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Restriction of Odd Degree Characters of $S_n$

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Abstract. Let $n$ and $k$ be natural numbers such that $2^k < n$. We study the restriction to $S_{n-2^k}$ of odd-degree irreducible characters of the symmetric group $S_n$. This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., Sémi. Lothar. Combin. 75 (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., J. Algebra 478 (2017), 271–282].

Key words: characters of symmetric groups; hooks in partitions

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1 Introduction

Let $n$ be a natural number, and let $\chi$ be an irreducible character of odd degree of the symmetric group $S_n$. Then there exists a unique odd-degree irreducible constituent of the restriction $\chi_{S_{n-1}}$. This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any $k \in \mathbb{N}$ such that $2^k < n$, there exists a unique odd-degree irreducible constituent $f^k_n(\chi)$ of $\chi_{S_{n-2^k}}$ appearing with odd multiplicity. The main goal of this article is to study for all $n,k \in \mathbb{N}$ the map

$$f^k_n: \text{Irr}_2(S_n) \longrightarrow \text{Irr}_2(S_{n-2^k}),$$

naturally defined by Theorem A of [3]. All our results are proved using a description of $f^k_n$ in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If $2^k$ appears in the binary expansion of $n$ we say that $2^k$ is a binary digit of $n$. Similarly we say that two natural numbers $m$ and $n$ are 2-disjoint if they do not have any common binary digit. On the other hand, if $m \leq n$ and all the binary digits of $m$ appear in the binary expansion of $n$, then we say that $m$ is a binary subsum of $n$. This will be denoted by $m \subseteq_2 n$. Let $\nu_2(n)$ be the exponent of the highest power of 2 dividing the integer $n$. 

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A question raised in [3] may be phrased as: For which \( n \) and \( k \) is \( f_k^n \) surjective? The authors showed that \( f_k^n \) is surjective whenever \( 2^k \) is a binary digit of \( n \), and they observed that otherwise \( f_k^n \) could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

**Theorem A.** Let \( n \in \mathbb{N} \), \( k \in \mathbb{N}_0 \) be such that \( 2^k < n \). Let \( d(n, k) = \nu_2 \left( \left\lfloor \frac{n}{2^k} \right\rfloor \right) \).

- If \( k = 0 \) then \( f_k^n \) is surjective if and only \( d(n, k) \leq 2 \).
- If \( k > 0 \) then \( f_k^n \) is surjective if and only \( d(n, k) \leq 1 \).

**Theorem A** is a consequence of Theorem 3.5 below, which describes the images of the maps \( f_k^n \).

For all \( n \in \mathbb{N} \), \( k \in \mathbb{N}_0 \) with \( 2^k < n \) and any \( \psi \in \text{Irr}_2(\mathfrak{S}_{n-2^k}) \) we define the set
\[
\mathcal{E}(\psi, 2^k) = \{ \chi \in \text{Irr}_2(\mathfrak{S}_n) \mid f_k^n(\chi) = \psi \},
\]
and set \( e(\psi, 2^k) = |\mathcal{E}(\psi, 2^k)| \). We show in Corollary 3.8 that the maps \( f_k^n \) are regular on their images. This means that for any \( \psi \) in the image of \( f_k^n \), the number \( e(\psi, 2^k) \) depends only on \( n \) and \( k \) and not on the specific \( \psi \). We also give a complete description of those \( \psi \in \text{Irr}_2(\mathfrak{S}_{n-2^k}) \) such that \( e(\psi, 2^k) = 0 \), in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote \( f_k^n \) just by \( f_k \), when the natural number \( n \) is clear from the context. Then, for \( k, \ell \in \mathbb{N}_0 \), \( k < \ell \), such that \( 2^k + 2^\ell \leq n \), we may ask: when is \( f_k f_\ell = f_\ell f_k \)? or more specifically: when is \( f_k^{n-2^k} f_\ell^{n-2^\ell} = f_\ell^{n-2^\ell} f_k^{n-2^k} \)? In [3, Proposition 4.3] it was proved that \( f_k f_\ell = f_\ell f_k \) whenever \( 2^\ell < n < 2^{\ell+1} \). This is the case \( \ell = t \) in our second main result, which answers the question completely.

**Theorem B.** Let \( n = 2^t + m \) where \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \leq t \) and \( 2^k + 2^\ell \leq n \). Then, with the exception of the case \( n = 6, k = 0, \ell = 1 \),
\[
f_k f_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.
\]

## 2 Notation and background

Let \( n \) be a natural number. We let \( \text{Irr}(\mathfrak{S}_n) \) denote the set of irreducible characters of \( \mathfrak{S}_n \) and \( \mathcal{P}(n) \) the set of partitions of \( n \). The notation \( \lambda \in \mathcal{P}(n) \) is sometimes replaced by \( \lambda \vdash n \) and we write \( |\lambda| = n \). There is a natural correspondence \( \lambda \leftrightarrow \chi^\lambda \) between \( \mathcal{P}(n) \) and \( \text{Irr}(\mathfrak{S}_n) \). We say then that \( \lambda \) labels \( \chi^\lambda \). We denote by \( \text{Irr}_2(\mathfrak{S}_n) \) the set of irreducible characters of \( \mathfrak{S}_n \) of odd degree. If \( \chi^\lambda \in \text{Irr}_2(\mathfrak{S}_n) \) we say that \( \chi^\lambda \) is an **odd character**, we call \( \lambda \) an **odd partition** of \( n \) and write \( \lambda \vdash_o n \). Also the empty partition will be considered as an odd partition.

