t_{i-1})\|$. For the second inequality, note that $\|m(t_i)l_n(t_{i-1})\| = r$. The third inequality follows by considering the combined speed of the man and the lion l_n and by observing that $\|m(t_{i-1})l_n(t_{i-1})\| \geq r - \Delta_n$. A similar argument using the inequality $\|m(t_{i-1})l_n(t_{i-1})\| \leq r + \Delta_n$ shows the fourth inequality. ◀

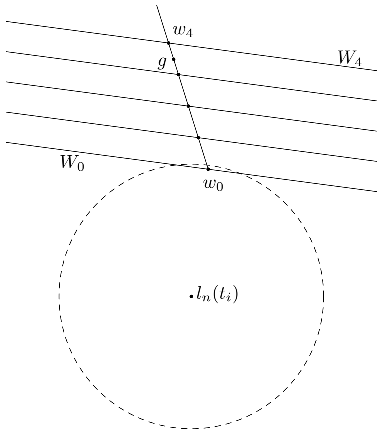


Figure 10 The distance between two consecutive of the parallel lines $W_0, \ldots, W_4$ is at least Δ_n, which proves that the man runs from $m(t_i) = w_0$ to w_4 unless g moves in the meantime.

▶ **Claim 10.** *An avoidance move is succeeded by an avoidance move or an escape move. When the man does an escape move, he will not do an avoidance move before he reaches $g(t)$ or $g(t)$ moves.*

Proof. Consider move i. We know from Claim 9 that if move $i-1$ was an avoidance move, then $\|m(t_i)l_n(t_i)\| \leq r + \Delta_n < r + (1+\varepsilon_n)\Delta_n$, so move i cannot be a free move.

For the second part of the statement, assume that move i is an escape move. Let $g := g(t_i)$. Let $w_0, \ldots, w_k$ be a sequence of points on the ray from $m(t_i)$ with direction to g such that $w_0 = m(t_i)$, $\|w_0 w_j\| = j(1+\varepsilon_n)\Delta_n$, and k is minimum such that either $g \in w_{k-1}w_k$ or $g(t') \neq g$ for some $t' \in [t_{i+k-1}, t_{i+k}]$. See Figure 10. Let W_0 and W_1 be the parallel lines defined in case B for move i. We define lines W_j for $j \geq 2$ to be parallel to W_0 and passing through w_j. We claim that for any $j \in \{0, \ldots, k-1\}$, the man moves from w_j to w_{j+1} during move $i+j$ using either an escape move or a free move. We prove this by induction on j. It holds for $j = 0$ by assumption, so assume it holds that $m(t_{i+j}) = w_j$ and that move $i+j-1$ was an escape move or a free move. Since the distance between consecutive lines W_j and W_{j+1} is at least Δ_n, it follows that $\mathrm{dist}(l_n(t_i), W_j) \geq r + (j-1)\Delta_n$ and hence that $\mathrm{dist}(l_n(t_{i+j}), W_j) \geq r - \Delta_n$. Now, if $\|m(t_{i+j})l_n(t_{i+j})\| < r + \Delta_n(1+\varepsilon_n)$, then the lines W_j and W_{j+1} are a witness that move $i+j$ is an escape move so that the man moves to w_{j+1}. Otherwise, move $i+j$ is a free move, in which case the man moves to w_{j+1}. Finally, since $g(t)$ moves or the man reaches g during move $i+k$, the statement holds. ◀

Define $\rho' := \rho + r + (3+\varepsilon_n)\Delta_n$ and $\tau := 6\pi r/\varepsilon_n$.

▶ **Claim 11.** *If move i is an avoidance move, one of the following events occurs before τ time has passed: (i) $g(t)$ moves, (ii) $\|m(t)g(t)\| < \rho'$, or (iii) the man makes an escape move.*

Proof Sketch. If the first two events do not occur, it follows from Claim 10 that the man keeps doing avoidance moves during this time. Let $\xi(t)$ resp. $\eta(t)$ denote the angle of the vector $\overrightarrow{l_n(t)m(t)}$ resp. $\overrightarrow{l_n(t)g(t_i)}$. A key observation is that if the difference in these angles is small, the man makes an escape move since then the lion and the goal g are roughly on opposite sides of the man. Showing that this difference eventually becomes small involves showing that η increases by at least 2π more than ξ after τ time so that at some point in time $t \in [t_i, t_i+\tau]$, vectors $\overrightarrow{l_n(t)m(t)}$ and $\overrightarrow{l_n(t)g(t_i)}$ have the same orientation. By Claim 9, the lion l_n never gets closer than ρ to $g(t_i)$ which implies that the change in η is small in any time interval $[t_j, t_{j+1}]$. Since the man keeps a minimum distance to the lion, it similarly follows that the change in ξ is small in $[t_j, t_{j+1}]$. Picking j to be the maximum such that $t_j \leq t$ gives $t - t_j \leq \Delta_n$ which implies that the difference in the two angles is small at time t_j at which point the man makes an escape move. Since $t_j \leq t + \tau$, the lemma follows. ◀

For $i \in \mathbb{N}_0$, define the *canonical interval* I_i as $I_i := [i\Delta_{n-1}, (i+1)\Delta_{n-1})$, i.e., I_i is the interval of time where the man would run from the i'th to the $(i+1)$'st corner on the polygonal line defined by the strategy M_{n-1}. We say that I_i *ends* at time $t = (i+1)\Delta_{n-1}$. Note that if $t \in I_i$, then $g(t) = M_{n-1}((i+1)\Delta_{n-1})$ and $g(t)$ moves when I_i ends.

As a consequence of Claim 10 and Claim 11, we get the following.

