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Restriction of Odd Degree Characters of $S_n$

Christine BESSENRODT†, Eugenio GIANNELLI ‡ and Jørn B. OLSSON §

† Institute for Algebra, Number Theory and Discrete Mathematics, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany
E-mail: bessen@math.uni-hannover.de

‡ Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge CB3 0WA, United Kingdom
E-mail: eg513@cam.ac.uk

§ Department of Mathematical Sciences, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark
E-mail: olsson@math.ku.dk

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Abstract. Let $n$ and $k$ be natural numbers such that $2^k < n$. We study the restriction to $S_{n-2^k}$ of odd-degree irreducible characters of the symmetric group $S_n$. This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., Sémin. Lothar. Combin. 75 (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., J. Algebra 478 (2017), 271–282].

Key words: characters of symmetric groups; hooks in partitions

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1 Introduction

Let $n$ be a natural number, and let $\chi$ be an irreducible character of odd degree of the symmetric group $S_n$. Then there exists a unique odd-degree irreducible constituent of the restriction $\chi_{S_n \rightarrow S_{n-1}}$. This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any $k \in \mathbb{N}$ such that $2^k < n$, there exists a unique odd-degree irreducible constituent $f^n_k(\chi)$ of $\chi_{S_{n-2^k}}$ appearing with odd multiplicity. The main goal of this article is to study for all $n, k \in \mathbb{N}$ the map

$$f^n_k : \text{Irr}_2(S_n) \rightarrow \text{Irr}_2(S_{n-2^k}),$$

naturally defined by Theorem A of [3]. All our results are proved using a description of $f^n_k$ in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If $2^k$ appears in the binary expansion of $n$ we say that $2^k$ is a binary digit of $n$. Similarly we say that two natural numbers $m$ and $n$ are 2-disjoint if they do not have any common binary digit. On the other hand, if $m \leq n$ and all the binary digits of $m$ appear in the binary expansion of $n$, then we say that $m$ is a binary subsum of $n$. This will be denoted by $m \subseteq_2 n$. Let $v_2(n)$ be the exponent of the highest power of 2 dividing the integer $n$. 

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A question raised in [3] may be phrased as: For which $n$ and $k$ is $f_k^n$ surjective? The authors showed that $f_k^n$ is surjective whenever $2^k$ is a binary digit of $n$, and they observed that otherwise $f_k^n$ could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

**Theorem A.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $d(n,k) = v_2\left(\left\lfloor \frac{n}{2^k}\right\rfloor\right)$.

- If $k = 0$ then $f_k^n$ is surjective if and only if $d(n,k) \leq 2$.
- If $k > 0$ then $f_k^n$ is surjective if and only if $d(n,k) \leq 1$.

Theorem A is a consequence of Theorem 3.5 below, which describes the images of the maps $f_k^n$. For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $2^k < n$ and any $\psi \in \text{Irr}_{2^k}(\mathfrak{S}_{n-2^k})$ we define the set

$$
E(\psi, 2^k) = \{\chi \in \text{Irr}_{2^k}(\mathfrak{S}_n) \mid f_k^n(\chi) = \psi\},
$$
and set $e(\psi, 2^k) = |E(\psi, 2^k)|$. We show in Corollary 3.8 that the maps $f_k^n$ are regular on their images. This means that for any $\psi$ in the image of $f_k^n$, the number $e(\psi, 2^k)$ depends only on $n$ and $k$ and not on the specific $\psi$. We also give a complete description of those $\psi \in \text{Irr}_{2^k}(\mathfrak{S}_{n-2^k})$ such that $e(\psi, 2^k) = 0$, in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote $f_k^n$ just by $f_k$, when the natural number $n$ is clear from the context. Then, for $k, \ell \in \mathbb{N}_0$, $k < \ell$, such that $2^k + 2^\ell \leq n$, we may ask: when is $f_k f_\ell = f_\ell f_k$? or more specifically: when is $f_k^n - 2^k f_\ell^n = f_\ell^n - 2^\ell f_k^n$? In [3, Proposition 4.3] it was proved that $f_k f_\ell = f_\ell f_k$ whenever $2^\ell < n < 2^{\ell+1}$. This is the case $\ell = t$ in our second main result, which answers the question completely.

**Theorem B.** Let $n = 2^\ell + m$ where $0 \leq m < 2^\ell$. Suppose that $k, \ell$ satisfy $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. Then, with the exception of the case $n = 6, k = 0, \ell = 1$,

$$f_k f_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.$$

## 2 Notation and background

Let $n$ be a natural number. We let $\text{Irr}(\mathfrak{S}_n)$ denote the set of irreducible characters of $\mathfrak{S}_n$ and $\mathcal{P}(n)$ the set of partitions of $n$. The notation $\lambda \in \mathcal{P}(n)$ is sometimes replaced by $\lambda \vdash n$ and we write $|\lambda| = n$. There is a natural correspondence $\lambda \leftrightarrow \chi^\lambda$ between $\mathcal{P}(n)$ and $\text{Irr}(\mathfrak{S}_n)$. We say then that $\lambda$ labels $\chi^\lambda$. We denote by $\text{Irr}_{2^k}(\mathfrak{S}_n)$ the set of irreducible characters of $\mathfrak{S}_n$ of odd degree. If $\chi^\lambda \in \text{Irr}_{2^k}(\mathfrak{S}_n)$ we say that $\chi^\lambda$ is an odd character, we call $\lambda$ an odd partition of $n$ and write $\lambda \vdash_o n$. Also the empty partition will be considered as an odd partition.

