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Heap-Construction Programs

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Abstract. This report together with an accompanying tar ball contains the source code of the programs described and benchmarked in the paper “Heap Construction—50 Years Later”. The programs in this package describe the state of the art in heap construction in 2016.

Keywords. Data structures, binary heaps, heap construction, algorithm engineering, element comparisons, element moves, cache misses, branch mispredictions

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The authors have tried to produce correct and useful programs, but no warranty of any kind should be assumed.

Release date

2016-11-04
Programs

In the paper “Heap Construction—50 Years Later”, the heap-construction programs considered were characterized as follows:

**stl:** The make_heap function that came with our g++ compiler. On closer inspection, it was seen to rely on the bottom-up sift_down policy [5, Exercise 5.2.3–18] (see also [8]). Two of the underlying subroutines passed elements by value, so this resulted in some unnecessary element moves.

**F:** Floyd’s Algol program [3] (F) converted into C++. In sift_down, this program employed the hole technique, so element swaps were not used.

**basic GM:** Our implementation of the algorithm of Gonnet and Munro [4] (GM) using a tournament tree. For an input of size $N$, the program used a tournament tree requiring $2N - 1$ extra space for indices and a temporary output area requiring $N$ extra space for elements.

**GM:** A space-efficient implementation of GM. This program could be configured to operate in-place ($O(1)$ extra space) or in-situ ($O(\lg N)$ extra space). Both versions used $O(1)$ extra space for elements. The program accepted a tuning parameter $\gamma$ and set the size of the bottom trees to the closest power of two larger than, or equal to, $\gamma \lg N/\lg \lg N$. By default, $\gamma = 32$. The in-place variant used a packed array that could store a sequence of integers of equal length compactly in memory. The in-situ variant stored the offsets in an integer array.

**MR:** A space-efficient implementation of the algorithm of McDiarmid and Reed [6] (MR). The program accepted a tuning parameter $\mu$ and set the size of the bottom trees to the closest power of two larger than, or equal to, $\mu \lg N$. By default, $\mu = 16$. As in the previous program, both cache and branch optimizations were applied. All the elements on the sift_down path were moved cyclically first after the final position of the new element was known as proposed in [7]. When making a bottom tree into a heap, the indices of the element array from the interval $[0..N)$ and those of the packed array from the interval $[0..S)$ were updated in tandem, which doubled the instruction count for index operations. Also this program could be configured to operate in-place (use a packed array of bit pairs) or in-situ (use an array of bytes).

The following optimization options were considered for Floyd’s program:

**opt1** [2]: We made sure that sift_down was always called with an odd $N$. This way, inside the inner loop, one easy-to-predict branch could be removed.

**opt2** [2]: We interpreted the result of an element comparison as an integer and used this value in normal index arithmetic. This way, inside the inner loop, the hard-to-predict branch in “if (condition) $j \leftarrow j + 1$” could be replaced with an assignment “$j \leftarrow j + (condition)$”.

**opt3** [2]: We did not make any element moves when the element at the root stayed in its original location.
opt₄ [1]: We visited the nodes in reverse depth-first order instead of reverse breadth-first order.

opt₅ [2]: We made the construction in a single loop by fusioning the two loops in `make_heap` and `sift_down`. Inside this loop, conditional moves were used, but two of the element moves were still made conditionally so that the number of element moves would not increase to 5\(N\). The outcome of these two conditional branches was predicted reasonably well so it was not worth avoiding these branches.

**Portability**

In the experiments reported in the paper “Heap Construction—50 Years Later”, in each experiment a C array was used to store the input elements. As in our pseudo-code, to access the element with rank \(i\) in an array \(A\) one would normally write \(A[i]\). However, the parameters given for `std::make_heap` are iterators and for them one has to write `*(A + i)` instead of `A[i]`. For arrays both expressions are valid and the semantics is the same. In this revised version all the programs have been made portable so that they can be used in the same way as C\++ standard-library function `make_heap`. However, be aware that in some cases a portable version can be a bit slower than a version using C arrays directly.

