Restriction of Odd Degree Characters of $S_n$

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Abstract. Let $n$ and $k$ be natural numbers such that $2^k < n$. We study the restriction to $S_{n-2^k}$ of odd-degree irreducible characters of the symmetric group $S_n$. This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., Sém. Lothar. Combin. 75 (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., J. Algebra 478 (2017), 271–282].

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1 Introduction

Let $n$ be a natural number, and let $\chi$ be an irreducible character of odd degree of the symmetric group $S_n$. Then there exists a unique odd-degree irreducible constituent of the restriction $\chi_{S_n - 1}$. This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any $k \in \mathbb{N}$ such that $2^k < n$, there exists a unique odd-degree irreducible constituent $f^k_n(\chi)$ of $\chi_{S_{n-2^k}}$ appearing with odd multiplicity. The main goal of this article is to study for all $n, k \in \mathbb{N}$ the map

$$f^k_n : \text{Irr}_2(S_n) \longrightarrow \text{Irr}_2(S_{n-2^k}),$$

naturally defined by Theorem A of [3]. All our results are proved using a description of $f^k_n$ in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If $2^k$ appears in the binary expansion of $n$ we say that $2^k$ is a binary digit of $n$. Similarly we say that two natural numbers $m$ and $n$ are 2-disjoint if they do not have any common binary digit. On the other hand, if $m \leq n$ and all the binary digits of $m$ appear in the binary expansion of $n$, then we say that $m$ is a binary subsum of $n$. This will be denoted by $m \subseteq_2 n$. Let $v_2(n)$ be the exponent of the highest power of 2 dividing the integer $n$.

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A question raised in [3] may be phrased as: For which $n$ and $k$ is $f^n_k$ surjective? The authors showed that $f^n_k$ is surjective whenever $2^k$ is a binary digit of $n$, and they observed that otherwise $f^n_k$ could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

**Theorem A.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $d(n,k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right)$.

- If $k = 0$ then $f^n_k$ is surjective if and only $d(n,k) \leq 2$.
- If $k > 0$ then $f^n_k$ is surjective if and only $d(n,k) \leq 1$.

Theorem A is a consequence of Theorem 3.5 below, which describes the images of the maps $f^n_k$. For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $2^k < n$ and any $\psi \in \text{Irr}_{2^k}(S_{n-2^k})$ we define the set

$$E(\psi, 2^k) = \{ \chi \in \text{Irr}_{2^k}(S_n) \mid f^n_k(\chi) = \psi \},$$

and set $e(\psi, 2^k) = |E(\psi, 2^k)|$. We show in Corollary 3.8 that the maps $f^n_k$ are regular on their images. This means that for any $\psi$ in the image of $f^n_k$, the number $e(\psi, 2^k)$ depends only on $n$ and $k$ and not on the specific $\psi$. We also give a complete description of those $\psi \in \text{Irr}_{2^k}(S_{n-2^k})$ such that $e(\psi, 2^k) = 0$, in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote $f^n_k$ just by $f_k$, when the natural number $n$ is clear from the context. Then, for $k, \ell \in \mathbb{N}_0$, $k < \ell$, such that $2^k + 2^\ell \leq n$, we may ask: when is $f_kf_\ell = f_\ell f_k$? or more specifically: when is $f^n_{k-2^k}f^n_\ell = f^n_{\ell-2^k}f^n_k$? In [3, Proposition 4.3] it was proved that $f_kf_\ell = f_\ell f_k$ whenever $2^\ell < n < 2^{\ell+1}$. This is the case $\ell = t$ in our second main result, which answers the question completely.

**Theorem B.** Let $n = 2^t + m$ where $0 \leq m < 2^t$. Suppose that $k, \ell$ satisfy $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. Then, with the exception of the case $n = 6$, $k = 0$, $\ell = 1$,

$$f_kf_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.$$  

### 2 Notation and background

Let $n$ be a natural number. We let $\text{Irr}(S_n)$ denote the set of irreducible characters of $S_n$ and $\mathcal{P}(n)$ the set of partitions of $n$. The notation $\lambda \in \mathcal{P}(n)$ is sometimes replaced by $\lambda \vdash n$ and we write $|\lambda| = n$. There is a natural correspondence $\lambda \leftrightarrow \chi^\lambda$ between $\mathcal{P}(n)$ and $\text{Irr}(S_n)$. We say then that $\lambda$ labels $\chi^\lambda$. We denote by $\text{Irr}_{2^k}(S_n)$ the set of irreducible characters of $S_n$ of odd degree. If $\chi^\lambda \in \text{Irr}_{2^k}(S_n)$ we say that $\chi^\lambda$ is an *odd character*, we call $\lambda$ an *odd partition* of $n$ and write $\lambda \vdash_o n$. Also the empty partition will be considered as an odd partition.

**Remark 2.1.** Let $n, k$ be such that $2^k < n$. In [3, Theorem A and Proposition 4.2] it is shown that the map $f^n_k : \text{Irr}_{2^k}(S_n) \to \text{Irr}_{2^k}(S_{n-2^k})$ may be described in terms of the odd partitions labelling the odd characters as follows:

$$f^n_k(\chi^\lambda) = \chi^\mu \iff \mu \vdash_o n - 2^k$$

can be obtained from $\lambda \vdash_o n$ by removing a $2^k$-hook.

