Abstract. Let \( n \) and \( k \) be natural numbers such that \( 2^k < n \). We study the restriction to \( S_{n-2^k} \) of odd-degree irreducible characters of the symmetric group \( S_n \). This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., Sémin. Lothar. Combin. 75 (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., J. Algebra 478 (2017), 271–282].

Key words: characters of symmetric groups; hooks in partitions

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1 Introduction

Let \( n \) be a natural number, and let \( \chi \) be an irreducible character of odd degree of the symmetric group \( S_n \). Then there exists a unique odd-degree irreducible constituent of the restriction \( \chi_{S_{n-1}} \). This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any \( k \in \mathbb{N} \) such that \( 2^k < n \), there exists a unique odd-degree irreducible constituent \( f_k^n(\chi) \) of \( \chi_{S_{n-2^k}} \) appearing with odd multiplicity. The main goal of this article is to study for all \( n, k \in \mathbb{N} \) the map

\[
 f_k^n: \text{Irr}_2(S_n) \rightarrow \text{Irr}_2(S_{n-2^k}),
\]

naturally defined by Theorem A of [3]. All our results are proved using a description of \( f_k^n \) in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If \( 2^k \) appears in the binary expansion of \( n \) we say that \( 2^k \) is a binary digit of \( n \). Similarly we say that two natural numbers \( m \) and \( n \) are 2-disjoint if they do not have any common binary digit. On the other hand, if \( m \leq n \) and all the binary digits of \( m \) appear in the binary expansion of \( n \), then we say that \( m \) is a binary subsum of \( n \). This will be denoted by \( m \subseteq_2 n \). Let \( \nu_2(n) \) be the exponent of the highest power of 2 dividing the integer \( n \).
A question raised in [3] may be phrased as: For which $n$ and $k$ is $f^n_k$ surjective? The authors showed that $f^n_k$ is surjective whenever $2^k$ is a binary digit of $n$, and they observed that otherwise $f^n_k$ could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

**Theorem A.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $d(n,k) = \nu_2(\left\lfloor \frac{n}{2^k} \right\rfloor)$.

- If $k = 0$ then $f^n_k$ is surjective if and only $d(n,k) \leq 2$.
- If $k > 0$ then $f^n_k$ is surjective if and only $d(n,k) \leq 1$.

Theorem A is a consequence of Theorem 3.5 below, which describes the images of the maps $f^n_k$. For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $2^k < n$ and any $\psi \in \text{Irr}_2(\mathfrak{S}_{n-2^k})$ we define the set

$$\mathcal{E}(\psi, 2^k) = \{ \chi \in \text{Irr}_2(\mathfrak{S}_n) \mid f^n_k(\chi) = \psi \},$$

and set $e(\psi, 2^k) = |\mathcal{E}(\psi, 2^k)|$. We show in Corollary 3.8 that the maps $f^n_k$ are regular on their images. This means that for any $\psi$ in the image of $f^n_k$, the number $e(\psi, 2^k)$ depends only on $n$ and $k$ and not on the specific $\psi$. We also give a complete description of those $\psi \in \text{Irr}_2(\mathfrak{S}_{n-2^k})$ such that $e(\psi, 2^k) = 0$, in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote $f^n_k$ just by $f_k$, when the natural number $n$ is clear from the context. Then, for $k, \ell \in \mathbb{N}_0$, $k < \ell$, such that $2^k + 2^\ell \leq n$, we may ask: when is $f_k f_\ell = f_\ell f_k$? or more specifically: when is $f^n_{n-2^k} f^n_\ell = f^n_{n-2^\ell} f^n_k$? In [3, Proposition 4.3] it was proved that $f_k f_\ell = f_\ell f_k$ whenever $2^{\ell} < n < 2^{\ell+1}$. This is the case $\ell = t$ in our second main result, which answers the question completely.

**Theorem B.** Let $n = 2^t + m$ where $0 \leq m < 2^t$. Suppose that $k$, $\ell$ satisfy $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. Then, with the exception of the case $n = 6$, $k = 0$, $\ell = 1$,

$$f_k f_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.$$

## 2 Notation and background

Let $n$ be a natural number. We let $\text{Irr}(\mathfrak{S}_n)$ denote the set of irreducible characters of $\mathfrak{S}_n$ and $\mathcal{P}(n)$ the set of partitions of $n$. The notation $\lambda \in \mathcal{P}(n)$ is sometimes replaced by $\lambda \vdash n$ and we write $|\lambda| = n$. There is a natural correspondence $\lambda \leftrightarrow \chi_\lambda$ between $\mathcal{P}(n)$ and $\text{Irr}(\mathfrak{S}_n)$. We say then that $\lambda$ labels $\chi_\lambda$. We denote by $\text{Irr}_2(\mathfrak{S}_n)$ the set of irreducible characters of $\mathfrak{S}_n$ of odd degree. If $\chi_\lambda \in \text{Irr}_2(\mathfrak{S}_n)$ we say that $\chi_\lambda$ is an odd character, we call $\lambda$ an odd partition of $n$ and write $\lambda \vdash_o n$. Also the empty partition will be considered as an odd partition.

