Restriction of Odd Degree Characters of $S_n$

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Abstract. Let $n$ and $k$ be natural numbers such that $2^k < n$. We study the restriction to $S_{n-2^k}$ of odd-degree irreducible characters of the symmetric group $S_n$. This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., Sémin. Lothar. Combin. 75 (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., J. Algebra 478 (2017), 271–282].

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1 Introduction

Let $n$ be a natural number, and let $\chi$ be an irreducible character of odd degree of the symmetric group $S_n$. Then there exists a unique odd-degree irreducible constituent of the restriction $\chi_{S_n}$. This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any $k \in \mathbb{N}$ such that $2^k < n$, there exists a unique odd-degree irreducible constituent $f^n_k(\chi)$ of $\chi_{S_{n-2^k}}$ appearing with odd multiplicity. The main goal of this article is to study for all $n, k \in \mathbb{N}$ the map

$$f^n_k: \text{Irr}_2(S_n) \rightarrow \text{Irr}_2(S_{n-2^k}),$$

naturally defined by Theorem A of [3]. All our results are proved using a description of $f^n_k$ in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If $2^k$ appears in the binary expansion of $n$ we say that $2^k$ is a binary digit of $n$. Similarly we say that two natural numbers $m$ and $n$ are 2-disjoint if they do not have any common binary digit. On the other hand, if $m \leq n$ and all the binary digits of $m$ appear in the binary expansion of $n$, then we say that $m$ is a binary subsum of $n$. This will be denoted by $m \subseteq_2 n$. Let $\nu_2(n)$ be the exponent of the highest power of 2 dividing the integer $n$.

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A question raised in [3] may be phrased as: For which \( n \) and \( k \) is \( f_k^n \) surjective? The authors showed that \( f_k^n \) is surjective whenever \( 2^k \) is a binary digit of \( n \), and they observed that otherwise \( f_k^n \) could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

**Theorem A.** Let \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) be such that \( 2^k < n \). Let \( d(n, k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor \right) \).

- If \( k = 0 \) then \( f_k^n \) is surjective if and only \( d(n, k) \leq 2 \).
- If \( k > 0 \) then \( f_k^n \) is surjective if and only \( d(n, k) \leq 1 \).

Theorem A is a consequence of Theorem 3.5 below, which describes the images of the maps \( f_k^n \). For all \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) with \( 2^k < n \) and any \( \psi \in \text{Irr}_{2^k}(\mathfrak{S}_{n-2^k}) \) we define the set

\[
\mathcal{E}(\psi, 2^k) = \{ \chi \in \text{Irr}_{2^k}(\mathfrak{S}_n) \mid f_k^n(\chi) = \psi \},
\]

and set \( e(\psi, 2^k) = |\mathcal{E}(\psi, 2^k)| \). We show in Corollary 3.8 that the maps \( f_k^n \) are regular on their images. This means that for any \( \psi \) in the image of \( f_k^n \), the number \( e(\psi, 2^k) \) depends only on \( n \) and \( k \) and not on the specific \( \psi \). We also give a complete description of those \( \psi \in \text{Irr}_{2^k}(\mathfrak{S}_{n-2^k}) \) such that \( e(\psi, 2^k) = 0 \), in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote \( f_k^n \) just by \( f_k \), when the natural number \( n \) is clear from the context. Then, for \( k, \ell \in \mathbb{N}_0, \ k < \ell \), such that \( 2^k + 2^{\ell} \leq n \), we may ask: when is \( f_k f_\ell = f_\ell f_k \)? or more specifically: when is \( f_k^{n-2^k} f_\ell^{n-2^\ell} = f_\ell^{n-2^\ell} f_k^{n-2^k} \)? In [3, Proposition 4.3] it was proved that \( f_k f_\ell = f_\ell f_k \) whenever \( 2^\ell < n < 2^{\ell+1} \). This is the case \( \ell = t \) in our second main result, which answers the question completely.

**Theorem B.** Let \( n = 2^t + m \) where \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \leq t \) and \( 2^k + 2^{\ell} \leq n \). Then, with the exception of the case \( n = 6, k = 0, \ell = 1 \),

\[ f_k f_\ell = f_\ell f_k \ if \ and \ only \ if \ 2^k > m \ or \ \ell = t. \]

## 2 Notation and background

Let \( n \) be a natural number. We let \( \text{Irr}(\mathfrak{S}_n) \) denote the set of irreducible characters of \( \mathfrak{S}_n \) and \( \mathcal{P}(n) \) the set of partitions of \( n \). The notation \( \lambda \in \mathcal{P}(n) \) is sometimes replaced by \( \lambda \vdash n \) and we write \( |\lambda| = n \). There is a natural correspondence \( \lambda \leftrightarrow \chi^\lambda \) between \( \mathcal{P}(n) \) and \( \text{Irr}(\mathfrak{S}_n) \). We say then that \( \lambda \) labels \( \chi^\lambda \). We denote by \( \text{Irr}_{2^k}(\mathfrak{S}_n) \) the set of irreducible characters of \( \mathfrak{S}_n \) of odd degree. If \( \chi^\lambda \in \text{Irr}_{2^k}(\mathfrak{S}_n) \) we say that \( \chi^\lambda \) is an odd character, we call \( \lambda \) an odd partition of \( n \) and write \( \lambda \vdash_o n \). Also the empty partition will be considered as an odd partition.

