Restriction of Odd Degree Characters of $S_n$

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Abstract. Let $n$ and $k$ be natural numbers such that $2^k < n$. We study the restriction to $S_{n-2^k}$ of odd-degree irreducible characters of the symmetric group $S_n$. This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., Sém. Lothar. Combin. 75 (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., J. Algebra 478 (2017), 271–282].

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1 Introduction

Let $n$ be a natural number, and let $\chi$ be an irreducible character of odd degree of the symmetric group $S_n$. Then there exists a unique odd-degree irreducible constituent of the restriction $\chi_{S_{n-1}}$. This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any $k \in \mathbb{N}$ such that $2^k < n$, there exists a unique odd-degree irreducible constituent $f^n_k(\chi)$ of $\chi_{S_{n-2^k}}$ appearing with odd multiplicity. The main goal of this article is to study for all $n, k \in \mathbb{N}$ the map

$$f^n_k : \text{Irr}^2(S_n) \longrightarrow \text{Irr}^2(S_{n-2^k}),$$

naturally defined by Theorem A of [3]. All our results are proved using a description of $f^n_k$ in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If $2^k$ appears in the binary expansion of $n$ we say that $2^k$ is a binary digit of $n$. Similarly we say that two natural numbers $m$ and $n$ are 2-disjoint if they do not have any common binary digit. On the other hand, if $m \leq n$ and all the binary digits of $m$ appear in the binary expansion of $n$, then we say that $m$ is a binary subsum of $n$. This will be denoted by $m \subseteq_2 n$. Let $v_2(n)$ be the exponent of the highest power of 2 dividing the integer $n$.

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A question raised in [3] may be phrased as: For which $n$ and $k$ is $f_k^n$ surjective? The authors showed that $f_k^n$ is surjective whenever $2^k$ is a binary digit of $n$, and they observed that otherwise $f_k^n$ could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

**Theorem A.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $d(n, k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor \right)$.

- If $k = 0$ then $f_k^n$ is surjective if and only $d(n, k) \leq 2$.
- If $k > 0$ then $f_k^n$ is surjective if and only $d(n, k) \leq 1$.

Theorem A is a consequence of Theorem 3.5 below, which describes the images of the maps $f_k^n$.

For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $2^k < n$ and any $\psi \in \text{Irr}_2(\mathfrak{S}_{n-2^k})$ we define the set

$$\mathcal{E}(\psi, 2^k) = \{ \chi \in \text{Irr}_2(\mathfrak{S}_n) \mid f_k^n(\chi) = \psi \},$$

and set $e(\psi, 2^k) = |\mathcal{E}(\psi, 2^k)|$. We show in Corollary 3.8 that the maps $f_k^n$ are regular on their images. This means that for any $\psi$ in the image of $f_k^n$, the number $e(\psi, 2^k)$ depends only on $n$ and $k$ and not on the specific $\psi$. We also give a complete description of those $\psi \in \text{Irr}_2(\mathfrak{S}_{n-2^k})$ such that $e(\psi, 2^k) = 0$, in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote $f_k^n$ just by $f_k$, when the natural number $n$ is clear from the context. Then, for $k, \ell \in \mathbb{N}_0$, $k < \ell$, such that $2^k + 2^\ell \leq n$, we may ask: when is $f_k f_\ell = f_\ell f_k$? or more specifically: when is $f_k^{n-2^k} f_\ell^n = f_\ell^{n-2^\ell} f_k^n$? In [3, Proposition 4.3] it was proved that $f_k f_\ell = f_\ell f_k$ whenever $2^\ell < n < 2^{\ell+1}$. This is the case $\ell = t$ in our second main result, which answers the question completely.

**Theorem B.** Let $n = 2^t + m$ where $0 \leq m < 2^t$. Suppose that $k, \ell$ satisfy $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. Then, with the exception of the case $n = 6, k = 0, \ell = 1$,

$$f_k f_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.$$

## 2 Notation and background

Let $n$ be a natural number. We let $\text{Irr}(\mathfrak{S}_n)$ denote the set of irreducible characters of $\mathfrak{S}_n$ and $\mathcal{P}(n)$ the set of partitions of $n$. The notation $\lambda \in \mathcal{P}(n)$ is sometimes replaced by $\lambda \vdash n$ and we write $|\lambda| = n$. There is a natural correspondence $\lambda \leftrightarrow \chi^\lambda$ between $\mathcal{P}(n)$ and $\text{Irr}(\mathfrak{S}_n)$. We say then that $\lambda$ labels $\chi^\lambda$. We denote by $\text{Irr}_2(\mathfrak{S}_n)$ the set of irreducible characters of $\mathfrak{S}_n$ of odd degree. If $\chi^\lambda \in \text{Irr}_2(\mathfrak{S}_n)$ we say that $\chi^\lambda$ is an odd character, we call $\lambda$ an odd partition of $n$ and write $\lambda \vdash_o n$. Also the empty partition will be considered as an odd partition.