Correspondingly we write (by abuse of notation) $f^n_k(\lambda) = \mu$. In fact when $\lambda$ is odd, there is only one $2^k$-hook of $\lambda$ whose removal leads again to an odd partition; we will refer to such a hook as an *odd hook* of $\lambda$. This combinatorial description of $f^n_k$ will be used throughout this paper, and we will regard $f^n_k$ also as a map between the corresponding sets of odd partitions. Also, for $\mu \vdash_o n - 2^k$ we set

$$e(\mu, 2^k) = e(\chi^\mu, 2^k).$$
We need some concepts and basic facts concerning hooks in partitions. For any integer \( e \in \mathbb{N} \) we denote by \( C_e(\lambda) \) and \( Q_e(\lambda) \) the \( e \)-core and the \( e \)-quotient of \( \lambda \), respectively. Then \( Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1}) \) is an \( e \)-tuple of partitions satisfying \( n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i| \). It is well-known that a partition is uniquely determined by its \( e \)-core and \( e \)-quotient (we refer the reader to \([6]\) or \([4, \text{Chapter 2.7}]\) for a detailed discussion on this topic).

Let \( \mathcal{H}_e(\lambda) \) be the set of hooks of \( \lambda \) having length divisible by \( e \), and let \( \mathcal{H}(Q_e(\lambda)) = \bigcup_{i=1}^{e-1} \mathcal{H}(\lambda_i) \).

As explained in \([6, \text{Theorem 3.3}]\), there is a bijection between \( \mathcal{H}_e(\lambda) \) and \( \mathcal{H}(Q_e(\lambda)) \) mapping hooks in \( \lambda \) of length \( ex \) to hooks in the quotient of length \( x \). Moreover, the bijection respects the process of hook removal. Namely, the partition \( \mu \) obtained by removing an \( ex \)-hook from \( \lambda \) is such that \( C_e(\mu) = C_e(\lambda) \) and the \( e \)-quotient of \( \mu \) is obtained by removing an \( x \)-hook from one of the partitions involved in \( Q_e(\lambda) \).

For \( e = 2 \) we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-quotient tower \( Q_2(\lambda) \) and the 2-core tower \( C_2(\lambda) \) of \( \lambda \). They have rows numbered by \( k \geq 0 \). The \( k \)-th row \( Q_2^{(k)}(\lambda) \) of \( Q_2(\lambda) \) contains \( 2^k \) partitions \( \lambda_i^{(k)} \), \( 0 \leq i \leq 2^k - 1 \), and the \( k \)-th row \( C_2^{(k)}(\lambda) \) of \( C_2(\lambda) \) contains the 2-cores of these partitions in the same order, i.e., \( C_2(\lambda^{(k)}) \), \( 0 \leq i \leq 2^k - 1 \). The 0th row of \( Q_2(\lambda) \) contains \( \lambda = \lambda_0^{(0)} \) itself, row 1 contains the partitions \( \lambda_0^{(1)}, \lambda_1^{(1)} \) occurring in the 2-quotient \( Q_2(\lambda) \), row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have \( Q_2(\lambda^{(k)}) = (\lambda_0^{(k+1)}, \lambda_1^{(k+1)}) \) for \( i \in \{0, 1, \ldots, 2^k - 1\} \). We remark that the \( 2^k \) partitions in \( Q_2^{(k)}(\lambda) \) are the same as those in the \( 2^k \)-quotient \( Q_{2^k}(\lambda) \) of \( \lambda \), but in a different order for \( k \geq 2 \).

We also introduce the \( k \)-data \( D_2^{(k)}(\lambda) \) of \( \lambda \). This is a table containing the following \( k+1 \) rows: the \( k \) rows \( C_2^{(j)}(\lambda), j = 0, \ldots, k-1 \), and in addition the row \( Q_2^{(k)}(\lambda) \).

**Remark 2.2.** A partition \( \lambda \) may be recovered from its 2-core tower. For \( k > 0 \), it may also be recovered from the knowledge of the \( k \)-data \( D_2^{(k)}(\lambda) \) of \( \lambda \), because the rows \( C_2^{(l)}(\lambda) \) with \( l \geq k \) of \( C_2(\lambda) \) consist of the 2-core towers of the partitions in \( Q_2^{(k)}(\lambda) \).

**Lemma 2.3.** Suppose that \( \lambda \vdash n - 2^k \) and \( \mu \vdash n \). The following are equivalent.

(i) \( \lambda \) is obtained from \( \mu \) by removing a \( 2^k \)-hook.

(ii) The \( k \)-data \( D_2^{(k)}(\mu) \) and \( D_2^{(k)}(\lambda) \) coincide, except that for one \( i \in \{0, \ldots, 2^k - 1\} \) \( \lambda_i^{(k)} \) is obtained from \( \mu_i^{(k)} \) by removing a 1-hook.