**Remark 2.1.** Let \( n, k \) be such that \( 2^k < n \). In [3, Theorem A and Proposition 4.2] it is shown that the map \( f_k^n: \text{Irr}_{2^k}(\mathfrak{S}_n) \to \text{Irr}_{2^k}(\mathfrak{S}_{n-2^k}) \) may be described in terms of the odd partitions labelling the odd characters as follows:

\[ f_k^n(\chi^\lambda) = \chi^{\mu} \Leftrightarrow \mu \vdash_o n - 2^k \] can be obtained from \( \lambda \vdash_o n \) by removing a \( 2^k \)-hook.

Correspondingly we write (by abuse of notation) \( f_k^n(\lambda) = \mu \). In fact when \( \lambda \) is odd, there is only one \( 2^k \)-hook of \( \lambda \) whose removal leads again to an odd partition; we will refer to such a hook as an odd hook of \( \lambda \). This combinatorial description of \( f_k^n \) will be used throughout this paper, and we will regard \( f_k^n \) also as a map between the corresponding sets of odd partitions. Also, for \( \mu \vdash_o n - 2^k \) we set \( e(\mu, 2^k) = e(\chi^\mu, 2^k) \).
Lemma 2.3. Suppose that $Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1})$ is an $e$-tuple of partitions satisfying $n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i|$. It is well-known that a partition is uniquely determined by its $e$-core and $e$-quotient (we refer the reader to [6] or [4, Chapter 2.7] for a detailed discussion on this topic).

Let $H_e(\lambda)$ be the set of hooks of $\lambda$ having length divisible by $e$, and let $\mathcal{H}(Q_e(\lambda)) = \cup_{i=1}^{e-1} H(\lambda_i)$. As explained in [6, Theorem 3.3], there is a bijection between $H_e(\lambda)$ and $\mathcal{H}(Q_e(\lambda))$ mapping hooks in $\lambda$ of length $ek$ to hooks in the quotient of length $k$. Moreover, the bijection respects the process of hook removal. Namely, the partition $\mu$ obtained by removing an $ek$-hook from $\lambda$ is such that $C_e(\mu) = C_e(\lambda)$ and the $e$-quotient of $\mu$ is obtained by removing an $x$-hook from one of the partitions involved in $Q_e(\lambda)$.

For $e = 2$ we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-quotient tower $Q_2(\lambda)$ and the 2-core tower $C_2(\lambda)$ of $\lambda$. They have rows numbered by $k \geq 0$. The $k$th row $Q_2^{(k)}(\lambda)$ of $Q_2(\lambda)$ contains $2^k$ partitions $\lambda_i^{(k)}$, $0 \leq i \leq 2^k - 1$, and the $k$th row $C_2^{(k)}(\lambda)$ of $C_2(\lambda)$ contains the 2-cores of these partitions in the same order, i.e., $C_2(\lambda_i^{(k)})$, $0 \leq i \leq 2^k - 1$. The 0th row of $Q_2(\lambda)$ contains $\lambda = \lambda_0^{(0)}$ itself, row 1 contains the partitions $\lambda_0^{(1)}$, $\lambda_1^{(1)}$ occurring in the 2-quotient $Q_2(\lambda)$, row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have $Q_2^{(1)}(\lambda_i^{(k)}) = (\lambda_{2i}^{(k+1)}, \lambda_{2i+1}^{(k+1)})$ for $i \in \{0, 1, \ldots, 2^k - 1\}$. We remark that the $2^k$ partitions in $Q_2^{(k)}(\lambda)$ are the same as those in the $2^k$-quotient $Q_{2^k}(\lambda)$ of $\lambda$, but in a different order for $k \geq 2$.

We also introduce the $k$-data $D_2^{(k)}(\lambda)$ of $\lambda$. This is a table containing the following $k+1$ rows: the $k$ rows $C_2^{(j)}(\lambda)$, $j = 0, \ldots, k - 1$, and in addition the row $Q_2^{(k)}(\lambda)$.

Remark 2.2. A partition $\lambda$ may be recovered from its 2-core tower. For $k > 0$, it may also be recovered from the knowledge of the $k$-data $D_2^{(k)}(\lambda)$ of $\lambda$, because the rows $C_2^{(l)}(\lambda)$ with $l \geq k$ of $C_2(\lambda)$ consist of the 2-core towers of the partitions in $Q_2^{(k)}(\lambda)$.

Lemma 2.3. Suppose that $\lambda \vdash n - 2^k$ and $\mu \vdash n$. The following are equivalent.

(i) $\lambda$ is obtained from $\mu$ by removing a $2^k$-hook.

(ii) The $k$-data $D_2^{(k)}(\mu)$ and $D_2^{(k)}(\lambda)$ coincide, except that for one $i \in \{0, \ldots, 2^k - 1\}$ $\lambda_i^{(k)}$ is obtained from $\mu_i^{(k)}$ by removing a 1-hook.