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Heap-construction programs

\textit{stl.h++}

```cpp
#include <iterator> // std::iterator_traits
#include <utility> // std::move

namespace stl {

    template <typename iterator, typename index, typename T, typename comparator>
    void push_heap(iterator a, index hole, index top, T value, comparator less) {
        index parent = (hole - 1) / 2;
        while (hole > top and less(*(a + parent), value)) {
            *(a + hole) = std::move(*(a + parent));
            hole = parent;
            parent = (hole - 1) / 2;
        }
        *(a + hole) = std::move(value);
    }

    template <typename iterator, typename index, typename T, typename comparator>
    void adjust_heap(iterator a, index hole, index n, T value, comparator less) {
        index const top = hole;
        index second = hole;
        while (second < (n - 1) / 2) {
            second = 2 + (second + 1);
            if (less(*(a + second), *(a + (second - 1)))) {
                second--;
            }
            *(a + hole) = std::move(*(a + second));
            hole = second;
        }
        if ((n bitand 1) == 0 and second == (n - 2) / 2) {
            second = 2 + (second + 1);
            *(a + hole) = std::move(*(a + (second - 1)));
            hole = second - 1;
        }
        stl::push_heap(a, hole, top, std::move(value), less);
    }

    template <typename iterator, typename comparator>
    void make_heap(iterator a, iterator past, comparator less) {
        using element = typename std::iterator_traits<iterator>::value_type;
        using index = typename std::iterator_traits<iterator>::difference_type;
        if (past - a < 2) {
            return;
        }
    }
}
```
f.h++

// Floyd's original Algol program translated into C++

#include <cstdint> // std::size_t
#include <iterator> // std::iterator_traits
#include <utility> // std::move

namespace f {

constexpr std::size_t root() { return 0; }

template<typename index>
index parent(index i) {
    return (i - 1) / 2;
}

template<typename index>
index left_child(index i) {
    return (i << 1) + 1;
}

template<typename iterator, typename index, typename comparator>
void sift_down(iterator a, index i, index n, comparator less) {
    using element = typename std::iterator_traits<iterator>::
        value_type;
    element x = std::move(*std::move(a + i));
    loop:
        index j = left_child(i);
        if (j < n) {
            if (j < (n - 1)) {
                if (less(*std::move(a + j), *std::move(a + (j + 1)))) {
                    j = j + 1;
                }
            }
            if (less(x, *(a + j))) {

        
}
36     *(a + i) = std::move(*(a + j));
37     i = j;
38     goto loop;
39   }
40   }
41   *(a + i) = std::move(x);
42 }
43 }
44 template <typename iterator, typename comparator>
45 void make_heap(iterator first, iterator past, comparator less) {
46   using index = std::size_t;
47   index const n = past - first;
48   if (n < 2) {
49     return;
50   }
51   index i = parent(n - 1);
52   while (true) {
53     sift_down(first, i, n, less);
54     if (i == root()) {
55       break;
56     }
57     --i;
58   }
59 }
60 }

opt1.h++

// Floyd's heap-construction program

// Optimization: one easy-to-predict branch removed

#include <algorithm> // std::make_heap
#include <cstddef>  // std::size_t
#include <iterator> // std::iterator_traits
#include <utility>  // std::move

namespace opt1 {

constexpr std::size_t root() {
  return 0;
}

template <typename index>
index parent(index i) {
  return (i - 1) / 2;
}

template <typename index>
index left_child(index i) {
  return (i << 1) + 1;
}
```cpp
template <typename iterator, typename index, typename comparator>
void sift_up(iterator a, index j, comparator less) {
  using element = typename std::iterator_traits<iterator>::value_type;
  element in = std::move(*(a + j));
  while (j > root()) {
    index i = parent(j);
    if (less(*a + i, in)) {
      *(a + j) = std::move(*(a + i));
      j = i;
    }
    else {
      break;
    }
  }
  *(a + j) = std::move(in);
}

template <typename iterator, typename index, typename comparator>
void sift_down(iterator a, index i, index n, comparator less) {
  using element = typename std::iterator_traits<iterator>::value_type;
  element in = std::move(*(a + i));
  loop:
    index j = left_child(i);
    if (j < n) {
      if (less(*(a + j), *(a + (j + 1)))) {
        j = j + 1;
      }
      if (less(in, *(a + j))) {
        *(a + i) = std::move(*(a + j));
        i = j;
        goto loop;
      }
    }
  *(a + i) = std::move(in);
}

template <typename iterator, typename comparator>
void make_heap(iterator first, iterator past, comparator less) {
  using index = std::size_t;
  index const n = past - first;
  if (n < 3) {
    std::make_heap(first, past, less);
    return;
  }
  index const m = (n bitand 1) ? n : n - 1;
  index i = parent(m - 1);
  while (true) {
    sift_down(first, i, m, less);
    if (i == root()) {
```
break;
   