**Remark 2.1.** Let $n, k$ be such that $2^k < n$. In [3, Theorem A and Proposition 4.2] it is shown that the map $f_k^n: \text{Irr}_{2^k}(\mathfrak{S}_n) \to \text{Irr}_{2^k}(\mathfrak{S}_{n-2^k})$ may be described in terms of the odd partitions labelling the odd characters as follows:

$$f_k^n(\chi^\lambda) = \chi^\mu \Leftrightarrow \mu \vdash_o n - 2^k \text{ can be obtained from } \lambda \vdash_o n \text{ by removing a } 2^k\text{-hook}.$$

Correspondingly we write (by abuse of notation) $f_k^n(\lambda) = \mu$. In fact when $\lambda$ is odd, there is only one $2^k$-hook of $\lambda$ whose removal leads again to an odd partition; we will refer to such a hook as an odd hook of $\lambda$. This combinatorial description of $f_k^n$ will be used throughout this paper, and we will regard $f_k^n$ also as a map between the corresponding sets of odd partitions. Also, for $\mu \vdash_o n - 2^k$ we set $e(\mu, 2^k) = e(\chi^\mu, 2^k)$. 
We need some concepts and basic facts concerning hooks in partitions. For any integer $e \in \mathbb{N}$ we denote by $C_e(\lambda)$ and $Q_e(\lambda)$ the $e$-core and the $e$-quotient of $\lambda$, respectively. Then $Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1})$ is an $e$-tuple of partitions satisfying $n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i|$. It is well-known that a partition is uniquely determined by its $e$-core and $e$-quotient (we refer the reader to [6] or [4, Chapter 2.7] for a detailed discussion on this topic).

Let $H_e(\lambda)$ be the set of hooks of $\lambda$ having length divisible by $e$, and let $H(Q_e(\lambda)) = \bigcup_{i=0}^{e-1} H(\lambda_i)$.

As explained in [6, Theorem 3.3], there is a bijection between $H_e(\lambda)$ and $H(Q_e(\lambda))$ mapping hooks in $\lambda$ of length $ex$ to hooks in the quotient of length $x$. Moreover, the bijection respects the process of hook removal. Namely, the partition $\mu$ obtained by removing an $ex$-hook from $\lambda$ is such that $C_e(\mu) = C_e(\lambda)$ and the $e$-quotient of $\mu$ is obtained by removing an $x$-hook from one of the partitions involved in $Q_e(\lambda)$.

For $e = 2$ we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-quotient tower $Q_2(\lambda)$ and the 2-core tower $C_2(\lambda)$ of $\lambda$. They have rows numbered by $k \geq 0$. The $k$th row $Q_2(\lambda)^{(k)}$ of $Q_2(\lambda)$ contains $2^k$ partitions $\lambda_i^{(k)}$, $0 \leq i \leq 2^k - 1$, and the $k$th row $C_2(\lambda)^{(k)}$ of $C_2(\lambda)$ contains the 2-cores of these partitions in the same order, i.e., $C_2(\lambda^{(k)}_i)$, $0 \leq i \leq 2^k - 1$.

The 0th row of $Q_2(\lambda)$ contains $\lambda = \lambda_0^{(0)}$ itself, row 1 contains the partitions $\lambda_0^{(1)}$, $\lambda_1^{(1)}$ occurring in the 2-quotient $Q_2(\lambda)$, row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have $Q_2(\lambda^{(k)}_i) = (\lambda^{(k+1)}_{2i}, \lambda^{(k+1)}_{2i+1})$ for $i \in \{0, 1, \ldots, 2^k - 1\}$. We remark that the $2^k$ partitions in $Q_2(\lambda)^{(k)}$ are the same as those in the $2^k$-quotient $Q_{2^k}(\lambda)$ of $\lambda$, but in a different order for $k \geq 2$.

We also introduce the $k$-data $D_2(\lambda)^{(k)}$ of $\lambda$. This is a table containing the following $k+1$ rows: the $k$ rows $C_2^{(j)}(\lambda)$, $j = 0, \ldots, k-1$, and in addition the row $Q_2^{(k)}(\lambda)$.

**Remark 2.2.** A partition $\lambda$ may be recovered from its 2-core tower. For $k > 0$, it may also be recovered from the knowledge of the $k$-data $D_2(\lambda)^{(k)}$ of $\lambda$, because the rows $C_2^{(l)}(\lambda)$ with $l \geq k$ of $C_2(\lambda)$ consist of the $2^k$-core towers of the partitions in $Q_2^{(k)}(\lambda)$.

**Lemma 2.3.** Suppose that $\lambda \vdash n - 2^k$ and $\mu \vdash n$. The following are equivalent.

(i) $\lambda$ is obtained from $\mu$ by removing a $2^k$-hook.

(ii) The $k$-data $D_2^{(k)}(\mu)$ and $D_2^{(k)}(\lambda)$ coincide, except that for one $i \in \{0, \ldots, 2^k - 1\}$ $\lambda_i^{(k)}$ is obtained from $\mu_i^{(k)}$ by removing a $1$-hook.