**Remark 2.1.** Let $n, k$ be such that $2^k < n$. In [3, Theorem A and Proposition 4.2] it is shown that the map $f^n_k : \text{Irr}_2(\mathfrak{S}_n) \rightarrow \text{Irr}_2(\mathfrak{S}_{n-2^k})$ may be described in terms of the odd partitions labelling the odd characters as follows:

$$f^n_k(\chi_\lambda) = \chi_\mu \leftrightarrow \mu \vdash_o n - 2^k \text{ can be obtained from } \lambda \vdash_o n \text{ by removing a } 2^k\text{-hook.}$$

Correspondingly we write (by abuse of notation) $f^n_k(\lambda) = \mu$. In fact when $\lambda$ is odd, there is only one $2^k$-hook of $\lambda$ whose removal leads again to an odd partition; we will refer to such a hook as an odd hook of $\lambda$. This combinatorial description of $f^n_k$ will be used throughout this paper, and we will regard $f^n_k$ also as a map between the corresponding sets of odd partitions. Also, for $\mu \vdash_o n - 2^k$ we set $e(\mu, 2^k) = e(\chi_\mu, 2^k)$.
We need some concepts and basic facts concerning hooks in partitions. For any integer \( e \in \mathbb{N} \) we denote by \( C_e(\lambda) \) and \( Q_e(\lambda) \) the \( e \)-core and the \( e \)-quotient of \( \lambda \), respectively. Then 
\[
Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1})
\]
is an \( e \)-tuple of partitions satisfying 
\[
n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i|.
\]
It is well-known that a partition is uniquely determined by its \( e \)-core and \( e \)-quotient (we refer the reader to [6] or [4, Chapter 2.7] for a detailed discussion on this topic).

Let \( \mathcal{H}_e(\lambda) \) be the set of hooks of \( \lambda \) having length divisible by \( e \), and let \( \mathcal{H}(Q_e(\lambda)) = \bigcup_{e=1}^{k-1} \mathcal{H}(\lambda_i) \).

As explained in [6, Theorem 3.3], there is a bijection between \( \mathcal{H}_e(\lambda) \) and \( \mathcal{H}(Q_e(\lambda)) \) mapping hooks in \( \lambda \) of length \( ex \) to hooks in the quotient of length \( x \). Moreover, the bijection respects the process of hook removal. Namely, the partition \( \mu \) obtained by removing an \( ex \)-hook from \( \lambda \) is such that \( C_e(\mu) = C_e(\lambda) \) and the \( e \)-quotient of \( \mu \) is obtained by removing an \( x \)-hook from one of the partitions involved in \( Q_e(\lambda) \).

For \( e = 2 \) we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-quotient tower \( Q_2(\lambda) \) and the 2-core tower \( C_2(\lambda) \) of \( \lambda \). They have rows numbered by \( k \geq 0 \).

The \( k \)th row \( Q_2^{(k)}(\lambda) \) of \( Q_2(\lambda) \) contains \( 2^k \) partitions \( \lambda^{(k)}_i \), \( 0 \leq i \leq 2^k - 1 \), and the \( k \)th row \( C_2^{(k)}(\lambda) \) of \( C_2(\lambda) \) contains the 2-cores of these partitions in the same order, i.e., \( C_2^{(k)}(\lambda^{(k)}_i) \), \( 0 \leq i \leq 2^k - 1 \).

The 0th row of \( Q_2(\lambda) \) contains \( \lambda = \lambda^{(0)}_0 \) itself, row 1 contains the partitions \( \lambda^{(1)}_0, \lambda^{(1)}_1 \) occurring in the 2-quotient \( Q_2(\lambda) \), row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have \( Q_2^{(k)}(\lambda^{(k)}_i) = (\lambda^{(k+1)}_{2i}, \lambda^{(k+1)}_{2i+1}) \) for \( i \in \{0, 1, \ldots, 2^k - 1\} \). We remark that the \( 2^k \) partitions in \( Q_2^{(k)}(\lambda) \) are the same as those in the \( 2^k \)-quotient \( Q_{2^k}(\lambda) \) of \( \lambda \), but in a different order for \( k \geq 2 \).

We also introduce the \( k \)-data \( D_2^{(k)}(\lambda) \) of \( \lambda \). This is a table containing the following \( k+1 \) rows: the \( k \) rows \( C_2^{(j)}(\lambda) \), \( j = 0, \ldots, k - 1 \), and in addition the row \( Q_2^{(k)}(\lambda) \).

**Remark 2.2.** A partition \( \lambda \) may be recovered from its 2-core tower. For \( k > 0 \), it may also be recovered from the knowledge of the \( k \)-data \( D_2^{(k)}(\lambda) \) of \( \lambda \), because the rows \( C_2^{(l)}(\lambda) \) with \( l \geq k \) of \( C_2(\lambda) \) consist of the 2-core towers of the partitions in \( Q_2^{(k)}(\lambda) \).

**Lemma 2.3.** Suppose that \( \lambda \vdash n - 2^k \) and \( \mu \vdash n \). The following are equivalent.

(i) \( \lambda \) is obtained from \( \mu \) by removing a \( 2^k \)-hook.

(ii) The \( k \)-data \( D_2^{(k)}(\mu) \) and \( D_2^{(k)}(\lambda) \) coincide, except that for one \( i \in \{0, \ldots, 2^k - 1\} \) \( \lambda^{(k)}_i \) is obtained from \( \mu^{(k)}_i \) by removing a 1-hook.

