Stability of the nonperturbative bosonic string vacuum

Ambjorn, J.; Makeenko, Y.

Published in:
Physics Letters B

DOI:
10.1016/j.physletb.2017.05.021

Publication date:
2017

Document version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Ambjorn, J., & Makeenko, Y. (2017). Stability of the nonperturbative bosonic string vacuum. *Physics Letters B*, *770*, 352-356. https://doi.org/10.1016/j.physletb.2017.05.021

Physics Letters B 770 (2017) 352–356

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Stability of the nonperturbative bosonic string vacuum

J. Ambjørn [a,b], Y. Makeenko [a,c,*]

[a] The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen, Denmark
[b] IMAPP, Radboud University, Heyendaalseweg 135, 6525 AJ, Nijmegen, The Netherlands
[c] Institute of Theoretical and Experimental Physics, B. Cheremushkinskaya 25, 117218 Moscow, Russia

ARTICLE INFO

Article history:
Received 27 March 2017
Received in revised form 8 May 2017
Accepted 8 May 2017
Available online 10 May 2017
Editor: N. Lambert

ABSTRACT

Quantization of the bosonic string around the classical, perturbative vacuum is not consistent for spacetime dimensions $2 < d < 26$. Recently we have showed that at large d there is another so-called mean-field vacuum. Here we extend this mean-field calculation to finite d and show that the corresponding mean-field vacuum is stable under quadratic fluctuations for $2 < d < 26$. We point out the analogy with the two-dimensional $O(N)$-symmetric sigma-model, where the $1/N$-vacuum is very close to the real vacuum state even for finite N, in contrast to the perturbative vacuum.

1. Introduction

The action of the Nambu–Goto string is the area of the string world sheet. It is highly nonlinear in the embedding-space coordinates. Making use of diffeomorphism invariance and fixing a gauge makes the action quadratic but nonlinearities are now hidden in the dependence of the cutoff on the metric induced at the world sheet.

If one uses the Polyakov formulation [1] of string theory, the embedding-space coordinates and the intrinsic world sheet metric are independent. The action is quadratic in the embedding-space coordinates and in the path integral one can in principle perform the integration over these coordinates. The dependence of this part of the path integral on the world sheet metric is determined by the conformal anomaly. In the conformal gauge this leads to the celebrated Liouville action whose solution [2] about the classical (perturbative) vacuum is consistent only for $d \le 2$. For $2 < d < 26$ the solution is not real which may indicate an instability of the vacuum.

In the work [3] we constructed another vacuum state of the Nambu–Goto string by introducing an independent intrinsic metric ρ_{ab} and the corresponding Lagrange multiplier λ^{ab} and then integrating over the d target-space coordinates X^μ. The corresponding effective action is a functional of ρ_{ab} and λ^{ab} which do not fluctuate in the mean-field approximation that becomes exact at large d. A vacuum state can be found by minimizing the effective action and it is a genuine quantum state because we have taken into account the quantum fluctuations of the target-space coordinates X^μ.

This approach is quite similar to the well-known introduction of a Lagrange multiplier field in the two-dimensional $O(N)$ sigma-model. In this model one integrates over fields $\vec{n}$ obeying the restriction $\vec{n}^2 = 1$. One gets rid of the constraint by introducing the Lagrange multiplier field u. After integration over $\vec{n}$ one obtains an effective action as a functional of u. The minimum of this effective action determines the exact vacuum state for infinite N. For finite N the quantum fluctuations of u have to be included, but they are small even at $N = 3$. The reason is roughly speaking that there is only one u, while the effective action is proportional to N, i.e. parametrically large, which is what is needed for a saddle point. That is not the case for the N fields $\vec{n}$ which fluctuate strongly. The perturbative vacuum $\vec{n}_{\rm cl} = (1, 0, \dots, 0)$ possesses an $O(N-1)$ symmetry and is far away from the genuine nonperturbative $O(N)$-symmetric vacuum, while the mean-field vacuum obtained via the Lagrange multiplier approach possesses the right symmetry and is close to the exact vacuum even at finite N. Fluctuations of u about the mean-field value are systematically tractable within the $1/N$-expansion.