**Proof.** A $2^k$-hook $H_0$ in $\mu$ corresponds in a canonical way to a $2^{k-1}$-hook $H_1$ in a partition in $Q_2^{(1)}(\mu)$, i.e., in row 1 of the 2-quotient tower $Q_2(\mu)$. Continuing we see that $H_0$ corresponds in a canonical way to a 1-hook $H_k$ in a partition $\mu_i^{(k)}$ in $Q_2^{(k)}(\mu)$, row $k$ of $Q_2(\mu)$. If $\lambda$ is obtained by removing $H_0$ from $\mu$, this corresponds to $\lambda_i^{(k)}$ being obtained by removing the 1-hook $H_k$ from $\mu_i^{(k)}$ (by repeated applications of [6, Theorem 3.3]). Apart from this the rows $Q_2^{(k)}(\mu)$ and $Q_2^{(k)}(\lambda)$ coincide. Note also that the rows $C_2^{(j)}(\mu)$ and $C_2^{(j)}(\lambda)$ coincide for $j = 0, \ldots, k-1$, since the removal of the hooks $H_j$ of even length do not change the 2-cores.

Odd-degree characters of $\mathfrak{S}_n$ and thus odd partitions were completely described in [5]. We restate this result in a language which is convenient for our purposes. We let $c_2^{(k)}(\lambda)$ be the sum of the cardinalities of the partitions in the $k$th row $C_2^{(k)}(\lambda)$ of $C_2(\lambda)$.

**Lemma 2.4 ([5]).** Let $\lambda$ be a partition. Then $\lambda$ is odd if and only if $c_2^{(k)}(\lambda) \leq 1$ for all $k \geq 0$.

It may be decided from the $k$-data $D_2^{(k)}(\lambda)$ whether $\lambda$ is odd. The case $k = 1$ of the following result appeared in [3, Lemma 4.1] and also in [1, Lemma 6].
Theorem 2.5. Let $\lambda \vdash n$, and let $k \geq 0$ be fixed. Consider $Q_2^{(k)}(\lambda) = (\lambda^{(k)}_i)$. Then $\lambda$ is odd if and only if the following conditions are all fulfilled:

(i) $c_2^{(j)}(\lambda) \leq 1$ for all $j < k$.

(ii) The partitions $\lambda^{(k)}_i$, $0 \leq i \leq 2^k - 1$, are all odd.

(iii) The numbers $|\lambda^{(k)}_i|$, $0 \leq i \leq 2^k - 1$, are pairwise 2-disjoint.

In this case $\sum_{i \geq 0} |\lambda^{(k)}_i| = \left\lfloor \frac{n}{2^k} \right\rfloor$.

Proof. This is proved by induction on $k \geq 0$, using Remark 2.2 and Lemma 2.4. $lacksquare$

We illustrate the result above by giving an example.

Example 2.6. Let $n = 15$ and take $\lambda = (5, 4, 2^2, 1^2) \vdash 15$. To decide whether $\lambda$ is odd, we choose $k = 2$ and compute the 2-data $D_2^{(2)}(\lambda)$. The 2-core is $C_2(\lambda) = (1)$, giving $C_2^{(0)}(\lambda) = ((1))$. Furthermore, the 2-quotient is $Q_2(\lambda) = ((2^2, 1^2), (1))$, and computing the 2-cores $C_2((2^2, 1^2)) = (0), C_2((1)) = (1)$, we obtain the next row: $C_2^{(1)}(\lambda) = ((0), (1))$. The 2-quotients are $Q_2((2^2, 1^2)) = ((1^2), (1)), Q_2((1)) = ((0), (0))$; hence the final row of the 2-data table is obtained as $Q_2^{(2)}(\lambda) = ((1^2), (1), (0), (0))$.

We visualize $D_2^{(2)}(\lambda)$ like this:

$C_2^{(0)}(\lambda)$: (1)
$C_2^{(1)}(\lambda)$: (0) (1)
$Q_2^{(2)}(\lambda)$: (12) (1) (0) (0)

Theorem 2.5 shows that $\lambda$ is odd and thus it contains a unique odd 4-hook. Again using the theorem, it is clear that removing this 4-hook corresponds to the second partition (1) in $Q_2^{(2)}(\lambda)$ being replaced by (0). Thus, removing the corresponding 4-hook of $\lambda$ we obtain the odd partition $\mu = (3, 2^4, 1^2) \vdash 11$ with the property that $D_2^{(2)}(\lambda)$ and $D_2^{(2)}(\mu)$ differ only in their final row.

Remark 2.7. Using the construction of partitions from their 2-cores and 2-quotients already mentioned, the criterion above can be applied to construct all odd partitions of $n$ with a specific $k$th row in the 2-quotient tower. For this, let $n, k \in \mathbb{N}$, and take any sequence of odd partitions $\nu_i$, $0 \leq i \leq 2^k - 1$, such that the numbers $|\nu_i|$ are pairwise 2-disjoint, and $\sum_{i \geq 0} |\nu_i| = \left\lfloor \frac{n}{2^k} \right\rfloor$.

