



Creating 3, 4, 6 and 10-dimensional spacetime from W3 symmetry

Ambjorn, J.; Watabiki, Y.

Published in:
Physics Letters B

DOI:
[10.1016/j.physletb.2017.04.051](https://doi.org/10.1016/j.physletb.2017.04.051)

Publication date:
2017

Document version
Publisher's PDF, also known as Version of record

Document license:
[CC BY](#)

Citation for published version (APA):
Ambjorn, J., & Watabiki, Y. (2017). Creating 3, 4, 6 and 10-dimensional spacetime from W3 symmetry. *Physics Letters B*, 770, 252-256. <https://doi.org/10.1016/j.physletb.2017.04.051>



Creating 3, 4, 6 and 10-dimensional spacetime from W_3 symmetry



J. Ambjørn^{a,b,*}, Y. Watabiki^c

^a The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

^b Institute for Mathematics, Astrophysics and Particle Physics (IMAPP), Radboud University Nijmegen, Heyendaalseweg 135, 6525 AJ, Nijmegen, The Netherlands

^c Tokyo Institute of Technology, Dept. of Physics, High Energy Theory Group, 2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan

ARTICLE INFO

Article history:

Received 28 March 2017

Accepted 24 April 2017

Available online 26 April 2017

Editor: J. Hisano

ABSTRACT

We describe a model where breaking of W_3 symmetry will lead to the emergence of time and subsequently of space. Surprisingly the simplest such models which lead to higher dimensional spacetimes are based on the four “magical” Jordan algebras of 3×3 Hermitian matrices with real, complex, quaternion and octonion entries, respectively. The simplest symmetry breaking leads to universes with spacetime dimensions 3, 4, 6, and 10.

© 2017 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

1. Introduction

String field theory is notoriously complicated, but there is a baby version, namely non-critical string field theory [1–3]. Non-critical string theory describes two-dimensional quantum gravity coupled to a conformal field theory with a central charge $c < 1$ and the corresponding string field theory aims to describe the dynamics of merging and splitting of such strings. For $c = 0$ the situation is particularly simple. One has creation and annihilation operators $\Psi^\dagger(L)$ and $\Psi(L)$ for spatial universes of length L . An even simpler string field theory exists, CDT string field theory [4]. The starting point is the continuum limit of 2d “causal dynamical triangulations” (CDT) [5,6], a limit which in the case of trivial spacetime topology is 2d quantum Horava–Lifshitz gravity [7] (for higher dimensional CDT which might also be related to Horava–Lifshitz gravity [8], see e.g. [9,10]). CDT string field theory describes the dynamics of topology changes of spacetime. The Hamiltonian involves terms like

$$\Psi^\dagger(L_1)\Psi^\dagger(L_2)\Psi(L_1 + L_2), \quad \Psi^\dagger(L_1 + L_2)\Psi(L_2)\Psi(L_1), \quad (1)$$

which describe the annihilation of a universe of length $L_1 + L_2$ and the creation of two universes of lengths L_1 and L_2 , or the reverse process [4]. The interaction term in the string field Hamiltonian thus contains products of three annihilation and creation operators. The same is true for standard non-critical string theory

which is related to a special kind of $W^{(3)}$ symmetry which ensures that the partition function can be viewed as a τ -function of certain coupling constants [11]. This led us to realize that one can obtain the CDT string field theory starting from a so-called $W^{(3)}$ Hamiltonian by symmetry breaking [12]. The $W^{(3)}$ Hamiltonian has a natural, so-called *absolute* vacuum and offers no obvious spacetime interpretation, but breaking the $W^{(3)}$ -symmetry led to a so-called *physical* vacuum and the emergence of time and one spatial dimension. The purpose of this article is to generalize the construction such that one can create universes with one time direction and higher dimensional spaces. The simplest symmetry breaking leads to spacetime dimensions $2 + 1$, $3 + 1$, $5 + 1$ and $9 + 1$.

In Sec. 2 we shortly review the $W^{(3)}$ Hamiltonian formalism introduced in [12] and in Sec. 3 we generalize it to include internal degrees of freedom. This leads to the introduction of the so-called magical Jordan algebras and it is the structures of these algebras which result in spacetime dimensions $2 + 1$, $3 + 1$, $5 + 1$ and $9 + 1$.