In the present Paper we construct a nonperturbative mean-field vacuum state for the Nambu–Goto string at finite d, show that it is energetically preferable to the perturbative classical vacuum and discuss two possible scaling limits. We calculate the effective action which governs fluctuations of ρ_{ab} and λ^{ab} about their mean-field values, repeating pretty much the original computation of the conformal anomaly in Ref. [1] for the Polyakov string. Fixing the conformal gauge we evaluate the determinants coming from path-integration over X^μ and ghosts, in order to compute the effective

* Corresponding author.
E-mail addresses: ambjorn@nbi.dk (J. Ambjørn), makeenko@nbi.dk (Y. Makeenko).

http://dx.doi.org/10.1016/j.physletb.2017.05.021
0370-2693/

action to quadratic order in $\delta\rho_{ab}$ and $\delta\lambda^{ab}$ at finite d. We show that it is positive definite for $2 < d < 26$, but becomes unstable in the stringy scaling limit for $d > 26$.

2. The mean-field vacuum

Let us consider a closed bosonic string in a target space with one compactified dimension of circumference β, whose world sheet wraps once around this compactified dimension. There is no tachyon with this setup if β is sufficient large for the classical energy squared to be larger than (minus) the tachyon mass squared. The Nambu–Goto action is given by the area of the embedded surface which we rewrite using a Lagrange multiplier λ^{ab} and an independent intrinsic metric ρ_{ab} as

$$K_0 \int d^2\omega \sqrt{\det \partial_a X \cdot \partial_b X}$$
$$= K_0 \int d^2\omega \sqrt{\det \rho} + \frac{K_0}{2} \int d^2\omega\, \lambda^{ab} (\partial_a X \cdot \partial_b X - \rho_{ab}). \tag{1}$$

We perform quantization by the path integral which goes over real $X^\mu(\omega)$ and $\rho_{ab}(\omega)$ and imaginary $\lambda^{ab}(\omega)$. We choose the world sheet coordinates ω_1 and ω_2 inside a $\omega_L \times \omega_\beta$ rectangle in the parameter space, when the classical solution $X^\mu_{\rm cl}$ minimizing the action (1) is ω-independent.

We integrate out quantum fluctuations of the fields X^μ by splitting $X^\mu = X^\mu_{\rm cl} + X^\mu_{\rm q}$ and then performing the Gaussian path integral over $X^\mu_{\rm q}$. We thus obtain the effective action governing the fields λ^{ab} and ρ_{ab},

$$S_{\rm eff} = K_0 \int d^2\omega \sqrt{\det \rho} + \frac{K_0}{2} \int d^2\omega\, \lambda^{ab} (\partial_a X_{\rm cl} \cdot \partial_b X_{\rm cl} - \rho_{ab})$$
$$+ \frac{d}{2} {\rm tr} \log(-\mathcal{O}),$$
$$\mathcal{O} := \frac{1}{\sqrt{\det \rho}} \partial_a \lambda^{ab} \partial_b. \tag{2}$$

The operator $\mathcal{O}$ reproduces the usual two-dimensional Laplacian for $\lambda^{ab} = \rho^{ab}\sqrt{\det \rho}$.

The action (2) is the effective action for path-integration over ρ_{ab} and λ^{ab}. Making use of diffeomorphism invariance one can choose the conformal gauge, diagonalizing $\rho_{ab} = \rho\delta_{ab}$. This produces the ghost determinant [1]

$$\mathcal{D}\rho_{ab} = \mathcal{D}\rho \det(-\mathcal{O}_{\rm gh}). \tag{3}$$

Here the operator

$$\mathcal{O}_{\rm gh} := \left(\Delta - \frac{1}{2}R\right)\delta^a_b = \left[\frac{1}{\rho}\partial^2 - \frac{1}{2\rho}(\partial^2 \log \rho)\right]\delta^a_b \tag{4}$$

acts on 2D vector functions obeying the mixed boundary conditions: Dirichlet for one component and Robin for the other. This produces the term

$$S_{\rm gh} = -{\rm tr} \log(-\mathcal{O}_{\rm gh}) \tag{5}$$

to be added to the right-hand side of Eq. (2).

It is easy to compute the determinants for constant fields $\rho_{ab} = \bar\rho\delta_{ab}$ and $\lambda^{ab} = \bar\lambda\delta^{ab}$. We may consider these as an ansatz for the values of ρ_{ab} and λ^{ab} minimizing the effective action. We shall then demonstrate that quadratic fluctuations around this minimum are stable, so it is indeed a solution minimizing the effective action.

