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Restriction of Odd Degree Characters of $\mathfrak{S}_n$

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Abstract. Let $n$ and $k$ be natural numbers such that $2^k < n$. We study the restriction to $\mathfrak{S}_{n-2^k}$ of odd-degree irreducible characters of the symmetric group $\mathfrak{S}_n$. This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., Sémin. Lothar. Combin. 75 (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., J. Algebra 478 (2017), 271–282].

Key words: characters of symmetric groups; hooks in partitions

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1 Introduction

Let $n$ be a natural number, and let $\chi$ be an irreducible character of odd degree of the symmetric group $\mathfrak{S}_n$. Then there exists a unique odd-degree irreducible constituent of the restriction $\chi_{\mathfrak{S}_{n-1}}$. This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any $k \in \mathbb{N}$ such that $2^k < n$, there exists a unique odd-degree irreducible constituent $f^n_k(\chi)$ of $\chi_{\mathfrak{S}_{n-2^k}}$ appearing with odd multiplicity. The main goal of this article is to study for all $n, k \in \mathbb{N}$ the map

$$f^n_k: \text{Irr}_2(\mathfrak{S}_n) \rightarrow \text{Irr}_2(\mathfrak{S}_{n-2^k}),$$

naturally defined by Theorem A of [3]. All our results are proved using a description of $f^n_k$ in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If $2^k$ appears in the binary expansion of $n$ we say that $2^k$ is a binary digit of $n$. Similarly we say that two natural numbers $m$ and $n$ are 2-disjoint if they do not have any common binary digit. On the other hand, if $m \leq n$ and all the binary digits of $m$ appear in the binary expansion of $n$, then we say that $m$ is a binary subsum of $n$. This will be denoted by $m \subseteq_2 n$. Let $\nu_2(n)$ be the exponent of the highest power of 2 dividing the integer $n$.

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A question raised in [3] may be phrased as: For which $n$ and $k$ is $f_k^n$ surjective? The authors showed that $f_k^n$ is surjective whenever $2^k$ is a binary digit of $n$, and they observed that otherwise $f_k^n$ could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

**Theorem A.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $d(n,k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor \right)$.

- If $k = 0$ then $f_k^n$ is surjective if and only if $d(n,k) \leq 2$.
- If $k > 0$ then $f_k^n$ is surjective if and only if $d(n,k) \leq 1$.

Theorem A is a consequence of Theorem 3.5 below, which describes the images of the maps $f_k^n$.

For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $2^k < n$ and any $\psi \in \text{Irr}_2(\mathfrak{S}_{n-2^k})$ we define the set

$$E(\psi, 2^k) = \{ \chi \in \text{Irr}_2(\mathfrak{S}_n) \mid f_k^n(\chi) = \psi \},$$

and set $e(\psi, 2^k) = |E(\psi, 2^k)|$. We show in Corollary 3.8 that the maps $f_k^n$ are regular on their images. This means that for any $\psi$ in the image of $f_k^n$, the number $e(\psi, 2^k)$ depends only on $n$ and $k$ and not on the specific $\psi$. We also give a complete description of those $\psi \in \text{Irr}_2(\mathfrak{S}_{n-2^k})$ such that $e(\psi, 2^k) = 0$, in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote $f_k^n$ just by $f_k$, when the natural number $n$ is clear from the context. Then, for $k, \ell \in \mathbb{N}_0$, $k < \ell$, such that $2^k + 2^\ell \leq n$, we may ask: when is $f_k f_\ell = f_\ell f_k$? or more specifically: when is $f_k^{n-2^k} f_\ell^{n-2^k} = f_\ell^{n-2^k} f_k^{n-2^k}$? In [3, Proposition 4.3] it was proved that $f_k f_\ell = f_\ell f_k$ whenever $2^\ell < n < 2^{\ell+1}$. This is the case $\ell = t$ in our second main result, which answers the question completely.

**Theorem B.** Let $n = 2^t + m$ where $0 \leq m < 2^t$. Suppose that $k, \ell$ satisfy $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. Then, with the exception of the case $n = 6$, $k = 0$, $\ell = 1$,

$$f_k f_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.$$
We need some concepts and basic facts concerning hooks in partitions. For any integer $e \in \mathbb{N}$ we denote by $C_e(\lambda)$ and $Q_e(\lambda)$ the $e$-core and the $e$-quotient of $\lambda$, respectively. Then $Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1})$ is an $e$-tuple of partitions satisfying $n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i|$. It is well-known that a partition is uniquely determined by its $e$-core and $e$-quotient (we refer the reader to [6] or [4, Chapter 2.7] for a detailed discussion on this topic).