**Proof.** A \( 2^k \)-hook \( H_0 \) in \( \mu \) corresponds in a canonical way to a \( 2^{k-1} \)-hook \( H_1 \) in a partition in \( Q_2^{(1)}(\mu) \), i.e., in row 1 of the 2-quotient tower \( Q_2(\mu) \). Continuing we see that \( H_0 \) corresponds in a canonical way to a 1-hook \( H_k \) in a partition \( \mu^{(k)} \) in \( Q_2^{(k)}(\mu) \), row \( k \) of \( Q_2(\mu) \). If \( \lambda \) is obtained by removing \( H_0 \) from \( \mu \), this corresponds to \( \lambda^{(k)}_i \) being obtained by removing the 1-hook \( H_k \) from \( \mu^{(k)}_i \) (by repeated applications of [6, Theorem 3.3]). Apart from this the rows \( Q_2^{(k)}(\mu) \) and \( Q_2^{(k)}(\lambda) \) coincide. Note also that the rows \( C_2^{(j)}(\mu) \) and \( C_2^{(j)}(\lambda) \) coincide for \( j = 0, \ldots, k - 1 \), since the removal of the hooks \( H_j \) of even length do not change the 2-cores.

Odd-degree characters of \( \mathfrak{S}_n \) and thus odd partitions were completely described in [5]. We restate this result in a language which is convenient for our purposes. We let \( c_2^{(k)}(\lambda) \) be the sum of the cardinalities of the partitions in the \( k \)th row \( C_2^{(k)}(\lambda) \) of \( C_2(\lambda) \).

**Lemma 2.4 ([5]).** Let \( \lambda \) be a partition. Then \( \lambda \) is odd if and only if \( c_2^{(k)}(\lambda) \leq 1 \) for all \( k \geq 0 \).

It may be decided from the \( k \)-data \( D_2^{(k)}(\lambda) \) whether \( \lambda \) is odd. The case \( k = 1 \) of the following result appeared in [3, Lemma 4.1] and also in [1, Lemma 6].
**Theorem 2.5.** Let \( \lambda \vdash n \), and let \( k \geq 0 \) be fixed. Consider \( Q_2^{(k)}(\lambda) = (\lambda_i^{(k)}) \). Then \( \lambda \) is odd if and only if the following conditions are all fulfilled:

(i) \( c_2^{(j)}(\lambda) \leq 1 \) for all \( j < k \).

(ii) The partitions \( \lambda_i^{(k)} \), \( 0 \leq i \leq 2^k - 1 \), are all odd.

(iii) The numbers \( |\lambda_i^{(k)}| \), \( 0 \leq i \leq 2^k - 1 \), are pairwise 2-disjoint.

In this case \( \sum_{i \geq 0} |\lambda_i^{(k)}| = \left\lfloor \frac{n}{2^k} \right\rfloor \).

**Proof.** This is proved by induction on \( k \geq 0 \), using Remark 2.2 and Lemma 2.4. \( \blacksquare \)

We illustrate the result above by giving an example.

**Example 2.6.** Let \( n = 15 \) and take \( \lambda = (5, 4, 2^2, 1^2) \vdash 15 \). To decide whether \( \lambda \) is odd, we choose \( k = 2 \) and compute the 2-data \( D_2^{(2)}(\lambda) \). The 2-core is \( C_2(\lambda) = (1) \), giving \( C_2^{(0)}(\lambda) = ((1)) \). Furthermore, the 2-quotient is \( Q_2(\lambda) = ((2^2, 1^2), (1)) \), and computing the 2-cores \( C_2((2^2, 1^2)) = (0), C_2((1)) = (1) \), we obtain the next row: \( C_2^{(1)}(\lambda) = ((0), (1)) \). The 2-quotients are \( Q_2((2^2, 1^2)) = ((1^2), (1)), Q_2((1)) = ((0), (0)) \); hence the final row of the 2-data table is obtained as \( Q_2^{(2)}(\lambda) = ((1^2), (1), (0), (0)) \).

We visualize \( D_2^{(2)}(\lambda) \) like this:

\[
\begin{array}{cccc}
C_2^{(0)}(\lambda): & (1) \\
C_2^{(1)}(\lambda): & (0) & (1) \\
Q_2^{(2)}(\lambda): & (1^2) & (1) & (0) & (0)
\end{array}
\]

Theorem 2.5 shows that \( \lambda \) is odd and thus it contains a unique odd 4-hook. Again using the theorem, it is clear that removing this 4-hook corresponds to the second partition \((1)\) in \( Q_2^{(2)} \) being replaced by \((0)\). Thus, removing the corresponding 4-hook of \( \lambda \) we obtain the odd partition \( \mu = (3, 2^4, 1^2) \vdash 11 \) with the property that \( D_2^{(2)}(\lambda) \) and \( D_2^{(2)}(\mu) \) differ only in their final row.

**Remark 2.7.** Using the construction of partitions from their 2-cores and 2-quotients already mentioned, the criterion above can be applied to construct all odd partitions of \( n \) with a specific \( k \)th row in the 2-quotient tower. For this, let \( n, k \in \mathbb{N} \), and take any sequence of odd partitions \( \nu_i \), \( 0 \leq i \leq 2^k - 1 \), such that the numbers \( |\nu_i| \) are pairwise 2-disjoint, and \( \sum_{i \geq 0} |\nu_i| = \left\lfloor \frac{n}{2^k} \right\rfloor \).