**Proof.** A \( 2^k \)-hook \( H_0 \) in \( \mu \) corresponds in a canonical way to a \( 2^{k-1} \)-hook \( H_1 \) in a partition in \( Q_0^{(1)}(\mu) \), i.e., in row 1 of the 2-quotient tower \( Q_2(\mu) \). Continuing we see that \( H_0 \) corresponds in a canonical way to a 1-hook \( H_k \) in a partition \( \mu_i^{(k)} \) in \( Q_2^{(k)}(\mu) \), row \( k \) of \( Q_2(\mu) \). If \( \lambda \) is obtained by removing \( H_0 \) from \( \mu \), this corresponds to \( \lambda_i^{(k)} \) being obtained by removing the 1-hook \( H_k \) from \( \mu_i^{(k)} \) (by repeated applications of \([6, \text{Theorem 3.3}]\)). Apart from this the rows \( Q_2^{(k)}(\mu) \) and \( Q_2^{(k)}(\lambda) \) coincide. Note also that the rows \( C_2^{(j)}(\mu) \) and \( C_2^{(j)}(\lambda) \) coincide for \( j = 0, \ldots, k-1 \), since the removal of the hooks \( H_j \) of even length do not change the 2-cores. 

Odd-degree characters of \( \mathfrak{S}_n \) and thus odd partitions were completely described in \([5]\). We restate this result in a language which is convenient for our purposes. We let \( c_2^{(k)}(\lambda) \) be the sum of the cardinalities of the partitions in the \( k \)-th row \( C_2^{(k)}(\lambda) \) of \( C_2(\lambda) \).

**Lemma 2.4 (5).** Let \( \lambda \) be a partition. Then \( \lambda \) is odd if and only if \( c_2^{(k)}(\lambda) \leq 1 \) for all \( k \geq 0 \).

It may be decided from the \( k \)-data \( D_2^{(k)}(\lambda) \) whether \( \lambda \) is odd. The case \( k = 1 \) of the following result appeared in \([3, \text{Lemma 4.1}]\) and also in \([1, \text{Lemma 6}]\).
Theorem 2.5. Let \( \lambda \vdash n \), and let \( k \geq 0 \) be fixed. Consider \( Q^{(k)}_2(\lambda) = (\lambda^{(k)}_i) \). Then \( \lambda \) is odd if and only if the following conditions are all fulfilled:

(i) \( c^{(j)}_2(\lambda) \leq 1 \) for all \( j < k \).

(ii) The partitions \( \lambda^{(k)}_i \), \( 0 \leq i \leq 2^k - 1 \), are all odd.

(iii) The numbers \( |\lambda^{(k)}_i| \), \( 0 \leq i \leq 2^k - 1 \), are pairwise 2-disjoint.

In this case \( \sum_{i \geq 0} |\lambda^{(k)}_i| = \lfloor \frac{n}{2^2} \rfloor \).

Proof. This is proved by induction on \( k \geq 0 \), using Remark 2.2 and Lemma 2.4. \( \blacksquare \)

We illustrate the result above by giving an example.

Example 2.6. Let \( n = 15 \) and take \( \lambda = (5, 4, 2^2, 1^2) \vdash 15 \). To decide whether \( \lambda \) is odd, we choose \( k = 2 \) and compute the 2-data \( \mathcal{D}^{(2)}_2(\lambda) \). The 2-core is \( C_2(\lambda) = (1) \), giving \( C^{(0)}_2(\lambda) = ((1)) \). Furthermore, the 2-quotient is \( Q_2(\lambda) = ((2^2, 1^2), (1)) \), and computing the 2-cores \( C_2((2^2, 1^2)) = (0), C_2((1)) = (1) \), we obtain the next row: \( C^{(1)}_2(\lambda) = ((0), (1)) \). The 2-quotients are \( Q_2((2^2, 1^2)) = ((1^2), (1)), Q_2((1)) = ((0), (0)) \); hence the final row of the 2-data table is obtained as \( Q^{(2)}_2(\lambda) = ((1^2), (1), (0), (0)) \).

We visualize \( \mathcal{D}^{(2)}_2(\lambda) \) like this:

\[
\begin{align*}
C^{(0)}_2(\lambda) : & \quad (1) \\
C^{(1)}_2(\lambda) : & \quad (0) & (1) \\
Q^{(2)}_2(\lambda) : & \quad (1^2) & (1) & (0) & (0)
\end{align*}
\]

Theorem 2.5 shows that \( \lambda \) is odd and thus it contains a unique odd 4-hook. Again using the theorem, it is clear that removing this 4-hook corresponds to the second partition \( (1) \) in \( Q^{(2)}_2(\lambda) \) being replaced by \( (0) \). Thus, removing the corresponding 4-hook of \( \lambda \) we obtain the odd partition \( \mu = (3, 2^2, 1^2) \vdash 11 \) with the property that \( \mathcal{D}^{(2)}_2(\lambda) \) and \( \mathcal{D}^{(2)}_2(\mu) \) differ only in their final row.

Remark 2.7. Using the construction of partitions from their 2-cores and 2-quotients already mentioned, the criterion above can be applied to construct all odd partitions of \( n \) with a specific \( k \)th row in the 2-quotient tower. For this, let \( n, k \in \mathbb{N} \), and take any sequence of odd partitions \( \nu_i \), \( 0 \leq i \leq 2^k - 1 \), such that the numbers \( |\nu_i| \) are pairwise 2-disjoint, and \( \sum_{i \geq 0} |\nu_i| = \lfloor \frac{n}{2^2} \rfloor \).

Then there are exactly \( \prod_{m \leq k \atop 2^m \leq 2^n} 2^m \) odd partitions \( \lambda \) of \( n \) with \( Q^{(k)}_2(\lambda) = (\nu_i) \), obtained by choosing one 2-core in row \( m \) of the \( k \)-data table to be \( (1) \), for each \( m < k \) such that \( 2^m \leq 2^n \).