Let $\mathcal{H}_e(\lambda)$ be the set of hooks of $\lambda$ having length divisible by $e$, and let $\mathcal{H}(Q_e(\lambda)) = \bigcup_{i=1}^{e-1} \mathcal{H}(\lambda_i)$. As explained in [6, Theorem 3.3], there is a bijection between $\mathcal{H}_e(\lambda)$ and $\mathcal{H}(Q_e(\lambda))$ mapping hooks in $\lambda$ of length $e\varepsilon x$ to hooks in the quotient of length $x$. Moreover, the bijection respects the process of hook removal. Namely, the partition $\mu$ obtained by removing an $e\varepsilon x$-hook from $\lambda$ is such that $C_e(\mu) = C_e(\lambda)$ and the $e$-quotient of $\mu$ is obtained by removing an $x$-hook from one of the partitions involved in $Q_e(\lambda)$.

For $e = 2$ we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-quotient tower $Q_2(\lambda)$ and the 2-core tower $C_2(\lambda)$ of $\lambda$. They have rows numbered by $k \geq 0$. The $k$th row of $Q_2^{(k)}(\lambda)$ of $Q_2(\lambda)$ contains $2^k$ partitions $\lambda_i^{(k)}$, $0 \leq i \leq 2^k - 1$, and the $k$th row of $C_2^{(k)}(\lambda)$ of $C_2(\lambda)$ contains the 2-cores of these partitions in the same order, i.e., $C_2(\lambda_i^{(k)})$, $0 \leq i \leq 2^k - 1$. The 0th row of $Q_2(\lambda)$ contains $\lambda = \lambda_0^{(0)}$ itself, row 1 contains the partitions $\lambda_0^{(1)}$, $\lambda_1^{(1)}$ occurring in the 2-quotient $Q_2(\lambda)$, row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have $Q_2(\lambda_i^{(k)}) = (\lambda_{2i}^{(k+1)}, \lambda_{2i+1}^{(k+1)})$ for $i \in \{0, 1, \ldots, 2^k - 1\}$. We remark that the $2^k$ partitions in $Q_2^{(k)}(\lambda)$ are the same as those in the $2^k$-quotient $Q_{2^k}(\lambda)$ of $\lambda$, but in a different order for $k \geq 2$.

We also introduce the $k$-data $D_2^{(k)}(\lambda)$ of $\lambda$. This is a table containing the following $k+1$ rows: the $k$ rows $C_2^{(j)}(\lambda)$, $j = 0, \ldots, k - 1$, and in addition the row $Q_2^{(k)}(\lambda)$.

**Remark 2.2.** A partition $\lambda$ may be recovered from its 2-core tower. For $k > 0$, it may also be recovered from the knowledge of the $k$-data $D_2^{(k)}(\lambda)$ of $\lambda$, because the rows $C_2^{(l)}(\lambda)$ with $l \geq k$ of $C_2(\lambda)$ consist of the 2-core towers of the partitions in $Q_2^{(k)}(\lambda)$.

**Lemma 2.3.** Suppose that $\lambda \vdash n - 2^k$ and $\mu \vdash n$. The following are equivalent.

1. $\lambda$ is obtained from $\mu$ by removing a $2^k$-hook.
2. The $k$-data $D_2^{(k)}(\mu)$ and $D_2^{(k)}(\lambda)$ coincide, except that for one $i \in \{0, \ldots, 2^k - 1\}$ $\lambda_i^{(k)}$ is obtained from $\mu_i^{(k)}$ by removing a 1-hook.

**Proof.** A $2^k$-hook $H_0$ in $\mu$ corresponds in a canonical way to a $2^{k-1}$-hook $H_1$ in a partition in $Q_2^{(1)}(\mu)$, i.e., in row 1 of the 2-quotient tower $Q_2(\mu)$. Continuing we see that $H_0$ corresponds in a canonical way to a 1-hook $H_k$ in a partition $\mu_k^{(k)}$ in $Q_2^{(k)}(\mu)$, row $k$ of $Q_2(\mu)$. If $\lambda$ is obtained by removing $H_0$ from $\mu$, this corresponds to $\lambda_i^{(k)}$ being obtained by removing the 1-hook $H_k$ from $\mu_i^{(k)}$ (by repeated applications of [6, Theorem 3.3]). Apart from this the rows $Q_2^{(k)}(\mu)$ and $Q_2^{(k)}(\lambda)$ coincide. Note also that the rows $C_2^{(j)}(\mu)$ and $C_2^{(j)}(\lambda)$ coincide for $j = 0, \ldots, k - 1$, since the removal of the hooks $H_j$ of even length do not change the 2-cores.

Odd-degree characters of $\mathfrak{S}_n$ and thus odd partitions were completely described in [5]. We restate this result in a language which is convenient for our purposes. We let $c_2^{(k)}(\lambda)$ be the sum of the cardinalities of the partitions in the $k$th row $C_2^{(k)}(\lambda)$ of $C_2(\lambda)$.

**Lemma 2.4 ([5]).** Let $\lambda$ be a partition. Then $\lambda$ is odd if and only if $c_2^{(k)}(\lambda) \leq 1$ for all $k \geq 0$.