**Remark 2.1.** Let \( n, k \) be such that \( 2^k < n \). In [3, Theorem A and Proposition 4.2] it is shown that the map \( f_k^n : \text{Irr}_2(\mathfrak{S}_n) \rightarrow \text{Irr}_2(\mathfrak{S}_{n-2^k}) \) may be described in terms of the odd partitions labelling the odd characters as follows:
\[
f_k^n(\chi^\lambda) = \chi^\mu \iff \lambda \vdash_o n - 2^k \text{ can be obtained from } \lambda \vdash_o n \text{ by removing a } 2^k\text{-hook}.
\]

Correspondingly we write (by abuse of notation) \( f_k^n(\lambda) = \mu \). In fact when \( \lambda \) is odd, there is only one \( 2^k\)-hook of \( \lambda \) whose removal leads again to an odd partition; we will refer to such a hook as an **odd hook** of \( \lambda \). This combinatorial description of \( f_k^n \) will be used throughout this paper, and we will regard \( f_k^n \) also as a map between the corresponding sets of odd partitions. Also, for \( \mu \vdash_o n - 2^k \) we set \( e(\mu, 2^k) = e(\chi^\lambda, 2^k) \).
We need some concepts and basic facts concerning hooks in partitions. For any integer \( e \in \mathbb{N} \) we denote by \( C_e(\lambda) \) and \( Q_e(\lambda) \) the \( e \)-core and the \( e \)-quotient of \( \lambda \), respectively. Then \( Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1}) \) is an \( e \)-tuple of partitions satisfying \( n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i| \). It is well-known that a partition is uniquely determined by its \( e \)-core and \( e \)-quotient (we refer the reader to [6] or [4, Chapter 2.7] for a detailed discussion on this topic).

Let \( \mathcal{H}_e(\lambda) \) be the set of hooks of \( \lambda \) having length divisible by \( e \), and let \( \mathcal{H}(Q_e(\lambda)) = \bigcup_{i=0}^{e-1} \mathcal{H}(\lambda_i) \).

As explained in [6, Theorem 3.3], there is a bijection between \( \mathcal{H}_e(\lambda) \) and \( \mathcal{H}(Q_e(\lambda)) \) mapping hooks in \( \lambda \) of length \( ex \) to hooks in the quotient of length \( x \). Moreover, the bijection respects the process of hook removal. Namely, the partition \( \mu \) obtained by removing an \( ex \)-hook from \( \lambda \) is such that \( C_e(\mu) = C_e(\lambda) \) and the \( e \)-quotient of \( \mu \) is obtained by removing an \( x \)-hook from one of the partitions involved in \( Q_e(\lambda) \).

For \( e = 2 \) we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-quotient tower \( Q_2(\lambda) \) and the 2-core tower \( C_2(\lambda) \) of \( \lambda \). They have rows numbered by \( k \geq 0 \).

The \( k \)th row \( Q_2^{(k)}(\lambda) \) of \( Q_2(\lambda) \) contains \( 2^k \) partitions \( \lambda_i^{(k)} \), \( 0 \leq i \leq 2^k - 1 \), and the \( k \)th row \( C_2^{(k)}(\lambda) \) of \( C_2(\lambda) \) contains the 2-cores of these partitions in the same order, i.e., \( C_2(\lambda_i^{(k)}) \), \( 0 \leq i \leq 2^k - 1 \). The 0th row of \( Q_2(\lambda) \) contains \( \lambda = \lambda_0^{(0)} \) itself, row 1 contains the partitions \( \lambda_0^{(1)}, \lambda_1^{(1)} \) occurring in the 2-quotient \( Q_2(\lambda) \), row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have \( Q_2(\lambda_i^{(k)}) = (\lambda_2i^{(k+1)}, \lambda_2(i+1)^{(k+1)}) \) for \( i \in \{0, 1, \ldots, 2^k - 1\} \). We remark that the \( 2^k \) partitions in \( Q_2^{(k)}(\lambda) \) are the same as those in the \( 2^k \)-quotient \( Q_{2^k}(\lambda) \) of \( \lambda \), but in a different order for \( k \geq 2 \).

We also introduce the \( k \)-data \( D_2^{(k)}(\lambda) \) of \( \lambda \). This is a table containing the following \( k + 1 \) rows: the \( k \) rows \( C_2^{(j)}(\lambda) \), \( j = 0, \ldots, k - 1 \), and in addition the row \( Q_2^{(k)}(\lambda) \).

**Remark 2.2.** A partition \( \lambda \) may be recovered from its 2-core tower. For \( k > 0 \), it may also be recovered from the knowledge of the \( k \)-data \( D_2^{(k)}(\lambda) \) of \( \lambda \), because the rows \( C_2^{(j)}(\lambda) \) with \( l \geq k \) of \( C_2(\lambda) \) consist of the 2-core towers of the partitions in \( Q_2^{(k)}(\lambda) \).

**Lemma 2.3.** Suppose that \( \lambda \vdash n - 2^k \) and \( \mu \vdash n \). The following are equivalent.

(i) \( \lambda \) is obtained from \( \mu \) by removing a \( 2^k \)-hook.