**Remark 2.1.** Let $n, k$ be such that $2^k < n$. In [3, Theorem A and Proposition 4.2] it is shown that the map $f_k^n : \text{Irr}_2(\mathfrak{S}_n) \to \text{Irr}_2(\mathfrak{S}_{n-2^k})$ may be described in terms of the odd partitions labelling the odd characters as follows:

$$f_k^n(\chi^\lambda) = \chi^\mu \iff \mu \vdash_o n - 2^k \text{ can be obtained from } \lambda \vdash_o n \text{ by removing a } 2^k\text{-hook}.$$

Correspondingly we write (by abuse of notation) $f_k^n(\lambda) = \mu$. In fact when $\lambda$ is odd, there is only one $2^k$-hook of $\lambda$ whose removal leads again to an odd partition; we will refer to such a hook as an odd hook of $\lambda$. This combinatorial description of $f_k^n$ will be used throughout this paper, and we will regard $f_k^n$ also as a map between the corresponding sets of odd partitions. Also, for $\mu \vdash_o n - 2^k$ we set $e(\mu, 2^k) = e(\chi^\mu, 2^k)$.
We need some concepts and basic facts concerning hooks in partitions. For any integer \( e \in \mathbb{N} \) we denote by \( C_e(\lambda) \) and \( Q_e(\lambda) \) the \( e \)-core and the \( e \)-quotient of \( \lambda \), respectively. Then \( Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1}) \) is an \( e \)-tuple of partitions satisfying \( n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i| \). It is well-known that a partition is uniquely determined by its \( e \)-core and \( e \)-quotient (we refer the reader to [6] or [4, Chapter 2.7] for a detailed discussion on this topic).

Let \( H_e(\lambda) \) be the set of hooks of \( \lambda \) having length divisible by \( e \), and let \( \mathcal{H}(Q_e(\lambda)) = \cup_{i=1}^{e-1} \mathcal{H}(Q_i) \).

As explained in [6, Theorem 3.3], there is a bijection between \( H_e(\lambda) \) and \( \mathcal{H}(Q_e(\lambda)) \) mapping hooks in \( \lambda \) of length \( ex \) to hooks in the quotient of length \( x \). Moreover, the bijection respects the process of hook removal. Namely, the partition \( \mu \) obtained by removing an \( ex \)-hook from \( \lambda \) is such that \( C_e(\mu) = C_e(\lambda) \) and the \( e \)-quotient of \( \mu \) is obtained by removing an \( x \)-hook from one of the partitions involved in \( Q_e(\lambda) \).

For \( e = 2 \) we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-quotient tower \( Q_2(\lambda) \) and the 2-core tower \( C_2(\lambda) \) of \( \lambda \). They have rows numbered by \( k \geq 0 \).

The \( k \)-th row \( Q_2(k)(\lambda) \) of \( Q_2(\lambda) \) contains \( 2^k \) partitions \( \lambda_i^{(k)} \), \( 0 \leq i \leq 2^k - 1 \), and the \( k \)-th row \( C_2(k)(\lambda) \) of \( C_2(\lambda) \) contains the 2-cores of these partitions in the same order, i.e., \( C_2(\lambda_i^{(k)}) \), \( 0 \leq i \leq 2^k - 1 \).

The 0th row of \( Q_2(\lambda) \) contains \( \lambda = \lambda_0^{(0)} \) itself, row 1 contains the partitions \( \lambda_0^{(1)}, \lambda_1^{(1)} \) occurring in the 2-quotient \( Q_2(\lambda) \), row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have \( Q_2(\lambda_i^{(k)}) = (\lambda_i^{(k+1)}, \lambda_{i+1}^{(k+1)}) \) for \( i \in \{0, 1, \ldots, 2^k - 1\} \). We remark that the \( 2^k \) partitions in \( Q_2(k)(\lambda) \) are the same as those in the \( 2^k \)-quotient \( Q_{2^k}(\lambda) \) of \( \lambda \), but in a different order for \( k \geq 2 \).

We also introduce the \( k \)-data \( D_2(k)(\lambda) \) of \( \lambda \). This is a table containing the following \( k+1 \) rows: the \( k \) rows \( C_2^{(j)}(\lambda) \), \( j = 0, \ldots, k - 1 \), and in addition the row \( Q_2(k)(\lambda) \).

**Remark 2.2.** A partition \( \lambda \) may be recovered from its 2-core tower. For \( k \geq 0 \), it may also be recovered from the knowledge of the \( k \)-data \( D_2(k)(\lambda) \) of \( \lambda \), because the rows \( C_2^{(l)}(\lambda) \) with \( l \geq k \) of \( C_2(\lambda) \) consist of the 2-core towers of the partitions in \( Q_2(k)(\lambda) \).

**Lemma 2.3.** Suppose that \( \lambda \vdash n - 2^k \) and \( \mu \vdash n \). The following are equivalent.

1. \( \lambda \) is obtained from \( \mu \) by removing a \( 2^k \)-hook.
2. The \( k \)-data \( D_2(k)(\mu) \) and \( D_2(k)(\lambda) \) coincide, except that for one \( i \in \{0, \ldots, 2^k - 1\} \) \( \lambda_i^{(k)} \) is obtained from \( \mu_i^{(k)} \) by removing a 1-hook.