   -- i;
   
   }
   
sift_up(first, n - 1, less);
   }
}

\texttt{opt2.h++}

// Floyd's heap-construction program

// Optimizations: 1) one easy-to-predict branch removed; 2) one
// hard-to-predict branch removed

#include <cstdlib> // std::size_t
#include <iterator> // std::iterator_traits
#include <utility> // std::move

namespace opt2 {

constexpr std::size_t root() { return 0; }


\texttt{template <typename index>

index parent(index i) {
   return (i - 1) / 2;

}}

\texttt{template <typename index>

index left_child(index i) {
   return (i << 1) + 1;

}}

\texttt{template <typename iterator, typename index, typename comparator>

void sift_up(iterator a, index i, comparator less) {
   using element = typename std::iterator_traits<iterator>::
   value_type;
   
element in = std::move(*a);
   while (j > root()) {
      index i = parent(j);
      if (less(*a, in)) {
         *a = std::move(*a + i);
         j = i;
      } else {
         break;
      }
   }
   *a = std::move(in);

}
template <typename iterator, typename index, typename comparator>
void sift_down(iterator a, index i, index n, comparator less) {
    using element = typename std::iterator_traits<iterator>::value_type;
    element in = std::move(*a + i);
    loop:
        index j = left_child(i);
        if (j < n) {
            j = j + less(*a + j), *(a + (j + 1)));
            if (less(in, *(a + j))) {
                *(a + i) = std::move(*(a + j));
                i = j;
                goto loop;
            }
        }
    *(a + i) = std::move(in);
}

template <typename iterator, typename comparator>
void make_heap(iterator first, iterator past, comparator less) {
    using index = std::size_t;
    index const n = past - first;
    if (n < 3) {
        std::make_heap(first, past, less);
        return;
    }
    index const m = (n bitand 1) ? n : n - 1;
    index i = parent(m - 1);
    while (true) {
        sift_down(first, i, m, less);
        if (i == root()) {
            break;
        }
        --i;
    }
    sift_up(first, n - 1, less);
}

opt3.h++

// Floyd's heap-construction program

// Optimizations: 1) one easy-to-predict branch removed; 2) one
// hard-to-predict branch removed; 3) no element moves done in
// sift_down if the new element can stay at the root

#include <algorithm> // std::make_heap
#include <stddef> // std::size_t
#include <iterator> // std::iterator_traits
#include <utility> // std::move
namespace opt3 {
    constexpr std::size_t root() {
        return 0;
    }

    template <typename index>
    index parent(index i) {
        return (i - 1) / 2;
    }

    template <typename index>
    index left_child(index i) {
        return (i << 1) + 1;
    }

    template <typename iterator, typename index, typename comparator>
    void sift_up(iterator a, index j, comparator less) {
        using element = typename std::iterator_traits<iterator>::value_type;
        index in = std::move(*a + j);
        while (j > root()) {
            index i = parent(j);
            if (less(*a + i, in)) {
                *(a + j) = std::move(*a + i);
                j = i;
            } else {
                break;
            }
        }
        *(a + j) = std::move(in);
    }

    template <typename iterator, typename index, typename comparator>
    void sift_down(iterator a, index i, index n, comparator less) {
        using element = typename std::iterator_traits<iterator>::value_type;
        index j = left_child(i);
        j = j + less(*(a + j), *(a + (j + 1)));
        if (not less(*(a + i), *(a + j))) {
            return;
        }
        element in = std::move(*(a + i));
        *(a + i) = std::move(*(a + j));
        loop:
        i = j;
        j = left_child(i);
        if (j < n) {
            j = j + less(*(a + j), *(a + (j + 1)));
        if (less(in, *(a + j))) {
            continue;
        }
    }
}
template<typename iterator, typename comparator>
void make_heap(iterator first, iterator past, comparator less) {
  using index = std::size_t;
  index const n = past - first;
  if (n < 3) {
    std::make_heap(first, past, less);
    return;
  }
  index const m = (n bitand 1) ? n : n - 1;
  index i = parent(m - 1);
  while (true) {
    sift_down(first, i, m, less);
    if (i == root()) {
      break;
    }
    --i;
  }
  sift_up(first, n - 1, less);
}