(ii) The \( k \)-data \( D_2^{(k)}(\mu) \) and \( D_2^{(k)}(\lambda) \) coincide, except that for one \( i \in \{0, \ldots, 2^k - 1\} \) \( \lambda_i^{(k)} \) is obtained from \( \mu_i^{(k)} \) by removing a 1-hook.

**Proof.** A \( 2^k \)-hook \( H_0 \) in \( \mu \) corresponds in a canonical way to a \( 2^{k-1} \)-hook \( H_1 \) in a partition in \( Q_2^{(1)}(\mu) \), i.e., in row 1 of the 2-quotient tower \( Q_2(\mu) \). Continuing we see that \( H_0 \) corresponds in a canonical way to a 1-hook \( H_k \) in a partition \( \mu_i^{(k)} \) in \( Q_2^{(k)}(\mu) \), row \( k \) of \( Q_2(\mu) \). If \( \lambda \) is obtained by removing \( H_0 \) from \( \mu \), this corresponds to \( \lambda_i^{(k)} \) being obtained by removing the 1-hook \( H_k \) from \( \mu_i^{(k)} \) (by repeated applications of [6, Theorem 3.3]). Apart from this the rows \( Q_2^{(k)}(\mu) \) and \( Q_2^{(k)}(\lambda) \) coincide. Note also that the rows \( C_2^{(j)}(\mu) \) and \( C_2^{(j)}(\lambda) \) coincide for \( j = 0, \ldots, k - 1 \), since the removal of the hooks \( H_j \) of even length do not change the 2-cores. \( \square \)

Odd-degree characters of \( \mathfrak{S}_n \) and thus odd partitions were completely described in [5]. We restate this result in a language which is convenient for our purposes. We let \( c_2^{(k)}(\lambda) \) be the sum of the cardinalities of the partitions in the \( k \)th row \( C_2^{(k)}(\lambda) \) of \( C_2(\lambda) \).

**Lemma 2.4** ([5]). Let \( \lambda \) be a partition. Then \( \lambda \) is odd if and only if \( c_2^{(k)}(\lambda) \leq 1 \) for all \( k \geq 0 \).

It may be decided from the \( k \)-data \( D_2^{(k)}(\lambda) \) whether \( \lambda \) is odd. The case \( k = 1 \) of the following result appeared in [3, Lemma 4.1] and also in [1, Lemma 6].
Theorem 2.5. Let $\lambda \vdash n$, and let $k \geq 0$ be fixed. Consider $Q_2^{(k)}(\lambda) = (\lambda_i^{(k)})$. Then $\lambda$ is odd if and only if the following conditions are all fulfilled:

(i) $c_2^{(j)}(\lambda) \leq 1$ for all $j < k$.

(ii) The partitions $\lambda_i^{(k)}$, $0 \leq i \leq 2^k - 1$, are all odd.

(iii) The numbers $|\lambda_i^{(k)}|$, $0 \leq i \leq 2^k - 1$, are pairwise $2$-disjoint.

In this case $\sum_{i \geq 0} |\lambda_i^{(k)}| = \left\lfloor \frac{n}{2^k} \right\rfloor$.

Proof. This is proved by induction on $k \geq 0$, using Remark 2.2 and Lemma 2.4. \hfill \blacksquare

We illustrate the result above by giving an example.

Example 2.6. Let $n = 15$ and take $\lambda = (5, 4, 2^2, 1^2) \vdash 15$. To decide whether $\lambda$ is odd, we choose $k = 2$ and compute the $2$-data $D_2^{(2)}(\lambda)$. The $2$-core is $C_2(\lambda) = (1)$, giving $C_2^{(0)}(\lambda) = ((1))$. Furthermore, the $2$-quotient is $Q_2(\lambda) = ((2^2, 1^2), (1))$, and computing the $2$-cores $C_2((2^2, 1^2)) = (0)$, $C_2((1)) = (1)$, we obtain the next row: $C_2((1)) = ((0), (1))$. The $2$-quotients are $Q_2((2^2, 1^2)) = ((1^2), (1))$, $Q_2((1)) = ((0), (0))$; hence the final row of the $2$-data table is obtained as $Q_2^{(2)}(\lambda) = ((1^2), (1), (0), (0))$.

We visualize $D_2^{(2)}(\lambda)$ like this:

$C_2^{(0)}(\lambda):$ (1)

$C_2^{(1)}(\lambda):$ (0) (1)

$Q_2^{(2)}(\lambda):$ (1) (0) (0)

Theorem 2.5 shows that $\lambda$ is odd and thus it contains a unique odd $4$-hook. Again using the theorem, it is clear that removing this $4$-hook corresponds to the second partition $(1)$ in $Q_2^{(2)}$ being replaced by $(0)$. Thus, removing the corresponding $4$-hook of $\lambda$ we obtain the odd partition $\mu = (3, 2^2, 1^2) \vdash 11$ with the property that $D_2^{(2)}(\lambda)$ and $D_2^{(2)}(\mu)$ differ only in their final row.

Remark 2.7. Using the construction of partitions from their $2$-cores and $2$-quotients already mentioned, the criterion above can be applied to construct all odd partitions of $n$ with a specific $k$th row in the $2$-quotient tower. For this, let $n, k \in \mathbb{N}$, and take any sequence of odd partitions $\nu_i$, $0 \leq i \leq 2^k - 1$, such that the numbers $|\nu_i|$ are pairwise $2$-disjoint, and $\sum_{i \geq 0} |\nu_i| = \left\lfloor \frac{n}{2^k} \right\rfloor$.