Proof. A $2^k$-hook $H_0$ in $\mu$ corresponds in a canonical way to a $2^{k-1}$-hook $H_1$ in a partition in $Q_2^{(1)}(\mu)$, i.e., in row 1 of the 2-quotient tower $Q_2(\mu)$. Continuing we see that $H_0$ corresponds in a canonical way to a 1-hook $H_k$ in a partition $\mu^{(k)}_i$ in $Q_2^{(k)}(\mu)$, row $k$ of $Q_2(\mu)$. If $\lambda$ is obtained by removing $H_0$ from $\mu$, this corresponds to $\lambda_i^{(k)}$ being obtained by removing the 1-hook $H_k$ from $\mu_i^{(k)}$ (by repeated applications of [6, Theorem 3.3]). Apart from this the rows $Q_2^{(k)}(\mu)$ and $Q_2^{(k)}(\lambda)$ coincide. Note also that the rows $C_2^{(j)}(\mu)$ and $C_2^{(j)}(\lambda)$ coincide for $j = 0, \ldots, k - 1$, since the removal of the hooks $H_j$ of even length do not change the 2-cores. \hfill \blacksquare

Odd-degree characters of $\mathfrak{S}_n$ and thus odd partitions were completely described in [5]. We restate this result in a language which is convenient for our purposes. We let $c_2^{(k)}(\lambda)$ be the sum of the cardinalities of the partitions in the $k$th row $C_2^{(k)}(\lambda)$ of $C_2(\lambda)$.

Lemma 2.4 ([5]). Let $\lambda$ be a partition. Then $\lambda$ is odd if and only if $c_2^{(k)}(\lambda) \leq 1$ for all $k \geq 0$.

It may be decided from the $k$-data $D_2^{(k)}(\lambda)$ whether $\lambda$ is odd. The case $k = 1$ of the following result appeared in [3, Lemma 4.1] and also in [1, Lemma 6].
Theorem 2.5. Let $\lambda \vdash n$, and let $k \geq 0$ be fixed. Consider $Q_2^{(k)}(\lambda) = (\lambda_i^{(k)})$. Then $\lambda$ is odd if and only if the following conditions are all fulfilled:

(i) $c_2^{(j)}(\lambda) \leq 1$ for all $j < k$.
(ii) The partitions $\lambda_i^{(k)}$, $0 \leq i \leq 2^k - 1$, are all odd.
(iii) The numbers $|\lambda_i^{(k)}|$, $0 \leq i \leq 2^k - 1$, are pairwise 2-disjoint.

In this case $\sum_{i \geq 0} |\lambda_i^{(k)}| = \lfloor \frac{n}{2^k} \rfloor$.

Proof. This is proved by induction on $k \geq 0$, using Remark 2.2 and Lemma 2.4.

We illustrate the result above by giving an example.

Example 2.6. Let $n = 15$ and take $\lambda = (5, 4, 2^2, 1^2) \vdash 15$. To decide whether $\lambda$ is odd, we choose $k = 2$ and compute the 2-data $D_2^{(2)}(\lambda)$. The 2-core is $C_2(\lambda) = (1)$, giving $C_2^{(0)}(\lambda) = ((1))$. Furthermore, the 2-quotient is $Q_2(\lambda) = ((2^2, 1^2), (1))$, and computing the 2-cores $C_2((2^2, 1^2)) = (0), C_2((1)) = (1)$, we obtain the next row: $C_2^{(1)}(\lambda) = ((0), (1))$. The 2-quotients are $Q_2((2^2, 1^2)) = ((1^2), (1), Q_2((1)) = ((0), (0)));$ hence the final row of the 2-data table is obtained as $Q_2^{(2)}(\lambda) = ((1^2), (1), (0), (0))$.

We visualize $D_2^{(2)}(\lambda)$ like this:

$C_2^{(0)}(\lambda): \quad (1)$
$C_2^{(1)}(\lambda): \quad (0) \quad (1)$
$Q_2^{(2)}(\lambda): \quad (1^2) \quad (1) \quad (0) \quad (0)$

Theorem 2.5 shows that $\lambda$ is odd and thus it contains a unique odd 4-hook. Again using the theorem, it is clear that removing this 4-hook corresponds to the second partition (1) in $Q_2^{(2)}$ being replaced by (0). Thus, removing the corresponding 4-hook of $\lambda$ we obtain the odd partition $\mu = (3, 2^4, 1^2) \vdash 11$ with the property that $D_2^{(2)}(\lambda)$ and $D_2^{(2)}(\mu)$ differ only in their final row.

Remark 2.7. Using the construction of partitions from their 2-cores and 2-quotients already mentioned, the criterion above can be applied to construct all odd partitions of $n$ with a specific $k$th row in the 2-quotient tower. For this, let $n, k \in \mathbb{N}$, and take any sequence of odd partitions $\nu_i$, $0 \leq i \leq 2^k - 1$, such that the numbers $|\nu_i|$ are pairwise 2-disjoint, and $\sum_{i \geq 0} |\nu_i| = \lfloor \frac{n}{2^k} \rfloor$.

Then there are exactly $\prod_{m<k} 2^m$ odd partitions $\lambda$ of $n$ with $Q_2^{(k)}(\lambda) = (\nu_i)$, obtained by choosing one 2-core in row $m$ of the $k$-data table to be (1), for each $m < k$ such that $2^m \leq 2n$.