The following easy consequence of Theorem 2.5 will be used repeatedly.

Lemma 2.8. Let \( 2^t \) be the largest binary digit of \( n \). A partition \( \lambda \) of \( n \) is odd if and only if \( \lambda \) contains a unique \( 2^t \)-hook and the partition obtained from \( \lambda \) by removing this \( 2^t \)-hook is an odd partition of \( n - 2^t \).

### 3 Surjectivity and regularity

The aim of this section is to study the images of the maps \( f^n_k \) for all \( n, k \) such that \( 2^k \leq n \). For this purpose we introduce the concept of \( d \)-good partitions (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when \( f^n_k \) is surjective) and to show that the maps \( f^n_k \) are always regular on their image (see Corollary 3.8).
Definition 3.1. Let \( d \geq 0 \). We call an odd partition \( \lambda \) \( d \)-good, if

(i) \( |\lambda| \equiv 2^d - 1 \mod 2^{d+1} \).

(ii) \( C_{2d}(\lambda) \) is a hook partition.

Let us remark that condition (i) may be reformulated as

\[ (i^*) \quad \nu_2(|\lambda| + 1) = d. \]

In particular, if \( \lambda \) is \( d \)-good, then \( |\lambda| \) is odd if and only if \( d > 0 \).

The relevance of \( d \)-good partitions in our context is illuminated by the following reformulation of [1, Theorem 2]:

Lemma 3.2. Let \( \lambda \vdash n \). Let \( d = \nu_2(n + 1) \). Then \( e(\lambda, 1) \neq 0 \) if and only if \( \lambda \) is \( d \)-good. In this case, \( e(\lambda, 1) = 1 \) if \( d = 0 \), and \( e(\lambda, 1) = 2 \) if \( d > 0 \).

Lemma 3.3. Let \( \lambda \) be an odd partition, and let \( d \geq 0 \). Then the following hold.

1. For \( d \leq 2 \), \( \lambda \) is \( d \)-good if and only if \( |\lambda| \equiv 2^d - 1 \mod 2^{d+1} \).
2. If \( \lambda \) is \( d \)-good, then \( C_{2d}(\lambda) \) is a partition of \( 2^d - 1 \).

Proof. If the odd partition \( \lambda \) is \( d \)-good, then \( |\lambda| = (2^d - 1) + m \) where the binary digits of \( m \) are at least \( 2^{d+1} \). The hooks of \( \lambda \) corresponding to the binary digits of \( m \) may be decomposed into \( 2^d \)-hooks and thus do not contribute to \( C_{2d}(\lambda) \). Thus \( |C_{2d}(\lambda)| = 2^d - 1 \). This shows (2).

For \( d = 0, 1, 2 \) we have \( |C_{2d}(\lambda)| = 0, 1 \) and 3, respectively. Since all partitions of 0, 1 and 3 are hook partitions, (1) follows.

Definition 3.4. If \( 2^k \leq n \), we define \( d(n, k) = \nu_2\left(\left\lceil \frac{n}{2^k} \right\rceil\right) \). Thus \( d(n, k) \) is the smallest integer \( d \geq 0 \) satisfying the condition \( 2^k + d \leq n \). In particular, \( d(n, k) = 0 \) if and only if \( 2^k \leq n \). Moreover, we may write \( \left\lceil \frac{n}{2^k} \right\rceil = 2^d(n, k) + m(n, k) \) where \( 2^d(n, k) + 1 \mid m(n, k) \).

As mentioned in the introduction, the results in [3] show that \( f_k^n \) is a surjective \((2^k\text{-to-1})\)-map whenever \( 2^k \leq n \), i.e., \( d(n, k) = 0 \). In the spirit of [1, Theorem 2], we now give a characterization of the image of the map \( f_k^n \) for all \( n, k \) such that \( 2^k < n \).

Theorem 3.5. Let \( n \in \mathbb{N} \), \( k \in \mathbb{N}_0 \) be such that \( 2^k < n \). Let \( \lambda \vdash n - 2^k \). Then \( e(\lambda, 2^k) \neq 0 \) if and only if there exists a \( d(n, k) \)-good partition in the \( k \)th row of \( \mathcal{Q}_2(\lambda) \). In this case, \( e(\lambda, 2^k) = 2^k \) if \( d(n, k) = 0 \), and \( e(\lambda, 2^k) = 2 \) if \( d(n, k) > 0 \).

Proof. If \( k = 0 \) then the statement follows from Lemma 3.2. Hence assume that \( k \geq 1 \). Let \( d = d(n, k) \). By assumption \( \left\lceil \frac{n}{2^k} \right\rceil = 2^d + m \), where the binary digits of \( m \) are at least \( 2^{d+1} \). Thus \( \left\lfloor \frac{n-2^k}{2^k} \right\rfloor = (2^d - 1) + m \).