**Proof.** A \( 2^k \)-hook \( H_0 \) in \( \mu \) corresponds in a canonical way to a \( 2^{k-1} \)-hook \( H_1 \) in a partition in \( Q_2^{(1)}(\mu) \), i.e., in row 1 of the 2-quotient tower \( Q_2(\mu) \). Continuing we see that \( H_0 \) corresponds in a canonical way to a 1-hook \( H_k \) in a partition \( \mu_i^{(k)} \) in \( Q_2^{(k)}(\mu) \), row \( k \) of \( Q_2(\mu) \). If \( \lambda \) is obtained by removing \( H_0 \) from \( \mu \), this corresponds to \( \lambda_i^{(k)} \) being obtained by removing the 1-hook \( H_k \) from \( \mu_i^{(k)} \) (by repeated applications of [6, Theorem 3.3]). Apart from this the rows \( Q_2^{(k)}(\mu) \) and \( Q_2^{(k)}(\lambda) \) coincide. Note also that the rows \( C_2^{(j)}(\mu) \) and \( C_2^{(j)}(\lambda) \) coincide for \( j = 0, \ldots, k - 1 \), since the removal of the hooks \( H_j \) of even length do not change the 2-cores.

Odd-degree characters of \( \mathfrak{S}_n \) and thus odd partitions were completely described in [5]. We restate this result in a language which is convenient for our purposes. We let \( c_2^{(k)}(\lambda) \) be the sum of the cardinalities of the partitions in the \( k \)-th row \( C_2^{(k)}(\lambda) \) of \( C_2(\lambda) \).

**Lemma 2.4 ([5]).** Let \( \lambda \) be a partition. Then \( \lambda \) is odd if and only if \( c_2^{(k)}(\lambda) \leq 1 \) for all \( k \geq 0 \).

It may be decided from the \( k \)-data \( D_2^{(k)}(\lambda) \) whether \( \lambda \) is odd. The case \( k = 1 \) of the following result appeared in [3, Lemma 4.1] and also in [1, Lemma 6].
Theorem 2.5. Let $\lambda \vdash n$, and let $k \geq 0$ be fixed. Consider $Q_2^{(k)}(\lambda) = (\lambda_i^{(k)})$. Then $\lambda$ is odd if and only if the following conditions are all fulfilled:

(i) $c_2^{(j)}(\lambda) \leq 1$ for all $j < k$.

(ii) The partitions $\lambda_i^{(k)}$, $0 \leq i \leq 2^k - 1$, are all odd.

(iii) The numbers $|\lambda_i^{(k)}|$, $0 \leq i \leq 2^k - 1$, are pairwise 2-disjoint.

In this case $\sum_{i \geq 0} |\lambda_i^{(k)}| = \left\lfloor \frac{n}{2^k} \right\rfloor$.

Proof. This is proved by induction on $k \geq 0$, using Remark 2.2 and Lemma 2.4. ♦

We illustrate the result above by giving an example.

Example 2.6. Let $n = 15$ and take $\lambda = (5, 4, 2^2, 1^2) \vdash 15$. To decide whether $\lambda$ is odd, we choose $k = 2$ and compute the 2-data $D_2^{(2)}(\lambda)$. The 2-core is $C_2(\lambda) = (1)$, giving $C_2^{(0)}(\lambda) = ((1))$. Furthermore, the 2-quotient is $Q_2(\lambda) = ((2^2, 1^2), (1))$, and computing the 2-cores $C_2((2^2, 1^2)) = (0)$, $C_2((1)) = (1)$, we obtain the next row: $C_2^{(1)}((1)) = (0), (1))$. The 2-quotients are $Q_2((2^2, 1^2)) = ((1^2), (1))$, $Q_2((1)) = ((0), (0))$; hence the final row of the 2-data table is obtained as $Q_2^{(2)}(\lambda) = ((1^2), (1), (0), (0))$.

We visualize $D_2^{(2)}(\lambda)$ like this:

$C_2^{(0)}(\lambda)$: (1)

$C_2^{(1)}(\lambda)$: (0) (1)

$Q_2^{(2)}(\lambda)$: (12) (1) (0) (0)

Theorem 2.5 shows that $\lambda$ is odd and thus it contains a unique odd 4-hook. Again using the theorem, it is clear that removing this 4-hook corresponds to the second partition (1) in $Q_2^{(2)}(\lambda)$ being replaced by (0). Thus, removing the corresponding 4-hook of $\lambda$ we obtain the odd partition $\mu = (3, 2^2, 1^2) \vdash 11$ with the property that $D_2^{(2)}(\lambda)$ and $D_2^{(2)}(\mu)$ differ only in their final row.

Remark 2.7. Using the construction of partitions from their 2-cores and 2-quotients already mentioned, the criterion above can be applied to construct all odd partitions of $n$ with a specific $k$th row in the 2-quotient tower. For this, let $n, k \in \mathbb{N}$, and take any sequence of odd partitions $\nu_i$, $0 \leq i \leq 2^k - 1$, such that the numbers $|\nu_i|$ are pairwise 2-disjoint, and $\sum_{i \geq 0} |\nu_i| = \left\lfloor \frac{n}{2^k} \right\rfloor$.

Then there are exactly $\prod_{m<k \atop 2m \leq 2n} 2^m$ odd partitions $\lambda$ of $n$ with $Q_2^{(k)}(\lambda) = (\nu_i)$, obtained by choosing one 2-core in row $m$ of the $k$-data table to be (1), for each $m < k$ such that $2^m \leq 2n$.