Then there are exactly $\prod_{m < k} 2^m$ odd partitions $\lambda$ of $n$ with $Q_2^{(k)}(\lambda) = (\nu_i)$, obtained by choosing one $2$-core in row $m$ of the $k$-data table to be $(1)$, for each $m < k$ such that $2^m \leq n$.

The following easy consequence of Theorem 2.5 will be used repeatedly.

Lemma 2.8. Let $2^t$ be the largest binary digit of $n$. A partition $\lambda$ of $n$ is odd if and only if $\lambda$ contains a unique $2^t$-hook and the partition obtained from $\lambda$ by removing this $2^t$-hook is an odd partition of $n - 2^t$.

3 Surjectivity and regularity

The aim of this section is to study the images of the maps $f_k^n$ for all $n, k$ such that $2^k \leq n$. For this purpose we introduce the concept of $d$-good partitions (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when $f_k^n$ is surjective) and to show that the maps $f_k^n$ are always regular on their image (see Corollary 3.8).
Definition 3.1. Let \( d \geq 0 \). We call an odd partition \( \lambda \) \( d \)-good, if

(i) \( |\lambda| \equiv 2^d - 1 \mod 2^{d+1} \).
(ii) \( C_{2^d}(\lambda) \) is a hook partition.

Let us remark that condition (i) may be reformulated as

\[(i^*) \, \nu_2(|\lambda| + 1) = d.\]

In particular, if \( \lambda \) is \( d \)-good, then \( |\lambda| \) is odd if and only if \( d > 0 \).

The relevance of \( d \)-good partitions in our context is illuminated by the following reformulation of [1, Theorem 2]:

Lemma 3.2. Let \( \lambda \vdash_o n \). Let \( d = \nu_2(n + 1) \). Then \( e(\lambda, 1) \neq 0 \) if and only if \( \lambda \) is \( d \)-good. In this case, \( e(\lambda, 1) = 1 \) if \( d = 0 \), and \( e(\lambda, 1) = 2 \) if \( d > 0 \).

Lemma 3.3. Let \( \lambda \) be an odd partition, and let \( d \geq 0 \). Then the following hold.

1. For \( d \leq 2 \), \( \lambda \) is \( d \)-good if and only if \( |\lambda| \equiv 2^d - 1 \mod 2^{d+1} \).
2. If \( \lambda \) is \( d \)-good, then \( C_{2^d}(\lambda) \) is a partition of \( 2^d - 1 \).

Proof. If the odd partition \( \lambda \) is \( d \)-good, then \( |\lambda| = (2^d - 1) + m \) where the binary digits of \( m \) are at least \( 2^{d+1} \). The hooks of \( \lambda \) corresponding to the binary digits of \( m \) may be decomposed into \( 2^d \)-hooks and thus do not contribute to \( C_{2^d}(\lambda) \). Thus \( |C_{2^d}(\lambda)| = 2^d - 1 \). This shows (2).

For \( d = 0, 1, 2 \) we have \( |C_{2^d}(\lambda)| = 0, 1 \) and 3, respectively. Since all partitions of 0, 1 and 3 are hook partitions, (1) follows.

Definition 3.4. If \( 2^k \leq n \), we define \( d(n, k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor \right) \). Thus \( d(n, k) \) is the smallest integer \( d \geq 0 \) satisfying the condition \( 2^{k+d} \subseteq n \). In particular, \( d(n, k) = 0 \) if and only if \( 2^k \subseteq n \).

Moreover, we may write \( \left\lfloor \frac{n}{2^k} \right\rfloor = 2^d(n, k) + m(n, k) \) where \( 2^{d(n, k)+1} \mid m(n, k) \).

As mentioned in the introduction, the results in [3] show that \( f^n_k \) is a surjective \((2^k\text{-to-}1)\)-map whenever \( 2^k \subseteq n \), i.e., \( d(n, k) = 0 \). In the spirit of [1, Theorem 2], we now give a characterization of the image of the map \( f^n_k \) for all \( n, k \) such that \( 2^k < n \).

Theorem 3.5. Let \( n \in \mathbb{N} \), \( k \in \mathbb{N}_0 \) be such that \( 2^k < n \). Let \( \lambda \vdash_o n - 2^k \). Then \( e(\lambda, 2^k) \neq 0 \) if and only if there exists a \( d(n, k) \)-good partition in the \( k \)-th row of \( \mathcal{Q}_2(\lambda) \). In this case, \( e(\lambda, 2^k) = 2^k \) if \( d(n, k) = 0 \), and \( e(\lambda, 2^k) = 2 \) if \( d(n, k) > 0 \).

Proof. If \( k = 0 \) then the statement follows from Lemma 3.2. Hence assume that \( k \geq 1 \). Let \( d = d(n, k) \). By assumption \( \left\lfloor \frac{n}{2^k} \right\rfloor = 2^d + m \), where the binary digits of \( m \) are at least \( 2^{d+1} \). Thus \( \left\lfloor \frac{n-2^k}{2^k} \right\rfloor = (2^d - 1) + m \).