▶ **Claim 12.** *If $t \in I_i$ and $\|m(t)g(t)\| \leq \rho'$, then for every $t' > t$, $t' \in I_i$, we have*

$$\|m(t')g(t)\| \leq \rho' + (1 + \varepsilon_n)\tau.$$

▶ **Claim 13.** *For any $i \in \mathbb{N}_0$ and at any time during the canonical interval I_i, the man is at distance at most $\rho' + 2(1 + \varepsilon_n)\tau$ away from the segment $M_{n-1}(I_i)$ and when I_i ends, the man is within distance $\rho' + (1 + \varepsilon_n)\tau$ from the endpoint $M_{n-1}((i+1)\Delta_{n-1})$ of the segment.*

Proof. We prove the claim by induction on i. To easily handle the base-case, we introduce an auxiliary canonical interval $I_{-1} = [-\Delta_{n-1}, 0)$ and assume that the lions and the man are standing at their initial positions during all of I_{-1}. The statement clearly holds for $i = -1$.

Assume inductively that the statement holds for I_{i-1} and consider the interval I_i. Let $g := M_{n-1}((i+1)\Delta_{n-1})$. The additional distance that the man runs during I_i when his speed is $1 + \varepsilon_n$ as compared to the speed $1 + \varepsilon_{n-1}$ is $\Delta_{n-1}(\varepsilon_n - \varepsilon_{n-1})$. It follows from the definition of r that

$$\Delta_{n-1}(\varepsilon_n - \varepsilon_{n-1}) \geq \rho + 2r + 3(1 + \varepsilon_n)\tau > \rho' + 3(1 + \varepsilon_n)\tau.$$

By the induction hypothesis, the man is within a distance of $\rho' + (1 + \varepsilon_n)\tau$ from $M_{n-1}(i\Delta_{n-1})$ at time $i\Delta_{n-1}$. Thus, his distance to g at the beginning of interval I_i is at most $\Delta_{n-1}(1 + \varepsilon_{n-1}) + \rho' + (1 + \varepsilon_n)\tau$, where $\Delta_{n-1}(1 + \varepsilon_{n-1})$ is the length of the interval $M_{n-1}(I_i)$. If the man does not do any avoidance moves during I_i, he runs straight to g, so it follows that he reaches g at time $(i+1)\Delta_{n-1} - 2\tau$ at the latest. Therefore, the statement is clearly true in this case.

Otherwise, let $t \in I_i$ be the first time of choice at which he does an avoidance move during I_i. If he is at distance at most ρ' from g at time t, the statement follows from Claim 12. Therefore, assume that the distance is more than ρ'. Then, we must have that $t < (i+1)\Delta_{n-1} - 2\tau$, since, if t was larger, he would already have reached g by the above discussion. Hence, Claim 11 gives that at some time $t' \leq t + \tau$, either

1. the man gets within a distance of ρ' from g, or
2. he does an escape move.

We first prove that in the interval $[t, t']$, the distance from the man to the segment $M_{n-1}(I_i)$ is at most $\rho' + 2(1 + \varepsilon_n)\tau$. To this end, note that his distance to the segment at time t is at most $\rho' + (1 + \varepsilon_n)\tau$. Thus, since $t' \leq t + \tau$, his distance at time t' can be at most $\rho' + 2(1 + \varepsilon_n)\tau$.

It remains to be proven that the man stays within distance $\rho' + 2(1 + \varepsilon_n)\tau$ from $M_{n-1}(I_i)$ after time t' and that he is at distance at most $\rho' + (1 + \varepsilon_n)\tau$ from g at time $(i+1)\Delta_{n-1}$. If we are in case 1, the statement follows from Claim 12, so assume case 2.

By Claim 10, the man will not do an avoidance move again after time t' until he reaches g or I_i ends. While he is running directly towards g, his distance to the segment $M_{n-1}(I_i)$ is decreasing, so it follows that the distance is always at most $\rho' + 2(1+\varepsilon_n)\tau$, as claimed. Since he was doing avoidance moves in a period of length at most τ before the escape move at time t', he can completely compensate for the delay caused by the avoidance moves in the same amount of time by running directly towards g. The total delay is therefore at most 2τ. Since he would reach g at time $(i+1)\Delta_{n-1} - 2\tau$ at the latest if he did not do any avoidance moves, it follows that he reaches g at time $(i+1)\Delta_{n-1}$ or earlier. The statement then follows from Claim 12. ◀

We are now ready to finish our proof of Theorem 8. Claim 13 implies that during interval I_i for any i, the distance from the man to any of the lions $l_1, \ldots, l_{n-1}$ is at most

$$\rho' + 2(1+\varepsilon_n)\tau < \rho + 2r + 2(1+\varepsilon_n)\tau \leq c_{n-1}/2.$$

Thus, these lions will not catch the man. By Claim 9, the distance to l_n is always at least $r - (3+\varepsilon_n)\Delta_n$. Therefore, we now define $c_n := \min\{c_{n-1}/2, r - (3+\varepsilon_n)\Delta_n\}$, and it holds that in the time interval $I := [i\Delta_n, (i+1)\Delta_n]$, the distance from any point on the segment $m(I)$ to any lion is at least c_n.

Claim 13 implies that at any time t and for any $i \in \{2, \ldots, n\}$, the distance $\|M_{i-1}(t)M_i(t)\|$ is bounded by some constant. It follows that $\|M_1(t)M_n(t)\| \leq d_n$ for some constant $d_n > 0$. Since $M_1(t)$ traverses a ray with constant speed $1 + \varepsilon/2 > 1$, the man eventually escapes the convex hull of the lions and that the distance diverges to ∞ as $t \to \infty$. This proves the theorem. ◀

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