The following easy consequence of Theorem 2.5 will be used repeatedly.

Lemma 2.8. Let $2^t$ be the largest binary digit of $n$. A partition $\lambda$ of $n$ is odd if and only if $\lambda$ contains a unique $2^t$-hook and the partition obtained from $\lambda$ by removing this $2^t$-hook is an odd partition of $n - 2^t$.

3 Surjectivity and regularity

The aim of this section is to study the images of the maps $f_k^n$ for all $n, k$ such that $2^k \leq n$. For this purpose we introduce the concept of $d$-good partitions (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when $f_k^n$ is surjective) and to show that the maps $f_k^n$ are always regular on their image (see Corollary 3.8).
Definition 3.1. Let $d \geq 0$. We call an odd partition $\lambda$ $d$-good, if
\begin{enumerate}[(i)]
  \item $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.
  \item $C_{2d}(\lambda)$ is a hook partition.
\end{enumerate}

Let us remark that condition (i) may be reformulated as
\begin{itemize}
  \item[(i*)] $\nu_2(|\lambda| + 1) = d$.
\end{itemize}

In particular, if $\lambda$ is $d$-good, then $|\lambda|$ is odd if and only if $d > 0$.

The relevance of $d$-good partitions in our context is illuminated by the following reformulation of [1, Theorem 2]:

Lemma 3.2. Let $\lambda \vdash_o n$. Let $d = \nu_2(n+1)$. Then $e(\lambda, 1) \neq 0$ if and only if $\lambda$ is $d$-good. In this case, $e(\lambda, 1) = 1$ if $d = 0$, and $e(\lambda, 1) = 2$ if $d > 0$.

Lemma 3.3. Let $\lambda$ be an odd partition, and let $d \geq 0$. Then the following hold.
\begin{enumerate}[(1)]
  \item For $d \leq 2$, $\lambda$ is $d$-good if and only if $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.
  \item If $\lambda$ is $d$-good, then $C_{2d}(\lambda)$ is a partition of $2^d - 1$.
\end{enumerate}

Proof. If the odd partition $\lambda$ is $d$-good, then $|\lambda| = (2^d - 1) + m$ where the binary digits of $m$ are at least $2^{d+1}$. The hooks of $\lambda$ corresponding to the binary digits of $m$ may be decomposed into $2^d$-hooks and thus do not contribute to $C_{2d}(\lambda)$. Thus $|C_{2d}(\lambda)| = 2^d - 1$. This shows (2).

As mentioned in the introduction, the results in [3] show that $f_k^n$ is a surjective ($2^k$-to-1)-map whenever $2^k \subseteq 2 n$, i.e., $d(n, k) = 0$. In the spirit of [1, Theorem 2], we now give a characterization of the image of the map $f_k^n$ for all $n, k$ such that $2^k < n$.

Theorem 3.5. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $\lambda \vdash_o n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a $d(n, k)$-good partition in the $k$th row of $Q_2(\lambda)$. In this case, $e(\lambda, 2^k) = 2^d$ if $d(n, k) = 0$, and $e(\lambda, 2^k) = 2$ if $d(n, k) > 0$.

Proof. If $k = 0$ then the statement follows from Lemma 3.2. Hence assume that $k \geq 1$. Let $d = d(n, k)$. By assumption $\lfloor \frac{n}{2^k} \rfloor = 2^d + m$, where the binary digits of $m$ are at least $2^{d+1}$. Thus $\lfloor \frac{n-2^k}{2^k} \rfloor = (2^d - 1) + m$.

Suppose first that $e(\lambda, 2^k) \neq 0$ and that $\mu \vdash_o n$ satisfies $f_k(\mu) = \lambda$. From Remark 2.1 and Lemma 2.3 we get that there exists an $i \in \{0, 1, \ldots, 2^k - 1\}$ such that $f_0(\mu_i^{(k)}) = \lambda_i^{(k)}$. Since $\mu_i^{(k)}$ and $\lambda_i^{(k)}$ are odd, we get $e(\lambda_i^{(k)}, 1) \neq 0$. We have that $|\lambda_i^{(k)}|$ and $|\mu_i^{(k)}|$ are both $2$-disjoint with $m_1 := \sum_{j \neq i} |\lambda_j^{(k)}| = \sum_{j \neq i} |\mu_j^{(k)}| \subseteq 2 \lfloor \frac{n-2^k}{2^k} \rfloor$, by Theorem 2.5. Since $m_1 \subseteq 2 \lfloor \frac{n-2^k}{2^k} \rfloor$ and $m_1 \subseteq 2 \lfloor \frac{n}{2^k} \rfloor$, we get $m_1 \subseteq 2 m$. Thus $|\lambda_i^{(k)}| = (2^d - 1) + m_2$ and $|\mu_i^{(k)}| = 2^d + m_2$, where $m_2 = m - m_1 \subseteq 2 m$.

In particular $\nu_2(|\lambda_i^{(k)}| + 1) = \nu_2(|\mu_i^{(k)}|) = d$. Then Lemma 3.2 shows that $\lambda_i^{(k)}$ is $d$-good.

