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Restriction of Odd Degree Characters of $S_n$

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Abstract. Let $n$ and $k$ be natural numbers such that $2^k < n$. We study the restriction to $S_{n-2^k}$ of odd-degree irreducible characters of the symmetric group $S_n$. This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., Sémin. Lothar. Combin. 75 (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., J. Algebra 478 (2017), 271–282].

Key words: characters of symmetric groups; hooks in partitions

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1 Introduction

Let $n$ be a natural number, and let $\chi$ be an irreducible character of odd degree of the symmetric group $S_n$. Then there exists a unique odd-degree irreducible constituent of the restriction $\chi|_{S_{n-1}}$. This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any $k \in \mathbb{N}$ such that $2^k < n$, there exists a unique odd-degree irreducible constituent $f^k_n(\chi)$ of $\chi|_{S_{n-2^k}}$ appearing with odd multiplicity. The main goal of this article is to study for all $n,k \in \mathbb{N}$ the map

$$f^k_n : \text{Irr}_2(S_n) \longrightarrow \text{Irr}_2(S_{n-2^k}),$$

naturally defined by Theorem A of [3]. All our results are proved using a description of $f^k_n$ in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If $2^k$ appears in the binary expansion of $n$ we say that $2^k$ is a binary digit of $n$. Similarly we say that two natural numbers $m$ and $n$ are 2-disjoint if they do not have any common binary digit. On the other hand, if $m \leq n$ and all the binary digits of $m$ appear in the binary expansion of $n$, then we say that $m$ is a binary subsum of $n$. This will be denoted by $m \subseteq_2 n$. Let $v_2(n)$ be the exponent of the highest power of 2 dividing the integer $n$.

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A question raised in [3] may be phrased as: For which \( n \) and \( k \) is \( f^n_k \) surjective? The authors showed that \( f^n_k \) is surjective whenever \( 2^k \) is a binary digit of \( n \), and they observed that otherwise \( f^n_k \) could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

**Theorem A.** Let \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) be such that \( 2^k < n \). Let \( d(n,k) = \nu_2 \left( \left\lfloor \frac{n}{2^k} \right\rfloor \right) \).

- If \( k = 0 \) then \( f^n_k \) is surjective if and only \( d(n,k) \leq 2 \).
- If \( k > 0 \) then \( f^n_k \) is surjective if and only \( d(n,k) \leq 1 \).

Theorem A is a consequence of Theorem 3.5 below, which describes the images of the maps \( f^n_k \). For all \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) with \( 2^k < n \) and any \( \psi \in \text{ Irr}_2(\mathcal{S}_{n-2^k}) \) we define the set

\[
\mathcal{E}(\psi,2^k) = \{ \chi \in \text{ Irr}_2(\mathcal{S}_n) \mid f^n_k(\chi) = \psi \},
\]

and set \( e(\psi,2^k) = |\mathcal{E}(\psi,2^k)| \). We show in Corollary 3.8 that the maps \( f^n_k \) are regular on their images. This means that for any \( \psi \) in the image of \( f^n_k \), the number \( e(\psi,2^k) \) depends only on \( n \) and \( k \) and not on the specific \( \psi \). We also give a complete description of those \( \psi \in \text{ Irr}_2(\mathcal{S}_{n-2^k}) \) such that \( e(\psi,2^k) = 0 \), in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote \( f^n_k \) just by \( f_k \), when the natural number \( n \) is clear from the context. Then, for \( k, \ell \in \mathbb{N}_0, k < \ell \), such that \( 2^k + 2^\ell \leq n \), we may ask: when is \( f_k f_\ell = f_\ell f_k \)? or more specifically: when is \( f_k^{n-2^k} f_\ell^n = f_\ell^{n-2^\ell} f_k^n \)? In [3, Proposition 4.3] it was proved that \( f_k f_\ell = f_\ell f_k \) whenever \( 2^\ell < n < 2^{\ell+1} \). This is the case \( \ell = t \) in our second main result, which answers the question completely.

**Theorem B.** Let \( n = 2^t + m \) where \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \leq t \) and \( 2^k + 2^\ell \leq n \). Then, with the exception of the case \( n = 6, k = 0, \ell = 1 \),

\[
f_k f_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.
\]

### 2 Notation and background

Let \( n \) be a natural number. We let \( \text{ Irr}(\mathcal{S}_n) \) denote the set of irreducible characters of \( \mathcal{S}_n \) and \( \mathcal{P}(n) \) the set of partitions of \( n \). The notation \( \lambda \in \mathcal{P}(n) \) is sometimes replaced by \( \lambda \vdash n \) and we write \( |\lambda| = n \). There is a natural correspondence \( \lambda \leftrightarrow \chi^\lambda \) between \( \mathcal{P}(n) \) and \( \text{ Irr}(\mathcal{S}_n) \). We say then that \( \lambda \) labels \( \chi^\lambda \). We denote by \( \text{ Irr}_2(\mathcal{S}_n) \) the set of irreducible characters of \( \mathcal{S}_n \) of odd degree. If \( \chi^\lambda \in \text{ Irr}_2(\mathcal{S}_n) \) we say that \( \chi^\lambda \) is an odd character, we call \( \lambda \) an odd partition of \( n \) and write \( \lambda \vdash_o n \). Also the empty partition will be considered as an odd partition.