Suppose first that \( e(\lambda, 2^k) \neq 0 \) and that \( \mu \vdash n \) satisfies \( f_k^e(\mu) = \lambda \). From Remark 2.1 and Lemma 2.3 we get that there exists an \( i \in \{0, 1, \ldots, 2^k - 1\} \) such that \( f_0(\mu_i^{(k)}) = \lambda_i^{(k)} \). Since \( \mu_i^{(k)} \) and \( \lambda_i^{(k)} \) are odd, we get \( e(\lambda_i^{(k)}, 1) \neq 0 \). We have that \( |\lambda_i^{(k)}| \) and \( |\mu_i^{(k)}| \) are both \( 2 \)-disjoint with \( m_1 := \sum_{j \neq i} |\lambda_j^{(k)}| = \sum_{j \neq i} |\mu_j^{(k)}| \subseteq 2 \left\lfloor \frac{n-2^k}{2^k} \right\rfloor \), by Theorem 2.5. Since \( m_1 \subseteq 2 \left\lfloor \frac{n-2^k}{2^k} \right\rfloor \) and \( m_1 \subseteq 2 \left\lfloor \frac{n}{2^k} \right\rfloor \), we get \( m_1 \leq 2 \). Thus \( |\lambda_i^{(k)}| = (2^d - 1) + m_2 \) and \( |\mu_i^{(k)}| = 2^d + m_2 \), where \( m_2 = m - m_1 \leq 2 \).

In particular \( \nu_2(|\lambda_i^{(k)}| + 1) = \nu_2(|\mu_i^{(k)}|) = d \). Then Lemma 3.2 shows that \( \lambda_i^{(k)} \) is \( d \)-good.

Conversely, if \( \lambda_i^{(k)} \) is a \( d \)-good partition for some \( i \in \{0, 1, \ldots, 2^k - 1\} \), then there exists a \( \mu^* \vdash n \) \( |\lambda_i^{(k)}| + 1 \) such that \( f_0(\mu^*) = \lambda_i^{(k)} \), by Lemma 3.2. We let \( \mu \) be the partition where the \( k \)-data \( D_2^{(k)}(\mu) \) and \( D_2^{(k)}(\lambda) \) coincide, except that \( \mu_i^{(k)} = \mu^* \). Since \( \lambda \) is odd and \( \lambda_i^{(k)} \) is \( d \)-good,
we know that $|\lambda_i^{(k)}| = (2^d - 1) + m'$ where $m' \subseteq m$, and $|\lambda_j^{(k)}| \subseteq m - m'$ for all $j \neq i$. Hence $|\mu'| = |\lambda_i^{(k)}| + 1 = 2^d + m'$ is 2-disjoint from all $|\lambda_j^{(k)}|$, $j \neq i$. Thus $\mu$ is an odd partition of $n$ by Theorem 2.5, and $f_k(\mu) = \lambda$ by Lemma 2.3 and Remark 2.1.

We conclude that $e(\lambda, 2^k) = \sum_{\lambda^{(k)} \text{d-good}} e(\lambda^{(k)}_i, 1)$. If $d = 0$ then $\lfloor \frac{n-2^k}{2^d} \rfloor$ is even. This implies that all $\lambda_i^{(k)}$ are of even cardinality and thus $d$-good. Thus $e_1^{(k)}(1) = 1$ for all $i$, and we get $e(\lambda, 2^k) = 2^k$. If $d > 0$ there is exactly one $\lambda_i^{(k)}$ in $Q_2^{(k)}(\lambda)$ of odd cardinality. Only this $\lambda_i^{(k)}$ may be $d$-good and then $e(\lambda, 2^k) = e_1^{(k)}(1) = 2$. Otherwise $e(\lambda, 2^k) = 0$.

**Corollary 3.6.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$, and let $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash o n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a partition $\lambda_i^{(k)}$ in the $k$th row of $Q_2(\lambda)$ such that $|\lambda_i^{(k)}| \equiv 2^d - 1 \mod 2^{d+1}$, and $C_{2d}(\lambda_i^{(k)})$ is a hook partition. In this case, $e(\lambda, 2^k) = 2^k$ if $d = 0$, and $e(\lambda, 2^k) = 2$ if $d > 0$.

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

**Corollary 3.7 (Theorem A).** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$.

- If $k = 0$ then $f^n_k$ is surjective if and only if $d(n, k) \leq 2$.
- If $k > 0$ then $f^n_k$ is surjective if and only if $d(n, k) \leq 1$.

**Proof.** By Theorem 3.5, $f^n_k$ is surjective if and only if for all $\lambda \vdash o n - 2^k$ we have that the $k$th row of $Q_2(\lambda)$ contains a $d(n, k)$-good partition $\lambda_i^{(k)}$. By Theorem 2.5 and Definition 3.4, for any $\lambda \vdash o n - 2^k$ we have $\sum_{j \geq 0} |\lambda_j^{(k)}| = \lfloor \frac{n-2^k}{2^d} \rfloor = (2^{d(n, k)} - 1) + m(n, k)$.

If $k = 0$ then $Q_2^{(0)}(\lambda)$ contains only $\lambda = \lambda_0^{(0)}$. Hence $f^n_0$ is surjective if and only all odd partitions of $n - 1$ are $d(n, 0)$-good. By Lemma 3.3(1), the latter condition holds when $d = d(n, 0) \leq 2$. On the other hand, if $d = \nu_2(n) > 2$, then $\lambda = (n - 5, 2, 2)$ is an odd partition of $n - 1$ by Theorem 2.5, but $C_6(\lambda)$ is not a hook, and hence $C_{2d}(\lambda)$ is not a hook. So $\lambda$ is not $d$-good, and thus $f^n_0$ is not surjective.