2. The $W^{(3)}$ Hamiltonian

The formal definition of a $W^{(3)}$ algebra in terms of operators α_n satisfying

$$[\alpha_m, \alpha_n] = m \delta_{m+n,0}, \quad (2)$$

is the following

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \frac{\alpha_n}{z^{n+1}}, \quad W^{(3)}(z) = \frac{1}{3} : \alpha(z)^3 : = \sum_{n \in \mathbb{Z}} \frac{W_n^{(3)}}{z^{n+3}}. \quad (3)$$

* Corresponding author.

E-mail addresses: ambjorn@nbi.dk (J. Ambjørn), watabiki@th.phys.titech.ac.jp (Y. Watabiki).

The normal ordering (\cdot) refers to the α_n operators (α_n to the left of α_m for $n > m$)¹ and we have

$$W_n^{(3)} = \frac{1}{3} \sum_{k,l,m} : \alpha_k \alpha_l \alpha_m : \delta_{k+l+m,n}. \quad (4)$$

We then define the “absolute vacuum” $|0\rangle$ by the following condition:

$$\alpha_n |0\rangle = 0, \quad n < 0, \quad (5)$$

and the so-called W -Hamiltonian \hat{H}_W by

$$\hat{H}_W := -W_{-2}^{(3)} = -\frac{1}{3} \sum_{k,l,m} : \alpha_k \alpha_l \alpha_m : \delta_{k+l+m,-2}. \quad (6)$$

Note that \hat{H}_W does not contain any coupling constants.

It was shown in [12] that by introducing a coherent state, which is an eigenstate of α_{-1} and α_{-3} and which we denoted the “physical” vacuum state $|\text{vac}\rangle$, \hat{H}_W was closely related to the CDT string field Hamiltonian \hat{H} . We thus defined

$$|\text{vac}\rangle \propto e^{\lambda_1 \alpha_1 + \lambda_3 \alpha_3} |0\rangle, \quad (7)$$

and we have

$$\alpha_{-1} |\text{vac}\rangle = \lambda_1 |\text{vac}\rangle, \quad \alpha_{-3} |\text{vac}\rangle = 3\lambda_3 |\text{vac}\rangle. \quad (8)$$

The main point is the following: because $\langle \text{vac} | \alpha_n | \text{vac} \rangle$ is different from zero for $n = -1$ and $n = -3$, \hat{H}_W will now contain terms only involving two operators α_l . These terms can act like quadratic terms in \hat{H} . At the same time the cubic terms left in \hat{H}_W will act like the interaction terms in \hat{H} , resulting in joining and splitting of universes. Finally, the expectation values of α_{-1} and α_{-3} determine the coupling constants of \hat{H} . More precisely one has [12]

$$\hat{H}_W \propto \hat{H} + c_4 \phi_4^\dagger + c_2 \phi_2^\dagger \quad (9)$$

where \hat{H} is the CDT string field Hamiltonian. c_4 and c_2 are constants. The creation operators ϕ_n^\dagger are the α_n , $n > 0$, while annihilation operators ϕ_n are related to α_n , $n < 0$, except that ϕ_1 and ϕ_3 are shifted by eigenvalues given in eq. (8), such that $\phi_n |\text{vac}\rangle = 0$. \hat{H} is normal ordered such that $\hat{H} |\text{vac}\rangle = 0$.

By breaking the $W^{(3)}$ symmetry one can thus obtain CDT string field theory except for one important point: the vacuum is not stable. The terms $c_4 \phi_4^\dagger + c_2 \phi_2^\dagger$ cause universes of infinitesimal length to be created and the non-interacting part of \hat{H} , which explicitly can be written as

$$\hat{H}_0 = - \sum_{l=1}^{\infty} \phi_{l+1}^\dagger l \phi_l + \mu \sum_{l=2}^{\infty} \phi_{l-1}^\dagger l \phi_l, \quad (10)$$

might expand such an infinitesimal length space to macroscopic size. The relation between the operators ϕ_l , ϕ_l^\dagger and the operators $\Psi(L)$, $\Psi^\dagger(L)$ which annihilate and create spatial universes of macroscopic length L is as follows

$$\Psi^\dagger(L) = \sum_{l=0}^L \frac{L!}{l!} \phi_l^\dagger. \quad (11)$$

When expressed in terms of $\Psi(L)$ and $\Psi^\dagger(L)$ the Hamiltonian (10) can be written as

$$\hat{H}_0 = \int_0^\infty dL \Psi^\dagger(L) H_0 \Psi(L), \quad H_0 = -L \frac{\partial^2}{\partial L^2} + \mu L, \quad (12)$$