Then there are exactly \( \prod_{m < k, 2^m \leq 2^n} 2^m \) odd partitions \( \lambda \) of \( n \) with \( Q_2^{(k)}(\lambda) = (\nu_i) \), obtained by choosing one 2-core in row \( m \) of the \( k \)-data table to be \((1)\), for each \( m < k \) such that \( 2^m \leq 2^n \).

The following easy consequence of Theorem 2.5 will be used repeatedly.

**Lemma 2.8.** Let \( 2^t \) be the largest binary digit of \( n \). A partition \( \lambda \) of \( n \) is odd if and only if \( \lambda \) contains a unique \( 2^t \)-hook and the partition obtained from \( \lambda \) by removing this \( 2^t \)-hook is an odd partition of \( n - 2^t \).

### 3 Surjectivity and regularity

The aim of this section is to study the images of the maps \( f_k^n \) for all \( n, k \) such that \( 2^k \leq n \). For this purpose we introduce the concept of \( d \)-good partitions (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when \( f_k^n \) is surjective) and to show that the maps \( f_k^n \) are always regular on their image (see Corollary 3.8).
**Definition 3.1.** Let \( d \geq 0 \). We call an odd partition \( \lambda \) \( d \)-good, if

1. \( |\lambda| \equiv 2^d - 1 \mod 2^{d+1} \).
2. \( C_{2d}(\lambda) \) is a hook partition.

Let us remark that condition (i) may be reformulated as

\[(i^*) \quad \nu_2(|\lambda| + 1) = d.\]

In particular, if \( \lambda \) is \( d \)-good, then \( |\lambda| \) is odd if and only if \( d > 0 \).

The relevance of \( d \)-good partitions in our context is illuminated by the following reformulation of [1, Theorem 2]:

**Lemma 3.2.** Let \( \lambda \vdash_o n \). Let \( d = \nu_2(n + 1) \). Then \( e(\lambda, 1) \neq 0 \) if and only if \( \lambda \) is \( d \)-good. In this case, \( e(\lambda, 1) = 1 \) if \( d = 0 \), and \( e(\lambda, 1) = 2 \) if \( d > 0 \).

**Lemma 3.3.** Let \( \lambda \) be an odd partition, and let \( d \geq 0 \). Then the following hold.

1. (For \( d \leq 2 \), \( \lambda \) is \( d \)-good if and only if \( |\lambda| \equiv 2^d - 1 \mod 2^{d+1} \).
2. (If \( \lambda \) is \( d \)-good, then \( C_{2d}(\lambda) \) is a partition of \( 2^d - 1 \).

**Proof.** If the odd partition \( \lambda \) is \( d \)-good, then \( |\lambda| = (2^d - 1) + m \) where the binary digits of \( m \) are at least \( 2^{d+1} \). The hooks of \( \lambda \) corresponding to the binary digits of \( m \) may be decomposed into \( 2^d \)-hooks and thus do not contribute to \( C_{2d}(\lambda) \). Thus \( |C_{2d}(\lambda)| = 2^d - 1 \). This shows (2).

For \( d = 0, 1, 2 \) we have \( |C_{2d}(\lambda)| = 0, 1 \) and \( 3 \), respectively. Since all partitions of \( 0, 1 \) and \( 3 \) are hook partitions, (1) follows.

**Definition 3.4.** If \( 2^k \leq n \), we define \( d(n, k) = \nu_2(\lfloor \frac{n}{2^k} \rfloor) \). Thus \( d(n, k) \) is the smallest integer \( d \geq 0 \) satisfying the condition \( 2^{k+d} \leq 2 n \). In particular, \( d(n, k) = 0 \) if and only if \( 2^k \leq 2 n \). Moreover, we may write \( \lfloor \frac{n}{2^k} \rfloor = 2^{d(n,k)} + m(n,k) \) where \( 2^{d(n,k)+1} | m(n,k) \).

As mentioned in the introduction, the results in [3] show that \( f_k^n \) is a surjective \((2^k\text{-to-}1)\)-map whenever \( 2^k \leq 2 n \), i.e., \( d(n, k) = 0 \). In the spirit of [1, Theorem 2], we now give a characterization of the image of the map \( f_k^n \) for all \( n, k \) such that \( 2^k < n \).

**Theorem 3.5.** Let \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) be such that \( 2^k < n \). Let \( \lambda \vdash_o n-2^k \). Then \( e(\lambda, 2^k) \neq 0 \) if and only if there exists a \( d(n,k) \)-good partition in the \( k \)th row of \( Q_2(\lambda) \). In this case, \( e(\lambda, 2^k) = 2k \) if \( d(n,k) = 0 \), and \( e(\lambda, 2^k) = 2 \) if \( d(n,k) > 0 \).

**Proof.** If \( k = 0 \) then the statement follows from Lemma 3.2. Hence assume that \( k \geq 1 \). Let \( d = d(n, k) \). By assumption \( \lfloor \frac{n}{2^k} \rfloor = 2^d + m \), where the binary digits of \( m \) are at least \( 2^{d+1} \). Thus \( \lfloor \frac{n-2^k}{2^k} \rfloor = (2^d - 1) + m \).