Computing the determinants for constant $\rho_{ab} = \bar\rho\delta_{ab}$ and $\lambda^{ab} = \bar\lambda\delta^{ab}$, we obtain for $L \gg \beta$ the well-known result

$$S_{\rm eff} + S_{\rm gh} = \frac{K_0}{2}\bar\lambda\left(\frac{L^2}{\omega_L^2} + \frac{\beta^2}{\omega_\beta^2}\right)\omega_L\omega_\beta - \frac{\pi(d-2)}{6}\frac{\omega_L}{\omega_\beta}$$
$$+ \left(K_0 - K_0\bar\lambda - \frac{d\Lambda^2}{2\bar\lambda} + \Lambda^2\right)\bar\rho\,\omega_L\omega_\beta, \tag{6}$$

where Λ^2 cuts off eigenvalues of the operators involved (which are parametrization-independent), the cutoff of integration over the proper time being precisely $(4\pi\Lambda^2)^{-1}$.

The minimum of (6) with respect to $\bar\rho$, $\bar\lambda$ and ω_β is reached at

$$\bar\lambda = C := \frac{1}{2} + \frac{\Lambda^2}{2K_0} + \sqrt{\left(\frac{1}{2} + \frac{\Lambda^2}{2K_0}\right)^2 - \frac{d\Lambda^2}{2K_0}}, \tag{7a}$$

$$\bar\rho = \frac{L}{\omega_L\omega_\beta}\frac{\left(\beta^2 - \frac{\pi(d-2)}{6K_0C}\right)}{\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0C}}}\frac{C}{\left(2C - 1 - \frac{\Lambda^2}{K_0}\right)}, \tag{7b}$$

$$\omega_\beta = \frac{\omega_L}{L}\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0C}}. \tag{7c}$$

The value of (6) at the minimum determines the energy of the ground state

$$E_0 = K_0C\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0C}}. \tag{8}$$

It is explicitly seen from this formula that the energy is not tachyonic if β is large enough for the difference under the square root to be positive.

The solution (7), (8) reproduces the one of Ref. [3] as $d \to \infty$ (when $K_0 \sim d$) and generalizes it to finite d. Equations (7) describe a nonperturbative vacuum in the mean-field approximation, where we disregard fluctuations of λ^{ab} and ρ_{ab} about the saddle-point values $\bar\lambda^{ab}$ and $\bar\rho_{ab}$. Note that C as given in (7a) takes values between 1 and

$$C_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right) \tag{9}$$

(monotonically changing from 1/2 to 1 with d decreasing from ∞ to 2) as K_0 decreases from infinity to

$$K_* = \left(d - 1 + \sqrt{d^2 - 2d}\right)\Lambda^2. \tag{10}$$

C would start from its classical value 1, if one were performed a perturbative expansion in $1/K_0$, i.e. about the classical (perturbative) vacuum. This is also true for (7a) and (7b) which would start out with their classical values. However, as described in Sect. 4 the continuum nonperturbative vacuum is approached as $K_0 \to K_*$ and correspondingly $C \to C_*$.

3. Instability of the classical vacuum

It is clear that the ground-state energy (8) is always smaller for $d > 2$ than its classical value $K_0\beta$ because $C < 1$. For this reason the mean-field vacuum (7) is energetically preferable to the perturbative, classical vacuum which is thus unstable.

To understand this instability, it is instructive to compute an "effective potential", like in the studies of symmetry breaking in quantum field theory. For this purpose we add to the action (1) the source term

$$S_{\rm src} = \frac{K_0}{2}\int d^2\omega\, j^{ab}\rho_{ab} \tag{11}$$

and define the partition function $Z[j]$ in the presence of the source by path integration over the fields. Repeating easily the above mean-field computation for constant $j^{ab} = j\delta^{ab}$, we find

$$\bar{\rho}(j) \equiv -\frac{1}{K_0 L\beta}\frac{\partial \log Z[j]}{\partial j} = \frac{1}{2} + \frac{1+j+\frac{\Lambda^2}{K_0}}{\sqrt{\left(1+j+\frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}} \tag{12}$$

for $\omega_L = L$ and $\omega_\beta = \beta \gg 1\sqrt{K_0}$, reproducing then (7b) for $j=0$.

The effective potential $\Gamma(\bar{\rho})$ is defined in the standard way by the Lagrange transformation

$$\Gamma[\bar{\rho}] \equiv -\frac{1}{K_0 L\beta}\left(\log Z[j] + \frac{K_0}{2}\int \mathrm{d}^2\omega\, j^{ab}\bar{\rho}_{ab}(j)\right). \tag{13}$$

Solving Eq. (12) for j we obtain

$$j(\bar{\rho}) = -1 - \frac{\Lambda^2}{K_0} + \sqrt{\frac{d\Lambda^2}{2K_0}}\frac{(2\bar{\rho}-1)}{\sqrt{\bar{\rho}(\bar{\rho}-1)}}, \tag{14}$$

which results in

$$\Gamma(\bar{\rho}) = \left(1+\frac{\Lambda^2}{K_0}\right)\bar{\rho} - \sqrt{\frac{2d\Lambda^2}{K_0}\bar{\rho}(\bar{\rho}-1)} \tag{15}$$

in the mean-field approximation. Note that

$$-\frac{\partial\Gamma(\bar{\rho})}{\partial\bar{\rho}} = j(\bar{\rho}) \tag{16}$$

with $j(\bar{\rho})$ given by Eq. (14) as it should.