The following easy consequence of Theorem 2.5 will be used repeatedly.

Lemma 2.8. Let $2^t$ be the largest binary digit of $n$. A partition $\lambda$ of $n$ is odd if and only if $\lambda$ contains a unique $2^t$-hook and the partition obtained from $\lambda$ by removing this $2^t$-hook is an odd partition of $n - 2^t$.

3 Surjectivity and regularity

The aim of this section is to study the images of the maps $f_k^n$ for all $n, k$ such that $2^k \leq n$. For this purpose we introduce the concept of $d$-good partitions (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when $f_k^n$ is surjective) and to show that the maps $f_k^n$ are always regular on their image (see Corollary 3.8).
Definition 3.1. Let $d \geq 0$. We call an odd partition $\lambda$ $d$-good, if

(i) $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.
(ii) $C_{2^d}(\lambda)$ is a hook partition.

Let us remark that condition (i) may be reformulated as

(i*) $\nu_2(|\lambda| + 1) = d$.

In particular, if $\lambda$ is $d$-good, then $|\lambda|$ is odd if and only if $d > 0$.

The relevance of $d$-good partitions in our context is illuminated by the following reformulation of [1, Theorem 2]:

Lemma 3.2. Let $\lambda \vdash_o n$. Let $d = \nu_2(n + 1)$. Then $e(\lambda, 1) \neq 0$ if and only if $\lambda$ is $d$-good. In this case, $e(\lambda, 1) = 1$ if $d = 0$, and $e(\lambda, 1) = 2$ if $d > 0$.

Lemma 3.3. Let $\lambda$ be an odd partition, and let $d \geq 0$. Then the following hold.

(1) For $d \leq 2$, $\lambda$ is $d$-good if and only if $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.
(2) If $\lambda$ is $d$-good, then $C_{2^d}(\lambda)$ is a partition of $2^d - 1$.

Proof. If the odd partition $\lambda$ is $d$-good, then $|\lambda| = (2^d - 1) + m$ where the binary digits of $m$ are at least $2^{d+1}$. The hooks of $\lambda$ corresponding to the binary digits of $m$ may be decomposed into $2^d$-hooks and thus do not contribute to $C_{2^d}(\lambda)$. Thus $|C_{2^d}(\lambda)| = 2^d - 1$. This shows (2). For $d = 0, 1, 2$ we have $|C_{2^d}(\lambda)| = 0, 1$ and $3$, respectively. Since all partitions of $0, 1$ and $3$ are hook partitions, (1) follows.

Definition 3.4. If $2^k \leq n$, we define $d(n, k) = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Thus $d(n, k)$ is the smallest integer $d \geq 0$ satisfying the condition $2^{k+d} \subseteq n$. In particular, $d(n, k) = 0$ if and only if $2^k \subseteq n$. Moreover, we may write $\lfloor \frac{n}{2^k} \rfloor = 2^{d(n, k)} + m(n, k)$ where $2^{d(n, k)+1} | m(n, k)$.

As mentioned in the introduction, the results in [3] show that $f^n_k$ is a surjective $(2^k$-to-$1$)-map whenever $2^k \subseteq n$, i.e., $d(n, k) = 0$. In the spirit of [1, Theorem 2], we now give a characterization of the image of the map $f^n_k$ for all $n, k$ such that $2^k < n$.

Theorem 3.5. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $\lambda \vdash_o n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a $d(n, k)$-good partition in the $k$th row of $Q_2(\lambda)$. In this case, $e(\lambda, 2^k) = 2^k$ if $d(n, k) = 0$, and $e(\lambda, 2^k) = 2$ if $d(n, k) > 0$.

Proof. If $k = 0$ then the statement follows from Lemma 3.2. Hence assume that $k \geq 1$. Let $d = d(n, k)$. By assumption $\lfloor \frac{n}{2^k} \rfloor = 2^d + m$, where the binary digits of $m$ are at least $2^{d+1}$. Thus $\lfloor \frac{n-2^k}{2^k} \rfloor = (2^d - 1) + m$.

Suppose first that $e(\lambda, 2^k) \neq 0$ and that $\mu \vdash_o n$ satisfies $f^k_n(\mu) = \lambda$. From Remark 2.1 and Lemma 2.3 we get that there exists an $i \in \{0, 1, \ldots, 2^k - 1\}$ such that $f^k_0(\mu_i^{(k)}) = \lambda_i^{(k)}$. Since $\mu_i^{(k)}$ and $\lambda_i^{(k)}$ are odd, we get $e(\lambda_i^{(k)}, 1) \neq 0$. We have that $|\lambda_i^{(k)}|$ and $|\mu_i^{(k)}|$ are both 2-disjoint with $m_1 := \sum_{j \neq i} |\lambda_j^{(k)}| = \sum_{j \neq i} |\mu_j^{(k)}| \subseteq 2 \lfloor \frac{n}{2^k} \rfloor$, by Theorem 2.5. Since $m_1 \subseteq 2 \lfloor \frac{n-2^k}{2^k} \rfloor$ and $m_1 \subseteq 2 \lfloor \frac{n}{2^k} \rfloor$, we get $m_1 \subseteq m$. Thus $|\lambda_i^{(k)}| = (2^d - 1) + m_2$ and $|\mu_i^{(k)}| = 2^d + m_2$, where $m_2 = m - m_1 \subseteq m$. In particular $\nu_2(|\lambda_i^{(k)}| + 1) = \nu_2(|\mu_i^{(k)}|) = d$. Then Lemma 3.2 shows that $\lambda_i^{(k)}$ is $d$-good.