Suppose first that \( e(\lambda, 2^k) \neq 0 \) and that \( \mu \vdash_o n \) satisfies \( f^n_k(\mu) = \lambda \). From Remark 2.1 and Lemma 2.3 we get that there exists an \( i \in \{0, 1, \ldots, 2^k-1\} \) such that \( f^n_k(\mu_i(k)) = \lambda_i(k) \). Since \( \mu_i(k) \) and \( \lambda_i(k) \) are odd, we get \( e(\lambda_i(k), 1) \neq 0 \). We have that \( |\lambda_i(k)| \) and \( |\mu_i(k)| \) are both \( 2 \)-disjoint with \( m_1 := \sum_{j \neq i} |\lambda_j(k)| = \sum_{j \neq i} |\mu_j(k)| \subseteq \left\lfloor \frac{n-2^k}{2^k} \right\rfloor \), by Theorem 2.5. Since \( m_1 \subseteq \left\lfloor \frac{n-2^k}{2^k} \right\rfloor \) and \( m_1 \leq \left\lfloor \frac{n}{2^k} \right\rfloor \), we get \( m_1 \leq m \). Thus \( |\lambda_i(k)| = (2^d - 1) + m_2 \) and \( |\mu_i(k)| = 2^d + m_2 \), where \( m_2 = m - m_1 \leq m \).

In particular \( \nu_2(|\lambda_i(k)| + 1) = \nu_2(|\mu_i(k)|) = d \). Then Lemma 3.2 shows that \( \lambda_i(k) \) is \( d \)-good.

Conversely, if \( \lambda_i(k) \) is a \( d \)-good partition for some \( i \in \{0, 1, \ldots, 2^k-1\} \), then there exists a \( \mu^* \vdash_o |\lambda_i(k)| + 1 \) such that \( f^n_k(\mu^*) = \lambda_i(k) \), by Lemma 3.2. We let \( \mu \) be the partition where the \( k \)-data \( D^{(k)}_2(\mu) \) and \( D^{(k)}_2(\lambda) \) coincide, except that \( \mu_i(k) = \mu^* \). Since \( \lambda \) is odd and \( \lambda_i(k) \) is \( d \)-good,
we know that $|\lambda_i^{(k)}| = (2d - 1) + m'$ where $m' \subseteq 2 m$, and $|\lambda_j^{(k)}| \subseteq 2 m - m'$ for all $j \neq i$. Hence $|\mu'| = |\lambda_i^{(k)}| + 1 = 2d + m'$ is 2-disjoint from all $|\lambda_j^{(k)}|, j \neq i$. Thus $\mu$ is an odd partition of $n$ by Theorem 2.5, and $f_k(\mu) = \lambda$ by Lemma 2.3 and Remark 2.1.

We conclude that $e(\lambda, 2^k) = \sum_{\lambda_i^{(k)}-\text{d-good}} e(\lambda_i^{(k)}, 1)$. If $d = 0$ then $\lfloor \frac{n-2^k}{2^d} \rfloor$ is even. This implies that all $\lambda_i^{(k)}$ are of even cardinality and thus d-good. Thus $e(\lambda_i^{(k)}, 1) = 1$ for all $i$, and we get $e(\lambda, 2^k) = 2^k$. If $d > 0$ there is exactly one $\lambda_i^{(k)}$ in $Q_2^{(k)}(\lambda)$ of odd cardinality. Only this $\lambda_i^{(k)}$ may be d-good and then $e(\lambda, 2^k) = e(\lambda_i^{(k)}, 1) = 2$. Otherwise $e(\lambda, 2^k) = 0$.

**Corollary 3.6.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$, and let $d = \nu_2 \left( \lfloor \frac{n}{2^k} \rfloor \right)$. Let $\lambda \vdash_o n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a partition $\lambda_i^{(k)}$ in the $k$th row of $Q_2(\lambda)$ such that $|\lambda_i^{(k)}| \equiv 2^d - 1 \mod 2^{d+1}$, and $C_{2^d}(\lambda_i^{(k)})$ is a hook partition. In this case, $e(\lambda, 2^k) = 2^k$ if $d = 0$, and $e(\lambda, 2^k) = 2$ if $d > 0$.

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

**Corollary 3.7 (Theorem A).** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$.

- If $k = 0$ then $f_k^n$ is surjective if and only if $d(n, k) \leq 2$.
- If $k > 0$ then $f_k^n$ is surjective if and only if $d(n, k) \leq 1$.

**Proof.** By Theorem 3.5, $f_k^n$ is surjective if and only if for all $\lambda \vdash_o n - 2^k$ we have that the $k$th row of $Q_2(\lambda)$ contains a $d(n, k)$-good partition $\lambda_i^{(k)}$. By Theorem 2.5 and Definition 3.4, for any $\lambda \vdash_o n - 2^k$ we have $\sum_{j \geq 0} |\lambda_j^{(k)}| = \lfloor \frac{n-2^k}{2^d} \rfloor = (2d(n, k) - 1) + m(n, k)$.

If $k = 0$ then $Q_2^{(0)}(\lambda)$ contains only $\lambda = \lambda_0^{(0)}$. Hence $f_0^n$ is surjective if and only all odd partitions of $n - 1$ are $d(n, 0)$-good. By Lemma 3.3(1), the latter condition holds when $d = d(n, 0) \leq 2$. On the other hand, if $d = \nu_2(n) > 2$, then $\lambda = (n - 5, 2, 2)$ is an odd partition of $n - 1$ by Theorem 2.5, but $C_{5^2}(\lambda) = (3, 2, 2)$ is not a hook, and hence $C_{2^d}(\lambda)$ is not a hook. So $\lambda$ is not $d$-good, and thus $f_k^n$ is not surjective.