Conversely, if $\lambda_i^{(k)}$ is a $d$-good partition for some $i \in \{0, 1, \ldots, 2^k - 1\}$, then there exists a $\mu^* \vdash_o |\lambda_i^{(k)}| + 1$ such that $f_0(\mu^*) = \lambda_i^{(k)}$, by Lemma 3.2. We let $\mu$ be the partition where the $k$-data $D_2^{(k)}(\mu)$ and $D_2^{(k)}(\lambda)$ coincide, except that $\mu_i^{(k)} = \mu^*$. Since $\lambda$ is odd and $\lambda_i^{(k)}$ is $d$-good,
we know that \(|\lambda_i^{(k)}| = (2^d - 1) + m'| where \(m' \subseteq m\), and \(|\lambda_j^{(k)}| \subseteq m - m' for all \(j \neq i\). Hence \(|\mu'| = |\lambda_i^{(k)}| + 1 = 2^d + m' is 2-disjoint from all \(|\lambda_j^{(k)}|, j \neq i\). Thus \(\mu\) is an odd partition of \(n\) by Theorem 2.5, and \(f_k(\mu) = \lambda\) by Lemma 2.3 and Remark 2.1.

We conclude that \(e(\lambda, 2^k) = \sum_{\lambda^{(k)}_i \text{-good}} e(\lambda^{(k)}_i, 1)\). If \(d = 0\) then \(\lfloor \frac{n-2^k}{2^d} \rfloor\) is even. This implies that all \(\lambda_i^{(k)}\) are of even cardinality and thus \(d\)-good. Thus \(e(\lambda^{(k)}_i, 1) = 1, i\) and we get \(e(\lambda, 2^k) = 2^k\). If \(d > 0\) there is exactly one \(\lambda_i^{(k)}\) in \(Q_2^{(k)}(\lambda)\) of odd cardinality. Only this \(\lambda_i^{(k)}\) may be \(d\)-good and then \(e(\lambda, 2^k) = e(\lambda^{(k)}_i, 1) = 2\). Otherwise \(e(\lambda, 2^k) = 0\). ■

**Corollary 3.6.** Let \(n \in \mathbb{N}, k \in \mathbb{N}_0\) be such that \(2^k < n\), and let \(d = \nu_2\left(\frac{n}{2^k}\right)\). Let \(\lambda \vdash_o n - 2^k\).

Then \(e(\lambda, 2^k) \neq 0\) if and only if there exists a partition \(\lambda_i^{(k)}\) in the \(k\)th row of \(Q_2(\lambda)\) such that \(|\lambda_i^{(k)}| \equiv 2^d - 1 \mod 2^{d+1}\), and \(C_{2^d}(\lambda_i^{(k)})\) is a hook partition. In this case, \(e(\lambda, 2^k) = 2^k\) if \(d = 0\), and \(e(\lambda, 2^k) = 2\) if \(d > 0\).

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

**Corollary 3.7 (Theorem A).** Let \(n \in \mathbb{N}, k \in \mathbb{N}_0\) be such that \(2^k < n\).

- If \(k = 0\) then \(f^n_k\) is surjective if and only if \(d(n, k) \leq 2\).
- If \(k > 0\) then \(f^n_k\) is surjective if and only if \(d(n, k) \leq 1\).

**Proof.** By Theorem 3.5, \(f^n_k\) is surjective if and only if for all \(\lambda \vdash_o n - 2^k\) we have that the \(k\)th row of \(Q_2(\lambda)\) contains a \(d(n, k)\)-good partition \(\lambda_i^{(k)}\). By Theorem 2.5 and Definition 3.4, for any \(\lambda \vdash_o n - 2^k\) we have \(\sum_{j \geq 0} |\lambda_j^{(k)}| = \lfloor \frac{n-2^k}{2^d} \rfloor = \left(2^{d(n,k)} - 1\right) + m(n,k)\).

If \(k = 0\) then \(Q_2^{(0)}(\lambda)\) contains only \(\lambda = \lambda_0^{(0)}\). Hence \(f^n_0\) is surjective if and only all odd partitions of \(n - 1\) are \(d(n, 0)\)-good. By Lemma 3.3(1), the latter condition holds when \(d = d(n, 0) \leq 2\). On the other hand, if \(d = \nu_2(n) > 2\), then \(\lambda = (n - 5, 2, 2)\) is an odd partition of \(n - 1\) by Theorem 2.5, but \(C_8(\lambda) = (3, 2, 2)\) is not a hook, and hence \(C_{2^d}(\lambda)\) is not a hook. So \(\lambda\) is not \(d\)-good, and thus \(f^n_0\) is not surjective.