It may be decided from the $k$-data $D_2^{(k)}(\lambda)$ whether $\lambda$ is odd. The case $k = 1$ of the following result appeared in [3, Lemma 4.1] and also in [1, Lemma 6].
Theorem 2.5. Let \( \lambda \vdash n \), and let \( k \geq 0 \) be fixed. Consider \( Q_2^{(k)}(\lambda) = (\lambda_i^{(k)}) \). Then \( \lambda \) is odd if and only if the following conditions are all fulfilled:

(i) \( c_2^{(j)}(\lambda) \leq 1 \) for all \( j < k \).
(ii) The partitions \( \lambda_i^{(k)} \), \( 0 \leq i \leq 2^k - 1 \), are all odd.
(iii) The numbers \( |\lambda_i^{(k)}| \), \( 0 \leq i \leq 2^k - 1 \), are pairwise 2-disjoint.

In this case \( \sum_{i \geq 0} |\lambda_i^{(k)}| = \left\lfloor \frac{n}{2^k} \right\rfloor \).

Proof. This is proved by induction on \( k \geq 0 \), using Remark 2.2 and Lemma 2.4.

We illustrate the result above by giving an example.

Example 2.6. Let \( n = 15 \) and take \( \lambda = (5,4,2^2,1^2) \vdash 15 \). To decide whether \( \lambda \) is odd, we choose \( k = 2 \) and compute the 2-data \( D_2^{(2)}(\lambda) \). The 2-core is \( C_2(\lambda) = (1) \), giving \( C_2^{(0)}(\lambda) = ((1)) \).

Furthermore, the 2-quotient is \( Q_2(\lambda) = ((2^2,1^2),(1)) \), and computing the 2-cores \( C_2(\lambda(2^2,1^2)) = (0) \), \( C_2(1) = (1) \), we obtain the next row: \( C_2^{(1)}(\lambda) = ((0),(1)) \). The 2-quotients are \( Q_2(2^2,1^2) = ((1^2),(1)) \), \( Q_2((1)) = ((0),(0)) \); hence the final row of the 2-data table is obtained as \( Q_2^{(2)}(\lambda) = ((1^2),(1),(0),(0)) \).

We visualize \( D_2^{(2)}(\lambda) \) like this:

\[
\begin{align*}
C_2^{(0)}(\lambda): & \quad (1) \\
C_2^{(1)}(\lambda): & \quad (0) \quad (1) \\
Q_2^{(2)}(\lambda): & \quad (1^2) \quad (1) \quad (0) \quad (0)
\end{align*}
\]

Theorem 2.5 shows that \( \lambda \) is odd and thus it contains a unique odd 4-hook. Again using the theorem, it is clear that removing this 4-hook corresponds to the second partition \( (1) \) in \( Q_2^{(2)}(\lambda) \) being replaced by \( (0) \). Thus, removing the corresponding 4-hook of \( \lambda \) we obtain the odd partition \( \mu = (3,2^2,1^2) \vdash 11 \) with the property that \( D_2^{(2)}(\lambda) \) and \( D_2^{(2)}(\mu) \) differ only in their final row.

Remark 2.7. Using the construction of partitions from their 2-cores and 2-quotients already mentioned, the criterion above can be applied to construct all odd partitions of \( n \) with a specific \( k \)th row in the 2-quotient tower. For this, let \( n,k \in \mathbb{N} \), and take any sequence of odd partitions \( \nu_i \), \( 0 \leq i \leq 2^k - 1 \), such that the numbers \( |\nu_i| \) are pairwise 2-disjoint, and \( \sum_{i \geq 0} |\nu_i| = \left\lfloor \frac{n}{2^k} \right\rfloor \).

Then there are exactly \( \prod_{m \leq k} 2^m \) odd partitions \( \lambda \) of \( n \) with \( Q_2^{(k)}(\lambda) = (\nu_i) \), obtained by choosing one 2-core in row \( m \) of the \( k \)-data table to be \( (1) \), for each \( m < k \) such that \( 2^m \subseteq 2 \).

The following easy consequence of Theorem 2.5 will be used repeatedly.

Lemma 2.8. Let \( 2^t \) be the largest binary digit of \( n \). A partition \( \lambda \) of \( n \) is odd if and only if \( \lambda \) contains a unique \( 2^t \)-hook and the partition obtained from \( \lambda \) by removing this \( 2^t \)-hook is an odd partition of \( n - 2^t \).

3 Surjectivity and regularity

The aim of this section is to study the images of the maps \( f_k^n \) for all \( n, k \) such that \( 2^k \leq n \). For this purpose we introduce the concept of \( d \)-good partitions (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when \( f_k^n \) is surjective) and to show that the maps \( f_k^n \) are always regular on their image (see Corollary 3.8).
Definition 3.1. Let \( d \geq 0 \). We call an odd partition \( \lambda \) \( d \)-good, if

(i) \(|\lambda| \equiv 2^d - 1 \mod 2^{d+1}\).

(ii) \( C_{2^d}(\lambda) \) is a hook partition.

Let us remark that condition (i) may be reformulated as

(i*) \( \nu_2(|\lambda| + 1) = d \).