Near the classical vacuum when $0 < \bar{\rho} - 1 \ll 1$ the potential (15) decreases with increasing $\bar{\rho}$ because of the second term with the negative sign, which demonstrates an instability of the classical vacuum. If $K_0 > K_*$ given by Eq. (10), the potential (15) linearly increases with $\bar{\rho}$ for large $\bar{\rho}$ and thus has a (stable) minimum at

$$\bar{\rho}(0) = \frac{1}{2} + \frac{1+\frac{\Lambda^2}{K_0}}{2\sqrt{\left(1+\frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}} \tag{17}$$

which is the same as (7b) for $\beta \gg 1/\sqrt{K_0}$. Near the minimum we have

$$\Gamma(\bar{\rho}) = C + \frac{K_0}{2d\Lambda^2}\left[\left(1+\frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}\right]^{3/2}[\bar{\rho}-\bar{\rho}(0)]^2 + \mathcal{O}\left([\bar{\rho}-\bar{\rho}(0)]^3\right). \tag{18}$$

The coefficient in front of the quadratic term is positive for $K_0 > K_*$ which explicitly demonstrates stability of the minimum.

We thus conclude that the effective potential $\Gamma(\bar{\rho})$ is lower for the (stable) mean-field minimum (17) than for the perturbative, classical vacuum $\bar{\rho} = 1$. The latter is therefore unstable. It looks like a dynamical symmetry breaking in quantum field theory that generates a nontrivial world sheet metric (17). This also determines the averaged induced metric because

$$\langle \partial_a X \cdot \partial_b X\rangle = \bar{\rho}_{ab} \tag{19}$$

in the mean-field approximation.

4. Scaling limit and renormalization

The renormalization of the formulas (7b), (8) can be performed quite similarly to the one discussed in Ref. [3] where we had $K_* = 2d\Lambda^2$ and $C_* = 1/2$ at large d. In [3] we discussed two possibilities for renormalization, one led to what we called "Gulliver's world", and it is the renormalization one has been using when one regularized the string theory on a hyper-cubic lattice [4,5] or via dynamical triangulations [6] in d dimensions. The other possibility led to what we denoted the "Lilliputian world", and it is the renormalization where we reproduce some of the standard continuum string theory results.

In both cases we define a renormalized string tension K_R by

$$K_R = K_0\sqrt{\left(1+\frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}} = K_0\left(2C - 1 - \frac{\Lambda^2}{K_0}\right) \tag{20}$$

and insist that it stays finite in the limit $\Lambda \to \infty$. This requirement corresponds to the following scaling behavior of K_0 for $\Lambda \to \infty$:

$$K_0 \to K_* + \frac{K_R^2}{2\Lambda^2\sqrt{d^2-2d}}. \tag{21}$$

With this scaling we have for $\Lambda \to \infty$ that

$$\left(K_0 - \frac{d\Lambda^2}{2C^2}\right) \to \frac{K_R}{C_*}, \qquad 2C - 1 - \frac{\Lambda^2}{K_0} \to \frac{K_R}{K_*}, \tag{22}$$

where C_* and K_* given by Eqs. (9) and (10) are positive functions of d for $2 < d < \infty$.

As described in [3] the difference between the "Gulliver" and the "Lilliputian" renormalizations was that in the lattice approach we did not have the freedom to perform further renormalization, while in the "Lilliputian" case we could perform an additional "background field" renormalization of the external lengths L and β:

$$L_R = L\sqrt{\frac{C}{2C-1-\frac{\Lambda^2}{K_0}}}, \qquad \beta_R = \beta\sqrt{\frac{C}{2C-1-\frac{\Lambda^2}{K_0}}}. \tag{23}$$

By insisting that L_R and β_R remain finite when $\Lambda \to \infty$, rather than L and β do as in lattice string theory, it follows from (22) that L and β go to zero in the scaling limit, thus creating a small (Lilliputian) world from the point of view of the (Gulliver) lattice people.