Now assume $k \geq 1$. Then $Q_2^{(k)}(\lambda)$ contains at least two odd partitions. If $d(n, k) \geq 2$ then any $d(n, k)$-good partition $\mu$ satisfies $3 \subseteq 2^{d(n, k) - 1} \subseteq |\mu|$. Write $\lfloor \frac{n-2^k}{2^d} \rfloor = 1 + m_1$ where $m_1$ is even. Applying Remark 2.7, take any $\lambda \vdash o n - 2^k$ such that $|\lambda_0^{(k)}| = 1$ and $\lambda_1^{(k)}$ is an odd partition with $|\lambda_1^{(k)}| = m_1$. Then no partition in $Q_2^{(k)}(\lambda)$ is $d(n, k)$-good. Thus $f^n_k$ is not surjective. On the other hand, if $d(n, k) = 0$ then $2^k \subseteq n$ and $f^n_k$ is surjective [3, Proposition 4.5]. If $d(n, k) = 1$ then $\lfloor \frac{n-2^k}{2^d} \rfloor = 1 + m(n, k)$, where $4 \mid m(n, k)$. Thus any $Q_2^{(k)}(\lambda)$ contains a partition with odd cardinality; this partition is 1-good, by Lemma 3.3. Again $f^n_k$ is surjective.

It is an immediate consequence of Theorem 3.5 that $f^n_k$ is regular on its image for all relevant choices of $n, k$ such that $2^k < n$. We have:

**Corollary 3.8.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$; set $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash o n - 2^k$. Then

$$e(\lambda, 2^k) = \begin{cases} 2^k & \text{if } d = 0; \\ 2 & \text{if } d > 0, \text{ and the } k\text{th row of } Q_2(\lambda) \text{ contains a } d\text{-good partition}; \\ 0 & \text{otherwise}. \end{cases}$$
Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take $k = 2$ above. For $n > 2^2$ we first compute $d(n, k) = \nu_2(\left\lfloor \frac{n}{2^k} \right\rfloor)$, and then consider odd partitions of $n - 4$ and their 4-extensions. For $n = 6$, $d(6, 2) = 0$. Thus $e(2, 4) = 4$. The odd 4-extensions of $(2)$ are $(6), (3^2), (2^2, 1^2), (2, 1^4)$. For $n = 10$, $d(10, 2) = 1$. In this case, $e(\lambda, 4) = 2$ for all odd partitions $\lambda$ of 6. For instance, the odd 4-extensions of $(6)$ are $(10)$ and $(6, 3, 1)$. For $n = 19$, $d(19, 2) = 2$. Example 2.6 shows that for $\lambda = (5, 4, 2^2, 1^2) \vdash_o 15$ there is no 2-good partition in $Q_2(2)\lambda$, hence $e(\lambda, 4) = 0$.

4 Deciding commutativity of the maps $f_k$ and $f_\ell$

Let $n \in \mathbb{N}$, and suppose that $0 \leq k < \ell$ satisfy $2^k + 2^\ell \leq n$. As stated in the introduction, we want to complete the discussion of the commutativity of the maps $f_k$ and $f_\ell$. Since the relevant $n$ will always be apparent for the maps $f_k^n$ in this section, we just write $f_k$.

We write $(n; k, \ell) \in T$ if for all $\lambda \vdash_o n$ we have $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$. Otherwise we write $(n; k, \ell) \in F$.

In this section we will prove Theorem B, which may be reformulated as follows.

**Theorem 4.1.** Let $n = 2^t + m$ where $0 \leq m < 2^t$. Suppose that $k, \ell$ satisfy $0 \leq k < \ell$ and $2^k + 2^\ell \leq n$. Then with the exception of $(6; 0, 1)$

$$(n; k, \ell) \in F \text{ if and only if } \ell < t \text{ and } 2^k \leq m.$$  

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two extreme cases, where $f_k$ and $f_\ell$ commute.

In the case $\ell = t$ we have the following result as a reformulation of [3, Proposition 4.3].

**Lemma 4.2.** Let $n = 2^t + m$ with $0 \leq m < 2^t$. If $2^k \leq m$, then $(n; k, t) \in T$.

It is also known that in the case where $n$ is a power of 2, the maps $f_k$ and $f_\ell$ commute [3, Remark 4.4], and we include a short proof here.

**Lemma 4.3.** If $n = 2^t$ then $(n; k, \ell) \in T$ for all $k, \ell$.

**Proof.** If $0 \leq b \leq a$ are integers then the binomial coefficient $\binom{a}{b}$ is odd if and only if $b \subseteq_2 a$, by Lucas’ theorem. The odd partitions of $2^t$ are exactly the hook partitions $(2^t - b, 1^b)$, $0 \leq b \leq 2^t - 1$, of degree $(2^t - 1)_b$. Hence for $k \in \{0, 1, \ldots, t - 1\}$ we have

$$f_k(\lambda) = \begin{cases} (2^t - b - 2^k, 1^b) & \text{if } 2^k \nsubseteq_2 b, \\ (2^t - b, 1^{b-2^k}) & \text{if } 2^k \subseteq_2 b. \end{cases}$$

It follows that for any $k, \ell < t$ and odd partition $\lambda$ of $2^t$, we have $f_\ell f_k(\lambda) = f_k f_\ell(\lambda)$.