**Remark 2.1.** Let \( n, k \) be such that \( 2^k < n \). In [3, Theorem A and Proposition 4.2] it is shown that the map \( f^n_k : \text{ Irr}_2(\mathcal{S}_n) \to \text{ Irr}_2(\mathcal{S}_{n-2^k}) \) may be described in terms of the odd partitions labelling the odd characters as follows:

\[
f^n_k(\chi^\lambda) = \chi^\mu \iff \mu \vdash_o n - 2^k \text{ can be obtained from } \lambda \vdash_o n \text{ by removing a } 2^k \text{-hook.}
\]

Correspondingly we write (by abuse of notation) \( f^n_k(\lambda) = \mu \). In fact when \( \lambda \) is odd, there is only one \( 2^k \)-hook of \( \lambda \) whose removal leads again to an odd partition; we will refer to such a hook as an odd hook of \( \lambda \). This combinatorial description of \( f^n_k \) will be used throughout this paper, and we will regard \( f^n_k \) also as a map between the corresponding sets of odd partitions. Also, for \( \mu \vdash_o n - 2^k \) we set \( e(\mu,2^k) = e(\chi^\mu,2^k) \).
We need some concepts and basic facts concerning hooks in partitions. For any integer \( e \in \mathbb{N} \) we denote by \( C_e(\lambda) \) and \( Q_e(\lambda) \) the \( e \)-core and the \( e \)-quotient of \( \lambda \), respectively. Then \( Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1}) \) is an \( e \)-tuple of partitions satisfying \( n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i| \). It is well-known that a partition is uniquely determined by its \( e \)-core and \( e \)-quotient (we refer the reader to [6] or [4, Chapter 2.7] for a detailed discussion on this topic).

Let \( H_e(\lambda) \) be the set of hooks of \( \lambda \) having length divisible by \( e \), and let \( \mathcal{H}(Q_e(\lambda)) = \bigcup_{i=1}^{e-1} \mathcal{H}(\lambda_i) \).

As explained in [6, Theorem 3.3], there is a bijection between \( H_e(\lambda) \) and \( \mathcal{H}(Q_e(\lambda)) \) mapping hooks in \( \lambda \) of length \( ex \) to hooks in the quotient of length \( x \). Moreover, the bijection respects the process of hook removal. Namely, the partition \( \mu \) obtained by removing an \( ex \)-hook from \( \lambda \) is such that \( C_e(\mu) = C_e(\lambda) \) and the \( e \)-quotient of \( \mu \) is obtained by removing an \( x \)-hook from one of the partitions involved in \( Q_e(\lambda) \).

For \( e = 2 \) we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-quotient tower \( Q_2(\lambda) \) and the 2-core tower \( C_2(\lambda) \) of \( \lambda \). They have rows numbered by \( k \geq 0 \). The \( k \)th row \( Q_2^k(\lambda) \) of \( Q_2(\lambda) \) contains \( 2^k \) partitions \( \lambda_i^{(k)} \), \( 0 \leq i \leq 2^k - 1 \), and the \( k \)th row \( C_2^k(\lambda) \) of \( C_2(\lambda) \) contains the 2-cores of these partitions in the same order, i.e., \( C_2(\lambda_i^{(k)}) \), \( 0 \leq i \leq 2^k - 1 \). The 0th row of \( Q_2(\lambda) \) contains \( \lambda = \lambda_0^{(0)} \) itself, row 1 contains the partitions \( \lambda_0^{(1)}, \ldots, \lambda_{2^k-1}^{(1)} \) occurring in the 2-quotient \( Q_2(\lambda) \), row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have \( Q_2(\lambda_i^{(k)}) = (\lambda_{2i}^{(k+1)}, \lambda_{2i+1}^{(k+1)}) \) for \( i \in \{0, 1, \ldots, 2^k - 1\} \). We remark that the \( 2^k \) partitions in \( Q_2^k(\lambda) \) are the same as those in the \( 2^k \)-quotient \( Q_2^k(\lambda) \) of \( \lambda \), but in a different order for \( k \geq 2 \).

We also introduce the \( k \)-data \( D_2^k(\lambda) \) of \( \lambda \). This is a table containing the following \( k+1 \) rows: the \( k \) rows \( C_2^{(j)}(\lambda) \), \( j = 0, \ldots, k-1 \), and in addition the row \( Q_2^k(\lambda) \).

**Remark 2.2.** A partition \( \lambda \) may be recovered from its 2-core tower. For \( k > 0 \), it may also be recovered from the knowledge of the \( k \)-data \( D_2^k(\lambda) \) of \( \lambda \), because the rows \( C_2^{(i)}(\lambda) \) with \( l \geq k \) of \( C_2(\lambda) \) consist of the 2-core towers of the partitions in \( Q_2^k(\lambda) \).

**Lemma 2.3.** Suppose that \( \lambda \vdash n - 2^k \) and \( \mu \vdash n \). The following are equivalent.

(i) \( \lambda \) is obtained from \( \mu \) by removing a \( 2^k \)-hook.