Suppose first that \( e(\lambda, 2^k) \neq 0 \) and that \( \mu \vdash_o n \) satisfies \( f_k(\mu) = \lambda \). From Remark 2.1 and Lemma 2.3 we get that there exists an \( i \in \{0, 1, \ldots, 2^k - 1\} \) such that \( f_0(\mu_i^{(k)}) = \lambda_i^{(k)} \). Since \( \mu_i^{(k)} \) and \( \lambda_i^{(k)} \) are odd, we get \( e(\lambda_i^{(k)}, 1) \neq 0 \). We have that \( \lambda_i^{(k)} \) and \( |\mu_i^{(k)}| \) are both \( 2 \)-disjoint with \( m_1 := \sum_{j \neq i} |\lambda_j^{(k)}| = \sum_{j \neq i} |\mu_j^{(k)}| \leq 2 \lfloor \frac{n-2^k}{2^k} \rfloor \), by Theorem 2.5. Since \( m_1 \subseteq 2 \lfloor \frac{n-2^k}{2^k} \rfloor \) and \( m_1 \subseteq 2 \lfloor \frac{n}{2^k} \rfloor \), we get \( m_1 \leq 2 m \). Thus \( |\lambda_i^{(k)}| = (2^d - 1) + m_2 \) and \( |\mu_i^{(k)}| = 2^d + m_2 \), where \( m_2 = m - m_1 \leq 2 m \).

In particular \( \nu_2(|\lambda_i^{(k)}| + 1) = \nu_2(|\mu_i^{(k)}|) = d \). Then Lemma 3.2 shows that \( \lambda_i^{(k)} \) is \( d \)-good.

Conversely, if \( \lambda_i^{(k)} \) is a \( d \)-good partition for some \( i \in \{0, 1, \ldots, 2^k - 1\} \), then there exists a \( \mu^* \vdash_o |\lambda_i^{(k)}| + 1 \) such that \( f_0(\mu^*) = \lambda_i^{(k)} \), by Lemma 3.2. We let \( \mu \) be the partition where the \( k \)-data \( D_2^{(k)}(\mu) \) and \( D_2^{(k)}(\lambda) \) coincide, except that \( \mu_i^{(k)} = \mu^* \). Since \( \lambda \) is odd and \( \lambda_i^{(k)} \) is \( d \)-good,
we know that $|\lambda_i^{(k)}| = (2^d - 1) + m'$ where $m' \subseteq_2 m$, and $|\lambda_j^{(k)}| \subseteq_2 m - m'$ for all $j \neq i$. Hence $|\mu'| = |\lambda_i^{(k)}| + 1 = 2^d + m'$ is 2-disjoint from all $|\lambda_j^{(k)}|, j \neq i$. Thus $\mu$ is an odd partition of $n$ by Theorem 2.5, and $f_k(\mu) = \lambda$ by Lemma 2.3 and Remark 2.1.

We conclude that $e(\lambda, 2^k) = \sum_{d \text{-good}} e(\lambda_i^{(k)}, 1)$. If $d = 0$ then $\lfloor \frac{n - 2^k}{2^k} \rfloor$ is even. This implies that all $\lambda_i^{(k)}$ are of even cardinality and thus $d$-good. Thus $e(\lambda_i^{(k)}, 1) = 1$ for all $i$, and we get $e(\lambda, 2^k) = 2^k$. If $d > 0$ there is exactly one $\lambda_i^{(k)}$ in $Q_2^{(k)}(\lambda)$ of odd cardinality. Only this $\lambda_i^{(k)}$ may be $d$-good and then $e(\lambda, 2^k) = e(\lambda_i^{(k)}, 1) = 2$. Otherwise $e(\lambda, 2^k) = 0$. □

**Corollary 3.6.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$, and let $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash_0 n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a partition $\lambda_i^{(k)}$ in the $k$th row of $Q_2(\lambda)$ such that $|\lambda_i^{(k)}| \equiv 2^d - 1 \pmod{2^{d+1}}$, and $C_{2d}(\lambda_i^{(k)})$ is a hook partition. In this case, $e(\lambda, 2^k) = 2^k$ if $d = 0$, and $e(\lambda, 2^k) = 2$ if $d > 0$.

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

**Corollary 3.7 (Theorem A).** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$.

- If $k = 0$ then $f^n_k$ is surjective if and only if $d(n, k) \leq 2$.
- If $k > 0$ then $f^n_k$ is surjective if and only if $d(n, k) \leq 1$.

**Proof.** By Theorem 3.5, $f^n_k$ is surjective if and only if for all $\lambda \vdash_0 n - 2^k$ we have that the $k$th row of $Q_2(\lambda)$ contains a $(d(n, k), 0)$-good partition $\lambda_i^{(k)}$. By Theorem 2.5 and Definition 3.4, for any $\lambda \vdash_0 n - 2^k$ we have $\sum_{j \geq 0} |\lambda_j^{(k)}| = \lfloor \frac{n - 2^k}{2^k} \rfloor = (2^{d(n, k)} - 1) + m(n, k)$.

If $k = 0$ then $Q_2^{(0)}(\lambda)$ contains only $\lambda = \lambda_0^{(0)}$. Hence $f^n_0$ is surjective if and only all odd partitions of $n - 1$ are $d(n, 0)$-good. By Lemma 3.3(1), the latter condition holds when $d = d(n, 0) \leq 2$. On the other hand, if $d = \nu_2(n) > 2$, then $\lambda = (n - 5, 2, 2)$ is an odd partition of $n - 1$ by Theorem 2.5, but $C_8(\lambda) = (3, 2, 2)$ is not a hook, and hence $C_{2d}(\lambda)$ is not a hook. So $\lambda$ is not $d$-good, and thus $f^n_0$ is not surjective.