where the two terms on the rhs of eq. (12) corresponds to the two terms on the rhs of eq. (10). It should now be clear why we denote the two terms on the rhs of eq. (10) the kinetic and the cosmological term, respectively. H_0 is the original CDT Hamiltonian [5] for the evolution of a single 2d universe, without topology changes. If $\mu > 0$ a universe starting with zero (or more precisely infinitesimal) length will have a (unnormalized) wave function and a corresponding expectation value of the size of space at time T :

$$G(L, T) = \mu L \frac{e^{-\sqrt{\mu} L \coth(\sqrt{\mu} T)}}{\sinh^2(\sqrt{\mu} T)} \quad \langle L \rangle = \frac{1}{\sqrt{\mu}} \tanh(\sqrt{\mu} T). \quad (13)$$

If $\mu < 0$ the corresponding equations become ($\tilde{\mu} = -\mu$):

$$G(L, T) = \tilde{\mu} L \frac{e^{-\sqrt{\tilde{\mu}} L \cot(\sqrt{\tilde{\mu}} T)}}{\sin^2(\sqrt{\tilde{\mu}} T)} \quad \langle L \rangle = \frac{1}{\sqrt{\tilde{\mu}}} \tan(\sqrt{\tilde{\mu}} T). \quad (14)$$

In this case the wave function only belongs to the Hilbert space of H_0 for $0 < T < \pi/(2\sqrt{\tilde{\mu}})$. At $T = \pi/(2\sqrt{\tilde{\mu}})$ the universe has expanded to infinite size.

3. Generalization to higher dimensions

The above creation of space and time, described in [12], is limited to one space and one time dimension. In order to create d -dimensional space we introduce an internal index a , $a = 1, \dots, d$ and consider the corresponding extended $W^{(3)}$ algebra. The classification of such $W^{(3)}$ algebras is closely related to the classification of Jordan algebras (see [14] for a review of W algebras and their relations to Jordan algebras) and surprisingly it turns out that only the four so-called magical Jordan algebras allow us to make symmetry breakings which lead to CDT-like Hamiltonians of the kind considered above. We will discuss the reason for that elsewhere [13], and here we will just review how one defines the four magical Jordan algebras and the corresponding $W^{(3)}$ Hamiltonians.

Let $H_3(\mathbb{F})$ denote the 3×3 Hermitian matrices with entries in \mathbb{F} , where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} (the real numbers, the complex numbers, the quaternions and the octonions). The $H_3(\mathbb{F})$'s are real vector spaces of dimensions 6, 9, 15 and 27, and they are Jordan algebras when one defines the algebra multiplication of two elements as the anti-commutator of the corresponding matrices:

$$X \circ Y := \frac{1}{2} \{X, Y\}. \quad (15)$$

If one defines the scalar product on $H_3(\mathbb{F})$ by

$$\langle X, Y \rangle = \frac{1}{2} \text{Tr}(X \circ Y), \quad (16)$$

it has an orthogonal decomposition

$$H_3(\mathbb{F}) = \mathbb{R} \cdot 1_{3 \times 3} \oplus \tilde{H}_3(\mathbb{F}), \quad (17)$$

where $\tilde{H}_3(\mathbb{F})$ denotes the traceless matrices. The $\tilde{H}_3(\mathbb{F})$'s are real vector spaces of dimensions $d = 5, 8, 14, 26$, respectively. Let E_a [$a = 1, \dots, d$] denote an orthonormal basis of the vector space $\tilde{H}_3(\mathbb{F})$. The structure constant of the Jordan algebra can be defined as

$$d_{abc} := \frac{1}{2} \text{Tr}((E_a \circ E_b) \circ E_c). \quad (18)$$

The structure constants are invariant under the action of the automorphism groups of the algebras, which are $SO(3)$, $SU(3)$, $USp(6)$

¹ We remark that this ordering is opposite to the standard ordering one would use in conformal field theory. One can obtain the conventional ordering by the so-called \star -operation [3].

and F_4 respectively. In the case of $\tilde{H}_3(\mathbb{C})$ (equal as a real vector space to the Lie algebra $su(3)$) we can choose as the E_a the standard Gell-Mann matrices λ_a of $su(3)$, which satisfy the anti-commutation relation

$$\{\lambda_a, \lambda_b\} = \frac{4}{3} \delta_{ab} \cdot 1_{3 \times 3} + \frac{1}{2} \sum_c d_{abc} \lambda_c \quad (19)$$