Now assume \(k \geq 1\). Then \(Q_2^{(k)}(\lambda)\) contains at least two odd partitions. If \(d(n, k) \geq 2\) then any \(d(n, k)\)-good partition \(\mu\) satisfies \(3 \subseteq 2^{d(n,k)} - 1 \subseteq |\mu|\). Write \(\lfloor \frac{n-2^k}{2^d} \rfloor = 1 + m_1\) where \(m_1\) is even. Applying Remark 2.7, take any \(\lambda \vdash_o n - 2^k\) such that \(|\lambda_0^{(k)}| = 1\) and \(\lambda_1^{(k)}\) is an odd partition with \(|\lambda_1^{(k)}| = m_1\). Then no partition in \(Q_2^{(k)}(\lambda)\) is \(d(n, k)\)-good. Thus \(f^n_k\) is not surjective. On the other hand, if \(d(n, k) = 0\) then \(2^k \subseteq n\) and \(f^n_k\) is surjective [3, Proposition 4.5]. If \(d(n, k) = 1\) then \(\lfloor \frac{n-2^k}{2^d} \rfloor = 1 + m(n, k)\), where \(4 \mid m(n, k)\). Thus any \(Q_2^{(k)}(\lambda)\) contains a partition with odd cardinality; this partition is 1-good, by Lemma 3.3. Again \(f^n_k\) is surjective. ■

It is an immediate consequence of Theorem 3.5 that \(f^n_k\) is regular on its image for all relevant choices of \(n, k\) such that \(2^k < n\). We have:

**Corollary 3.8.** Let \(n \in \mathbb{N}, k \in \mathbb{N}_0\) be such that \(2^k < n\); set \(d = \nu_2\left(\frac{n}{2^k}\right)\). Let \(\lambda \vdash_o n - 2^k\). Then

\[
e(\lambda, 2^k) = \begin{cases} 2^k & \text{if } d = 0; \\ 2 & \text{if } d > 0, \text{ and the } k\text{th row of } Q_2(\lambda) \text{ contains a } d\text{-good partition}; \\ 0 & \text{otherwise}. \end{cases}
\]
Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take \( k = 2 \) above. For \( n > 2^2 \) we first compute \( d(n, k) = \nu_2\left(\frac{n}{2^k}\right) \), and then consider odd partitions of \( n - 4 \) and their 4-extensions. For \( n = 6 \), \( d(6, 2) = 0 \). Thus \( e(2, 4) = 4 \). The odd 4-extensions of \( (2) \) are \((6), (3^2), (2^2, 1^2), (2, 1^4)\). For \( n = 10 \), \( d(10, 2) = 1 \). In this case, \( e(\lambda, 4) = 2 \) for all odd partitions \( \lambda \) of 6. For instance, the odd 4-extensions of \( (6) \) are \((10)\) and \((6, 3, 1)\). For \( n = 19 \), \( d(19, 2) = 2 \). Example 2.6 shows that for \( \lambda = (5, 4, 2^2, 1^2) \vdash_o 15 \) there is no 2-good partition in \( Q_2(\lambda) \), hence \( e(\lambda, 4) = 0 \).

4 Deciding commutativity of the maps \( f_k \) and \( f_\ell \)

Let \( n \in \mathbb{N} \), and suppose that \( 0 \leq k < \ell \) satisfy \( 2^k + 2^\ell \leq n \). As stated in the introduction, we want to complete the discussion of the commutativity of the maps \( f_k \) and \( f_\ell \). Since the relevant \( n \) will always be apparent for the maps \( f_k^n \) in this section, we just write \( f_k \).

We write \( (n; k, \ell) \in \mathcal{T} \) if for all \( \lambda \vdash_o n \) we have \( f_k f_\ell(\lambda) = f_\ell f_k(\lambda) \). Otherwise we write \( (n; k, \ell) \in \mathcal{F} \).

In this section we will prove Theorem B, which may be reformulated as follows.

Theorem 4.1. Let \( n = 2^t + m \) where \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). Then with the exception of \((6; 0, 1)\)

\( (n; k, \ell) \in \mathcal{F} \) if and only if \( \ell < t \) and \( 2^k \leq m \).

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two extreme cases, where \( f_k \) and \( f_\ell \) commute.

In the case \( \ell = t \) we have the following result as a reformulation of [3, Proposition 4.3].

Lemma 4.2. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). If \( 2^k \leq m \), then \((n; k, t) \in \mathcal{T} \).

It is also known that in the case where \( n \) is a power of 2, the maps \( f_k \) and \( f_\ell \) commute [3, Remark 4.4], and we include a short proof here.

Lemma 4.3. If \( n = 2^t \) then \((n; k, \ell) \in \mathcal{T} \) for all \( k, \ell \).

Proof. If \( 0 \leq b \leq a \) are integers then the binomial coefficient \( \binom{a}{b} \) is odd if and only if \( b \subseteq 2 a \), by Lucas’ theorem. The odd partitions of \( 2^t \) are exactly the hook partitions \( (2^t - b, 1^b) \), \( 0 \leq b \leq 2^t - 1 \), of degree \( (2^t - 1) \). Hence for \( k \in \{0, 1, \ldots, t - 1\} \) we have

\[
f_k(\lambda) = \begin{cases} (2^t - b - 2^k, 1^b) & \text{if } 2^k \not\subseteq b, \\ (2^t - b, 1^{b-2^k}) & \text{if } 2^k \subseteq b. \end{cases}
\]

It follows that for any \( k, \ell < t \) and odd partition \( \lambda \) of \( 2^t \), we have \( f_\ell f_k(\lambda) = f_k f_\ell(\lambda) \).