(ii) The \( k \)-data \( D_2^k(\mu) \) and \( D_2^k(\lambda) \) coincide, except that for one \( i \in \{0, \ldots, 2^k - 1\} \) \( \lambda_i^{(k)} \) is obtained from \( \mu_i^{(k)} \) by removing a 1-hook.

**Proof.** A \( 2^k \)-hook \( H_0 \) in \( \mu \) corresponds in a canonical way to a \( 2^{k-1} \)-hook \( H_1 \) in a partition in \( Q_2^{(1)}(\mu) \), i.e., in row 1 of the 2-quotient tower \( Q_2(\mu) \). Continuing we see that \( H_0 \) corresponds in a canonical way to a 1-hook \( H_k \) in a partition \( \mu_i^{(k)} \) in \( Q_2^{(k)}(\mu) \), row \( k \) of \( Q_2(\mu) \). If \( \lambda \) is obtained by removing \( H_0 \) from \( \mu \), this corresponds to \( \lambda_i^{(k)} \) being obtained by removing the 1-hook \( H_k \) from \( \mu_i^{(k)} \) (by repeated applications of [6, Theorem 3.3]). Apart from this the rows \( Q_2^{(k)}(\mu) \) and \( Q_2^{(k)}(\lambda) \) coincide. Note also that the rows \( C_2^{(j)}(\mu) \) and \( C_2^{(j)}(\lambda) \) coincide for \( j = 0, \ldots, k-1 \), since the removal of the hooks \( H_j \) of even length do not change the 2-cores.

Odd-degree characters of \( \mathfrak{S}_n \) and thus odd partitions were completely described in [5]. We restate this result in a language which is convenient for our purposes. We let \( c_2^k(\lambda) \) be the sum of the cardinalities of the partitions in the \( k \)th row \( C_2^k(\lambda) \) of \( C_2(\lambda) \).

**Lemma 2.4 ([5]).** Let \( \lambda \) be a partition. Then \( \lambda \) is odd if and only if \( c_2^k(\lambda) \leq 1 \) for all \( k \geq 0 \).

It may be decided from the \( k \)-data \( D_2^k(\lambda) \) whether \( \lambda \) is odd. The case \( k = 1 \) of the following result appeared in [3, Lemma 4.1] and also in [1, Lemma 6].
**Theorem 2.5.** Let $\lambda \vdash n$, and let $k \geq 0$ be fixed. Consider $Q_2^{(k)}(\lambda) = (\lambda^{(k)}_i)$. Then $\lambda$ is odd if and only if the following conditions are all fulfilled:

(i) $c_2^{(j)}(\lambda) \leq 1$ for all $j < k$.

(ii) The partitions $\lambda^{(k)}_i$, $0 \leq i \leq 2^k - 1$, are all odd.

(iii) The numbers $|\lambda^{(k)}_i|$, $0 \leq i \leq 2^k - 1$, are pairwise 2-disjoint.

In this case $\sum_{i \geq 0} |\lambda^{(k)}_i| = \left\lceil \frac{n}{2^k} \right\rceil$.

**Proof.** This is proved by induction on $k \geq 0$, using Remark 2.2 and Lemma 2.4. \hfill \blacksquare

We illustrate the result above by giving an example.

**Example 2.6.** Let $n = 15$ and take $\lambda = (5,4,2^2,1^2) \vdash 15$. To decide whether $\lambda$ is odd, we choose $k = 2$ and compute the 2-data $D_2^{(2)}(\lambda)$. The 2-core is $C_2(\lambda) = (1)$, giving $C_2^{(0)}(\lambda) = ((1))$. Furthermore, the 2-quotient is $Q_2(\lambda) = ((2^2,1^2),(1))$, and computing the 2-cores $C_2((2^2,1^2)) = (0), C_2((1)) = (1)$, we obtain the next row: $C_2^{(1)}(\lambda) = ((0),(1))$. The 2-quotients are $Q_2((2^2,1^2)) = ((1^2),(1)), Q_2((1)) = ((0),(0))$; hence the final row of the 2-data table is obtained as $Q_2^{(2)}(\lambda) = ((1^2),(1),(0),(0))$.

We visualize $D_2^{(2)}(\lambda)$ like this:

\[
\begin{align*}
C_2^{(0)}(\lambda): & \quad (1) \\
C_2^{(1)}(\lambda): & \quad (0) \quad (1) \\
Q_2^{(2)}(\lambda): & \quad (1^2) \quad (1) \quad (0) \quad (0)
\end{align*}
\]

Theorem 2.5 shows that $\lambda$ is odd and thus it contains a unique odd 4-hook. Again using the theorem, it is clear that removing this 4-hook corresponds to the second partition (1) in $Q_2^{(2)}(\lambda)$ being replaced by (0). Thus, removing the corresponding 4-hook of $\lambda$ we obtain the odd partition $\mu = (3,2^2,1^2) \vdash 11$ with the property that $D_2^{(2)}(\lambda)$ and $D_2^{(2)}(\mu)$ differ only in their final row.