λ_a^R , $a = 1, 3, 4, 6, 8$ are 5 real, symmetric matrices and they form the 5-dimensional basis E_a for $\tilde{H}_3(\mathbb{R})$. $A_b = -i\lambda_b$, $b = 2, 5, 7$ are real antisymmetric matrices. When multiplied by i they form together with the λ_a^R matrices the 8-dimensional basis E_a for $\tilde{H}_3(\mathbb{C})$, as already mentioned. If the A_b matrices are multiplied by i, j, k , the generalized imaginary quaternions numbers they form together with the 5 λ_a^R the 14-dimensional basis E_a of $\tilde{H}_3(\mathbb{H})$. Finally, if the A_b matrices are multiplied by the 7 generalized imaginary octonion numbers $i, j, k, \bar{i}, \bar{j}, \bar{k}, \ell$ they form together with the λ_a^R matrices the 26-dimensional basis E_a for $\tilde{H}_3(\mathbb{O})$.

The generalization of (2)–(6) is now straightforward. We define the current

$$\alpha(z) = \sum_a \alpha^{(a)}(z) E_a, \quad \alpha^{(a)}(z) = \sum_{n \in \mathbb{Z}} \frac{\alpha_n^{(a)}}{z^{n+1}}, \quad (20)$$

$$W^{(3)}(z) = \frac{1}{3} : \text{Tr}((\alpha(z) \circ \alpha(z)) \circ \alpha(z)) : = \sum_{n \in \mathbb{Z}} \frac{W_n^{(3)}}{z^{n+3}}, \quad (21)$$

where the commutation relations are

$$[\alpha_m^{(a)}, \alpha_n^{(b)}] = m \delta_{m+n,0} \delta_{a,b}, \quad (22)$$

and we find for the $W^{(3)}$ Hamiltonian the expression

$$\hat{H}_W := -W_{-2}^{(3)} = -\frac{1}{3} \sum_{k,l,m,a,b,c} d_{abc} : \alpha_k^{(a)} \alpha_l^{(b)} \alpha_m^{(c)} : \delta_{k+l+m,-2}. \quad (23)$$

Again, this model only allows a spacetime interpretation after choosing a specific coherent state. There are some interesting choices, but here we will only discuss the simplest ones, namely some choices of breaking in the 8-direction and the 3-direction. Instead of (7) and (8) we can choose

$$|\text{vac}\rangle_8 \propto e^{\lambda_1^{(8)} \alpha_1^{(8)} + \lambda_3^{(8)} \alpha_3^{(8)}} |0\rangle, \quad (24)$$

and we have

$$\alpha_{-1}^{(8)} |\text{vac}\rangle_8 = \lambda_1^{(8)} |\text{vac}\rangle_8, \quad \alpha_{-3}^{(8)} |\text{vac}\rangle_8 = 3\lambda_3^{(8)} |\text{vac}\rangle_8. \quad (25)$$

When one looks at the coefficients d_{ab8} in order to obtain the non-interacting part of the Hamiltonian (the equivalent of \hat{H}_0 given by (10)), firstly it is observed that only coefficients d_{aa8} are different from zero. This implies that the non-interacting part of the Hamiltonian is diagonal in the “space” indices a . Next, the only coefficient d_{aaa} which is different from zero is d_{888} . Thus the only $\phi^{(a)}$ field which has a cubic self-interaction is the 8-field. The d_{aa8} have the following values

$$d_{aa8} = \frac{1}{\sqrt{3}} \text{ or } -\frac{1}{2\sqrt{3}}, \quad d_{888} = -\frac{1}{\sqrt{3}}. \quad (26)$$

If we use the vacuum $|\text{vac}\rangle_8$, the two groups will result in Hamiltonians with opposite signs. Let us assume that the symmetry breaking is chosen such that the non-interacting Hamiltonians with $d_{aa8} > 0$ will correspond to the CDT Hamiltonians of the form (12), only carrying now an index a . The Hamiltonians with negative d_{aa8} will now have a negative kinetic term. We have sev-