If we do not renormalize the external lengths L and β, it follows from (7b) that $\bar{\rho}$ diverges in the scaling limit $\Lambda \to \infty$. This is a reflection of the fact that by integrating out the quantum fluctuations of X_q^μ in the decomposition $X^\mu = X_{cl}^\mu + X_q^\mu$ the typical quantum world sheet will have an infinite area. However, the renormalization (23) brings this back to a finite value in the limit $\Lambda \to \infty$ since we then obtain the metric

$$\bar{\rho}_R = \frac{L_R}{\omega_L\omega_\beta}\frac{\left(\beta_R^2 - \frac{\pi(d-2)}{6K_R}\right)}{\sqrt{\beta_R^2 - \frac{\pi(d-2)}{3K_R}}}. \tag{24}$$

Similarly, the renormalized mean-field ground state energy becomes finite

$$E_R = K_R\sqrt{\beta_R^2 - \frac{\pi(d-2)}{3K_R}}, \tag{25}$$

reproducing the well-known Alvarez–Arvis formula.

In the next Sections we will study the stability of the mean-field vacuum (7), both when the "lattice" renormalization and "string theory" renormalization are used.

5. 2D determinants and the Seeley expansion

Two-dimensional determinants diverge and have to be regularized. A standard regularization via the proper time is defined by

$$\log\det(-\mathcal{O})\Big|_{\rm reg}=\operatorname{tr}\log(-\mathcal{O})\Big|_{\rm reg}=-\int_{a^2}^{\infty}\frac{\mathrm{d}\tau}{\tau}\operatorname{tr}\mathrm{e}^{\tau\mathcal{O}},$$

$$a^2\equiv\frac{1}{4\pi\Lambda^2} \tag{26}$$

with $\mathcal{O}$ given in Eq. (2) or Eq. (4).

The standard computation of the (proper-time regularized) determinants of 2D operators is based on the formula

$$-\frac{1}{2}\rho_{ab}(\omega)\frac{\delta}{\delta\rho_{ab}(\omega)}\operatorname{tr}\log(-\mathcal{O})\Big|_{\rm reg}=\left\langle\omega|\mathrm{e}^{a^2\mathcal{O}}|\omega\right\rangle, \tag{27}$$

where one substitutes the expansion in a^2 of the matrix element of the heat kernel operator on the right-hand side, known as the Seeley expansion. To two leading orders it is well-known [7,8] for the bulk part. The boundary terms are also known [9,10] for our case of the Dirichlet (or Robin) boundary conditions, but we shall not need them for $L\gg\beta$ so below we only write the bulk terms. We then have [7–10]

$$\left\langle\omega\middle|\mathrm{e}^{a^2\rho^{-1}\partial_a\lambda^{ab}\partial_b}\middle|\omega\right\rangle$$
$$=\frac{1}{4\pi a^2}\frac{\rho}{\lambda}+\frac{1}{4\pi}\left[-\frac{1}{6}\partial_a^2\ln\rho-\frac{1}{3}\partial_a^2\ln\lambda-\frac{1}{4}(\partial_a\ln\lambda)^2\right]$$
$$+\mathcal{O}(a^2) \tag{28}$$

for diagonal $\lambda^{ab}=\lambda\delta^{ab}$ and $\rho_{ab}=\rho\delta_{ab}$, while the general case can be obtained by making use of diffeomorphism invariance.

Given Eqs. (28) and (27), we can restore the effective action to quadratic order in fluctuations, except for the term $(\delta\lambda)^2$ whose variation with respect to ρ vanishes. We can directly compute this term (as well as the terms $(\delta\rho)^2$ and $\delta\rho\delta\lambda$) by the standard technique of calculating the Coleman–Weinberg potential to quadratic order. For this purpose we expand the regularized determinant to the second order in $\delta\lambda$ (and $\delta\rho$) and obtain

$$A_{\lambda\lambda}(p)=\frac{1}{2\bar\lambda^2}\int_{\bar\lambda a^2/\bar\rho}^{\infty}\mathrm{d}\tau\int_0^{\tau}\mathrm{d}\sigma\int\frac{d^2k}{(2\pi)^2}$$
$$\times k_a(k+p)_a\mathrm{e}^{-\sigma k^2}(k+p)_bk_be^{-(\tau-\sigma)(k+p)^2} \tag{29}$$

for the appropriate coefficient of the quadratic form coming from the determinant. Equation (29) is applicable in our case of one compactified dimension for $\beta\gg\sqrt{1/K_0}$. Otherwise, an additional (Lüscher) term appears from the difference between the integral and the discrete sum over k_2. It is explicitly written in Eq. (6) for constant λ and ρ.