**Remark 2.7.** Using the construction of partitions from their 2-cores and 2-quotients already mentioned, the criterion above can be applied to construct all odd partitions of $n$ with a specific $k$th row in the 2-quotient tower. For this, let $n, k \in \mathbb{N}$, and take any sequence of odd partitions $\nu_i$, $0 \leq i \leq 2^k - 1$, such that the numbers $|\nu_i|$ are pairwise 2-disjoint, and $\sum_{i \geq 0} |\nu_i| = \left\lceil \frac{n}{2^k} \right\rceil$.

Then there are exactly $\prod_{m \leq k} 2^m$ odd partitions $\lambda$ of $n$ with $Q_2^{(k)}(\lambda) = (\nu_i)$, obtained by choosing one 2-core in row $m$ of the $k$-data table to be (1), for each $m < k$ such that $2^m \leq 2 n$.

The following easy consequence of Theorem 2.5 will be used repeatedly.

**Lemma 2.8.** Let $2^t$ be the largest binary digit of $n$. A partition $\lambda$ of $n$ is odd if and only if $\lambda$ contains a unique $2^t$-hook and the partition obtained from $\lambda$ by removing this $2^t$-hook is an odd partition of $n - 2^t$.

### 3 Surjectivity and regularity

The aim of this section is to study the images of the maps $f^n_k$ for all $n, k$ such that $2^k \leq n$. For this purpose we introduce the concept of $d$-good partitions (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when $f^n_k$ is surjective) and to show that the maps $f^n_k$ are always regular on their image (see Corollary 3.8).
Definition 3.1. Let $d \geq 0$. We call an odd partition $\lambda$ $d$-good, if

(i) $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.

(ii) $C_{2d}(\lambda)$ is a hook partition.

Let us remark that condition (i) may be reformulated as

(i*) $\nu_2(|\lambda| + 1) = d$.

In particular, if $\lambda$ is $d$-good, then $|\lambda|$ is odd if and only if $d > 0$.

The relevance of $d$-good partitions in our context is illuminated by the following reformulation of [1, Theorem 2]:

Lemma 3.2. Let $\lambda \vdash_o n$. Let $d = \nu_2(n+1)$. Then $e(\lambda, 1) \neq 0$ if and only if $\lambda$ is $d$-good. In this case, $e(\lambda, 1) = 1$ if $d = 0$, and $e(\lambda, 1) = 2$ if $d > 0$.

Lemma 3.3. Let $\lambda$ be an odd partition, and let $d \geq 0$. Then the following hold.

1. For $d \leq 2$, $\lambda$ is $d$-good if and only if $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.

2. If $\lambda$ is $d$-good, then $C_{2d}(\lambda)$ is a partition of $2^d - 1$.

Proof. If the odd partition $\lambda$ is $d$-good, then $|\lambda| = (2^d - 1) + m$ where the binary digits of $m$ are at least $2^{d+1}$. The hooks of $\lambda$ corresponding to the binary digits of $m$ may be decomposed into $2^d$-hooks and thus do not contribute to $C_{2d}(\lambda)$. Thus $|C_{2d}(\lambda)| = 2^d - 1$. This shows (2). For $d = 0, 1, 2$ we have $|C_{2d}(\lambda)| = 0, 1$ and $3$, respectively. Since all partitions of 0, 1 and 3 are hook partitions, (1) follows.

Definition 3.4. If $2^k \leq n$, we define $d(n, k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor \right)$. Thus $d(n, k)$ is the smallest integer $d \geq 0$ satisfying the condition $2^{k+d} \leq n$. In particular, $d(n, k) = 0$ if and only if $2^k \leq n$. Moreover, we may write $\left\lfloor \frac{n}{2^k} \right\rfloor = 2^d(n,k) + m(n,k)$ where $2^{d(n,k)+1} | m(n,k)$.

As mentioned in the introduction, the results in [3] show that $f_k^n$ is a surjective $(2^k$-to-1)-map whenever $2^k \leq n$, i.e., $d(n,k) = 0$. In the spirit of [1, Theorem 2], we now give a characterization of the image of the map $f_k^n$ for all $n, k$ such that $2^k < n$.

Theorem 3.5. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $\lambda \vdash_o n-2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a $d(n,k)$-good partition in the $k$th row of $Q_2(\lambda)$. In this case, $e(\lambda, 2^k) = 2^k$ if $d(n,k) = 0$, and $e(\lambda, 2^k) = 2$ if $d(n,k) > 0$.

Proof. If $k = 0$ then the statement follows from Lemma 3.2. Hence assume that $k \geq 1$. Let $d = d(n, k)$. By assumption $\left\lfloor \frac{n}{2^k} \right\rfloor = 2^d + m$, where the binary digits of $m$ are at least $2^{d+1}$. Thus $\left\lfloor \frac{n-2^k}{2^k} \right\rfloor = (2^d - 1) + m$.