Now assume $k \geq 1$. Then $Q_2^{(k)}(\lambda)$ contains at least two odd partitions. If $d(n, k) \geq 2$ then any $(d(n, k), 0)$-good partition $\mu$ satisfies $3 \subseteq_2 2^{d(n, k)} - 1 \subseteq_2 |\mu|$. Write $\lfloor \frac{n - 2^k}{2^k} \rfloor = 1 + m_1$ where $m_1$ is even. Applying Remark 2.7, take any $\lambda \vdash_0 n - 2^k$ such that $|\lambda^{(k)}_0| = 1$ and $\lambda^{(k)}_1$ is an odd partition with $|\lambda^{(k)}_1| = m_1$. Then no partition in $Q_2^{(k)}(\lambda)$ is $(d(n, k), 0)$-good. Thus $f^n_k$ is not surjective. On the other hand, if $d(n, k) = 0$ then $2^k \subseteq_2 n$ and $f^n_k$ is surjective [3, Proposition 4.5]. If $d(n, k) = 1$ then $\lfloor \frac{n - 2^k}{2^k} \rfloor = 1 + m(n, k)$, where $4 \mid m(n, k)$. Thus any $Q_2^{(k)}(\lambda)$ contains a partition with odd cardinality; this partition is 1-good, by Lemma 3.3. Again $f^n_k$ is surjective. □

It is an immediate consequence of Theorem 3.5 that $f^n_k$ is regular on its image for all relevant choices of $n, k$ such that $2^k < n$. We have:

**Corollary 3.8.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$; set $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash_0 n - 2^k$.

Then

$$e(\lambda, 2^k) = \begin{cases} 2^k & \text{if } d = 0; \\ 2 & \text{if } d > 0, \text{ and the } k\text{th row of } Q_2(\lambda) \text{ contains a } d\text{-good partition}; \\ 0 & \text{otherwise}. \end{cases}$$
Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take \( k = 2 \) above. For \( n > 2^2 \) we first compute \( d(n, k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) \), and then consider odd partitions of \( n - 4 \) and their 4-extensions. For \( n = 6, d(6, 2) = 0 \). Thus \( e(2, 4) = 4 \). The odd 4-extensions of \( (2) \) are \((6), (3^2), (2^2, 1^2), (2, 1^4)\). For \( n = 10, d(10, 2) = 1 \). In this case, \( e(\lambda, 4) = 2 \) for all odd partitions \( \lambda \) of 6. For instance, the odd 4-extensions of \( (6) \) are \((10)\) and \((6, 3, 1)\). For \( n = 19, d(19, 2) = 2 \). Example 2.6 shows that for \( \lambda = (5, 4, 2^2, 1^2) \) \( \vdash_0 15 \) there is no 2-good partition in \( Q_2^{(2)}(\lambda) \), hence \( e(\lambda, 4) = 0 \).

4 Deciding commutativity of the maps \( f_k \) and \( f_\ell \)

Let \( n \in \mathbb{N} \), and suppose that \( 0 \leq k < \ell \) satisfy \( 2^k + 2^\ell \leq n \). As stated in the introduction, we want to complete the discussion of the commutativity of the maps \( f_k \) and \( f_\ell \). Since the relevant \( n \) will always be apparent for the maps \( f_k^n \) in this section, we just write \( f_k \).

We write \((n; k, \ell) \in T\) if for all \( \lambda \vdash_o n \) we have \( f_k f_\ell(\lambda) = f_\ell f_k(\lambda) \). Otherwise we write \((n; k, \ell) \in F\).

In this section we will prove Theorem B, which may be reformulated as follows.

Theorem 4.1. Let \( n = 2^t + m \) where \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). Then with the exception of \((6; 0, 1)\)

\((n; k, \ell) \in F\) if and only if \( \ell < t \) and \( 2^k \leq m \).

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two extreme cases, where \( f_k \) and \( f_\ell \) commute.

In the case \( \ell = t \) we have the following result as a reformulation of [3, Proposition 4.3].

Lemma 4.2. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). If \( 2^k \leq m \), then \((n; k, t) \in T\).

It is also known that in the case where \( n \) is a power of 2, the maps \( f_k \) and \( f_\ell \) commute [3, Remark 4.4], and we include a short proof here.

Lemma 4.3. If \( n = 2^t \) then \((n; k, \ell) \in T\) for all \( k, \ell \).

Proof. If \( 0 \leq b \leq a \) are integers then the binomial coefficient \( \binom{a}{b} \) is odd if and only if \( b \subseteq_2 a \), by Lucas’ theorem. The odd partitions of \( 2^t \) are exactly the hook partitions \( (2^t - b, 1^b) \), \( 0 \leq b \leq 2^t - 1 \), of degree \( (2^{t-1}) \). Hence for \( k \in \{0, 1, \ldots, t-1\} \) we have

\[
 f_k(\lambda) = \begin{cases} 
 (2^t - b - 2^k, 1^b) & \text{if } 2^k \nsubseteq b, \\
 (2^t - b, 1^{b-2^k}) & \text{if } 2^k \subseteq b. 
\end{cases}
\]

It follows that for any \( k, \ell < t \) and odd partition \( \lambda \) of \( 2^t \), we have \( f_\ell f_k(\lambda) = f_k f_\ell(\lambda) \). □

Lemma 4.4. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). If \( m < 2^k \) then \((n; k, \ell) \in T\).