Then there are exactly $\prod_{m \leq k} 2^m$ odd partitions $\lambda$ of $n$ with $Q_2^{(k)}(\lambda) = (\nu_i)$, obtained by choosing one 2-core in row $m$ of the $k$-data table to be (1), for each $m < k$ such that $2^m \subseteq 2n$.

The following easy consequence of Theorem 2.5 will be used repeatedly.

Lemma 2.8. Let $2^t$ be the largest binary digit of $n$. A partition $\lambda$ of $n$ is odd if and only if $\lambda$ contains a unique $2^t$-hook and the partition obtained from $\lambda$ by removing this $2^t$-hook is an odd partition of $n - 2^t$.

3 Surjectivity and regularity

The aim of this section is to study the images of the maps $f_k^n$ for all $n, k$ such that $2^k \leq n$. For this purpose we introduce the concept of $d$-good partitions (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when $f_k^n$ is surjective) and to show that the maps $f_k^n$ are always regular on their image (see Corollary 3.8).
**Definition 3.1.** Let \( d \geq 0 \). We call an odd partition \( \lambda \) \( d \)-good, if

(i) \( |\lambda| \equiv 2^d - 1 \mod 2^{d+1} \).

(ii) \( C_{2d}(\lambda) \) is a hook partition.

Let us remark that condition (i) may be reformulated as

\[(i^*) \ \nu_2(|\lambda| + 1) = d.\]

In particular, if \( \lambda \) is \( d \)-good, then \( |\lambda| \) is odd if and only if \( d > 0 \).

The relevance of \( d \)-good partitions in our context is illuminated by the following reformulation of \([1, \text{Theorem 2}]\):

**Lemma 3.2.** Let \( \lambda \vdash_o n \). Let \( d = \nu_2(n + 1) \). Then \( e(\lambda, 1) \neq 0 \) if and only if \( \lambda \) is \( d \)-good. In this case, \( e(\lambda, 1) = 1 \) if \( d = 0 \), and \( e(\lambda, 1) = 2 \) if \( d > 0 \).

**Lemma 3.3.** Let \( \lambda \) be an odd partition, and let \( d \geq 0 \). Then the following hold.

1. For \( d \leq 2 \), \( \lambda \) is \( d \)-good if and only if \( |\lambda| \equiv 2^d - 1 \mod 2^{d+1} \).

2. If \( \lambda \) is \( d \)-good, then \( C_{2d}(\lambda) \) is a partition of \( 2^d - 1 \).

**Proof.** If the odd partition \( \lambda \) is \( d \)-good, then \( |\lambda| = (2^d - 1) + m \) where the binary digits of \( m \) are at least \( 2^{d+1} \). The hooks of \( \lambda \) corresponding to the binary digits of \( m \) may be decomposed into \( 2^{d} \)-hooks and thus do not contribute to \( C_{2d}(\lambda) \). Thus \( |C_{2d}(\lambda)| = 2^d - 1 \). This shows (2). For \( d = 0, 1, 2 \) we have \( |C_{2d}(\lambda)| = 0, 1, 3 \), respectively. Since all partitions of 0, 1 and 3 are hook partitions, (1) follows.

**Definition 3.4.** If \( 2^k \leq n \), we define \( d(n, k) = \nu_2\left(\frac{n}{2^k}\right) \). Thus \( d(n, k) \) is the smallest integer \( d \geq 0 \) satisfying the condition \( 2^{k+d} \subseteq n \). In particular, \( d(n, k) = 0 \) if and only if \( 2^k \subseteq n \). Moreover, we may write \( \left\lfloor \frac{n}{2^k} \right\rfloor = 2^{d(n,k)} + m(n,k) \) where \( 2^{d(n,k)+1} \mid m(n,k) \).

As mentioned in the introduction, the results in \([3]\) show that \( f_k^n \) is a surjective \((2^k\text{-to-1})\)-map whenever \( 2^k \subseteq n \), i.e., \( d(n,k) = 0 \). In the spirit of \([1, \text{Theorem 2}]\), we now give a characterization of the image of the map \( f_k^n \) for all \( n, k \) such that \( 2^k < n \).

**Theorem 3.5.** Let \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) be such that \( 2^k < n \). Let \( \lambda \vdash_o n - 2^k \). Then \( e(\lambda, 2^k) \neq 0 \) if and only if there exists a \( d(n,k) \)-good partition in the \( k \)th row of \( O_2(\lambda) \). In this case, \( e(\lambda, 2^k) = 2^k \) if \( d(n,k) = 0 \), and \( e(\lambda, 2^k) = 2 \) if \( d(n,k) > 0 \).

**Proof.** If \( k = 0 \) then the statement follows from Lemma 3.2. Hence assume that \( k \geq 1 \). Let \( d = d(n,k) \). By assumption \( \left\lfloor \frac{n}{2^k} \right\rfloor = 2^d + m \), where the binary digits of \( m \) are at least \( 2^{d+1} \). Thus \( \left\lfloor \frac{n-2^k}{2^k} \right\rfloor = (2^d - 1) + m \).