Conversely, if $\lambda_i^{(k)}$ is a $d$-good partition for some $i \in \{0, 1, \ldots, 2^k - 1\}$, then there exists a $\mu^* \vdash_o |\lambda_i^{(k)}| + 1$ such that $f^k_0(\mu^*) = \lambda_i^{(k)}$, by Lemma 3.2. We let $\mu$ be the partition where the $k$-data $D_2^{(k)}(\mu)$ and $D_2^{(k)}(\lambda)$ coincide, except that $\mu_i^{(k)} = \mu^*$. Since $\lambda$ is odd and $\lambda_i^{(k)}$ is $d$-good,
we know that $|\lambda_i^{(k)}| = (2^d - 1) + m'$ where $m' \subseteq m$, and $|\lambda_j^{(k)}| \subseteq 2m - m'$ for all $j \neq i$. Hence $|\mu'| = |\lambda_i^{(k)}| + 1 = 2^d + m'$ is 2-disjoint from all $|\lambda_j^{(k)}|$, $j \neq i$. Thus $\mu$ is an odd partition of $n$ by Theorem 2.5, and $f_k(\mu) = \lambda$ by Lemma 2.3 and Remark 2.1.

We conclude that $e(\lambda, 2^k) = \sum_{\lambda^{(k)}_{d}\text{-good}} e(\lambda^{(k)}_i, 1)$. If $d = 0$ then $\left\lfloor \frac{n - 2^k}{2} \right\rfloor$ is even. This implies that all $\lambda_i^{(k)}$ are of even cardinality and thus $d$-good. Thus $e(\lambda_i^{(k)}_i, 1) = 1$ for all $i$, and we get $e(\lambda, 2^k) = 2^k$. If $d > 0$ there is exactly one $\lambda_i^{(k)}$ in $Q_2^{(k)}(\lambda)$ of odd cardinality. Only this $\lambda_i^{(k)}$ may be $d$-good and then $e(\lambda, 2^k) = e(\lambda^{(k)}_i, 1) = 2$. Otherwise $e(\lambda, 2^k) = 0$. 

\section*{Corollary 3.6.} Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$, and let $d = \nu_2\left(\frac{n}{2^k}\right)$. Let $\lambda \vdash_o n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a partition $\lambda_i^{(k)}$ in the $k$th row of $Q_2(\lambda)$ such that $|\lambda_i^{(k)}| \equiv 2^d - 1 \mod 2^d + 1$, and $C_{2^d}(\lambda_i^{(k)})$ is a hook partition. In this case, $e(\lambda, 2^k) = 2^k$ if $d = 0$, and $e(\lambda, 2^k) = 2$ if $d > 0$.

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

\section*{Corollary 3.7 (Theorem A).} Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$.

- If $k = 0$ then $f^n_k$ is surjective if and only if $d(n, k) \leq 2$.
- If $k > 0$ then $f^n_k$ is surjective if and only if $d(n, k) \leq 1$.

\section*{Proof.} By Theorem 3.5, $f^n_k$ is surjective if and only if for all $\lambda \vdash_o n - 2^k$ we have that the $k$th row of $Q_2(\lambda)$ contains a $d(n, k)$-good partition $\lambda_i^{(k)}$. By Theorem 2.5 and Definition 3.4, for any $\lambda \vdash_o n - 2^k$ we have $\sum_{j \geq 0} |\lambda_j^{(k)}| = \left\lfloor \frac{n - 2^k}{2^k} \right\rfloor = (2^{d(n, k)} - 1) + m(n, k)$.

If $k = 0$ then $Q_2^{(0)}(\lambda)$ contains only $\lambda = \lambda_0^{(0)}$. Hence $f^n_0$ is surjective if and only all odd partitions of $n - 1$ are $d(n, 0)$-good. By Lemma 3.3(1), the latter condition holds when $d = d(n, 0) \leq 2$. On the other hand, if $d = \nu_2(n) > 2$, then $\lambda = (n - 5, 2, 2)$ is an odd partition of $n - 1$ by Theorem 2.5, but $C_8(\lambda) = (3, 2, 2)$ is not a hook, and hence $C_{2^d}(\lambda)$ is not a hook. So $\lambda$ is not $d$-good, and thus $f^n_0$ is not surjective.