Integrating over σ and τ, we obtain

$$A_{\lambda\lambda}(p)=\frac{1}{2\bar\lambda^2}\int\frac{d^2k}{(2\pi)^2}[k_a(k+p)_a]^2$$
$$\times\left[\frac{\mathrm{e}^{-a^2\bar\lambda(k+p)^2/\bar\rho}}{(k+p)^2}-\frac{\mathrm{e}^{-a^2\bar\lambda k^2/\bar\rho}}{k^2}\right]\frac{1}{(k+p)^2-k^2}. \tag{30}$$

Expanding in a^2, we then find

$$A_{\lambda\lambda}=-\frac{\bar\rho}{4\pi a^2\bar\lambda}-\frac{p^2}{16\pi}\log\left(\frac{c\bar\lambda p^2a^2}{\bar\rho}\right), \tag{31}$$

where c is a (non-universal) constant.

Analogously, for the ghost determinant we have from the Seeley expansion the standard result

$$\operatorname{tr}\log\left\{\left[-\frac{1}{\rho}\partial^2+\frac{1}{2\rho}(\partial^2\log\rho)\right]\delta^a_b\right\}$$
$$=-\Lambda^2\int d^2\omega\,\rho-\frac{13}{48\pi}\int d^2\omega\,(\partial_a\log\rho)^2, \tag{32}$$

where we write only the bulk term, so it does not depend on the boundary conditions.

Combining all together, we obtain the effective action to quadratic order in fluctuations

$$\delta S_2=-\left(K_0-\frac{d\Lambda^2}{2\bar\lambda^2}\right)\bar\rho\bar\lambda\int d^2\omega\,\frac{\delta\rho}{\bar\rho}\frac{\delta\lambda}{\bar\lambda}-\frac{d\Lambda^2\bar\rho}{2\bar\lambda}\int d^2\omega\left(\frac{\delta\lambda}{\bar\lambda}\right)^2$$
$$+\frac{(26-d)}{96\pi}\int d^2\omega\left(\frac{\partial_a\delta\rho}{\bar\rho}\right)^2$$
$$-\frac{d}{24\pi}\int d^2\omega\left(\frac{\partial_a\delta\rho}{\bar\rho}\right)\left(\frac{\partial_a\delta\lambda}{\bar\lambda}\right)$$
$$+\frac{d}{32\pi}\int\frac{d^2p}{(2\pi)^2}\left(\frac{\delta\lambda(p)}{\bar\lambda}\right)\left(\frac{\delta\lambda(-p)}{\bar\lambda}\right)p^2\log\left(\frac{\Lambda^2\bar\rho}{cp^2\bar\lambda}\right). \tag{33}$$

Notice the last term on the right-hand side is normal (and therefore regularization dependent) rather than anomalous as the third and fourth terms are.

6. Stability of the effective action to quadratic order

In the previous Section we have performed the computation assuming that $\lambda^{ab}=\lambda\delta^{ab}$. In order to justify this assumption, let us consider the divergent part of the effective action for nondiagonal λ^{ab}

$$S_{\rm div}=\int d^2\omega\left[\frac{K_0}{2}\lambda^{ab}\partial_aX_{\rm cl}\cdot\partial_bX_{\rm cl}+K_0\rho\left(1-\frac{1}{2}\lambda^{aa}\right)\right.$$
$$\left.-\frac{d\Lambda^2}{2}\frac{\rho}{\sqrt{\det\lambda}}+\Lambda^2\rho\right],\quad\lambda^{aa}=\lambda^{11}+\lambda^{22}. \tag{34}$$

The divergent part of Eq. (6) above is the same as Eq. (34) for constant $\lambda^{ab}=\bar\lambda\delta^{ab}$ and $\rho=\bar\rho$.

Expanding to quadratic order

$$\sqrt{\det(\bar\lambda\delta^{ab}+\delta\lambda^{ab})}=\lambda+\frac{1}{2}\delta\lambda^{aa}-\delta\lambda_2+\mathcal{O}\left((\delta\lambda)^3\right),$$
$$\delta\lambda_2=\frac{1}{8\bar\lambda}(\delta\lambda_{11}-\delta\lambda_{22})^2+\frac{1}{2\bar\lambda}(\delta\lambda_{12})^2, \tag{35}$$

we find from (34) for $\bar\lambda=C$

$$S^{(2)}_{\rm div}=-\frac{d\Lambda^2\bar\rho}{2C}\int d^2\omega\,\delta\lambda_2-\left(K_0-\frac{d\Lambda^2}{2C^2}\right)\int d^2\omega\,\delta\rho\frac{\delta\lambda^{aa}}{2}$$
$$-\frac{d\Lambda^2\bar\rho}{2C^3}\int d^2\omega\left(\frac{\delta\lambda^{aa}}{2}\right)^2. \tag{36}$$