eral options when addressing the negative Hamiltonians. The state $|l\rangle^{(a)} = (\phi_1^{(a)})^\dagger |\text{vac}\rangle$ has macroscopic length $L = 0$. This wave function is basically l -times the derivative of $\delta(L)$. In the case where the internal index a is such that the kinetic part of the Hamiltonian $\hat{H}_0^{(a)}$ is positive, the time evolution of such an initial state, created from the vacuum by the creation operators in (9) (with internal index a), will have the time evolution shown in eqs. (13) or (14) (ignoring the cubic interaction terms in the Hamiltonian). In some time interval the wave functions thus belong to the Hilbert space of Hamiltonian. However, if the internal index a is such that the kinetic term of $\hat{H}_0^{(a)}$ is negative, the time evolution does not give us an acceptable wave function (it is obtained from (13) by changing the sign of T). Thus we can choose to insist that a macroscopic state with macroscopic length is never created in this way for modes where d_{aa8} is negative. Let us first accept this viewpoint and assume that macroscopic directions with $d_{aa8} < 0$ are not excited by acting with $\phi_1^{(a)\dagger}$ on $|0\rangle_8$.

When we then look at the four magical algebras, we have for $\tilde{H}_3(\mathbb{R})$ two a where $d_{aa8} = 1/\sqrt{3}$. For $\tilde{H}_3(\mathbb{C})$ we have three a 's where $d_{aa8} = 1/\sqrt{3}$. For $\tilde{H}_3(\mathbb{H})$ we have five a 's where $d_{aa8} = 1/\sqrt{3}$ and finally for $\tilde{H}_3(\mathbb{O})$ we have nine a 's where $d_{aa8} = 1/\sqrt{3}$. The symmetry breaking corresponds to breaking the automorphism group of the $W^{(3)}$ algebras from $SO(3)$ to $SO(2)$, from $SU(3)$ to $SO(3)$, from $USp(6)$ to $SO(5)$ and finally from F_4 to $SO(9)$. The extended spacetime dimensions will be (including the time) $2 + 1$, $3 + 1$, $5 + 1$ and $9 + 1$ which are the dimensions of the classical superstrings (the quantum superstring only survives in the ten-dimensional case).

However, it is possible to take another point of view. If we consider the modes α_l or ϕ_l as the fundamental variables, it is up to us to find a continuum length interpretation. For the positive Hamiltonian (11) was fine. If we obtain a negative Hamiltonian with this prescription, we are free to change (11) to

$$\Psi^\dagger(L) = \sum_{l=0} \frac{(-L)^l}{l!} \phi_l^\dagger. \quad (27)$$

This will simply change the sign of L and thus the sign of the Hamiltonian since it is linear in L . With such a change for the a 's where the kinetic part of \hat{H} is negative, all directions now have a positive kinetic Hamiltonian and all directions now have the potential to develop a macroscopic length. However, there will not isotropy between the directions anymore since the values of $|d_{aa8}|$ will fall in three groups so the global symmetry in space will be more complicated.

Let us finally mention that we can refine the symmetry breaking by multiplying (24) by a coherent state operator in the 3-direction

$$|\text{vac}\rangle_{8,3} \propto e^{\lambda_1^{(3)} \alpha_1^{(3)}} |\text{vac}\rangle_8. \quad (28)$$

With this choice, the kinetic terms will be as before, positive and negative. If we set the L assignment mentioned above such that all kinetic terms become positive, the mass matrix takes a form such that we have 2, 3, 5 or 9 space directions which follow eq. (14), i.e. expand to infinity, and the rest of the space directions, 2, 4, 8 or 16, stay bounded by a fixed radius, like in eq. (13). Especially, in the case of $\tilde{H}_3(\mathbb{O})$, the number of space directions expanding to infinity is 9 and the number of space directions staying compact is 16. The reader might recall that these numbers of space dimensions (non-compact and compact) are also encountered in the case of the heterotic string. Thus, by imposing an additional symmetry breaking in the 3-direction of the internal space we have obtained symmetry breaking patterns which resemble the first ones discussed, where we simply disregarded directions with negative

kinetic terms. A more detailed analysis of the symmetry breaking pattern will appear elsewhere [13].

4. Discussion

The starting point for introducing the above mentioned model was that even the baby versions of string field theory, the non-critical string field theories, tell us surprising little about the actual creation of the universe. We wanted a starting point from which space and time could emerge, maybe (illustrating our lack of creativity) by some symmetry breaking. Such a toy model was presented in [12] and by construction it was a 1 + 1 dimensional model. We have now tried to generalize this approach to higher dimensions, to create some dimension enhancement mechanism, somewhat inspired by string theory where the dimension of space-time is “just” given by the number of Gaussian fields X^a which appears in the string action. Since our starting point was a $W^{(3)}$ algebra, we were naturally led to $W^{(3)}$ algebras with intrinsic symmetry. These are related to Jordan algebras and to our surprise, only the four so-called magical Jordan algebras seemed to lead to a simple generalization of the one-component model studied in [12]. Of course a large number of questions need to be clarified. Let us discuss a number of the issues.