In particular, if \( \lambda \) is \( d \)-good, then \(|\lambda|\) is odd if and only if \( d > 0 \).

The relevance of \( d \)-good partitions in our context is illuminated by the following reformulation of [1, Theorem 2]:

Lemma 3.2. Let \( \lambda \vdash_o n \). Let \( d = \nu_2(n + 1) \). Then \( e(\lambda, 1) \neq 0 \) if and only if \( \lambda \) is \( d \)-good. In this case, \( e(\lambda, 1) = 1 \) if \( d = 0 \), and \( e(\lambda, 1) = 2 \) if \( d > 0 \).

Lemma 3.3. Let \( \lambda \) be an odd partition, and let \( d \geq 0 \). Then the following hold.

1. For \( d \leq 2 \), \( \lambda \) is \( d \)-good if and only if \(|\lambda| \equiv 2^d - 1 \mod 2^{d+1}\).

2. If \( \lambda \) is \( d \)-good, then \( C_{2^d}(\lambda) \) is a partition of \( 2^d - 1 \).

Proof. If the odd partition \( \lambda \) is \( d \)-good, then \(|\lambda| = (2^d - 1) + m\) where the binary digits of \( m \) are at least \( 2^{d+1}\). The hooks of \( \lambda \) corresponding to the binary digits of \( m \) may be decomposed into \( 2^d \)-hooks and thus do not contribute to \( C_{2^d}(\lambda) \). Thus \(|C_{2^d}(\lambda)| = 2^d - 1\). This shows (2).

As mentioned in the introduction, the results in [3] show that \( f^n_k \) is a surjective \((2^k\)-to-1\)-map whenever \( 2^k \subseteq \mathbb{N} \), i.e., \( d(n, k) = 0 \). In the spirit of [1, Theorem 2], we now give a characterization of the image of the map \( f^n_k \) for all \( n, k \) such that \( 2^k < n \).

Theorem 3.5. Let \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) be such that \( 2^k < n \). Let \( \lambda \vdash_o n - 2^k \). Then \( e(\lambda, 2^k) \neq 0 \) if and only if there exists a \( d(n, k) \)-good partition in the \( k \)th row of \( Q_2(\lambda) \). In this case, \( e(\lambda, 2^k) = 2^k \) if \( d(n, k) = 0 \), and \( e(\lambda, 2^k) = 2 \) if \( d(n, k) > 0 \).

Proof. If \( k = 0 \) then the statement follows from Lemma 3.2. Hence assume that \( k \geq 1 \). Let \( d = d(n, k) \). By assumption \( \left\lfloor \frac{n}{2^k} \right\rfloor = 2^d + m \), where the binary digits of \( m \) are at least \( 2^{d+1}\). Thus \( \left\lfloor \frac{n - 2^k}{2^k} \right\rfloor = (2^d - 1) + m \).

Suppose first that \( e(\lambda, 2^k) \neq 0 \) and that \( \mu \vdash_o n \) satisfies \( f_2^k(\mu) = \lambda \). From Remark 2.1 and Lemma 2.3 we get that there exists an \( i \in \{0, 1, \ldots, 2^k - 1\} \) such that \( f_0(\mu_i^{(k)}) = \lambda_i^{(k)} \). Since \( \mu_i^{(k)} \) and \( \lambda_i^{(k)} \) are odd, we get \( e(\lambda_i^{(k)}, 1) \neq 0 \). We have that \( |\lambda_i^{(k)}| \) and \( |\mu_i^{(k)}| \) are both \( 2 \)-disjoint with

\[
m_1 := \sum_{j \neq i} |\lambda_j^{(k)}| = \sum_{j \neq i} |\mu_j^{(k)}| \subseteq \left[ \frac{n - 2^k}{2^k} \right],
\]

by Theorem 2.5. Since \( m_1 \subseteq \left[ \frac{n - 2^k}{2^k} \right] \) and \( m_1 \subseteq \left[ \frac{n}{2^k} \right] \),

we get \( m_1 \subseteq 2m \). Thus \( |\lambda_i^{(k)}| = (2^d - 1) + m_2 \) and \( |\mu_i^{(k)}| = 2^d + m_2 \), where \( m_2 = m - m_1 \subseteq 2m \).

In particular \( \nu_2(|\lambda_i^{(k)}| + 1) = \nu_2(|\mu_i^{(k)}|) = d \). Then Lemma 3.2 shows that \( \lambda_i^{(k)} \) is \( d \)-good.