Suppose first that \( e(\lambda, 2^k) \neq 0 \) and that \( \mu \vdash_o n \) satisfies \( f_k(\mu) = \lambda \). From Remark 2.1 and Lemma 2.3 we get that there exists an \( i \in \{0, 1, \ldots, 2^k - 1\} \) such that \( f_0(\mu_i^{(k)}) = \lambda_i^{(k)} \). Since \( \mu_i^{(k)} \) and \( \lambda_i^{(k)} \) are odd, we get \( e(\lambda_i^{(k)}, 1) \neq 0 \). We have that \( |\lambda_i^{(k)}| \) and \( |\mu_i^{(k)}| \) are both 2-disjoint with \( m_1 := \sum_{j \neq i} |\lambda_j^{(k)}| = \sum_{j \neq i} |\mu_j^{(k)}| \subseteq \left\lfloor \frac{n-2^k}{2^k} \right\rfloor \), by Theorem 2.5. Since \( m_1 \subseteq \left\lfloor \frac{n-2^k}{2^k} \right\rfloor \) and \( m_1 \subseteq \left\lfloor \frac{n}{2^k} \right\rfloor \), we get \( m_1 \subseteq 2m \). Thus \( |\lambda_i^{(k)}| = (2^d - 1) + m_2 \) and \( |\mu_i^{(k)}| = 2^d + m_2 \), where \( m_2 = m - m_1 \subseteq m \). In particular \( \nu_2(|\lambda_i^{(k)}| + 1) = \nu_2(|\mu_i^{(k)}|) = d \). Then Lemma 3.2 shows that \( \lambda_i^{(k)} \) is \( d \)-good.

Conversely, if \( \lambda_i^{(k)} \) is a \( d \)-good partition for some \( i \in \{0, 1, \ldots, 2^k - 1\} \), then there exists a \( \mu^* \vdash_o |\lambda_i^{(k)}| + 1 \) such that \( f_0(\mu^*) = \lambda_i^{(k)} \), by Lemma 3.2. We let \( \mu \) be the partition where the \( k \)th-data \( D_2^{(k)}(\mu) \) and \( D_2^{(k)}(\lambda) \) coincide, except that \( \mu_i^{(k)} = \mu^* \). Since \( \lambda \) is odd and \( \lambda_i^{(k)} \) is \( d \)-good,
we know that $|\lambda_i^{(k)}| = (2^d - 1) + m'$ where $m' \subseteq_2 m$, and $|\lambda_j^{(k)}| \subseteq_2 m - m'$ for all $j \neq i$. Hence $|\mu^*| = |\lambda_i^{(k)}| + 1 = 2^d + m'$ is 2-disjoint from all $|\lambda_j^{(k)}|, j \neq i$. Thus $\mu$ is an odd partition of $n$ by Theorem 2.5, and $f_k(\mu) = \lambda$ by Lemma 2.3 and Remark 2.1.

We conclude that $e(\lambda, 2^k) = \sum_{\lambda_i^{(k)} \text{-good}} e(\lambda_i^{(k)}, 1)$. If $d = 0$ then $\lfloor \frac{n - 2^k}{2^k} \rfloor$ is even. This implies that all $\lambda_i^{(k)}$ are of even cardinality and thus $d$-good. Thus $e(\lambda_i^{(k)}, 1) = 1$ for all $i$, and we get $e(\lambda, 2^k) = 2^k$. If $d > 0$ there is exactly one $\lambda_i^{(k)}$ in $Q_2^{(k)}(\lambda)$ of odd cardinality. Only this $\lambda_i^{(k)}$ may be $d$-good and then $e(\lambda, 2^k) = e(\lambda_i^{(k)}, 1) = 2$. Otherwise $e(\lambda, 2^k) = 0$.

**Corollary 3.6.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$, and let $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash_o n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a partition $\lambda_i^{(k)}$ in the $k$th row of $Q_2(\lambda)$ such that $|\lambda_i^{(k)}| \equiv 2^d - 1 \mod 2^{d+1}$, and $C_{2^d}(\lambda_i^{(k)})$ is a hook partition. In this case, $e(\lambda, 2^k) = 2^k$ if $d = 0$, and $e(\lambda, 2^k) = 2$ if $d > 0$.

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

**Corollary 3.7 (Theorem A).** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$.

- If $k = 0$ then $f_n^k$ is surjective if and only if $d(n, k) \leq 2$.
- If $k > 0$ then $f_n^k$ is surjective if and only if $d(n, k) \leq 1$.

**Proof.** By Theorem 3.5, $f_n^k$ is surjective if and only if for all $\lambda \vdash_o n - 2^k$ we have that the $k$th row of $Q_2(\lambda)$ contains a $d(n, k)$-good partition $\lambda_i^{(k)}$. By Theorem 2.5 and Definition 3.4, for any $\lambda \vdash_o n - 2^k$ we have $\sum_{j \geq 0} |\lambda_j^{(k)}| = \left\lfloor \frac{n - 2^k}{2^k} \right\rfloor = (2^{d(n, k)} - 1) + m(n, k)$.