The first term on the right-hand side of Eq. (36) plays a very important role for dynamics of quadratic fluctuations. Because the path integral over λ^{ab} goes parallel to imaginary axis, i.e. $\delta\lambda^{ab}$ is pure imaginary, the first term is always *positive*. Moreover, its exponential plays the role of a (functional) delta-function as $\Lambda\to\infty$, forcing $\delta\lambda^{ab}=\delta\lambda\,\delta^{ab}$. The last two terms on the right-hand side of Eq. (36) then reproduce the first two terms in (33).

From Eq. (33) for the effective action to the second order in fluctuations we find the following quadratic form:

$$\delta S_2=\int\frac{d^2p}{(2\pi)^2}\left[A_{\rho\rho}\frac{\delta\rho(p)\delta\rho(-p)}{\bar\rho^2}+2A_{\rho\lambda}\frac{\delta\rho(p)\delta\lambda(-p)}{\bar\rho\bar\lambda}\right.$$
$$\left.+A_{\lambda\lambda}\frac{\delta\lambda(p)\delta\lambda(-p)}{\bar\lambda^2}\right] \tag{37}$$

with

$$A_{ij} = \begin{bmatrix} \frac{(26-d)p^2}{96\pi} & -\frac{1}{2}\left(K_0 - \frac{d\Lambda^2}{2C^2}\right)\bar{\rho}C - \frac{dp^2}{48\pi} \\ -\frac{1}{2}\left(K_0 - \frac{d\Lambda^2}{2C^2}\right)\bar{\rho}C - \frac{dp^2}{48\pi} & -A \end{bmatrix}, \tag{38}$$

where

$$A = \frac{d\Lambda^2\bar{\rho}}{2C} + \frac{dp^2}{32\pi}\log(cp^2/\Lambda^2\bar{\rho}). \tag{39}$$

For $p^2 \ll \Lambda^2\bar{\rho}$, we can drop the second term on the right-hand side of Eq. (39), so A becomes constant. For $p^2 \gtrsim \Lambda^2\bar{\rho}$, A depends on p^2 but remains positive.

Since $\delta\lambda(\omega)$ is pure imaginary, i.e. $\delta\lambda(-p) = -\delta\lambda^*(p)$, we find for the determinant associated with the matrix in Eq. (38)

$$D = \left[\frac{1}{2}\left(K_0 - \frac{d\Lambda^2}{2C^2}\right)\bar{\rho}C + \frac{dp^2}{48\pi}\right]^2 + \frac{(26-d)p^2}{96\pi}A, \tag{40}$$

and the propagators corresponding to the action (37) are given by

$$\left\langle\phi_i^*(p)\phi_j(p)\right\rangle = \frac{A_{ij}}{D}, \qquad \phi_i = \left(\frac{\delta\rho}{\bar{\rho}}, \frac{\delta\lambda}{\bar{\lambda}}\right). \tag{41}$$

For generic $K_0 > d\Lambda^2/2C^2$ the first term in (40) dominates and we have a trivial stability of fluctuations for any d: nothing propagates. However, we are really interested in the scaling regime (21), where K_R is finite as $\Lambda \to \infty$ and because of the scaling (22) we now have two situations (1) lattice scaling where $\bar{\rho} \sim \Lambda^2$ and (2) string scaling where $\bar{\rho}_R$ is finite according to Eq. (24) for $\Lambda \to \infty$. In the latter case we can disregard the first term in D and the off-diagonal elements of the matrix (38), so that

$$\frac{1}{\bar{\rho}_R^2}\left\langle\delta\rho_R(p)\delta\rho_R(-p)\right\rangle = \frac{48\pi}{(26-d)p^2}. \tag{42}$$

It is positive for $d < 26$, but becomes negative for $d > 26$ which may indicate a negative-norm state. In the first case we obtain

$$\frac{1}{\bar{\rho}^2}\left\langle\delta\rho(p)\delta\rho(-p)\right\rangle = \frac{48\pi}{(26-d)}\frac{1}{(p^2+m^2)},$$
$$m^2 \propto \frac{K_R^2\bar{\rho}}{(26-d)d\Lambda^2}, \tag{43}$$

the mass being positive and finite as $\Lambda \to \infty$ for $d < 26$.