Conversely, if \( \lambda_i^{(k)} \) is a \( d \)-good partition for some \( i \in \{0, 1, \ldots, 2^k - 1\} \), then there exists a \( \mu^* \vdash_o |\lambda_i^{(k)}| + 1 \) such that \( f_0(\mu^*) = \lambda_i^{(k)} \), by Lemma 3.2. We let \( \mu \) be the partition where the \( k \)-data \( D_2^{(k)}(\mu) \) and \( D_2^{(k)}(\lambda) \) coincide, except that \( \mu_1^{(k)} = \mu^* \). Since \( \lambda \) is odd and \( \lambda_i^{(k)} \) is \( d \)-good,
we know that $|\lambda_i^{(k)}| = (2^d - 1) + m'$ where $m' \subseteq 2 m$, and $|\lambda_j^{(k)}| \subseteq 2 m - m'$ for all $j \neq i$. Hence $|\mu'| = |\lambda_i^{(k)}| + 1 = 2^d + m'$ is 2-disjoint from all $|\lambda_j^{(k)}|, j \neq i$. Thus $\mu$ is an odd partition of $n$ by Theorem 2.5, and $f_k(\mu) = \lambda$ by Lemma 2.3 and Remark 2.1.

We conclude that $e(\lambda, 2^k) = \sum_{\lambda_i^{(k)} \text{d-good}} e(\lambda_i^{(k)}, 1)$. If $d = 0$ then $\left\lceil \frac{n-2^k}{2^k} \right\rceil$ is even. This implies that all $\lambda_i^{(k)}$ are of even cardinality and thus d-good. Thus $e(\lambda_i^{(k)}, 1) = 1$ for all $i$, and we get $e(\lambda, 2^k) = 2^k$. If $d > 0$ there is exactly one $\lambda_i^{(k)}$ in $Q_2^{(k)}(\lambda)$ of odd cardinality. Only this $\lambda_i^{(k)}$ may be d-good and then $e(\lambda, 2^k) = e(\lambda_i^{(k)}, 1) = 2$. Otherwise $e(\lambda, 2^k) = 0$.

Corollary 3.6. Let $n \in \mathbb{N}, k \in \mathbb{N}_0$ be such that $2^k < n$, and let $d = \nu_2\left(\left\lceil \frac{n}{2^k} \right\rceil\right)$. Let $\lambda \vdash_o n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a partition $\lambda_i^{(k)}$ in the $k$th row of $Q_2(\lambda)$ such that $|\lambda_i^{(k)}| \equiv 2^d - 1 \mod 2^{d+1}$, and $C_{2^d}(\lambda_i^{(k)})$ is a hook partition. In this case, $e(\lambda, 2^k) = 2^k$ if $d = 0$, and $e(\lambda, 2^k) = 2$ if $d > 0$.

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

Corollary 3.7 (Theorem A). Let $n \in \mathbb{N}, k \in \mathbb{N}_0$ be such that $2^k < n$.

- If $k = 0$ then $f_k^n$ is surjective if and only if $d(n, k) \leq 2$.
- If $k > 0$ then $f_k^n$ is surjective if and only if $d(n, k) \leq 1$.

Proof. By Theorem 3.5, $f_k^n$ is surjective if and only if for all $\lambda \vdash_o n - 2^k$ we have that the $k$th row of $Q_2(\lambda)$ contains a $d(n, k)$-good partition $\lambda_i^{(k)}$. By Theorem 2.5 and Definition 3.4, for any $\lambda \vdash_o n - 2^k$ we have $\sum_{j \geq 0} \lfloor \frac{n-2^k}{2^j} \rfloor = 2d(n,k) - 1 + m(n, k)$.

If $k = 0$ then $Q_2^{(0)}(\lambda)$ contains only $\lambda = \lambda_0^{(0)}$. Hence $f_0^n$ is surjective if and only all odd partitions of $n - 1$ are $d(n, 0)$-good. By Lemma 3.3(1), the latter condition holds when $d = d(n, 0) \leq 2$. On the other hand, if $d = \nu_2(n) > 2$, then $\lambda = (n - 5, 2, 2)$ is an odd partition of $n - 1$ by Theorem 2.5, but $C_{2^d}(\lambda) = (3, 2, 2)$ is not a hook, and hence $C_{2^d}(\lambda)$ is not a hook. So $\lambda$ is not even, and thus $f_0^n$ is not surjective.

Now assume $k \geq 1$. Then $Q_2^{(k)}(\lambda)$ contains at least two odd partitions. If $d(n, k) \geq 2$ then any $d(n, k)$-good partition $\mu$ satisfies $3 \subseteq 2d(n,k) - 1 \subseteq |\mu|$. Write $\left\lceil \frac{n-2^k}{2^k} \right\rceil = 1 + m_1$ where $m_1$ is even. Applying Remark 2.7, take any $\lambda \vdash_o n - 2^k$ such that $|\lambda_0^{(k)}| = 1$ and $\lambda_1^{(k)}$ is an odd partition with $|\lambda_1^{(k)}| = m_1$. Then no partition in $Q_2^{(k)}(\lambda)$ is $d(n, k)$-good. Thus $f_k^n$ is not surjective. On the other hand, if $d(n, k) = 0$ then $2^k \subseteq 2$ and $f_k^n$ is surjective [3, Proposition 4.5]. If $d(n, k) = 1$ then $\left\lceil \frac{n-2^k}{2^k} \right\rceil = 1 + m(n, k)$, where $4 \mid m(n, k)$. Thus any $Q_2^{(k)}(\lambda)$ contains a partition with odd cardinality; this partition is 1-good, by Lemma 3.3. Again $f_k^n$ is surjective.