Proof. We use induction on \( k \geq 0 \). For \( k = 0 \) we have \( m = 0 \) and the claim follows from Lemma 4.3. Suppose that \( k \geq 1 \) and that the claim has been proved up to \( k - 1 \). Let \( \lambda \vdash_o n \). Odd hooks of length \( 2^k \) and \( 2^\ell \) in \( \lambda \) correspond to odd hooks of length \( 2^{k-1} \) and \( 2^{\ell-1} \) in the 2-quotient \( Q_2(\lambda) = (\lambda_0, \lambda_1) \) of \( \lambda \). From Theorem 2.5 we deduce that \( |\lambda_0| \) and \( |\lambda_1| \) are 2-disjoint binary subsums of \( \left\lfloor \frac{n}{2^k} \right\rfloor \), so one of them contains \( 2^{\ell-1} \), say \( |\lambda_0| \); then \( |\lambda_1| \leq \left\lfloor \frac{m}{2^{k-1}} \right\rfloor < 2^{k-1} < 2^{\ell-1} \). Thus the odd \( 2^{k-1} \)-hook in \( Q_2(\lambda) \) has to be in \( \lambda_0 \). Therefore

\[
 Q_2(f_k(\lambda)) = (f_{k-1}(\lambda_0), \lambda_1).
\]
Applying $f_\ell$, the odd $2^{\ell-1}$-hook cannot be in $\lambda_1$, hence

$$Q_2(f_\ell f_k(\lambda)) = (f_{\ell-1} f_{k-1}(\lambda_0), \lambda_1).$$

In particular, we know that $|\lambda_0| \geq 2^{\ell-1} + 2^{k-1}$. Also $|\lambda_0| + |\lambda_1| = \left\lfloor \frac{n}{2^\ell} \right\rfloor = 2^{\ell-1} + \left\lfloor \frac{n}{2^k} \right\rfloor$. We have already seen that $2^{\ell-1}$ is the largest binary digit of $|\lambda_0|$; furthermore $|\lambda_0| - 2^{\ell-1}$ is a binary subsum of $\left\lfloor \frac{n}{2^k} \right\rfloor < 2^{k-1}$. We may therefore apply the inductive hypothesis to $\lambda_0$ to get $f_{\ell-1} f_{k-1}(\lambda_0) = f_{k-1} f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_k f_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$.

Lemmas 4.2 and 4.4 show that the only if part of the theorem is true. We now turn to the if part. We start by proving the statement for $k = 0$ and use this as part of an inductive argument.

**Lemma 4.5.** Let $n = 2^\ell + m$ with $0 < m < 2^\ell$. If $0 < \ell < t$ then $(n; 0, \ell) \in \mathcal{F}$, with the exception of $(6; 0, 1)$.

**Proof.** The result is easily checked for $n \leq 8$, which includes the exception $(6; 0, 1)$. So we assume that $t \geq 3$.

**Case 1:** $2^\ell < m$. Then $m \geq 3$, since $\ell > 0$. Consider the partition $\lambda = (m, m, 1^a) \vdash n$ where $a = n - 2m = 2^\ell - m$. The $(1,1)$-hook length of $\lambda$ is $2^{\ell+1}$. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Removing the $(2,1)$-hook hook we get the odd partition $(m)$, so $\lambda$ is odd, by Lemma 2.8. We claim that

$$f_0(\lambda) = (m, m, 1^{a-1}).$$

Indeed we cannot have $f_0(\lambda) = (m, m - 1, 1^a)$ because this partition does not have a hook of length $2^\ell$, and thus it is not odd. Now

$$f_\ell(f_0(\lambda)) = f_\ell(m, m, 1^{a-1}) = (m, m - 2^\ell, 1^{a-1})$$

since $(m, m, 1^{a-1-2^\ell})$ and $(m - 1, m - 2^\ell + 1, 1^{a-1})$ both do not have a hook of length $2^\ell$ and thus are not odd (again by Lemma 2.8).

On the other hand,

$$f_\ell(\lambda) = (m - 1, m - (2^\ell - 1), 1^a).$$

Indeed, the other candidates for $f_\ell(\lambda)$, which are $(m, m - 2^\ell, 1^a)$ and $(m, m, 1^{a-2^\ell})$, do not have hooks of length $2^\ell$. Then

$$f_0(f_\ell(\lambda)) = f_0(m - 1, m - (2^\ell - 1), 1^a) = (m - 1, m - 2^\ell, 1^a).$$

This follows (again) by observing that all the other partitions of $n - 2^\ell - 1$ obtained from $(m - 1, m - (2^\ell - 1), 1^a)$ by removing a node do not have hooks of length $2^\ell$. Thus $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