Firstly, we have not at all discussed any dynamical mechanism leading to the symmetry breaking of our $W^{(3)}$ model. If it is spontaneous, we have not yet found any natural mechanism which would lead to such symmetry breaking. However, one can imagine other ways of realizing the symmetry breaking. It is possible to have an interaction such that

$$|\text{vac}\rangle_{\lambda_3, \lambda_1} \longleftrightarrow |\text{vac}\rangle_{\lambda'_3, \lambda'_1} \otimes |\text{vac}\rangle_{\lambda''_3, \lambda''_1}. \quad (29)$$

Details of how to implement this will be published elsewhere [13]. If we denote the theory described by \hat{H}_W as a “first quantized theory”, it would represent a “second quantization”, in the sense that we then introduce a quantum theory for the λ 's, which were before just c-numbers which labeled the different “vacua” and thus different coupling constants of the universes created. Via an interaction which allows the process (29), the absolute vacuum $|0\rangle = |\text{vac}\rangle_{0,0}$ can become a physical vacuum $|\text{vac}\rangle_{\lambda_3, \lambda_1}$ and then time will emerge, as described in [12]. After the emergence of time \hat{H}_W can trigger the creation of macroscopic space.

Secondly, how should we really think about the $W^{(3)}$ symmetry? The four magical Jordan algebras lead to four classical $W^{(3)}$ symmetries. However, it is not easy to promote these symmetries to quantum symmetries [15]. The commutators $[W_m^{(3)}, W_n^{(3)}]$ may lead to $W^{(4)}$ operators. For a number of $W^{(3)}$ algebras these $W^{(4)}$ operators can be rewritten in a consistent way as a product of $W^{(2)}$ operators (i.e. Virasoro algebra operators), and we have a closed $W^{(2)}$, $W^{(3)}$ algebra realized via the free bosonic currents $\alpha(z)$ defined as in (3). However, for the four magical algebras this does not work. These algebras then have to be viewed as embedded in some larger algebras. Whether it should be the $W^{(\infty)}$ algebra, which contains all higher spin components $W^{(N)}$, or, as suggested in [16], one should use a different decomposition, is not known. Following the line of thinking in [16], there seems to be an interesting algebraic structure related to the magical Jordan algebras, even at the quantum level. This is due to some interesting algebraic properties of the structure constants d_{abc} for the magical Jordan algebras. Naively, the extended W algebra consists of

$$W^{(2|\alpha, \beta)}(z) = \frac{1}{2} \sum_{a,b} \delta_{ab} : \partial^\alpha \alpha^{(a)}(z) \partial^\beta \alpha^{(b)}(z) :,$$

$$W^{(3|\alpha, \beta, \gamma)}(z) = \frac{1}{3} \sum_{a,b,c} d_{abc} : \partial^\alpha \alpha^{(a)}(z) \partial^\beta \alpha^{(b)}(z) \partial^\gamma \alpha^{(c)}(z) :,$$

$$\begin{aligned} W^{(4|\alpha, \beta, \gamma, \delta)}(z) &= \frac{1}{4} \sum_{a,b,c,d,e} d_{abe} d_{cde} : \\ &\partial^\alpha \alpha^{(a)}(z) \partial^\beta \alpha^{(b)}(z) \partial^\gamma \alpha^{(c)}(z) \partial^\delta \alpha^{(d)}(z) :, \\ &\vdots \end{aligned} \quad (30)$$

where $\alpha, \beta, \gamma, \delta, \dots$ run $0, 1, 2, \dots$ and indicate the number of derivatives of z . However, some generators are not independent because of special properties of structure constants. As examples of such properties we mention

$$\begin{aligned} \sum_b d_{abb} &= 0 & \sum_{d,e,f} d_{ade} d_{bef} d_{cfd} &= -\frac{d-2}{12} d_{abc}, \\ \sum_e d_{abe} d_{cde} &= \frac{6}{d+2} \sum_{e,f,g,h} d_{aef} d_{bfg} d_{cgh} d_{dhe} &= \frac{1}{3} \delta_{ac} \delta_{bd}, \end{aligned} \quad (31)$$

where the indices with underline is symmetrized. The implications of relations like the ones listed in (31) will be discussed elsewhere [13].