If $k = 0$ then $Q_2^{(0)}(\lambda)$ contains only $\lambda = \lambda_0^{(0)}$. Hence $f_0^k$ is surjective if and only all odd partitions of $n - 1$ are $d(n, 0)$-good. By Lemma 3.3(1), the latter condition holds when $d = d(n, 0) \leq 2$. On the other hand, if $d = \nu_2(n) > 2$, then $\lambda = (n - 5, 2, 2)$ is an odd partition of $n - 1$ by Theorem 2.5, but $C_8(\lambda) = (3, 2, 2)$ is not a hook, and hence $C_{2^d}(\lambda)$ is not a hook. So $\lambda$ is not $d$-good, and thus $f_0^k$ is not surjective.

Now assume $k \geq 1$. Then $Q_2^{(k)}(\lambda)$ contains at least two odd partitions. If $d(n, k) \geq 2$ then any $d(n, k)$-good partition $\mu$ satisfies $3 \subseteq_2 2^{d(n, k) - 1} \subseteq_2 |\mu|$. Write $\left\lfloor \frac{n - 2^k}{2^k} \right\rfloor = 1 + m_1$ where $m_1$ is even. Applying Remark 2.7, take any $\lambda \vdash_o n - 2^k$ such that $|\lambda_0^{(k)}| = 1$ and $\lambda_1^{(k)}$ is an odd partition with $|\lambda_1^{(k)}| = m_1$. Then no partition in $Q_2^{(k)}(\lambda)$ is $d(n, k)$-good. Thus $f_0^k$ is not surjective. On the other hand, if $d(n, k) = 0$ then $2^k \subseteq n$ and $f_0^k$ is surjective [3, Proposition 4.5]. If $d(n, k) = 1$ then $\left\lfloor \frac{n - 2^k}{2^k} \right\rfloor = 1 + m(n, k)$, where $4 \mid m(n, k)$. Thus any $Q_2^{(k)}(\lambda)$ contains a partition with odd cardinality; this partition is 1-good, by Lemma 3.3. Again $f_0^k$ is surjective.

It is an immediate consequence of Theorem 3.5 that $f_n^k$ is regular on its image for all relevant choices of $n, k$ such that $2^k < n$. We have:

**Corollary 3.8.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$; set $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash_o n - 2^k$. Then

$$
eq \begin{cases} 
2 & \text{if } d > 0, \text{ and the } k\text{th row of } Q_2(\lambda) \text{ contains a } d\text{-good partition}; \\
0 & \text{otherwise.}
\end{cases}$$
Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take $k = 2$ above. For $n > 2^2$ we first compute $d(n, k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor \right)$, and then consider odd partitions of $n - 4$ and their 4-extensions. For $n = 6$, $d(6, 2) = 0$. Thus $e(2, 4) = 4$. The odd 4-extensions of (2) are (6), $(3^2)$, $(2^2, 1^2)$, $(2, 1^4)$. For $n = 10$, $d(10, 2) = 1$. In this case, $e(\lambda, 4) = 2$ for all odd partitions $\lambda$ of 6. For instance, the odd 4-extensions of (6) are (10) and $(6, 3, 1)$. For $n = 19$, $d(19, 2) = 2$. Example 2.6 shows that for $\lambda = (5, 4, 2^2, 1^2) \vdash_o 15$ there is no 2-good partition in $Q_2^2(\lambda)$, hence $e(\lambda, 4) = 0$.

4 Deciding commutativity of the maps $f_k$ and $f_\ell$

Let $n \in \mathbb{N}$, and suppose that $0 \leq k < \ell$ satisfy $2^k + 2^\ell \leq n$. As stated in the introduction, we want to complete the discussion of the commutativity of the maps $f_k$ and $f_\ell$. Since the relevant $n$ will always be apparent for the maps $f_k^n$ in this section, we just write $f_k$.

We write $(n; k, \ell) \in T$ if for all $\lambda \vdash_o n$ we have $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$. Otherwise we write $(n; k, \ell) \in F$.

In this section we will prove Theorem B, which may be reformulated as follows.

Theorem 4.1. Let $n = 2^t + m$ where $0 \leq m < 2^t$. Suppose that $k$, $\ell$ satisfy $0 \leq k < \ell$ and $2^k + 2^\ell \leq n$. Then with the exception of $(6; 0, 1)$

$$(n; k, \ell) \in F \text{ if and only if } \ell < t \text{ and } 2^k \leq m.$$ 

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two extreme cases, where $f_k$ and $f_\ell$ commute.

In the case $\ell = t$ we have the following result as a reformulation of [3, Proposition 4.3].

Lemma 4.2. Let $n = 2^t + m$ with $0 \leq m < 2^t$. If $2^k \leq m$, then $(n; k, t) \in T$.

It is also known that in the case where $n$ is a power of 2, the maps $f_k$ and $f_\ell$ commute [3, Remark 4.4], and we include a short proof here.

Lemma 4.3. If $n = 2^t$ then $(n; k, \ell) \in T$ for all $k$, $\ell$.