Suppose first that $e(\lambda, 2^k) \neq 0$ and that $\mu \vdash_o n$ satisfies $f_k(\mu) = \lambda$. From Remark 2.1 and Lemma 2.3 we get that there exists an $i \in \{0, 1, \ldots, 2^k - 1\}$ such that $f_0(\mu_i^{(k)}) = \lambda_i^{(k)}$. Since $\mu_i^{(k)}$ and $\lambda_i^{(k)}$ are odd, we get $e(\lambda_i^{(k)}, 1) \neq 0$. We have that $|\lambda_i^{(k)}|$ and $|\mu_i^{(k)}|$ are both 2-disjoint with $m_1 := \sum_{j \neq i} |\lambda_j^{(k)}| = \sum_{j \neq i} |\mu_j^{(k)}| \leq 2 \left\lfloor \frac{n-2^k}{2^k} \right\rfloor$, by Theorem 2.5. Since $m_1 \leq 2\left\lfloor \frac{n-2^k}{2^k} \right\rfloor$ and $m_1 \leq 2 \left\lfloor \frac{n}{2^k} \right\rfloor$, we get $m_1 \leq m$. Thus $|\lambda_i^{(k)}| = (2^d - 1) + m$ and $|\mu_i^{(k)}| = 2^d + m_2$, where $m_2 = m - m_1 \leq 2 m$. In particular $\nu_2(|\lambda_i^{(k)}| + 1) = \nu_2(|\mu_i^{(k)}|) = d$. Then Lemma 3.2 shows that $\lambda_i^{(k)}$ is $d$-good.

Conversely, if $\lambda_i^{(k)}$ is a $d$-good partition for some $i \in \{0, 1, \ldots, 2^k - 1\}$, then there exists a $\mu^* \vdash_o |\lambda_i^{(k)}| + 1$ such that $f_0(\mu^*) = \lambda_i^{(k)}$, by Lemma 3.2. We let $\mu$ be the partition where the $k$-data $D_2^{(k)}(\mu)$ and $D_2^{(k)}(\lambda)$ coincide, except that $\mu_i^{(k)} = \mu^*$. Since $\lambda$ is odd and $\lambda_i^{(k)}$ is $d$-good,
we know that $|\lambda^{(k)}_i| = (2^d - 1) + m'$ where $m' \subseteq_2 m$, and $|\lambda^{(k)}_{j'}| \subseteq_2 m - m'$ for all $j \neq i$. Hence $|\mu'| = |\lambda^{(k)}_i| + 1 = 2^d + m'$ is 2-disjoint from all $|\lambda^{(k)}_{j'}|$, $j \neq i$. Thus $\mu$ is an odd partition of $n$ by Theorem 2.5, and $f_k(\mu) = \lambda$ by Lemma 2.3 and Remark 2.1.

We conclude that $e(\lambda, 2^k) = \sum_{\lambda^{(k)}_i \text{d-good}} e(\lambda^{(k)}_i, 1)$. If $d = 0$ then $\lfloor \frac{n-2^k}{2^d} \rfloor$ is even. This implies that all $\lambda^{(k)}_i$ are of even cardinality and thus $d$-good. Thus $e(\lambda^{(k)}_i, 1) = 1$ for all $i$, and we get $e(\lambda, 2^k) = 2^k$. If $d > 0$ there is exactly one $\lambda^{(k)}_i$ in $Q_2^{(k)}(\lambda)$ of odd cardinality. Only this $\lambda^{(k)}_i$ may be $d$-good and then $e(\lambda, 2^k) = e(\lambda^{(k)}_i, 1) = 2$. Otherwise $e(\lambda, 2^k) = 0$. ■

**Corollary 3.6.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$, and let $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash_n n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a partition $\lambda^{(k)}_i$ in the $k$th row of $Q_2^{(k)}(\lambda)$ such that $|\lambda^{(k)}_i| \equiv 2^d - 1 \pmod{2^{d+1}}$, and $C_{2d}(\lambda^{(k)}_i)$ is a hook partition. In this case, $e(\lambda, 2^k) = 2^k$ if $d = 0$, and $e(\lambda, 2^k) = 2$ if $d > 0$.

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

**Corollary 3.7 (Theorem A).** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$.

- If $k = 0$ then $f^n_k$ is surjective if and only if $d(n, k) \leq 2$.
- If $k > 0$ then $f^n_k$ is surjective if and only if $d(n, k) \leq 1$.

**Proof.** By Theorem 3.5, $f^n_k$ is surjective if and only if for all $\lambda \vdash_n n - 2^k$ we have that the $k$th row of $Q_2^{(k)}(\lambda)$ contains a $d(n, k)$-good partition $\lambda^{(k)}_i$. By Theorem 2.5 and Definition 3.4, for any $\lambda \vdash_n n - 2^k$ we have $\sum_{j \geq 0} |\lambda^{(k)}_j| = \lfloor \frac{n-2^k}{2^d} \rfloor = (2^{d(n,k)} - 1) + m(n, k)$.

If $k = 0$ then $Q_2^{(0)}(\lambda)$ contains only $\lambda = \lambda^{(0)}_0$. Hence $f^n_0$ is surjective if and only all odd partitions of $n - 1$ are $d(n, 0)$-good. By Lemma 3.3(1), the latter condition holds when $d(d(n, 0) \leq 2$. On the other hand, if $d = \nu_2(n) > 2$, then $\lambda = (n - 5, 2, 2)$ is an odd partition of $n - 1$ by Theorem 2.5, but $C_{2d}(\lambda) = (3, 2, 2)$ is not a hook, and hence $C_{2d}(\lambda)$ is not a hook. So $\lambda$ is not $d$-good, and thus $f^n_0$ is not surjective.