Until now we have just treated the different operators $\alpha^{(a)}$ as indexed with a “flavor”. The unbroken symmetry ($SO(2)$, $SO(3)$, $SO(5)$ and $SO(9)$) allows us to transform these flavors dimensions into each other. However, in the four cases we want space to be viewed as a 2, 3, 5 and 9 dimensional connected continuum, respectively. Preferable, we want to be able to talk about these spaces as topological spaces, e.g. 2, 3, 5 or 9 dimensional tori where the concept of neighborhoods or maybe even distances make some sense. This might happen dynamically via the cubic interaction, a possibility we find intriguing. Consider the simplest situation: $\tilde{H}_3(\mathbb{R})$, and symmetry breaking in the 8-direction. We have $d_{118} = d_{338} = 1/\sqrt{3}$ and $d_{888} = -1/\sqrt{3}$. Thus, according to one of the point of views presented above, space in the 8-direction will have no extension, but a 1-space and a 3-space can be glued together at a “point” via 8-space. A set of such wormholes of 8-space, each of which has infinitesimal length and connects one point of 1-space and one point of 3-space, forms two-dimensional coordinates. We can image such a “knitting” taking place everywhere and in this sense the interaction via the 8-direction mode is what will create for us the genuine concept of a two-dimensional space for $\tilde{H}_3(\mathbb{R})$. Similar considerations apply for the higher dimensional spaces coming from $\tilde{H}_3(\mathbb{C})$, $\tilde{H}_3(\mathbb{H})$ and $\tilde{H}_3(\mathbb{O})$. The knitting mechanism has the potential to form the space into a higher dimensional torus. Clearly this idea need to be substantiated by more explicit calculations.

If we take the point of view that we allow “negative” L in the sense discussed above, all directions now have an extension, also the 8-direction used for the “knitting”, and if we want such a “knitting” picture to make sense we have create large extended spaces and small spaces of “Planck size” and the 8-direction have to be of Planck size. This can be done by having positive and negative “cosmological” terms (μ and $\tilde{\mu}$ in the notation of eqs. (13) and (14)) and insisting that the scale $1/\sqrt{\mu}$ should be viewed as the Planck scale. Then, depending on the symmetry breaking mass matrix, a number of dimensions will be of Planck size, while the others will expand infinitely. Of course this picture is even more challenging than the picture where some flavors could not be associated with any spatial extension, since the flavors with macroscopic spatial extension acquire this extension within Planck time (see eq. (14)). One needs a mechanism which slows down this expansion and in this context it is natural to think about Coleman’s mechanism for lowering the cosmological constant [17]. Again we clearly need explicit calculations to substantiate any claims, but contrary to Coleman’s situation we actually have a model where

questions of baby universe creation and annihilation can be answered by calculations, although going beyond perturbation theory when it comes to the creation and annihilation might be difficult (some calculation to all order exists in CDT string field theory [18]).

As always, a model for the Big Bang and for the creation of the universe from nothing creates more questions than it answers. This is also the case for this model. However, it is an explicit model where hopefully explicit calculations can be performed, and it might be that some cosmological predictions do not require the full solution of the model and thus can be used to falsify the model when compared to observations. Alternatively they might be encouraging and then provide further motivation for studying the model. As an example we have very preliminary indications that the model can provide an explanation of dark energy which is not related to any “bare” cosmological constant which might appear in the process of symmetry breaking of the $W^{(3)}$ algebra. This will be published elsewhere, once we feel more confident of a number of other features of the model.

Acknowledgements

JA and YW acknowledge support from the ERC-Advance grant 291092, “Exploring the Quantum Universe” (EQU).