Proof. If $0 \leq b \leq a$ are integers then the binomial coefficient $\binom{a}{b}$ is odd if and only if $b \subseteq_2 a$, by Lucas’ theorem. The odd partitions of $2^t$ are exactly the hook partitions $(2^t - b, 1^b)$, $0 \leq b \leq 2^t - 1$, of degree $(\binom{2^t - 1}{b})$. Hence for $k \in \{0, 1, \ldots, t - 1\}$ we have

$$f_k(\lambda) = \begin{cases} (2^t - b - 2^k, 1^b) & \text{if } 2^k \not\subseteq_2 b, \\ (2^t - b, 1^{b - 2^k}) & \text{if } 2^k \subseteq_2 b. \end{cases}$$

It follows that for any $k, \ell < t$ and odd partition $\lambda$ of $2^t$, we have $f_\ell f_k(\lambda) = f_k f_\ell(\lambda)$. ■

Lemma 4.4. Let $n = 2^t + m$ with $0 \leq m < 2^t$. Suppose that $k$, $\ell$ satisfy $0 \leq k \leq \ell$ and $2^k + 2^\ell \leq n$. If $m < 2^k$ then $(n; k, \ell) \in T$.

Proof. We use induction on $k \geq 0$. For $k = 0$ we have $m = 0$ and the claim follows from Lemma 4.3. Suppose that $k \geq 1$ and that the claim has been proved up to $k - 1$. Let $\lambda \vdash_o n$. Odd hooks of length $2^k$ and $2^\ell$ in $\lambda$ correspond to odd hooks of length $2^{k-1}$ and $2^{\ell-1}$ in the 2-quotient $Q_2(\lambda) = (\lambda_0, \lambda_1)$ of $\lambda$. From Theorem 2.5 we deduce that $|\lambda_0|$ and $|\lambda_1|$ are 2-disjoint binary subsums of $\left\lfloor \frac{n}{2^k} \right\rfloor$, so one of them contains $2^{k-1}$, say $|\lambda_0|$; then $|\lambda_1| \leq \left\lfloor \frac{m}{2^{k-1}} \right\rfloor < 2^{k-1} < 2^{\ell-1}$.

Thus the odd $2^{k-1}$-hook in $Q_2(\lambda)$ has to be in $\lambda_0$. Therefore

$$Q_2(f_k(\lambda)) = (f_{k-1}(\lambda_0), \lambda_1).$$
Applying $f_\ell$, the odd $2^{\ell-1}$-hook cannot be in $\lambda_1$, hence

$$Q_2(f_\ell f_k(\lambda)) = (f_{\ell-1} f_{k-1}(\lambda_0), \lambda_1)).$$

In particular, we know that $|\lambda_0| \geq 2^{\ell-1} + 2^{k-1}$. Also $|\lambda_0| + |\lambda_1| = \left\lfloor \frac{n}{2} \right\rfloor = 2^{\ell-1} + \left\lfloor \frac{n}{2} \right\rfloor$. We have already seen that $2^{\ell-1}$ is the largest binary digit of $|\lambda_0|$; furthermore $|\lambda_0| - 2^{\ell-1}$ is a binary subsum of $\left\lfloor \frac{n}{2} \right\rfloor < 2^{k-1}$. We may therefore apply the inductive hypothesis to $\lambda_0$ to get $f_{\ell-1} f_{k-1}(\lambda_0) = f_{k-1} f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_k f_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$. $\square$

Lemmas 4.2 and 4.4 show that the only if part of the theorem is true. We now turn to the if part. We start by proving the statement for $k = 0$ and use this as part of an inductive argument.

**Lemma 4.5.** Let $n = 2^t + m$ with $0 < m < 2^t$. If $0 < \ell < t$ then $(n; 0, \ell) \in \mathcal{F}$, with the exception of $(6; 0, 1)$.

**Proof.** The result is easily checked for $n \leq 8$, which includes the exception $(6; 0, 1)$. So we assume that $t \geq 3$.

Case 1: $2^t < m$. Then $m \geq 3$, since $\ell > 0$. Consider the partition $\lambda = (m, m, 1^a) \vdash n$ where $a = n - 2m = 2^{t-1} - m$. The $(1,1)$-hook length of $\lambda$ is $2^t + 1$. The $(2,1)$-hook length of $\lambda$ is $2^t$. Removing the $(2,1)$-hook hook we get the odd partition $(m)$, so $\lambda$ is odd, by Lemma 2.8. We claim that

$$f_0(\lambda) = (m, m, 1^{a-1}).$$

Indeed we cannot have $f_0(\lambda) = (m, m - 1, 1^a)$ because this partition does not have a hook of length $2^t$, and thus it is not odd. Now

$$f_\ell(f_0(\lambda)) = f_\ell(m, m, 1^{a-1}) = (m, m - 2^t, 1^{a-1})$$

since $(m, m, 1^{a-1-2^t})$ and $(m - 1, m - 2^t + 1, 1^{a-1})$ both do not have a hook of length $2^t$ and thus are not odd (again by Lemma 2.8).