**Case 2:** $m < 2^\ell$. Consider the partition $\lambda = (n - 2^\ell, m + 1, 1^a)$, where $a = 2^\ell - (m + 1)$. Note that $n - 2^\ell \geq m + 1$ since $\ell < t$ by assumption, and that $a \geq 0$. The $(1,1)$-hook length of $\lambda$ is $n - m = 2^\ell$. Removing this hook we get the odd partition $(m)$, so $\lambda$ is odd. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Now

$$f_0(\lambda) = (n - 2^\ell, m, 1^a)$$

since the other candidates do not have hooks of length $2^\ell$. Then

$$f_\ell(f_0(\lambda)) = f_\ell(n - 2^\ell, m, 1^a) = \mu,$$
where \( \mu \) is obtained from \( f_0(\lambda) \) by removing a \( 2^\ell \)-hook in the first row. (There are only hooks of length \( < 2^\ell \) in the other rows.) In fact, \( \mu = (n - 2^{\ell + 1}, m, 1^\alpha) \) since \( n - 2^{\ell + 1} \geq n - 2^\ell = m \). Thus \( f_\ell(f_0(\lambda)) \) has at least 2 parts. On the other hand

\[
f_\ell(\lambda) = (n - 2^\ell)
\]

since this odd partition is obtained from the odd partition \( \lambda \) by removing a \( 2^\ell \)-hook (the one in (2,1)). It follows that

\[
f_0(f_\ell(\lambda)) = (n - 2^\ell - 1)
\]

and again \( f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda)) \).

**Case 3:** \( m = 2^\ell \). Then \( n = 2^\ell + 2^\ell \). If \( \ell \geq 2 \) then choose \( \lambda = (2^\ell, 2^\ell - 1, 1) \). The \( (1,2) \)-hook length of \( \lambda \) is \( 2^\ell \); thus \( \lambda \) is an odd partition since removing this \( 2^\ell \)-hook gives an odd partition \( (2^\ell - 2, 1, 1) \) of \( 2^\ell \). We have \( f_0(\lambda) = (2^\ell, 2^\ell - 2, 1) \) since the other candidates are not odd. Then

\[
f_\ell(f_0(\lambda)) = (2^\ell - 2^\ell, 2^\ell - 2, 1).
\]

The \( (2,1) \)-hook length of \( \lambda \) is \( 2^\ell \), so \( f_\ell(\lambda) = (2^\ell) \) and

\[
f_0(f_\ell(\lambda)) = (2^\ell - 1),
\]

showing \( f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda)) \).

On the other hand, if \( \ell = 1 \) then choose \( \lambda = (2^\ell - 2, 2, 2) \vdash_o 2^\ell + 2 = n \). Since \( t \geq 3 \), it is now easy to show that \( f_1(f_0(\lambda)) = (2^\ell - 4, 2, 1) \). On the other hand we see that \( f_0(f_1(\lambda)) \) is a hook partition of \( 2^\ell - 1 = n - 3 \) and therefore is not equal to \( f_1(f_0(\lambda)) \).

**Lemma 4.6.** If \( (n; k, \ell) \in \mathcal{F} \) then also \( (2n; k + 1, \ell + 1) \in \mathcal{F} \) and \( (2n + 1; k + 1, \ell + 1) \in \mathcal{F} \).

**Proof.** Let the odd partition \( \mu \) of \( n \) satisfy \( f_k f_\ell(\mu) \neq f_\ell f_k(\mu) \). Let \( \lambda \) be a partition of \( 2n \) or \( 2n + 1 \) having \( 2 \)-quotient \( Q_2(\lambda) = (\mu, (0)) \). Then \( \lambda \) is odd, by Theorem 2.5. We have

\[
Q_2(f_{k+1}f_{t+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_\ell f_k(\mu), (0)) = Q_2(f_{t+1}f_{k+1}(\lambda)),
\]

so that \( f_{k+1}f_{t+1}(\lambda) \neq f_{t+1}f_{k+1}(\lambda) \).

We are now ready to conclude this section with the proof of Theorem B.

**Proof of Theorem 4.1.** The only if part follows from Lemmas 4.2 and 4.4. To prove the if part we use induction on \( k \geq 0 \). If \( k = 0 \), then the statement follows from Lemma 4.5. Let \( k > 1 \) and suppose that the assertion is true up to and including \( k - 1 \). To show that \( (n; k, \ell) \in \mathcal{F} \) it suffices to prove \( (\left\lfloor \frac{n}{2} \right\rfloor; k - 1, \ell - 1) \in \mathcal{F} \), by Lemma 4.6. We are assuming \( n = 2^\ell + m \), \( 0 \leq m < 2^\ell \), \( 0 \leq k < \ell \leq t \) and \( 2^k + 2^\ell \leq n \). This implies \( \left\lfloor \frac{n}{2} \right\rfloor = 2^{t-1} + \left\lfloor \frac{m}{2} \right\rfloor \), \( 0 \leq \left\lfloor \frac{m}{2} \right\rfloor < 2^{t-1} \) and \( 2^{k-1} + 2^{t-1} \leq \left\lfloor \frac{n}{2} \right\rfloor \). We may apply the inductive hypothesis to get \( (\left\lfloor \frac{n}{2} \right\rfloor; k-1, \ell-1) \in \mathcal{F} \), and then \( (n; k, \ell) \in \mathcal{F} \) except when \( (\left\lfloor \frac{n}{2} \right\rfloor; k-1, \ell-1) = (6; 0, 1) \). In that case we are considering (12;1,2) or (13;1,2) which are both in \( \mathcal{F} \), by direct computation (consider for example \( (6, 4, 2) \vdash_o 12 \) and \( (6, 4, 3) \vdash_o 13 \), respectively).

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