Lemma 4.4. Let \( n = 2^t + m \) with \( 0 \leq m < 2^t \). Suppose that \( k, \ell \) satisfy \( 0 \leq k < \ell \) and \( 2^k + 2^\ell \leq n \). If \( m < 2^k \) then \((n; k, \ell) \in \mathcal{T} \).

Proof. We use induction on \( k \geq 0 \). For \( k = 0 \) we have \( m = 0 \) and the claim follows from Lemma 4.3. Suppose that \( k \geq 1 \) and that the claim has been proved up to \( k - 1 \). Let \( \lambda \vdash_o n \). Odd hooks of length \( 2^k \) and \( 2^\ell \) in \( \lambda \) correspond to odd hooks of length \( 2^{k-1} \) and \( 2^{\ell-1} \) in the 2-quotient \( Q_2(\lambda) = (\lambda_0, \lambda_1) \) of \( \lambda \). From Theorem 2.5 we deduce that \( |\lambda_0| \) and \( |\lambda_1| \) are 2-disjoint binary subsums of \( \left\lfloor \frac{n}{2} \right\rfloor \), so one of them contains \( 2^{k-1} \), say \( |\lambda_0| \); then \( |\lambda_1| \leq \left\lfloor \frac{m}{2} \right\rfloor < 2^{k-1} < 2^{\ell-1} \). Thus the odd \( 2^{k-1} \)-hook in \( Q_2(\lambda) \) has to be in \( \lambda_0 \). Therefore

\[
Q_2(f_k(\lambda)) = (f_{k-1}(\lambda_0), \lambda_1).
\]
Applying $f_{\ell}$, the odd $2^{\ell-1}$-hook cannot be in $\lambda_1$, hence

$$Q_2(f_{\ell}f_k(\lambda)) = (f_{\ell-1}f_{k-1}(\lambda_0), \lambda_1)).$$

In particular, we know that $|\lambda_0| \geq 2^{\ell-1} + 2^{k-1}$. Also $|\lambda_0| + |\lambda_1| = 2^{\ell-1} + |\frac{n}{2}|$. We have already seen that $2^{\ell-1}$ is the largest binary digit of $|\lambda_0|$; furthermore $|\lambda_0| - 2^{\ell-1}$ is a binary subsum of $|\frac{n}{2}| < 2^{k-1}$. We may therefore apply the inductive hypothesis to $\lambda_0$ to get $f_{\ell-1}f_{k-1}(\lambda_0) = f_{k-1}f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_kf_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_kf_\ell(\lambda) = f_\ell f_k(\lambda)$. ■

Lemmas 4.2 and 4.4 show that the only if part of the theorem is true. We now turn to the if part. We start by proving the statement for $k = 0$ and use this as part of an inductive argument.

**Lemma 4.5.** Let $n = 2^\ell + m$ with $0 < m < 2^\ell$. If $0 < \ell < t$ then $(n; 0, \ell) \in \mathcal{F}$, with the exception of $(6; 0, 1)$.

**Proof.** The result is easily checked for $n \leq 8$, which includes the exception $(6; 0, 1)$. So we assume that $t \geq 3$.

**Case 1:** $2^\ell < m$. Then $m \geq 3$, since $\ell > 0$. Consider the partition $\lambda = (m, m, 1^a) \vdash n$ where $a = n - 2m = 2^\ell - m$. The $(1,1)$-hook length of $\lambda$ is $2^\ell + 1$. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Removing the $(2,1)$-hook hook we get the odd partition $(m)$, so $\lambda$ is odd, by Lemma 2.8. We claim that

$$f_0(\lambda) = (m, m, 1^{a-1}).$$

Indeed we cannot have $f_0(\lambda) = (m, m-1, 1^a)$ because this partition does not have a hook of length $2^\ell$, and thus it is not odd. Now

$$f_\ell(f_0(\lambda)) = f_\ell(m, m, 1^{a-1}) = (m, m - 2^\ell, 1^{a-1})$$

since $(m, m, 1^{a-1-2^\ell})$ and $(m - 1, m - 2^\ell + 1, 1^{a-1})$ both do not have a hook of length $2^\ell$ and thus are not odd (again by Lemma 2.8).

On the other hand,

$$f_\ell(\lambda) = (m - 1, m - (2^\ell - 1), 1^a).$$

Indeed, the other candidates for $f_\ell(\lambda)$, which are $(m, m - 2^\ell, 1^a)$ and $(m, m, 1^{a-2^\ell})$, do not have hooks of length $2^\ell$. Then

$$f_0(f_\ell(\lambda)) = f_0(m - 1, m - (2^\ell - 1), 1^a) = (m - 1, m - 2^\ell, 1^a).$$

This follows (again) by observing that all the other partitions of $n - 2^\ell - 1$ obtained from $(m - 1, m - (2^\ell - 1), 1^a)$ by removing a node do not have hooks of length $2^\ell$. Thus $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