Now assume $k \geq 1$. Then $Q_2^{(k)}(\lambda)$ contains at least two odd partitions. If $d(n, k) \geq 2$ then any $d(n, k)$-good partition $\mu$ satisfies $3 \subseteq 2d(n, k) - 1 \subseteq |\mu|$. Write $\left\lfloor \frac{n - 2^k}{2^k} \right\rfloor = 1 + m_1$ where $m_1$ is even. Applying Remark 2.7, take any $\lambda \vdash_o n - 2^k$ such that $|\lambda_0^{(k)}| = 1$ and $\lambda_1^{(k)}$ is an odd partition with $|\lambda_1^{(k)}| = m_1$. Then no partition in $Q_2^{(k)}(\lambda)$ is $d(n, k)$-good. Thus $f^n_k$ is not surjective. On the other hand, if $d(n, k) = 0$ then $2^k \subseteq n$ and $f^n_k$ is surjective [3, Proposition 4.5]. If $d(n, k) = 1$ then $\left\lfloor \frac{n - 2^k}{2^k} \right\rfloor = 1 + m(n, k)$, where $4 \mid m(n, k)$. Thus any $Q_2^{(k)}(\lambda)$ contains a partition with odd cardinality; this partition is 1-good, by Lemma 3.3. Again $f^n_k$ is surjective.

It is an immediate consequence of Theorem 3.5 that $f^n_k$ is regular on its image for all relevant choices of $n, k$ such that $2^k < n$. We have:

\section*{Corollary 3.8.} Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$; set $d = \nu_2\left(\frac{n}{2^k}\right)$. Let $\lambda \vdash_o n - 2^k$. Then

$$e(\lambda, 2^k) = \begin{cases} 2^k & \text{if } d = 0; \\ 2 & \text{if } d > 0, \text{ and the } k\text{th row of } Q_2(\lambda) \text{ contains a } d\text{-good partition;} \\ 0 & \text{otherwise.} \end{cases}$$
Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take \( k = 2 \) above. For \( n > 2^2 \) we first compute \( d(n,k) = \nu_2\left(\left\lceil \frac{n}{2^k} \right\rceil\right) \), and then consider odd partitions of \( n - 4 \) and their 4-extensions. For \( n = 6 \), \( d(6,2) = 0 \). Thus \( e((2),4) = 4 \). The odd 4-extensions of \((2)\) are \((6), (3^2), (2^2, 1^2), (2, 1^4)\). For \( n = 10 \), \( d(10,2) = 1 \). In this case, \( e(\lambda,4) = 2 \) for all odd partitions \( \lambda \) of \( 6 \). For instance, the odd 4-extensions of \((6)\) are \((10)\) and \((6,3,1)\). For \( n = 19 \), \( d(19,2) = 2 \). Example 2.6 shows that for \( \lambda = (5, 4, 2^2, 1^2) \vdash_o 15 \) there is no 2-good partition in \( Q_2(\lambda) \), hence \( e(\lambda,4) = 0 \).

4 Deciding commutativity of the maps \( f_k \) and \( f_\ell \)

Let \( n \in \mathbb{N} \), and suppose that \( 0 \leq k < \ell \) satisfy \( 2^k + 2^\ell \leq n \). As stated in the introduction, we want to complete the discussion of the commutativity of the maps \( f_k \) and \( f_\ell \). Since the relevant \( n \) will always be apparent for the maps \( f_k^n \) in this section, we just write \( f_k \).

We write \( (n; k, \ell) \in T \) if for all \( \lambda |-o n \) we have \( f_k f_\ell(\lambda) = f_\ell f_k(\lambda) \). Otherwise we write \( (n; k, \ell) \in F \).

In this section we will prove Theorem B, which may be reformulated as follows.

Theorem 4.1. Let \( n = 2^t + m \) where \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 < k < \ell \) and \( 2^k + 2^\ell \leq n \). Then with the exception of \((6; 0,1)\)

\[ (n; k, \ell) \in F \text{ if and only if } \ell < t \text{ and } 2^k \leq m. \]

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two extreme cases, where \( f_k \) and \( f_\ell \) commute.

In the case \( \ell = t \) we have the following result as a reformulation of [3, Proposition 4.3].

Lemma 4.2. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). If \( 2^k \leq m \), then \( (n; k, t) \in T \).

It is also known that in the case where \( n \) is a power of 2, the maps \( f_k \) and \( f_\ell \) commute [3, Remark 4.4], and we include a short proof here.

Lemma 4.3. If \( n = 2^t \) then \( (n; k, \ell) \in T \) for all \( k, \ell \).

Proof. If \( 0 \leq b \leq a \) are integers then the binomial coefficient \( \binom{a}{b} \) is odd if and only if \( b \subseteq 2 a \), by Lucas’ theorem. The odd partitions of \( 2^t \) are exactly the hook partitions \( (2^t - b, 1^b) \), \( 0 \leq b \leq 2^t - 1 \), of degree \( 2^{t-1} \). Hence for \( k \in \{0, 1, \ldots, t-1\} \) we have

\[ f_k(\lambda) = \begin{cases} (2^t - b - 2^k, 1^b) & \text{if } 2^k \not\subseteq 2 b, \\ (2^t - b, 1^{b-2^k}) & \text{if } 2^k \subseteq 2 b. \end{cases} \]

It follows that for any \( k, \ell < t \) and odd partition \( \lambda \) of \( 2^t \), we have \( f_\ell f_k(\lambda) = f_k f_\ell(\lambda) \).

Lemma 4.4. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). If \( m < 2^k \) then \( (n; k, \ell) \in T \).