Now assume $k \geq 1$. Then $Q_2^{(k)}(\lambda)$ contains at least two odd partitions. If $d(n, k) \geq 2$ then any $d(n, k)$-good partition $\mu$ satisfies $3 \subseteq_2 2^{d(n,k)} - 1 \subseteq_2 |\mu|$. Write $\lfloor \frac{n-2^k}{2^d} \rfloor = 1 + m_1$ where $m_1$ is even. Applying Remark 2.7, take any $\lambda \vdash_n n - 2^k$ such that $|\lambda^{(k)}_0| = 1$ and $\lambda^{(k)}_1$ is an odd partition with $|\lambda^{(k)}_1| = m_1$. Then no partition in $Q_2^{(k)}(\lambda)$ is $d(n, k)$-good. Thus $f^n_k$ is not surjective. On the other hand, if $d(n, k) = 0$ then $2^k \subseteq_2 n$ and $f^n_k$ is surjective [3, Proposition 4.5]. If $d(n, k) = 1$ then $\lfloor \frac{n-2^k}{2^k} \rfloor = 1 + m(n, k)$, where $4 \mid m(n, k)$. Thus any $Q_2^{(k)}(\lambda)$ contains a partition with odd cardinality; this partition is 1-good, by Lemma 3.3. Again $f^n_k$ is surjective. ■

It is an immediate consequence of Theorem 3.5 that $f^n_k$ is regular on its image for all relevant choices of $n, k$ such that $2^k < n$. We have:

**Corollary 3.8.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$; set $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash_n n - 2^k$. Then $e(\lambda, 2^k) =$

$\begin{cases} 2^k & \text{if } d = 0; \\
2 & \text{if } d > 0, \text{ and the } k\text{th row of } Q_2(\lambda) \text{ contains a } d\text{-good partition}; \\
0 & \text{otherwise.} \end{cases}$
Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take $k = 2$ above. For $n > 2^2$ we first compute $d(n, k) = \nu_2\left(\binom{n}{2k}\right)$, and then consider odd partitions of $n - 4$ and their 4-extensions. For $n = 6$, $d(6, 2) = 0$. Thus $e(2, 4) = 4$. The odd 4-extensions of (2) are (6), (3$^2$), (2$^2$, 1$^2$), (2, 1$^4$). For $n = 10$, $d(10, 2) = 1$. In this case, $e(\lambda, 4) = 2$ for all odd partitions $\lambda$ of 6. For instance, the odd 4-extensions of (6) are (10) and (6,3,1). For $n = 19$, $d(19, 2) = 2$. Example 2.6 shows that for $\lambda = (5, 4, 2^2, 1^2)$ ⊥o 15 there is no 2-good partition in $Q_2(\lambda)$, hence $e(\lambda, 4) = 0$.

4 Deciding commutativity of the maps $f_k$ and $f_\ell$

Let $n \in \mathbb{N}$, and suppose that $0 \leq k < \ell$ satisfy $2^k + 2^\ell \leq n$. As stated in the introduction, we want to complete the discussion of the commutativity of the maps $f_k$ and $f_\ell$. Since the relevant $n$ will always be apparent for the maps $f_k^n$ in this section, we just write $f_k$.

We write $(n; k, \ell) \in T$ if for all $\lambda \vdash_o n$ we have $f_k f_\ell (\lambda) = f_\ell f_k (\lambda)$. Otherwise we write $(n; k, \ell) \in \mathcal{F}$.

In this section we will prove Theorem B, which may be reformulated as follows.

Theorem 4.1. Let $n = 2^t + m$ where $0 \leq m < 2^t$. Suppose that $k$, $\ell$ satisfy $0 \leq k < \ell$ and $2^k + 2^\ell \leq n$. Then with the exception of (6;0,1)

$$(n; k, \ell) \in T \text{ if and only if } \ell < t \text{ and } 2^k \leq m.$$  

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two extreme cases, where $f_k$ and $f_\ell$ commute.

In the case $\ell = t$ we have the following result as a reformulation of [3, Proposition 4.3].

Lemma 4.2. Let $n = 2^t + m$ with $0 \leq m < 2^t$. If $2^k \leq m$, then $(n; k, t) \in T$.

It is also known that in the case where $n$ is a power of 2, the maps $f_k$ and $f_\ell$ commute [3, Remark 4.4], and we include a short proof here.

Lemma 4.3. If $n = 2^t$ then $(n; k, \ell) \in T$ for all $k$, $\ell$.

Proof. If $0 \leq b \leq a$ are integers then the binomial coefficient $\binom{a}{b}$ is odd if and only if $b \subseteq_2 a$, by Lucas’ theorem. The odd partitions of $2^t$ are exactly the hook partitions $(2^t - b, 1^b)$, $0 \leq b \leq 2^t - 1$, of degree $(2^{t-1})$. Hence for $k \in \{0, 1, \ldots, t - 1\}$ we have

$$f_k (\lambda) = \begin{cases} (2^t - b - 2^k, 1^b) & \text{if } 2^k \not\subseteq_2 b, \\ (2^t - b, 1^{b-2^k}) & \text{if } 2^k \subseteq_2 b. \end{cases}$$

It follows that for any $k, \ell < t$ and odd partition $\lambda$ of $2^t$, we have $f_\ell f_k (\lambda) = f_k f_\ell (\lambda)$. 