namespace opt4 {

  constexpr std::size_t root() {
    return 0;
  }

  template<typename index>
  index parent(index i) {
    return (i - 1) / 2;
  }

} // opt4.h++
template <typename index>
index left_child(index i) {
    return (i << 1) + 1;
}

template <typename iterator, typename index, typename comparator>
void sift_up(iterator a, index j, comparator less) {
    using element = typename std::iterator_traits<iterator>::value_type;
    element in = std::move(*(a + j));
    while (j > root()) {
        index i = parent(j);
        if (less(*(a + i), in)) {
            *(a + j) = std::move(*(a + i));
            j = i;
        } else {
            break;
        }
    }
    *(a + j) = std::move(in);
}

template <typename iterator, typename index, typename comparator>
void sift_down_1_3(iterator a, index i, index n, comparator less) {
    index j = left_child(i);
    j = j + less(*(a + j), *(a + (j + 1)));
    if (not less(*(a + i), *(a + j))) {
        return;
    }
    std::swap(*(a + i), *(a + j));
}

template <typename iterator, typename index, typename comparator>
void sift_down(iterator a, index i, index n, comparator less) {
    using element = typename std::iterator_traits<iterator>::value_type;
    index j = left_child(i);
    j = j + less(*(a + j), *(a + (j + 1)));
    if (not less(*(a + i), *(a + j))) {
        return;
    }
    element in = std::move(*(a + i));
    loop:
        *(a + i) = std::move(*(a + j));
        i = j;
        j = left_child(j);
if (j < n) {
    j = j + less(*a + j, *(a + (j + 1)));
    if (less(in, *(a + j))) {
        goto loop;
    }
}

*(a + i) = std::move(in);

 template<typename iterator, typename comparator>
 void make_heap(iterator first, iterator past, comparator less) {
     using index = std::size_t;
     index const n = past - first;
     if (n < 5) {
         std::make_heap(first, past, less);
         return;
     }
     index const m = (n bitand 1) ? n : n - 1;
     index j = (1 << ilogb(m)) - 2;
     index const i = parent(j + 1);
     while (j ≥ i) {
         sift_down_1_3(first, j, m, less);
         index z = j;
         while ((z bitand 1) == 1) {
             z = parent(z);
             sift_down(first, z, m, less);
         }
         --j;
     }
     sift_up(first, n - 1, less);
 }

 opt5.h++

// A program that constructs a binary heap in a single loop
// Optimizations: 1) branches avoided in general; 2) a compiler
// should use conditional moves, not conditional branches;
// 3) construction is done in a single loop to avoid the branch
// mispredictions caused by the inner loop when stepping out of it
#include <algorithm> // std::make_heap
#include <cstdlib> // std::size_t
#include <iterator> // std::iterator_traits
#include <utility> // std::move
namespace opt5 {
    constexpr std::size_t root() {
        return 0;
    }
}
template<
type name index\>
index parent(index i) {
    return (i - 1) / 2;
}

template<
type name index\>
index left_child(index i) {
    return (i << 1) + 1;
}

template<
type name iterator, type name index, type name comparator\>
void sift_up(iterator a, index j, comparator less) {
    using element = typename std::iterator_traits<iterator>::
        value_type;
    element in = std::move(*(a + j));
    while ((j > root())) {  
        index i = parent(j);
        if (less(*(a + i), in)) {
            *(a + j) = std::move(*(a + i));
            j = i;
        } else {
            break;
        }
    } *(a + j) = std::move(in);
}

template<
type name iterator, type name comparator\>
void make_heap(iterator first, iterator past, comparator less) {
    using index = std::size_t;
    using element = typename std::iterator_traits<iterator>::
        value_type;
    index const n = past - first;
    if (n < 3) {
        std::make_heap(first, past, less);
        return;
    }
    index const m = (n bitand 1) ? n : n - 1;
    index i = parent(m - 1);
    index j = i;
    index hole = j;
    element in = std::move(*(first + j));
    while (true) {
        hole = (i == j) ? j : hole;
        if (i == j) {
            in = std::move(*(first + j));
        }
        j = left_child(j);
        j += less(*(first + j), *(first + (j + 1)));
        *(first + hole) = std::move(*(first + j));
    }
hole = less(in, *(first + j)) ? j : hole;
bool done = (left_child(j) ≥ m);
if (done) {
    *(first + hole) = std::move(in);
} if (done and i == root()) {
    break;
} i = (done) ? i - 1 : i;
j = (done) ? i : j;
sift_up(first, n - 1, less);
}