It is an immediate consequence of Theorem 3.5 that $f_k^n$ is regular on its image for all relevant choices of $n, k$ such that $2^k < n$. We have:

Corollary 3.8. Let $n \in \mathbb{N}, k \in \mathbb{N}_0$ be such that $2^k < n$; set $d = \nu_2\left(\left\lceil \frac{n}{2^k} \right\rceil\right)$. Let $\lambda \vdash_o n - 2^k$. Then

$$e(\lambda, 2^k) = \begin{cases} 2 & \text{if } d > 0, \text{ and the } k\text{th row of } Q_2(\lambda) \text{ contains a } d\text{-good partition;} \\ 0 & \text{otherwise.} \end{cases}$$
Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take \( k = 2 \) above. For \( n > 2^2 \) we first compute \( d(n, k) = \nu_2 \left( \left\lfloor \frac{n}{2^k} \right\rfloor \right) \), and then consider odd partitions of \( n - 4 \) and their 4-extensions. For \( n = 6, \) \( d(6, 2) = 0 \). Thus \( e(2, 4) = 4 \). The odd 4-extensions of \( (2) \) are \((6), (3^2), (2^2, 1^2), (2, 1^4)\). For \( n = 10, \) \( d(10, 2) = 1 \). In this case, \( e(\lambda, 4) = 2 \) for all odd partitions \( \lambda \) of \( 6 \). For instance, the odd 4-extensions of \( (6) \) are \((10)\) and \((6, 3, 1)\). For \( n = 19, \) \( d(19, 2) = 2 \). Example 2.6 shows that for \( \lambda = (5, 4, 2^2, 1^2) \vdash_o 15 \) there is no 2-good partition in \( Q_2(\lambda) \), hence \( e(\lambda, 4) = 0 \).

4 Deciding commutativity of the maps \( f_k \) and \( f_\ell \)

Let \( n \in \mathbb{N} \), and suppose that \( 0 \leq k < \ell \) satisfy \( 2^k + 2^\ell \leq n \). As stated in the introduction, we want to complete the discussion of the commutativity of the maps \( f_k \) and \( f_\ell \). Since the relevant \( n \) will always be apparent for the maps \( f_k^n \) in this section, we just write \( f_k \).

We write \((n; k, \ell) \in \mathcal{T} \) if for all \( \lambda \vdash_o n \) we have \( f_k f_\ell (\lambda) = f_\ell f_k (\lambda) \). Otherwise we write \((n; k, \ell) \in \mathcal{F} \).

In this section we will prove Theorem B, which may be reformulated as follows.

Theorem 4.1. Let \( n = 2^t + m \) where \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). Then with the exception of \((6; 0, 1)\)

\[
(n; k, \ell) \in \mathcal{F} \text{ if and only if } \ell < t \text{ and } 2^k \leq m.
\]

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two extreme cases, where \( f_k \) and \( f_\ell \) commute.

In the case \( \ell = t \) we have the following result as a reformulation of [3, Proposition 4.3].

Lemma 4.2. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). If \( 2^k \leq m \), then \((n; k, t) \in \mathcal{T} \).

It is also known that in the case where \( n \) is a power of 2, the maps \( f_k \) and \( f_\ell \) commute [3, Remark 4.4], and we include a short proof here.

Lemma 4.3. If \( n = 2^t \) then \((n; k, \ell) \in \mathcal{T} \) for all \( k, \ell \).

Proof. If \( 0 \leq b \leq a \) are integers then the binomial coefficient \( \binom{a}{b} \) is odd if and only if \( b \subseteq_2 a \), by Lucas’ theorem. The odd partitions of \( 2^t \) are exactly the hook partitions \( (2^t - b, 1^b) \), \( 0 \leq b \leq 2^t - 1 \), of degree \( \left( \frac{2^{t-1}}{b} \right) \). Hence for \( k \in \{0, 1, \ldots, t - 1\} \) we have

\[
f_k (\lambda) = \begin{cases} 
(2^t - b - 2^k, 1^b) & \text{if } 2^k \nsubseteq_2 b, \\
(2^t - b - 1^{b-2^k}) & \text{if } 2^k \subseteq_2 b.
\end{cases}
\]

It follows that for any \( k, \ell < t \) and odd partition \( \lambda \) of \( 2^t \), we have \( f_\ell f_k (\lambda) = f_k f_\ell (\lambda) \).

Lemma 4.4. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). If \( m < 2^k \) then \((n; k, \ell) \in \mathcal{T} \).

Proof. We use induction on \( k \geq 0 \). For \( k = 0 \) we have \( m = 0 \) and the claim follows from Lemma 4.3. Suppose that \( k \geq 1 \) and that the claim has been proved up to \( k - 1 \). Let \( \lambda \vdash_o n \).