Lemma 4.4. Let $n = 2^t + m$ with $0 \leq m < 2^t$. Suppose that $k$, $\ell$ satisfy $0 \leq k < \ell$ and $2^k + 2^\ell \leq n$. If $m < 2^k$ then $(n; k, \ell) \in T$.

Proof. We use induction on $k \geq 0$. For $k = 0$ we have $m = 0$ and the claim follows from Lemma 4.3. Suppose that $k \geq 1$ and that the claim has been proved up to $k - 1$. Let $\lambda \vdash_o n$. Odd hooks of length $2^k$ and $2^\ell$ in $\lambda$ correspond to odd hooks of length $2^{k-1}$ and $2^{\ell-1}$ in the 2-quotient $Q_2(\lambda) = (\lambda_0, \lambda_1)$ of $\lambda$. From Theorem 2.5 we deduce that $|\lambda_0|$ and $|\lambda_1|$ are 2-disjoint binary subsums of $\left\lfloor \frac{n}{2} \right\rfloor$, so one of them contains $2^{\ell-1}$, say $|\lambda_0|$; then $|\lambda_1| \leq \left\lfloor \frac{m}{2} \right\rfloor < 2^{k-1} < 2^{\ell-1}$. Thus the odd $2^{k-1}$-hook in $Q_2(\lambda)$ has to be in $\lambda_0$. Therefore

$$Q_2(f_k (\lambda)) = (f_{k-1}(\lambda_0), \lambda_1).$$
Applying $f_\ell$, the odd $2^{\ell-1}$-hook cannot be in $\lambda_1$, hence
\[
Q_2(f_\ell f_k(\lambda)) = (f_{\ell-1}f_{k-1}(\lambda_0), \lambda_1)).
\]
In particular, we know that $|\lambda_0| \geq 2^{\ell-1} + 2^{k-1}$. Also $|\lambda_0| + |\lambda_1| = \left\lceil \frac{n}{2} \right\rceil = 2^{\ell-1} + \left\lceil \frac{m}{2} \right\rceil$. We have already seen that $2^{\ell-1}$ is the largest binary digit of $|\lambda_0|$; furthermore $|\lambda_0| - 2^{\ell-1}$ is a binary subsum of $\left\lceil \frac{m}{2} \right\rceil < 2^{k-1}$. We may therefore apply the inductive hypothesis to $\lambda_0$ to get $f_{\ell-1}f_{k-1}(\lambda_0) = f_{k-1} f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_k f_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$.

Lemmas 4.2 and 4.4 show that the only if part of the theorem is true. We now turn to the if part. We start by proving the statement for $k = 0$ and use this as part of an inductive argument.

**Lemma 4.5.** Let $n = 2^t + m$ with $0 < m < 2^t$. If $0 < \ell < t$ then $(n; 0, \ell) \in \mathcal{F}$, with the exception of $(6; 0, 1)$.

**Proof.** The result is easily checked for $n \leq 8$, which includes the exception $(6; 0, 1)$. So we assume that $t \geq 3$.

*Case 1: $2^\ell < m$. Then $m \geq 3$, since $\ell > 0$. Consider the partition $\lambda = (m, m, 1^a) \vdash n$ where $a = n - 2m = 2^t - m$. The $(1,1)$-hook length of $\lambda$ is $2^t + 1$. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Removing the $(2,1)$-hook hook we get the odd partition $(m)$, so $\lambda$ is odd, by Lemma 2.8. We claim that
\[
f_0(\lambda) = (m, m, 1^{a-1}).
\]
Indeed we cannot have $f_0(\lambda) = (m, m - 1, 1^a)$ because this partition does not have a hook of length $2^\ell$, and thus it is not odd. Now
\[
f_\ell(f_0(\lambda)) = f_\ell(m, m, 1^{a-1}) = (m, m - 2^\ell, 1^{a-1})
\]
since $(m, m, 1^{a-1-2^\ell})$ and $(m - 1, m - 2^\ell + 1, 1^{a-1})$ both do not have a hook of length $2^\ell$ and thus are not odd (again by Lemma 2.8).

On the other hand,
\[
f_\ell(\lambda) = (m - 1, m - (2^\ell - 1), 1^a).
\]
Indeed, the other candidates for $f_\ell(\lambda)$, which are $(m, m - 2^\ell, 1^a)$ and $(m, m, 1^{a-2^\ell})$, do not have hooks of length $2^\ell$. Then
\[
f_0(f_\ell(\lambda)) = f_0(m - 1, m - (2^\ell - 1), 1^a) = (m - 1, m - 2^\ell, 1^a).
\]
This follows (again) by observing that all the other partitions of $n - 2^\ell - 1$ obtained from $(m - 1, m - (2^\ell - 1), 1^a)$ by removing a node do not have hooks of length $2^\ell$. Thus $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