In both cases λ stays localized even in the scaling limit, i.e. $\lambda(\omega) = \bar{\lambda}$. Thus only ρ fluctuates. This is similar to what is described in the book [11].

7. Discussion

We have constructed the nonperturbative mean-field vacuum of the Nambu–Goto string at finite d disregarding fluctuations of ρ_{ab} and λ^{ab}, which is an extension of the one [3] at large d. We have demonstrated the stability of this vacuum under fluctuations to quadratic order for $2 < d < 26$.

Because of the observed instability for $d > 26$, a question arises as to how to understand the expansion in fluctuations about the mean-field. Originally, we expected that it comes along with the expansion in $1/d$, like the $1/N$-expansion in the two-dimensional $O(N)$ sigma-model. This would be indeed the case if $\delta\lambda$ was real, but in our case of imaginary $\delta\lambda$ the action is no longer stable for $d > 26$. We can still make sense of the expansion about the mean-field for $2 < d < 26$ as a semiclassical WKB expansion about the nonperturbative "classical" vacuum, i.e. that of an expansion in the number of "quantum" loops. It is technically well-defined in the path-integral language by assuming that the diagonal part $\delta\lambda$ is real at large d.

It is possible to compute such a "quantum" correction to the mean-field values of C and of the energy of the string ground state. This should help to answer the long-standing question of whether or not the Alvarez–Arvis formula (25), which was derived historically by the canonical quantization of the bosonic string with the Dirichlet boundary conditions and reproduced by our approach in the mean-field approximation, is exact not only at $d = 26$ but also for $2 < d < 26$. The computation of such a correction will involve only the propagator $\langle\delta\rho\delta\rho\rangle$ given in Eq. (42). It would be most interesting to compute such a correction to the mean-field values.

Acknowledgements

The authors acknowledge support by the ERC-Advance grant 291092, "Exploring the Quantum Universe" (EQU). Y. M. thanks the Theoretical Particle Physics and Cosmology group at the Niels Bohr Institute for the hospitality.

References

[1] A.M. Polyakov, Quantum geometry of bosonic strings, Phys. Lett. B 103 (1981) 207.
[2] V.G. Knizhnik, A.M. Polyakov, A.B. Zamolodchikov, Fractal structure of 2D quantum gravity, Mod. Phys. Lett. A 3 (1988) 819;
J. Distler, H. Kawai, Conformal field theory and 2D quantum gravity, Nucl. Phys. B 321 (1989) 509;
F. David, Conformal field theories coupled to 2D gravity in the conformal gauge, Mod. Phys. Lett. A 3 (1988) 1651.
[3] J. Ambjorn, Y. Makeenko, String theory as a Lilliputian world, Phys. Lett. B 756 (2016) 142, arXiv:1601.00540;
Scaling behavior of regularized bosonic strings, Phys. Rev. D 93 (2016) 066007, arXiv:1510.03390.
[4] R. Giles, C.B. Thorn, A lattice approach to string theory, Phys. Rev. D 16 (1977) 366.
[5] B. Durhuus, J. Frohlich, T. Jonsson, Selfavoiding and planar random surfaces on the lattice, Nucl. Phys. B 225 (1983) 185;
Critical behavior in a model of planar random surfaces, Nucl. Phys. B 240 (1984) 453; Phys. Lett. B 137 (1984) 93.
[6] J. Ambjorn, B. Durhuus, J. Frohlich, Diseases of triangulated random surface models, and possible cures, Nucl. Phys. B 257 (1985) 433;
F. David, Planar diagrams, two-dimensional lattice gravity and surface models, Nucl. Phys. B 257 (1985) 45;
V.A. Kazakov, A.A. Migdal, I.K. Kostov, Critical properties of randomly triangulated planar random surfaces, Phys. Lett. B 157 (1985) 295.
[7] B. DeWitt, Dynamical theory of groups and fields, in: Les Houches, 1963, eq. (17.95).
[8] P.B. Gilkey, The spectral geometry of a Riemannian manifold, J. Differ. Geom. 10 (1975) 601.
[9] B. Durhuus, P. Olesen, J.L. Petersen, Polyakov's quantized string with boundary terms, Nucl. Phys. B 198 (1982) 157.
[10] O. Alvarez, Theory of strings with boundaries: fluctuations, topology and quantum geometry, Nucl. Phys. B 216 (1983) 125.
[11] A.M. Polyakov, Gauge Fields and Strings, Harwood Acad. Pub., 1987, p. 173, 174.