Odd hooks of length \( 2^k \) and \( 2^\ell \) in \( \lambda \) correspond to odd hooks of length \( 2^{k-1} \) and \( 2^{t-1} \) in the 2-quotient \( Q_2(\lambda) = (\lambda_0, \lambda_1) \) of \( \lambda \). From Theorem 2.5 we deduce that \(|\lambda_0|\) and \(|\lambda_1|\) are 2-disjoint binary subsums of \( \left\lfloor \frac{n}{2^t} \right\rfloor \), so one of them contains \( 2^{t-1} \), say \(|\lambda_0|\); then \(|\lambda_1| \leq \left\lfloor \frac{m}{2^t} \right\rfloor < 2^{k-1} < 2^{t-1} \).

Thus the odd \( 2^{k-1} \)-hook in \( Q_2(\lambda) \) has to be in \( \lambda_0 \). Therefore

\[
Q_2(f_k(\lambda)) = (f_{k-1}(\lambda_0), \lambda_1).
\]
Applying $f_\ell$, the odd $2^{\ell-1}$-hook cannot be in $\lambda_1$, hence
\[ Q_2(f_\ell f_k(\lambda)) = (f_{\ell-1}f_{k-1}(\lambda_0), \lambda_1)). \]

In particular, we know that $|\lambda_0| \geq 2^{\ell-1} + 2^{k-1}$. Also $|\lambda_0| + |\lambda_1| = \left\lfloor \frac{m}{2} \right\rfloor = 2^{\ell-1} + \left\lfloor \frac{m}{2} \right\rfloor$. We have already seen that $2^{\ell-1}$ is the largest binary digit of $|\lambda_0|$; furthermore $|\lambda_0| - 2^{\ell-1}$ is a binary subsum of $\left\lfloor \frac{m}{2} \right\rfloor < 2^{k-1}$. We may therefore apply the inductive hypothesis to $\lambda_0$ to get $f_{\ell-1}f_{k-1}(\lambda_0) = f_{k-1}f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_kf_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_kf_\ell(\lambda) = f_\ell f_k(\lambda)$. \qed

Lemmas 4.2 and 4.4 show that the only if part of the theorem is true. We now turn to the if part. We start by proving the statement for $k = 0$ and use this as part of an inductive argument.

**Lemma 4.5.** Let $n = 2^t + m$ with $0 < m < 2^t$. If $0 < \ell < t$ then $(n; 0, \ell) \in \mathcal{F}$, with the exception of $(6; 0, 1)$.

**Proof.** The result is easily checked for $n \leq 8$, which includes the exception $(6; 0, 1)$. So we assume that $t \geq 3$.

Case 1: $2^\ell < m$. Then $m \geq 3$, since $\ell > 0$. Consider the partition $\lambda = (m, m, 1^a) \vdash n$ where $a = n - 2m = 2^t - m$. The $(1, 1)$-hook length of $\lambda$ is $2^t + 1$. The $(2, 1)$-hook length of $\lambda$ is $2^t$.

Removing the $(2, 1)$-hook we get the odd partition $(m)$, so $\lambda$ is odd, by Lemma 2.8. We claim that
\[ f_0(\lambda) = (m, m, 1^{a-1}). \]

Indeed we cannot have $f_0(\lambda) = (m, m - 1, 1^a)$ because this partition does not have a hook of length $2^t$, and thus it is not odd. Now
\[ f_\ell(f_0(\lambda)) = f_\ell(m, m, 1^{a-1}) = (m, m - 2^t, 1^{a-1}) \]

since $(m, m, 1^{a-1-2^t})$ and $(m - 1, m - 2^t + 1, 1^{a-1})$ both do not have a hook of length $2^t$ and thus are not odd (again by Lemma 2.8).

On the other hand,
\[ f_\ell(\lambda) = (m - 1, m - (2^t - 1), 1^a). \]

Indeed, the other candidates for $f_\ell(\lambda)$, which are $(m, m - 2^t, 1^a)$ and $(m, m, 1^{a-2^t})$, do not have hooks of length $2^t$. Then
\[ f_0(f_\ell(\lambda)) = f_0(m - 1, m - (2^t - 1), 1^a) = (m - 1, m - 2^t, 1^a). \]

This follows (again) by observing that all the other partitions of $n - 2^t - 1$ obtained from $(m - 1, m - (2^t - 1), 1^a)$ by removing a node do not have hooks of length $2^t$. Thus $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