**Lemma 4.4.** Let $n = 2^t + m$ with $0 \leq m < 2^t$. Suppose that $k, \ell$ satisfy $0 \leq k < \ell$ and $2^k + 2^\ell \leq n$. If $m < 2^k$ then $(n; k, \ell) \in T$.

**Proof.** We use induction on $k \geq 0$. For $k = 0$ we have $m = 0$ and the claim follows from Lemma 4.3. Suppose that $k \geq 1$ and that the claim has been proved up to $k - 1$. Let $\lambda \vdash_o n$. Odd hooks of length $2^k$ and $2^\ell$ in $\lambda$ correspond to odd hooks of length $2^{k-1}$ and $2^{\ell-1}$ in the 2-quotient $Q_2(\lambda) = (\lambda_0, \lambda_1)$ of $\lambda$. From Theorem 2.5 we deduce that $|\lambda_0|$ and $|\lambda_1|$ are 2-disjoint binary subsums of $\left\lfloor \frac{n}{2} \right\rfloor$, so one of them contains $2^{t-1}$, say $|\lambda_0|$; then $|\lambda_1| \leq \left\lfloor \frac{m}{2} \right\rfloor < 2^{k-1} < 2^{t-1}$. Thus the odd $2^{k-1}$-hook in $Q_2(\lambda)$ has to be in $\lambda_0$. Therefore

$$Q_2(f_k(\lambda)) = (f_{k-1}(\lambda_0), \lambda_1).$$
Applying $f_\ell$, the odd $2^{\ell-1}$-hook cannot be in $\lambda_1$, hence
\[ Q_2(f_\ell f_k(\lambda)) = (f_{\ell-1} f_{k-1}(\lambda_0), \lambda_1)). \]
In particular, we know that $|\lambda_0| \geq 2^{\ell-1} + 2^{k-1}$. Also $|\lambda_0| + |\lambda_1| = \left\lceil \frac{n}{2} \right\rceil = 2^{\ell-1} + \left\lceil \frac{n}{2} \right\rceil$. We have already seen that $2^{\ell-1}$ is the largest binary digit of $|\lambda_0|$; furthermore $|\lambda_0| - 2^{\ell-1}$ is a binary subsum of $\left\lceil \frac{n}{2} \right\rceil < 2^{k-1}$. We may therefore apply the inductive hypothesis to $\lambda_0$ to get $f_{\ell-1} f_{k-1}(\lambda_0) = f_{k-1} f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_k f_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$. □

Lemmas 4.2 and 4.4 show that the only if part of the theorem is true. We now turn to the if part. We start by proving the statement for $k = 0$ and use this as part of an inductive argument.

**Lemma 4.5.** Let $n = 2^t + m$ with $0 < m < 2^t$. If $0 < \ell < t$ then $(n; 0, \ell) \in F$, with the exception of $(6; 0, 1)$.

**Proof.** The result is easily checked for $n \leq 8$, which includes the exception $(6; 0, 1)$. So we assume that $t \geq 3$.

**Case 1:** $2^\ell < m$. Then $m \geq 3$, since $\ell > 0$. Consider the partition $\lambda = (m, m, 1^a) \vdash n$ where $a = n - 2m = 2^\ell - m$. The $(1,1)$-hook length of $\lambda$ is $2^\ell + 1$. The $(2,1)$-hook length of $\lambda$ is $2^\ell$.

Removing the $(2,1)$-hook we get the odd partition $(m)$, so $\lambda$ is odd, by Lemma 2.8. We claim that
\[ f_0(\lambda) = (m, m, 1^{a-1}). \]

Indeed we cannot have $f_0(\lambda) = (m, m - 1, 1^a)$ because this partition does not have a hook of length $2^\ell$, and thus it is not odd. Now
\[ f_\ell(f_0(\lambda)) = f_\ell(m, m, 1^{a-1}) = (m, m - 2^\ell, 1^{a-1}) \]

since $(m, m, 1^{a-1-2^\ell})$ and $(m - 1, m - 2^\ell + 1, 1^{a-1})$ both do not have a hook of length $2^\ell$ and thus are not odd (again by Lemma 2.8).

On the other hand,
\[ f_\ell(\lambda) = (m - 1, m - (2^\ell - 1), 1^a). \]

Indeed, the other candidates for $f_\ell(\lambda)$, which are $(m, m - 2^\ell, 1^a)$ and $(m, m, 1^{a-2^\ell})$, do not have hooks of length $2^\ell$. Then
\[ f_0(f_\ell(\lambda)) = f_0(m - 1, m - (2^\ell - 1), 1^a) = (m - 1, m - 2^\ell, 1^a). \]

This follows (again) by observing that all the other partitions of $n - 2^\ell - 1$ obtained from $(m - 1, m - (2^\ell - 1), 1^a)$ by removing a node do not have hooks of length $2^\ell$. Thus $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