**Case 2:** $m < 2^\ell$. Consider the partition $\lambda = (n - 2^\ell, m + 1, 1^a)$, where $a = 2^\ell - (m + 1)$. Note that $n - 2^\ell \geq m + 1$ since $\ell < t$ by assumption, and that $a \geq 0$. The $(1,1)$-hook length of $\lambda$ is $n - m = 2^\ell$. Removing this hook we get the odd partition $(m)$, so $\lambda$ is odd. The $(2,1)$-hook length of $\lambda$ is $2^\ell$. Now

$$f_0(\lambda) = (n - 2^\ell, m, 1^a)$$

since the other candidates do not have hooks of length $2^\ell$. Then

$$f_\ell(f_0(\lambda)) = f_\ell(n - 2^\ell, m, 1^a) = \mu,$$
where $\mu$ is obtained from $f_0(\lambda)$ by removing a $2^\ell$-hook in the first row. (There are only hooks of length $< 2^\ell$ in the other rows.) In fact, $\mu = (n - 2^{\ell+1}, m, 1^a)$ since $n - 2^{\ell+1} \geq n - 2^\ell = m$. Thus $f_\ell(f_0(\lambda))$ has at least 2 parts. On the other hand
\[ f_\ell(\lambda) = (n - 2^\ell) \]

since this odd partition is obtained from the odd partition $\lambda$ by removing a $2^\ell$-hook (the one in $(2,1)$). It follows that
\[ f_0(f_\ell(\lambda)) = (n - 2^\ell - 1) \]

and again $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

**Case 3:** $m = 2^\ell$. Then $n = 2^\ell + 2^\ell$. If $\ell \geq 2$ then choose $\lambda = (2^\ell, 2^\ell - 1, 1)$. The $(1,2)$-hook length of $\lambda$ is $2^\ell$; thus $\lambda$ is an odd partition since removing this $2^\ell$-hook gives an odd partition $(2^\ell - 2, 1, 1)$ of $2^\ell$. We have $f_0(\lambda) = (2^\ell, 2^\ell - 2, 1)$ since the other candidates are not odd. Then
\[ f_\ell(f_0(\lambda)) = (2^\ell - 2^\ell, 2^\ell - 2, 1) \]

The $(2,1)$-hook length of $\lambda$ is $2^\ell$, so $f_\ell(\lambda) = (2^\ell)$ and
\[ f_0(f_\ell(\lambda)) = (2^\ell - 1), \]

showing $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

On the other hand, if $\ell = 1$ then choose $\lambda = (2^\ell - 2, 2, 2) \vdash_o 2^\ell + 2 = n$. Since $\ell \geq 3$, it is now easy to show that $f_\ell(f_0(\lambda)) = (2^\ell - 4, 2, 1)$. On the other hand we see that $f_0(f_\ell(\lambda))$ is a hook partition of $2^\ell - 1 = n - 3$ and therefore is not equal to $f_1(f_0(\lambda))$. \hfill \blacksquare

**Lemma 4.6.** If $(n; k, \ell) \in \mathcal{F}$ then also $(2n; k + 1, \ell + 1) \in \mathcal{F}$ and $(2n + 1; k + 1, \ell + 1) \in \mathcal{F}$.

**Proof.** Let the odd partition $\mu$ of $n$ satisfy $f_k f_\ell(\mu) \neq f_\ell f_k(\mu)$. Let $\lambda$ be a partition of $2n$ or $2n + 1$ having $2$-quotient $Q_2(\lambda) = (\mu, 0))$. Then $\lambda$ is odd, by Theorem 2.5. We have
\[ Q_2(f_{k+1} f_{\ell+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_\ell f_k(\mu), (0)) = Q_2(f_{\ell+1} f_{k+1}(\lambda)), \]

so that $f_{k+1} f_{\ell+1}(\lambda) \neq f_{\ell+1} f_{k+1}(\lambda)$. \hfill \blacksquare

We are now ready to conclude this section with the proof of Theorem B.

**Proof of Theorem 4.1.** The only if part follows from Lemmas 4.2 and 4.4. To prove the if part we use induction on $k \geq 0$. If $k = 0$, then the statement follows from Lemma 4.5. Let $k > 1$ and suppose that the assertion is true up to and including $k - 1$. To show that $(n; k, \ell) \in \mathcal{F}$ it suffices to prove $\left(\left[\frac{n}{2}\right]; k - 1, \ell - 1\right) \in \mathcal{F}$, by Lemma 4.6. We are assuming $n = 2^\ell + m$, $0 \leq m < 2^\ell$, $0 \leq \ell < k \leq t$ and $2^\ell + 2^\ell \leq n$. This implies $\left[\frac{n}{2}\right] = 2^{t-1} + \left[\frac{m}{2}\right]$, $0 \leq \left[\frac{m}{2}\right] < 2^{t-1}$ and $2^{k-1} + 2^{t-1} \leq \left[\frac{n}{2}\right]$. We may apply the inductive hypothesis to get $\left(\left[\frac{n}{2}\right]; k - 1, \ell - 1\right) \in \mathcal{F}$, and then $(n; k, \ell) \in \mathcal{F}$ except when $\left(\left[\frac{n}{2}\right]; k - 1, \ell - 1\right) = (6; 0, 1)$. In that case we are considering $(12; 1, 2)$ or $(13; 1, 2)$ which are both in $\mathcal{F}$, by direct computation (consider for example $(6, 4, 2) \vdash_o 12$ and $(6, 4, 3) \vdash_o 13$, respectively). \hfill \blacksquare

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