Proof. We use induction on \( k \geq 0 \). For \( k = 0 \) we have \( m = 0 \) and the claim follows from Lemma 4.3. Suppose that \( k \geq 1 \) and that the claim has been proved up to \( k - 1 \). Let \( \lambda |-o n \). Odd hooks of length \( 2^k \) and \( 2^\ell \) in \( \lambda \) correspond to odd hooks of length \( 2^{k-1} \) and \( 2^{\ell-1} \) in the 2-quotient \( Q_2(\lambda) = (\lambda_0, \lambda_1) \) of \( \lambda \). From Theorem 2.5 we deduce that \( |\lambda_0| \) and \( |\lambda_1| \) are 2-disjoint binary subsums of \( \left\lceil \frac{m}{2} \right\rceil \), so one of them contains \( 2^{\ell-1} \), say \( \lambda_0 \); then \( |\lambda_1| \leq \left\lceil \frac{m}{2} \right\rceil < 2^{k-1} < 2^{\ell-1} \). Thus the odd \( 2^{k-1} \)-hook in \( Q_2(\lambda) \) has to be in \( \lambda_0 \). Therefore

\[ Q_2(f_k(\lambda)) = (f_{k-1}(\lambda_0), \lambda_1). \]
Applying \( f_\ell \), the odd \( 2^{\ell-1} \)-hook cannot be in \( \lambda_1 \), hence
\[
Q_2(f_\ell f_k(\lambda)) = (f_{\ell-1}f_{k-1}(\lambda_0), \lambda_1)).
\]

In particular, we know that \( |\lambda_0| \geq 2^{\ell-1} + 2^{k-1} \). Also \( |\lambda_0| + |\lambda_1| = \left\lfloor \frac{n}{2} \right\rfloor = 2^{\ell-1} + \left\lfloor \frac{n}{2} \right\rfloor \). We have already seen that \( 2^{\ell-1} \) is the largest binary digit of \( |\lambda_0| \); furthermore \( |\lambda_0| - 2^{\ell-1} \) is a binary subsum of \( \left\lfloor \frac{n}{2} \right\rfloor < 2^{k-1} \). We may therefore apply the inductive hypothesis to \( \lambda_0 \) to get \( f_{\ell-1}f_{k-1}(\lambda_0) = f_{k-1}f_{\ell-1}(\lambda_0) \). This implies that \( Q_2(f_k f_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda)) \) and thus \( f_k f_\ell(\lambda) = f_\ell f_k(\lambda) \). □

Lemmas 4.2 and 4.4 show that the only if part of the theorem is true. We now turn to the if part. We start by proving the statement for \( k = 0 \) and use this as part of an inductive argument.

**Lemma 4.5.** Let \( n = 2^t + m \) with \( 0 < m < 2^t \). If \( 0 < \ell < t \) then \((n;0,\ell) \in \mathcal{F} \), with the exception of \((6;0,1)\).

**Proof.** The result is easily checked for \( n \leq 8 \), which includes the exception \((6;0,1)\). So we assume that \( t \geq 3 \).

*Case 1: \( 2^\ell < m \).* Then \( m \geq 3 \), since \( \ell > 0 \). Consider the partition \( \lambda = (m,m,1^a) \vdash n \) where \( a = n - 2m = 2^\ell - m \). The \((1,1)\)-hook length of \( \lambda \) is \( 2^\ell + 1 \). The \((2,1)\)-hook length of \( \lambda \) is \( 2^\ell \).

Removing the \((2,1)\)-hook hook we get the odd partition \((m)\), so \( \lambda \) is odd, by Lemma 2.8. We claim that
\[
f_0(\lambda) = (m,m,1^{a-1}).
\]

Indeed we cannot have \( f_0(\lambda) = (m,m-1,1^a) \) because this partition does not have a hook of length \( 2^\ell \), and thus it is not odd. Now
\[
f_\ell(f_0(\lambda)) = f_\ell(m,m,1^{a-1}) = (m,m-2^\ell,1^{a-1})
\]
since \((m,m,1^{a-1}-2^\ell)\) and \((m-1,m-2^\ell+1,1^{a-1})\) both do not have a hook of length \( 2^\ell \) and thus are not odd (again by Lemma 2.8).

On the other hand,
\[
f_\ell(\lambda) = (m-1,m-(2^\ell-1),1^a).
\]

Indeed, the other candidates for \( f_\ell(\lambda) \), which are \((m,m-2^\ell,1^a)\) and \((m,m,1^{a-2^\ell})\), do not have hooks of length \( 2^\ell \). Then
\[
f_0(f_\ell(\lambda)) = f_0(m-1,m-(2^\ell-1),1^a) = (m-1,m-2^\ell,1^a).
\]

This follows (again) by observing that all the other partitions of \( n-2^\ell-1 \) obtained from \((m-1,m-(2^\ell-1),1^a)\) by removing a node do not have hooks of length \( 2^\ell \). Thus \( f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda)) \).

*Case 2: \( m < 2^\ell \).* Consider the partition \( \lambda = (n-2^\ell,m+1,1^a) \), where \( a = 2^\ell - (m + 1) \). Note that \( n-2^\ell \geq m + 1 \) since \( \ell < t \) by assumption, and that \( a \geq 0 \). The \((1,1)\)-hook length of \( \lambda \) is \( n-m = 2^t \). Removing this hook we get the odd partition \((m)\), so \( \lambda \) is odd. The \((2,1)\)-hook length of \( \lambda \) is \( 2^\ell \).