basic_gm.h++

// Gonnet & Munro
#include <algorithm> // std::copy std::make_heap
#include <cstddef> // std::size_t
#include <functional> // std::less
#include <iterator> // std::iterator_traits
namespace basic_gm {
    constexpr std::size_t root() {
        return 0;
    }

    template <typename index>
    index parent(index i) {
        return (i - 1) / 2;
    }

    template <typename index>
    index left_child(index i) {
        return 2 * i + 1;
    }

    template <typename index>
    index right_child(index i) {
        return 2 * i + 2;
    }

    template <typename index>
    index odd(index i) {
        return (i bitand 1) == 1;
    }

    template <typename index>
    index even(index i) {
template <typename index>
index sibling(index i) {
    return i + odd(i) - even(i);
}

template <typename index>
bool is_inside(index i, index n) {
    return i < n;
}

template <typename integer>
bool is_power_of_2(integer n) {
    return __builtin_popcount(n) == 1;
}

template <typename iterator, typename index, typename comparator>
void sift_down(iterator a, index i, index n, comparator less) {
    using element = typename std::iterator_traits<iterator>::value_type;
    element x = std::move(*(a + i));
    index j = left_child(i);
    while (is_inside(j, n)) {
        if (is_inside(j + 1, n)) {
            j = j + less(*(a + j), *(a + (j + 1)));
        }
        if (!less(x, *(a + j))) {
            break;
        }
    }
    *(a + i) = std::move(*(a + j));
    i = j;
    j = left_child(i);
    *(a + i) = std::move(x);
}

template <typename iterator, typename index, typename comparator>
void make_heap(iterator a, index K, index N, comparator less) {
    if (!is_inside(left_child(K), N)) {
        return;
    }
    index L = K;
    while (is_inside(left_child(L), N)) {
        L = left_child(L);
    }
    index R = K;
    while (is_inside(right_child(R), N)) {
        R = right_child(R);
    }
    if (R < L) {
R = right_child(R);

} 
do {
    L = parent(L);
    R = parent(R);
    index J = R;
    while (true) {
        sift_down(a, J, N, less);
        if (J == L) {
            break;
        }
        J -= 1;
    } while (L ≠ R);

} 

//template <typename input, typename index, typename output>
void copy_heap(input a, index k, index h, output o) {
    index i = k;
    while (true) {
        index j = i;
        while (j ≤ k) {
            *(o + j) = std::move(*(a + j));
            ++j;
        }
        if (h == 0) {
            return;
        }
        --h;
        i = left_child(i);
        k = right_child(k);
    }
}

//template <typename tournament, typename index>
void populate_tournament(tournament t, index n, index K, index L, index R, index E) {
    index i = parent(n);
    t[i] = E;
    ++i;
    index I = R;
    while (I ≥ L) {
        t[i] = I;
        ++i;
        index Z = I;
        while (odd(Z) and Z ≠ K) {
            Z = parent(Z);
            t[i] = Z;
            ++i;
        }
        --I;
    }
}
```cpp
}  

template <typename tournament, typename index, typename input,  
typename comparator>
void run_tournament(tournament t, index n, input a, comparator less  
) {  
    index i = parent(n - 1);  
    while (true) {  
        index C = t[left_child(i)];  
        index D = t[right_child(i)];  
        t[i] = less(*(a + C), *(a + D)) ? D : C;  
        if (i == root()) {  
            break;  
        }  
        -- i;  
    }  
}  

template <typename tournament, typename index, typename input,  
typename comparator>
void update(tournament t, index j, index h, index E, input a,  
comparator less) {  
    index M = t[j]; // location of the old champion  
    index k = j;  
    do {  
        k = left_child(k);  
        k = k + (t[k] != M);  
        -- h;  
    } while (h != 0);  
    t[k] = E;  
    do {  
        k = parent(k);  
        index C = t[left_child(k)];  
        index D = t[right_child(k)];  
        t[k] = less(*(a + C), *(a + D)) ? D : C;  
    } while (k != j);  
}  

template <typename tournament, typename index, typename input,  
typename output, typename comparator>
index handle_base_case(tournament t, index j, input a, output o,  
index J, comparator less) {  
    index champion = t[j];  
    *(o + J) = std::move(*(a + champion));  
    index k = left_child(j);  
    index l = k + (t[k] == champion);  
    index w = sibling(l);  
    index L = left_child(J);  
    index W = right_child(J);  
    index second = t[l];  
    *(o + L) = std::move(*(a + second));  
    k = left_child(l);  
    return champion;  
}  
```
template<typename tournament, typename index, typename input, 
     typename output, typename comparator>