Case 2: $m < 2^t$. Consider the partition $\lambda = (n - 2^t, m + 1, 1^a)$, where $a = 2^t - (m + 1)$. Note that $n - 2^t \geq m + 1$ since $\ell < t$ by assumption, and that $a \geq 0$. The $(1, 1)$-hook length of $\lambda$ is $n - m = 2^t$. Removing this hook we get the odd partition $(m)$, so $\lambda$ is odd. The $(2, 1)$-hook length of $\lambda$ is $2^t$. Now
\[ f_0(\lambda) = (n - 2^t, m, 1^a) \]

since the other candidates do not have hooks of length $2^t$. Then
\[ f_\ell(f_0(\lambda)) = f_\ell(n - 2^t, m, 1^a) = \mu, \]
where \( \mu \) is obtained from \( f_0(\lambda) \) by removing a \( 2^\ell \)-hook in the first row. (There are only hooks of length \(< 2^i \) in the other rows.) In fact, \( \mu = (n - 2^{\ell+1}, m, 1^a) \) since \( n - 2^{\ell+1} \geq n - 2^\ell = m \). Thus \( f_\ell(f_0(\lambda)) \) has at least 2 parts. On the other hand

\[
f_\ell(\lambda) = (n - 2^\ell)
\]

since this odd partition is obtained from the odd partition \( \lambda \) by removing a \( 2^\ell \)-hook (the one in (2,1)). It follows that

\[
f_0(f_\ell(\lambda)) = (n - 2^\ell - 1)
\]

and again \( f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda)) \).

**Case 3:** \( m = 2^\ell \). Then \( n = 2^\ell + 2^\ell \). If \( \ell \geq 2 \) then choose \( \lambda = (2^\ell, 2^\ell - 1, 1) \). The \((1,2)\)-hook length of \( \lambda \) is \( 2^\ell \); thus \( \lambda \) is an odd partition since removing this \( 2^\ell \)-hook gives an odd partition \((2^\ell - 2, 1, 1)\) of \( 2^\ell \). We have \( f_0(\lambda) = (2^\ell, 2^\ell - 2, 1) \) since the other candidates are not odd. Then

\[
f_\ell(f_0(\lambda)) = (2^\ell - 2^\ell, 2^\ell - 2, 1).
\]

The \((2,1)\)-hook length of \( \lambda \) is \( 2^\ell \), so \( f_\ell(\lambda) = (2^\ell) \) and

\[
f_0(f_\ell(\lambda)) = (2^\ell - 1),
\]

showing \( f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda)) \).

On the other hand, if \( \ell = 1 \) then choose \( \lambda = (2^\ell - 2, 2, 2 \upharpoonright_o 2^\ell + 2 = n \). Since \( t \geq 3 \), it is now easy to show that \( f_1(f_0(\lambda)) = (2^\ell - 4, 2, 1) \). On the other hand we see that \( f_0(f_1(\lambda)) \) is a hook partition of \( 2^\ell - 1 = n - 3 \) and therefore is not equal to \( f_1(f_0(\lambda)) \).

**Lemma 4.6.** If \((n; k, \ell) \in \mathcal{F} \) then also \((2n; k + 1, \ell + 1) \in \mathcal{F} \) and \((2n + 1; k + 1, \ell + 1) \in \mathcal{F} \).

**Proof.** Let the odd partition \( \mu \) of \( n \) satisfy \( f_k f_\ell(\mu) \neq f_\ell f_k(\mu) \). Let \( \lambda \) be a partition of \( 2n \) or \( 2n + 1 \) having 2-quotient \( Q_2(\lambda) = (\mu, (0)) \). Then \( \lambda \) is odd, by Theorem 2.5. We have

\[
Q_2(f_{k+1}f_{\ell+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_\ell f_k(\mu), (0)) = Q_2(f_{\ell+1}f_{k+1}(\lambda)),
\]

so that \( f_{k+1}f_{\ell+1}(\lambda) \neq f_{\ell+1}f_{k+1}(\lambda) \).

We are now ready to conclude this section with the proof of Theorem B.

**Proof of Theorem 4.1.** The only if part follows from Lemmas 4.2 and 4.4. To prove the if part we use induction on \( k \geq 0 \). If \( k = 0 \), then the statement follows from Lemma 4.5. Let \( k > 1 \) and suppose that the assertion is true up to and including \( k - 1 \). To show that \((n; k, \ell) \in \mathcal{F} \) it suffices to prove \((\lceil \frac{n}{2} \rceil; k - 1, \ell - 1) \in \mathcal{F} \), by Lemma 4.6. We are assuming \( n = 2^\ell + m \), \( 0 \leq m < 2^\ell \), \( 0 \leq k < \ell \leq t \) and \( 2^k + 2^\ell \leq n \). This implies \( \lceil \frac{n}{2} \rceil = 2^{\ell-1} + \lceil \frac{m}{2} \rceil \), \( 0 \leq \lceil \frac{m}{2} \rceil < 2^{\ell-1} \) and \( 2^{k-1} + 2^{\ell-1} \leq \lceil \frac{n}{2} \rceil \). We may apply the inductive hypothesis to get \((\lceil \frac{n}{2} \rceil; k-1, \ell-1) \in \mathcal{F} \), and then \((n; k, \ell) \in \mathcal{F} \) except when \((\lceil \frac{n}{2} \rceil; k - 1, \ell - 1) = (6; 0, 1) \). In that case we are considering \((12;1,2) \) or \((13;1,2) \) which are both in \( \mathcal{F} \), by direct computation (consider for example \((6,4,2) \upharpoonright_o 12 \) and \((6,4,3) \upharpoonright_o 13 \), respectively).

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