Now assume $k \geq 1$. Then $Q_2^{(k)}(\lambda)$ contains at least two odd partitions. If $d(n, k) \geq 2$ then any $d(n, k)$-good partition $\mu$ satisfies $3 \subseteq 2^{d(n, k) - 1} \subseteq |\mu|$. Write $\lfloor \frac{n-2^k}{2^d} \rfloor = 1 + m_1$ where $m_1$ is even. Applying Remark 2.7, take any $\lambda \vdash_o n - 2^k$ such that $|\lambda_0^{(k)}| = 1$ and $\lambda_1^{(k)}$ is an odd partition with $|\lambda_1^{(k)}| = m_1$. Then no partition in $Q_2^{(k)}(\lambda)$ is $d(n, k)$-good. Thus $f_k^n$ is not surjective. On the other hand, if $d(n, k) = 0$ then $2^k \subseteq n$ and $f_k^n$ is surjective [3, Proposition 4.5]. If $d(n, k) = 1$ then $\lfloor \frac{n-2^k}{2^d} \rfloor = 1 + m(n, k)$, where $4 \mid m(n, k)$. Thus any $Q_2^{(k)}(\lambda)$ contains a partition with odd cardinality; this partition is 1-good, by Lemma 3.3. Again $f_k^n$ is surjective.

It is an immediate consequence of Theorem 3.5 that $f_k^n$ is regular on its image for all relevant choices of $n, k$ such that $2^k < n$. We have:

**Corollary 3.8.** Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$; set $d = \nu_2 \left( \lfloor \frac{n}{2^k} \rfloor \right)$. Let $\lambda \vdash_o n - 2^k$. Then

$$e(\lambda, 2^k) = \begin{cases} 2^k & \text{if } d = 0; \\ 2 & \text{if } d > 0, \text{ and the } k\text{th row of } Q_2(\lambda) \text{ contains a } d\text{-good partition}; \\ 0 & \text{otherwise.} \end{cases}$$
Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take \( k = 2 \) above. For \( n > 2^2 \) we first compute \( d(n, k) = \nu_2\left(\left[\frac{n}{2^k}\right]\right) \), and then consider odd partitions of \( n - 4 \) and their 4-extensions. For \( n = 6 \), \( d(6, 2) = 0 \). Thus \( e((2), 4) = 4 \). The odd 4-extensions of (2) are (6), (3\(^2\)), (2\(^2\), 1\(^2\)), (2, 1\(^4\)). For \( n = 10 \), \( d(10, 2) = 1 \). In this case, \( e(\lambda, 4) = 2 \) for all odd partitions \( \lambda \) of 6. For instance, the odd 4-extensions of (6) are (10) and (6, 3, 1). For \( n = 19 \), \( d(19, 2) = 2 \). Example 2.6 shows that for \( \lambda = (5, 4, 2^2, 1^2) \vdash_\omega 15 \) there is no 2-good partition in \( Q_2((2)) \), hence \( e(\lambda, 4) = 0 \).

4 Deciding commutativity of the maps \( f_k \) and \( f_\ell \)

Let \( n \in \mathbb{N} \), and suppose that \( 0 \leq k < \ell \) satisfy \( 2^k + 2^\ell \leq n \). As stated in the introduction, we want to complete the discussion of the commutativity of the maps \( f_k \) and \( f_\ell \). Since the relevant \( n \) will always be apparent for the maps \( f_k^n \) in this section, we just write \( f_k \).

We write \( (n; k, \ell) \in T \) if for all \( \lambda \vdash_\omega n \) we have \( f_k f_\ell (\lambda) = f_\ell f_k (\lambda) \). Otherwise we write \( (n; k, \ell) \in \mathcal{F} \).

In this section we will prove Theorem B, which may be reformulated as follows.

Theorem 4.1. Let \( n = 2^t + m \) where \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). Then with the exception of \((6; 0, 1)\)

\[
(n; k, \ell) \in \mathcal{F} \text{ if and only if } \ell < t \text{ and } 2^k \leq m.
\]

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two extreme cases, where \( f_k \) and \( f_\ell \) commute.

In the case \( \ell = t \) we have the following result as a reformulation of [3, Proposition 4.3].

Lemma 4.2. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). If \( 2^k \leq m \), then \( (n; k, t) \in T \).

It is also known that in the case where \( n \) is a power of 2, the maps \( f_k \) and \( f_\ell \) commute [3, Remark 4.4], and we include a short proof here.

Lemma 4.3. If \( n = 2^t \) then \( (n; k, \ell) \in T \) for all \( k, \ell \).

Proof. If \( 0 \leq b \leq a \) are integers then the binomial coefficient \( \binom{a}{b} \) is odd if and only if \( b \subseteq 2 \bigcap a \), by Lucas’ theorem. The odd partitions of \( 2^t \) are exactly the hook partitions \( (2^t - b, 1^b) \), \( 0 \leq b \leq 2^t - 1 \), of degree \( \binom{2^t - 1}{b} \). Hence for \( k \in \{0, 1, \ldots, t - 1\} \) we have

\[
f_k (\lambda) = \begin{cases} (2^t - b - 2^k, 1^b) & \text{if } 2^k \not\subseteq 2 \bigcap b, \\ (2^t - b, 1^{b-2^k}) & \text{if } 2^k \subseteq 2 \bigcap b. \end{cases}
\]

It follows that for any \( k, \ell < t \) and odd partition \( \lambda \) of \( 2^t \), we have \( f_\ell f_k (\lambda) = f_k f_\ell (\lambda) \).