References

- [1] N. Ishibashi, H. Kawai, Phys. Lett. B 314 (1993) 190, arXiv:hep-th/9307045, Phys. Lett. B 322 (1994) 67, arXiv:hep-th/9312047, Phys. Lett. B 352 (1995) 75, arXiv:hep-th/9503134.
- [2] Y. Watabiki, Nucl. Phys. B 441 (1995) 119, arXiv:hep-th/9401096, Phys. Lett. B 346 (1995) 46, arXiv:hep-th/9407058.
- [3] J. Ambjørn, Y. Watabiki, Int. J. Mod. Phys. A 12 (1997) 4257, arXiv:hep-th/9604067.
- [4] J. Ambjørn, R. Loll, Y. Watabiki, W. Westra, S. Zohren, J. High Energy Phys. 0805 (2008) 032, arXiv:0802.0719 [hep-th].
- [5] J. Ambjørn, R. Loll, Nucl. Phys. B 536 (1998) 407–434, arXiv:hep-th/9805108.
- [6] J. Ambjørn, R. Loll, W. Westra, S. Zohren, J. High Energy Phys. 0712 (2007) 017, arXiv:0709.2784 [gr-qc];
- J. Ambjørn, R. Loll, Y. Watabiki, W. Westra, S. Zohren, Phys. Lett. B 665 (2008) 252, arXiv:0804.0252 [hep-th];
- J. Ambjørn, T.G. Budd, J. Phys. A, Math. Theor. 46 (2013) 315201, arXiv:1302.1763 [hep-th].
- [7] J. Ambjørn, L. Glaser, Y. Sato, Y. Watabiki, Phys. Lett. B 722 (2013) 172, arXiv:1302.6359 [hep-th].
- [8] P. Hořava, Phys. Rev. D 79 (2009) 084008, arXiv:0901.3775.
- [9] J. Ambjørn, A. Görlich, J. Jurkiewicz, R. Loll, Phys. Rev. Lett. 100 (2008) 091304, arXiv:0712.2485, Phys. Rev. D 78 (2008) 063544, arXiv:0807.4481, Phys. Lett. B 690 (2010) 420–426, arXiv:1001.4581;
- J. Ambjørn, J. Jurkiewicz, R. Loll, Phys. Rev. Lett. 93 (2004) 131301, arXiv:hep-th/0404156, Phys. Rev. D 72 (2005) 064014, arXiv:hep-th/0505154, Phys. Lett. B 607 (2005) 205–213, arXiv:hep-th/0411152;
- J. Ambjørn, A. Gorlich, S. Jordan, J. Jurkiewicz, R. Loll, Phys. Lett. B 690 (2010) 413, arXiv:1002.3298 [hep-th];
- J. Ambjørn, S. Jordan, J. Jurkiewicz, R. Loll, Phys. Rev. Lett. 107 (2011) 211303, arXiv:1108.3932 [hep-th], Phys. Rev. D 85 (2012) 124044, arXiv:1205.1229 [hep-th].
- [10] J. Ambjørn, A. Goerlich, J. Jurkiewicz, R. Loll, Phys. Rep. 519 (2012) 127, arXiv:1203.3591 [hep-th].
- [11] M. Fukuma, H. Kawai, R. Nakayama, Int. J. Mod. Phys. A 6 (1991) 1385, Commun. Math. Phys. 143 (1992) 371;
- A. Mironov, A. Morozov, Phys. Lett. B 252 (1990) 47;
- A. Marshakov, A. Mironov, A. Morozov, Mod. Phys. Lett. A 7 (1992) 1345, arXiv:hep-th/9201010.
- [12] J. Ambjørn, Y. Watabiki, Phys. Lett. B 749 (2015) 149, <http://dx.doi.org/10.1016/j.physletb.2015.07.067>, arXiv:1505.04353 [hep-th].
- [13] J. Ambjørn, Y. Watabiki, in press.
- [14] L.J. Romans, Nucl. Phys. B 352 (1991) 829, [http://dx.doi.org/10.1016/0550-3213\(91\)90108-A](http://dx.doi.org/10.1016/0550-3213(91)90108-A).
- [15] N. Mohammadi, Mod. Phys. Lett. A 6 (1991) 2977, <http://dx.doi.org/10.1142/S0217732391003481>;
- J.M. Figueroa-O'Farrill, Phys. Lett. B 326 (1994) 89, [http://dx.doi.org/10.1016/0370-2693\(94\)91197-5](http://dx.doi.org/10.1016/0370-2693(94)91197-5), arXiv:hep-th/9401108;
- S.A. Lyakhovich, A.A. Sharapov, arXiv:hep-th/9411167.
- [16] C.M. Hull, Trieste 1992, Proceedings, High Energy Physics and Cosmology, 76-142 and London Queen Mary and Westfield Coll. – QMW-93-02 (93/02) 54 pp. (305933), arXiv:hep-th/9302110.
- [17] S.R. Coleman, Nucl. Phys. B 310 (1988) 643, [http://dx.doi.org/10.1016/0550-3213\(88\)90097-1](http://dx.doi.org/10.1016/0550-3213(88)90097-1).
- [18] J. Ambjørn, R. Loll, W. Westra, S. Zohren, Phys. Lett. B 678 (2009) 227, arXiv:0905.2108 [hep-th].