Now
\[
f_0(\lambda) = (n-2^\ell,m,1^a)
\]

since the other candidates do not have hooks of length \( 2^\ell \). Then
\[
f_\ell(f_0(\lambda)) = f_\ell(n-2^\ell,m,1^a) = \mu,
\]
where $\mu$ is obtained from $f_0(\lambda)$ by removing a $2^{k}$-hook in the first row. (There are only hooks of length $< 2^l$ in the other rows.) In fact, $\mu = (n - 2^{l+1}, m, 1^a)$ since $n - 2^{l+1} \geq n - 2^l = m$. Thus $f_\ell(f_0(\lambda))$ has at least 2 parts. On the other hand

$$f_\ell(\lambda) = (n - 2^{l})$$

since this odd partition is obtained from the odd partition $\lambda$ by removing a $2^{l}$-hook (the one in (2, 1)). It follows that

$$f_0(f_\ell(\lambda)) = (n - 2^{l} - 1)$$

and again $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

Case 3: $m = 2^{l}$. Then $n = 2^{l} + 2^{l}$. If $l \geq 2$ then choose $\lambda = (2^{l}, 2^{l} - 1, 1)$. The $(1, 2)$-hook length of $\lambda$ is $2^{l}$; thus $\lambda$ is an odd partition since removing this $2^{l}$-hook gives an odd partition $(2^{l} - 2, 1, 1)$ of $2^{l}$. We have $f_0(\lambda) = (2^{l}, 2^{l} - 2, 1)$ since the other candidates are not odd. Then

$$f_\ell(f_0(\lambda)) = (2^{l} - 2^{l}, 2^{l} - 2, 1).$$

The $(2, 1)$-hook length of $\lambda$ is $2^{l}$, so $f_\ell(\lambda) = (2^{l})$ and

$$f_0(f_\ell(\lambda)) = (2^{l} - 1),$$

showing $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

On the other hand, if $\ell = 1$ then choose $\lambda = (2^{l} - 2, 2, 2) \vdash_o 2^{l} + 2 = n$. Since $t \geq 3$, it is now easy to show that $f_1(f_0(\lambda)) = (2^{l} - 4, 2, 1)$. On the other hand we see that $f_0(f_1(\lambda))$ is a hook partition of $2^{l} - 1 = n - 3$ and therefore is not equal to $f_1(f_0(\lambda))$. \hfill $\blacksquare$

**Lemma 4.6.** If $(n; k, \ell) \in \mathcal{F}$ then also $(2n; k + 1, \ell + 1) \in \mathcal{F}$ and $(2n + 1; k + 1, \ell + 1) \in \mathcal{F}$.

**Proof.** Let the odd partition $\mu$ of $n$ satisfy $f_k f_\ell(\mu) \neq f_\ell f_k(\mu)$. Let $\lambda$ be a partition of $2n$ or $2n + 1$ having 2-quotient $Q_2(\lambda) = (\mu, (0))$. Then $\lambda$ is odd, by Theorem 2.5. We have

$$Q_2(f_{k+1} f_{\ell+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_\ell f_k(\mu), (0)) = Q_2(f_{\ell+1} f_{k+1}(\lambda)),$$

so that $f_{k+1} f_{\ell+1}(\lambda) \neq f_{\ell+1} f_{k+1}(\lambda)$. \hfill $\blacksquare$

We are now ready to conclude this section with the proof of Theorem B.

**Proof of Theorem 4.1.** The only if part follows from Lemmas 4.2 and 4.4. To prove the if part we use induction on $k \geq 0$. If $k = 0$, then the statement follows from Lemma 4.5. Let $k > 1$ and suppose that the assertion is true up to and including $k - 1$. To show that $(n; k, \ell) \in \mathcal{F}$ it suffices to prove $((\frac{n}{2}; k - 1, \ell - 1) \in \mathcal{F}$, by Lemma 4.6. We are assuming $n = 2^{l} + m$, $0 \leq m < 2^{l}$, $0 \leq k < \ell \leq l$ and $2^{k} + 2^{l} \leq n$. This implies $\left\lfloor \frac{n}{2} \right\rfloor = 2^{l-\ell} + \left\lfloor \frac{m}{2} \right\rfloor$, $0 \leq \left\lfloor \frac{m}{2} \right\rfloor < 2^{l-\ell}$ and $2^{k-1} + 2^{\ell-1} \leq \left\lfloor \frac{n}{2} \right\rfloor$. We may apply the inductive hypothesis to get $((\frac{n}{2}; k - 1, \ell - 1) \in \mathcal{F}$, and then $(n; k, \ell) \in \mathcal{F}$ except when $((\frac{n}{2}; k - 1, \ell - 1) = (6; 0, 1)$. In that case we are considering $((12; 1, 2)$ or $((13; 1, 2)$ which are both in $\mathcal{F}$, by direct computation (consider for example $(6, 4, 2) \vdash_o 12$ and $(6, 4, 3) \vdash_o 13$, respectively). \hfill $\blacksquare$

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