Lemma 4.4. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). If \( m < 2^k \) then \( (n; k, \ell) \in T \).

Proof. We use induction on \( k \geq 0 \). For \( k = 0 \) we have \( m = 0 \) and the claim follows from Lemma 4.3. Suppose that \( k \geq 1 \) and that the claim has been proved up to \( k - 1 \). Let \( \lambda \vdash_\omega n \). Odd hooks of length \( 2^k \) and \( 2^\ell \) in \( \lambda \) correspond to odd hooks of length \( 2^{k-1} \) and \( 2^{\ell-1} \) in the 2-quotient \( Q_2(\lambda) = (\lambda_0, \lambda_1) \) of \( \lambda \). From Theorem 2.5 we deduce that \( |\lambda_0| \) and \( |\lambda_1| \) are 2-disjoint binary subsums of \( \left[\frac{n}{2}\right] \), so one of them contains \( 2^{k-1} \), say \( |\lambda_0| \); then \( |\lambda_1| \leq \left[\frac{m}{2}\right] < 2^{k-1} < 2^{\ell-1} \). Thus the odd \( 2^{k-1} \)-hook in \( Q_2(\lambda) \) has to be in \( \lambda_0 \). Therefore

\[
Q_2(f_k (\lambda)) = (f_{k-1}(\lambda_0), \lambda_1).
\]
Applying $f_\ell$, the odd $2^{\ell-1}$-hook cannot be in $\lambda_1$, hence

$$Q_2(f_\ell f_k(\lambda)) = (f_{\ell-1} f_{k-1}(\lambda_0), \lambda_1)).$$

In particular, we know that $|\lambda_0| \geq 2^{\ell-1} + 2^{k-1}$. Also $|\lambda_0| + |\lambda_1| = \left\lfloor \frac{n}{2} \right\rfloor = 2^{\ell-1} + \left\lfloor \frac{n}{2} \right\rfloor$. We have already seen that $2^\ell$ is the largest binary digit of $|\lambda_0|$; furthermore $|\lambda_0| - 2^{\ell-1}$ is a binary subsum of $\left\lfloor \frac{n}{2} \right\rfloor < 2^{k-1}$. We may therefore apply the inductive hypothesis to $\lambda_0$ to get $f_{\ell-1} f_{k-1}(\lambda_0) = f_{k-1} f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_k f_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$.

Lemmas 4.2 and 4.4 show that the only if part of the theorem is true. We now turn to the if part. We start by proving the statement for $k = 0$ and use this as part of an inductive argument.

**Lemma 4.5.** Let $n = 2^\ell + m$ with $0 < m < 2^\ell$. If $0 < \ell < t$ then $(n, 0, \ell) \in \mathcal{F}$, with the exception of $(6; 0, 1)$.

**Proof.** The result is easily checked for $n \leq 8$, which includes the exception $(6; 0, 1)$. So we assume that $t \geq 3$.

**Case 1:** $2^\ell < m$. Then $m \geq 3$, since $\ell > 0$. Consider the partition $\lambda = (m, m, 1^a) \vdash n$ where $a = n - 2m = 2^\ell - m$. The $(1,1)$-hook length of $\lambda$ is $2^\ell + 1$. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Removing the $(2,1)$-hook hook we get the odd partition $(m)$, so $\lambda$ is odd, by Lemma 2.8. We claim that

$$f_0(\lambda) = (m, m, 1^{a-1}).$$

Indeed we cannot have $f_0(\lambda) = (m, m-1, 1^a)$ because this partition does not have a hook of length $2^\ell$, and thus it is not odd. Now

$$f_\ell(f_0(\lambda)) = f_\ell(m, m, 1^{a-1}) = (m, m-2^\ell, 1^{a-1})$$

since $(m, m, 1^{a-1-2^\ell})$ and $(m-1, m-2^\ell + 1, 1^{a-1})$ both do not have a hook of length $2^\ell$ and thus are not odd (again by Lemma 2.8).

On the other hand,

$$f_\ell(\lambda) = (m-1, m-2^\ell-1, 1^a).$$

Indeed, the other candidates for $f_\ell(\lambda)$, which are $(m, m-2^\ell, 1^a)$ and $(m, m, 1^{a-2^\ell})$, do not have hooks of length $2^\ell$. Then

$$f_0(f_\ell(\lambda)) = f_0(m-1, m-2^\ell-1, 1^a) = (m-1, m-2^\ell, 1^a).$$

This follows (again) by observing that all the other partitions of $n-2^\ell-1$ obtained from $(m-1, m-2^\ell-1, 1^a)$ by removing a node do not have hooks of length $2^\ell$. Thus $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