*Case 2: $m < 2^\ell$. Consider the partition $\lambda = (n - 2^\ell, m + 1, 1^a)$, where $a = 2^\ell - (m + 1)$. Note that $n - 2^\ell \geq m + 1$ since $\ell < t$ by assumption, and that $a \geq 0$. The $(1,1)$-hook length of $\lambda$ is $n - m = 2^\ell$. Removing this hook we get the odd partition $(m)$, so $\lambda$ is odd. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Now
\[
f_0(\lambda) = (n - 2^\ell, m, 1^a)
\]
since the other candidates do not have hooks of length $2^\ell$. Then
\[
f_\ell(f_0(\lambda)) = f_\ell(n - 2^\ell, m, 1^a) = \mu,
\]
where $\mu$ is obtained from $f_0(\lambda)$ by removing a $2^\ell$-hook in the first row. (There are only hooks of length $< 2^\ell$ in the other rows.) In fact, $\mu = (n - 2^{\ell+1}, m, 1^a)$ since $n - 2^{\ell+1} \geq n - 2^\ell = m$. Thus $f_\ell(f_0(\lambda))$ has at least 2 parts. On the other hand

$$f_\ell(\lambda) = (n - 2^\ell)$$

since this odd partition is obtained from the odd partition $\lambda$ by removing a $2^\ell$-hook (the one in $(2, 1)$). It follows that

$$f_0(f_\ell(\lambda)) = (n - 2^\ell - 1)$$

and again $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

Case 3: $m = 2^\ell$. Then $n = 2^\ell + 2^\ell$. If $\ell \geq 2$ then choose $\lambda = (2^\ell, 2^\ell - 1, 1)$. The $(1, 2)$-hook length of $\lambda$ is $2^\ell$; thus $\lambda$ is an odd partition since removing this $2^\ell$-hook gives an odd partition $(2^\ell - 2, 1, 1)$ of $2^\ell$. We have $f_0(\lambda) = (2^\ell, 2^\ell - 2, 1)$ since the other candidates are not odd. Then

$$f_\ell(f_0(\lambda)) = (2^\ell - 2^\ell, 2^\ell - 2, 1).$$

The $(2, 1)$-hook length of $\lambda$ is $2^\ell$, so $f_\ell(\lambda) = (2^\ell)$ and

$$f_0(f_\ell(\lambda)) = (2^\ell - 1),$$

showing $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

On the other hand, if $\ell = 1$ then choose $\lambda = (2^\ell - 2, 2, 2) \vdash_o 2^\ell + 2 = n$. Since $\ell \geq 3$, it is now easy to show that $f_1(f_0(\lambda)) = (2^\ell - 4, 2, 1)$. On the other hand we see that $f_0(f_1(\lambda))$ is a hook partition of $2^\ell - 1 = n - 3$ and therefore is not equal to $f_1(f_0(\lambda))$. 

**Lemma 4.6.** If $(n; k, \ell) \in F$ then also $(2n; k + 1, \ell + 1) \in F$ and $(2n + 1; k + 1, \ell + 1) \in F$.

**Proof.** Let the odd partition $\mu$ of $n$ satisfy $f_k f_\ell(\mu) \neq f_\ell f_k(\mu)$. Let $\lambda$ be a partition of $2n$ or $2n + 1$ having 2-quotient $Q_2(\lambda) = (\mu, (0))$. Then $\lambda$ is odd, by Theorem 2.5. We have

$$Q_2(f_{k+1}f_{\ell+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_\ell f_k(\mu), (0)) = Q_2(f_{\ell+1}f_{k+1}(\lambda)),$$

so that $f_{k+1}f_{\ell+1}(\lambda) \neq f_{\ell+1}f_{k+1}(\lambda)$. 

We are now ready to conclude this section with the proof of Theorem B.

**Proof of Theorem 4.1.** The only if part follows from Lemmas 4.2 and 4.4. To prove the if part we use induction on $k \geq 0$. If $k = 0$, then the statement follows from Lemma 4.5. Let $k > 1$ and suppose that the assertion is true up to and including $k - 1$. To show that $(n; k, \ell) \in F$ it suffices to prove $(\lfloor \frac{n}{2} \rfloor; k - 1, \ell - 1) \in F$, by Lemma 4.6. We are assuming $n = 2^\ell + m$, $0 \leq m < 2^\ell$, $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. This implies $\lfloor \frac{n}{2} \rfloor = 2^{t-1} + \lfloor \frac{m}{2} \rfloor$, $0 \leq \lfloor \frac{m}{2} \rfloor < 2^{t-1}$ and $2^{k-1} + 2^{t-1} \leq \lfloor \frac{n}{2} \rfloor$. We may apply the inductive hypothesis to get $(\lfloor \frac{n}{2} \rfloor; k - 1, \ell - 1) \in F$, and then $(n; k, \ell) \in F$ except when $(\lfloor \frac{n}{2} \rfloor; k - 1, \ell - 1) = (6; 0, 1)$. In that case we are considering $(12; 1, 2)$ or $(13; 1, 2)$ which are both in $F$, by direct computation (consider for example $(6, 4, 2) \vdash_o 12$ and $(6, 4, 3) \vdash_o 13$, respectively).

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References