**Case 2:** $m < 2^\ell$. Consider the partition $\lambda = (n - 2^\ell, m, 1^a)$, where $a = 2^\ell - (m + 1)$. Note that $n - 2^\ell \geq m + 1$ since $\ell < t$ by assumption, and that $a \geq 0$. The $(1,1)$-hook length of $\lambda$ is $n - m = 2^\ell$. Removing this hook we get the odd partition $(m)$, so $\lambda$ is odd. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Now
\[ f_0(\lambda) = (n - 2^\ell, m, 1^a) \]

since the other candidates do not have hooks of length $2^\ell$. Then
\[ f_\ell(f_0(\lambda)) = f_\ell(n - 2^\ell, m, 1^a) = \mu, \]

where $\mu$ is a binary subsum of $\left\lceil \frac{n}{2} \right\rceil < 2^{k-1}$. We may therefore apply the inductive hypothesis to $\lambda_0$ to get $f_{\ell-1} f_{k-1}(\lambda_0) = f_{k-1} f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_k f_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$. □
where $\mu$ is obtained from $f_0(\lambda)$ by removing a $2^\ell$-hook in the first row. (There are only hooks of length $<2^\ell$ in the other rows.) In fact, $\mu = (n - 2^{\ell+1}, m, 1^a)$ since $n - 2^{\ell+1} \geq n - 2^\ell = m$. Thus $f_\ell(f_0(\lambda))$ has at least 2 parts. On the other hand

$$f_\ell(\lambda) = (n - 2^\ell)$$

since this odd partition is obtained from the odd partition $\lambda$ by removing a $2^\ell$-hook (the one in $(2,1)$). It follows that

$$f_0(f_\ell(\lambda)) = (n - 2^\ell - 1)$$

and again $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

**Case 3:** $m = 2^\ell$. Then $n = 2^\ell + 2^\ell$. If $\ell \geq 2$ then choose $\lambda = (2^\ell, 2^\ell - 1, 1)$. The $(1,2)$-hook length of $\lambda$ is $2^\ell$; thus $\mu$ is an odd partition since removing this $2^\ell$-hook gives an odd partition $(2^\ell - 2, 1, 1)$ of $2^\ell$. We have $f_0(\lambda) = (2^\ell, 2^\ell - 2, 1)$ since the other candidates are not odd. Then

$$f_\ell(f_0(\lambda)) = (2^\ell - 2^\ell, 2^\ell - 2, 1).$$

The $(2,1)$-hook length of $\lambda$ is $2^\ell$, so $f_\ell(\lambda) = (2^\ell)$ and

$$f_0(f_\ell(\lambda)) = (2^\ell - 1),$$

showing $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

On the other hand, if $\ell = 1$ then choose $\lambda = (2^\ell - 2, 2, 2) \to 2^\ell + 2 = n$. Since $t \geq 3$, it is now easy to show that $f_\ell(f_0(\lambda)) = (2^\ell - 42, 1)$. On the other hand we see that $f_0(f_\ell(\lambda))$ is a hook partition of $2^\ell - 1 = n - 3$ and therefore is not equal to $f_\ell(f_0(\lambda))$. \hfill $\blacksquare$

**Lemma 4.6.** If $(n; k, \ell) \in \mathcal{F}$ then also $(2n; k+1, \ell+1) \in \mathcal{F}$ and $(2n+1; k+1, \ell+1) \in \mathcal{F}$.

**Proof.** Let the odd partition $\mu$ of $n$ satisfy $f_k f_\ell(\mu) \neq f_\ell f_k(\mu)$. Let $\lambda$ be a partition of $2n$ or $2n+1$ having 2-quotient $Q_2(\lambda) = (\mu, (0))$. Then $\lambda$ is odd, by Theorem 2.5. We have

$$Q_2(f_{k+1} f_{\ell+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_\ell f_k(\mu), (0)) = Q_2(f_{\ell+1} f_{k+1}(\lambda)),$$

so that $f_{k+1} f_{\ell+1}(\lambda) \neq f_{\ell+1} f_{k+1}(\lambda)$.

We are now ready to conclude this section with the proof of Theorem B.

**Proof of Theorem 4.1.** The *only if* part follows from Lemmas 4.2 and 4.4. To prove the *if* part we use induction on $k \geq 0$. If $k = 0$, then the statement follows from Lemma 4.5. Let $k > 1$ and suppose that the assertion is true up to and including $k - 1$. To show that $(n; k, \ell) \in \mathcal{F}$ it suffices to prove $((n; k, \ell) - 1, \ell - 1) \in \mathcal{F}$, by Lemma 4.6. We are assuming $n = 2^\ell + m$, $0 \leq m < 2^\ell$, $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. This implies $\left\lceil \frac{n}{2^\ell} \right\rceil = 2^{\ell-1} + \left\lceil \frac{m}{2^\ell} \right\rceil$, $0 \leq \left\lceil \frac{m}{2^\ell} \right\rceil < 2^{\ell-1}$ and $2^{k-1} + 2^{\ell-1} \leq \left\lceil \frac{n}{2^{\ell-1}} \right\rceil$. We may apply the inductive hypothesis to get $\left\lceil \frac{n}{2^\ell} \right\rceil; k-1, \ell-1) \in \mathcal{F}$, and then $(n; k, \ell) \in \mathcal{F}$ except when $\left(\left\lceil \frac{n}{2^{\ell-1}} \right\rceil; k-1, \ell-1\right) = (6; 0, 1)$. In that case we are considering $(12; 1, 2)$ or $(13; 1, 2)$ which are both in $\mathcal{F}$, by direct computation (consider for example $(6, 4, 2) \to 12$ and $(6, 4, 3) \to 13$, respectively). \hfill $\blacksquare$

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