On the other hand,

$$f_\ell(\lambda) = (m - 1, m - (2^t - 1), 1^a).$$

Indeed, the other candidates for $f_\ell(\lambda)$, which are $(m, m - 2^t, 1^a)$ and $(m, m, 1^{a-2^t})$, do not have hooks of length $2^t$. Then

$$f_0(f_\ell(\lambda)) = f_0(m - 1, m - (2^t - 1), 1^a) = (m - 1, m - 2^t, 1^a).$$

This follows (again) by observing that all the other partitions of $n - 2^t - 1$ obtained from $(m - 1, m - (2^t - 1), 1^a)$ by removing a node do not have hooks of length $2^t$. Thus $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

Case 2: $m < 2^t$. Consider the partition $\lambda = (n - 2^\ell, m + 1, 1^a)$, where $a = 2^\ell - (m + 1)$. Note that $n - 2^t \geq m + 1$ since $\ell < t$ by assumption, and that $a \geq 0$. The $(1,1)$-hook length of $\lambda$ is $n - m = 2^t$. Removing this hook we get the odd partition $(m)$, so $\lambda$ is odd. The $(2,1)$-hook length of $\lambda$ is $2^t$. Now

$$f_0(\lambda) = (n - 2^\ell, m, 1^a)$$

since the other candidates do not have hooks of length $2^t$. Then

$$f_\ell(f_0(\lambda)) = f_\ell(n - 2^\ell, m, 1^a) = \mu,$$
where $\mu$ is obtained from $f_0(\lambda)$ by removing a $2^\ell$-hook in the first row. (There are only hooks of length $< 2^\ell$ in the other rows.) In fact, $\mu = (n - 2^k + 1, m, 1^o)$ since $n - 2^k + 1 \geq n - 2^\ell = m$. Thus $f_\ell(f_0(\lambda))$ has at least 2 parts. On the other hand

$$f_\ell(\lambda) = (n - 2^\ell)$$

since this odd partition is obtained from the odd partition $\lambda$ by removing a $2^\ell$-hook (the one in $(2,1)$). It follows that

$$f_0(f_\ell(\lambda)) = (n - 2^\ell - 1)$$

and again $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

Case 3: $m = 2^\ell$. Then $n = 2^\ell + 2^\ell$. If $\ell \geq 2$ then choose $\lambda = (2^\ell, 2^\ell - 1, 1)$. The $(1,2)$-hook length of $\lambda$ is $2^\ell$; thus $\lambda$ is an odd partition since removing this $2^\ell$-hook gives an odd partition $(2^\ell - 2, 1, 1)$ of $2^\ell$. We have $f_0(\lambda) = (2^\ell, 2^\ell - 2, 1)$ since the other candidates are not odd. Then

$$f_\ell(f_0(\lambda)) = (2^\ell - 2^\ell, 2^\ell - 2, 1).$$

The $(2,1)$-hook length of $\lambda$ is $2^\ell$, so $f_\ell(\lambda) = (2^\ell)$ and

$$f_0(f_\ell(\lambda)) = (2^\ell - 1),$$

showing $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

On the other hand, if $\ell = 1$ then choose $\lambda = (2^\ell - 2, 2, 2) \vdash_0 2^\ell + 2 = n$. Since $\ell \geq 3$, it is now easy to show that $f_1(f_0(\lambda)) = (2^\ell - 4, 2, 1)$. On the other hand we see that $f_0(f_1(\lambda))$ is a hook partition of $2^\ell - 1 = n - 3$ and therefore is not equal to $f_1(f_0(\lambda))$. ■

**Lemma 4.6.** If $(n; k, \ell) \in \mathcal{F}$ then also $(2n; k + 1, \ell + 1) \in \mathcal{F}$ and $(2n + 1; k + 1, \ell + 1) \in \mathcal{F}$.

**Proof.** Let the odd partition $\mu$ of $n$ satisfy $f_k f_\ell(\mu) \neq f_\ell f_k(\mu)$. Let $\lambda$ be a partition of $2n$ or $2n + 1$ having 2-quotient $Q_2(\lambda) = (\mu, (0))$. Then $\lambda$ is odd, by Theorem 2.5. We have

$$Q_2(f_{k+1} f_{\ell+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_\ell f_k(\mu), (0)) = Q_2(f_{\ell+1} f_{k+1}(\lambda)),$$

so that $f_{k+1} f_{\ell+1}(\lambda) \neq f_{\ell+1} f_{k+1}(\lambda)$. ■

We are now ready to conclude this section with the proof of Theorem B.

**Proof of Theorem 4.1.** The only if part follows from Lemmas 4.2 and 4.4. To prove the if part we use induction on $k \geq 0$. If $k = 0$, then the statement follows from Lemma 4.5. Let $k > 1$ and suppose that the assertion is true up to and including $k - 1$. To show that $(n; k, \ell) \in \mathcal{F}$ it suffices to prove $((\ell - 1); k - 1, \ell - 1) \in \mathcal{F}$, by Lemma 4.6. We are assuming $n = 2^\ell + m$, $0 \leq m < 2^\ell$, $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. This implies $\left\lceil \frac{n}{2} \right\rceil = 2^{t-1} + \left\lceil \frac{m}{2} \right\rceil$, $0 \leq \left\lceil \frac{m}{2} \right\rceil < 2^{t-1}$ and $2^{k-1} + 2^{\ell-1} \leq \left\lceil \frac{n}{2} \right\rceil$. We may apply the inductive hypothesis to get $((\ell - 1); k - 1, \ell - 1) \in \mathcal{F}$, and then $(n; k, \ell) \in \mathcal{F}$ except when $((\ell - 1); k - 1, \ell - 1) = (6; 0, 1)$. In that case we are considering $(12; 1, 2)$ or $(13; 1, 2)$ which are both in $\mathcal{F}$, by direct computation (consider for example $(6, 4, 2) \vdash_0 12$ and $(6, 4, 3) \vdash_0 13$, respectively). ■

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