**Case 2:** $m < 2^\ell$. Consider the partition $\lambda = (n-2^\ell, m+1, 1^a)$, where $a = 2^\ell - (m+1)$. Note that $n-2^\ell \geq m+1$ since $\ell < t$ by assumption, and that $a \geq 0$. The $(1,1)$-hook length of $\lambda$ is $n-m = 2^\ell$. Removing this hook we get the odd partition $(m)$, so $\lambda$ is odd. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Now

$$f_0(\lambda) = (n-2^\ell, m, 1^a)$$

since the other candidates do not have hooks of length $2^\ell$. Then

$$f_\ell(f_0(\lambda)) = f_\ell(n-2^\ell, m, 1^a) = \mu,$$
where \( \mu \) is obtained from \( f_0(\lambda) \) by removing a \( 2^\ell \)-hook in the first row. (There are only hooks of length \(< 2^\ell \) in the other rows.) In fact, \( \mu = (n - 2^{\ell+1}, m, 1^q) \) since \( n - 2^{\ell+1} \geq n - 2^\ell = m \). Thus \( f_t(f_0(\lambda)) \) has at least 2 parts. On the other hand

\[
f_t(\lambda) = (n - 2^\ell)
\]

since this odd partition is obtained from the odd partition \( \lambda \) by removing a \( 2^\ell \)-hook (the one in \( (2,1) \)). It follows that

\[
f_0(f_t(\lambda)) = (n - 2^\ell - 1)
\]

and again \( f_0(f_t(\lambda)) \neq f_t(f_0(\lambda)) \).

**Case 3:** \( m = 2^\ell \). Then \( n = 2^\ell + 2^\ell \). If \( \ell \geq 2 \) then choose \( \lambda = (2^\ell, 2^\ell - 1, 1) \). The \((1,2)\)-hook length of \( \lambda \) is \( 2^\ell \); thus \( \lambda \) is an odd partition since removing this \( 2^\ell \)-hook gives an odd partition \((2^\ell - 2, 1, 1)\) of \( 2^\ell \). We have \( f_0(\lambda) = (2^\ell, 2^\ell - 2, 1) \) since the other candidates are not odd. Then

\[
f_t(f_0(\lambda)) = (2^\ell - 2^\ell, 2^\ell - 2, 1).\]

The \((2,1)\)-hook length of \( \lambda \) is \( 2^\ell \), so \( f_t(\lambda) = (2^\ell) \) and

\[
f_0(f_t(\lambda)) = (2^\ell - 1),
\]

showing \( f_0(f_t(\lambda)) \neq f_t(f_0(\lambda)) \).

On the other hand, if \( \ell = 1 \) then choose \( \lambda = (2^\ell - 2, 2, 2) \vdash_o 2^\ell + 2 = n \). Since \( t \geq 3 \), it is now easy to show that \( f_t(f_0(\lambda)) = (2^\ell - 4, 2, 1) \). On the other hand we see that \( f_0(f_t(\lambda)) \) is a hook partition of \( 2^\ell - 1 = n - 3 \) and therefore is not equal to \( f_t(f_0(\lambda)) \).

**Lemma 4.6.** If \( (n; k, \ell) \in \mathcal{F} \) then also \( (2n; k + 1, \ell + 1) \in \mathcal{F} \) and \( (2n + 1; k + 1, \ell + 1) \in \mathcal{F} \).

**Proof.** Let the odd partition \( \mu \) of \( n \) satisfy \( f_k f_t(\mu) \neq f_t f_k(\mu) \). Let \( \lambda \) be a partition of \( 2n \) or \( 2n + 1 \) having 2-quotient \( Q_2(\lambda) = (\mu, (0)) \). Then \( \lambda \) is odd, by Theorem 2.5. We have

\[
Q_2(f_{k+1} f_{\ell+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_t f_k(\mu), (0)) = Q_2(f_{\ell+1} f_{k+1}(\lambda)),
\]

so that \( f_{k+1} f_{\ell+1}(\lambda) \neq f_{\ell+1} f_{k+1}(\lambda) \).

We are now ready to conclude this section with the proof of Theorem B.

**Proof of Theorem 4.1.** The only if part follows from Lemmas 4.2 and 4.4. To prove the if part we use induction on \( k \geq 0 \). If \( k = 0 \), then the statement follows from Lemma 4.5. Let \( k > 1 \) and suppose that the assertion is true up to and including \( k - 1 \). To show that \( (n; k, \ell) \in \mathcal{F} \) it suffices to prove \( ([\frac{n}{2}] ; k - 1, \ell - 1) \in \mathcal{F} \), by Lemma 4.6. We are assuming \( n = 2^\ell + m \), \( 0 \leq m < 2^\ell \), \( 0 \leq k < \ell \leq t \) and \( 2^\ell + 2^\ell \leq n \). This implies \( [\frac{n}{2}] = 2^{\ell - 1} + [\frac{m}{2}] \), \( 0 \leq [\frac{m}{2}] < 2^{\ell - 1} \) and \( 2^{k-1} + 2^{\ell-1} \leq [\frac{n}{2}] \). We may apply the inductive hypothesis to get \( ([\frac{n}{2}] ; k - 1, \ell - 1) \in \mathcal{F} \), and then \( (n; k, \ell) \in \mathcal{F} \) except when \( ([\frac{n}{2}] ; k - 1, \ell - 1) = (6; 0, 1) \). In that case we are considering \((12;1,2)\) or \((13;1,2)\) which are both in \( \mathcal{F} \), by direct computation (consider for example \((6,4,2) \vdash_o 12 \) and \((6,4,3) \vdash_o 13 \) respectively).

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References