index convert_tournament(tournament t, index j, index h, input a, 
    output o, index J, comparator less) {
    // J: current output location in the heap
    // j: current root in the tournament tree
    if (h == 3) {
        return handle_base_case(t, j, a, o, J, less);
    }
    index champion = t[j];
    *(o + J) = std::move(*(a + champion));
    index k = left_child(j);
    k = k + (t[k] == champion);
    index excess = convert_tournament(t, k, h - 1, a, o, left_child(J), less);
    k = sibling(k);
    update(t, k, h - 1, excess, a, less);
    return convert_tournament(t, k, h - 1, a, o, right_child(J), less);
}
template<typename input, typename index, typename tournament, 
    typename output, typename comparator>
index make_heap(input a, index K, index N, index E, tournament t, 
    output o, comparator less) {
    if (not is_inside(K, N)) {
        return E;
    }
    index h = 0;
    index L = K;
    while (is_inside(left_child(L), N)) {
        ++h;
        L = left_child(L);
    }
    index height = h;
    index R = K;
    h = 0;
    while (is_inside(right_child(R), N)) {
        ++h;
        R = right_child(R);
    }
    index n = l << (height + 1);
    if (n < 8) {
        copy_heap(a, K, h, o);
        make_heap(o, K, N, less);
        return E;
    }
    populate_tournament(t, 2 * n - 1, K, L, R, E);
    run_tournament(t, 2 * n - 1, a, less);
    E = convert_tournament(t, root(), height + 1, a, o, K, less);
    return E;
}

template<typename iterator, typename comparator>
void make_heap(iterator a, iterator past, comparator less) {
    using index = std::size_t;
    using element = typename std::iterator_traits<iterator>::
        value_type;
    index const n = past - a;
    if (n < 2) {
        return;
    }
    index* t = (index*) malloc((4 * n) / 3 * sizeof(index));
    element* o = (element*) malloc(n * sizeof(element)); // no
    element moves
    index i = n; // invariant: everything moved up to this point
    if (odd(n)) {
        *(o + n - 1) = std::move(*(a + n - 1));
        i = n - 1;
    }
    do {
        index j = sibling(i);
        i = parent(i);
index excess = i;
excess = make_heap(a, j, n, excess, t, o, less);
o[i] = std::move(*a + excess);
} while (i != root());
i = n - 1;
do {
i = parent(i);
sift_down(o, i, n, less);
} while (i != root());
*(a + i) = std::move(*(o + i));
free(t);
free(o);
}

namespace gm {

constexpr std::size_t root() {
  return 0;
}

template <typename index>
index parent(index i) {
  return (i - 1) / 2;
}

template <typename index, typename height>
index ancestor(index i, height h) {
  return (i + 1) / (1 << h) - 1;
}
template<typename index>
index left_child(index i) {
    return 2 * i + 1;
}

template<typename index>
index right_child(index i) {
    return 2 * i + 2;
}

template<typename index>
index odd(index i) {
    return (i & 1) == 1;
}

template<typename index>
index even(index i) {
    return (i & 1) == 0;
}

template<typename index>
index sibling(index i) {
    return i + odd(i) - even(i);
}

template<typename index>
bool is_inside(index i, index n) {
    return i < n;
}

template<typename index>
bool general_case(index i, index n) {
    return left_child(left_child(left_child(i))) < parent(n);
}

std::size_t ilogb(std::size_t x) {
    asm("bsr \%0, \%0\n" : "r"(x)
          : "0" (x)
          );
    return x;
}

template<typename offset, typename index>
index offset_to_index(offset i, index J) {
    if (index(i) != index(0)) {
        index h = ilogb(index(i));
        return (1 << h) * J + index(i) - 1;
    }
    return parent(J);
template<typename iterator, typename index, typename comparator>
void sift_down(iterator a, index i, index n, comparator less) {
  using element = typename std::iterator_traits<iterator>::
                  value_type;
  element x = std::move(*(a + i));
  index j = left_child(i);
  while (is_inside(j, n)) {
    if (is_inside(j + 1, n)) {
      j = j + less(*(a + j), *(a + (j + 1)));
    }
    if (!less(x, *(a + j))) {
      break;
    }
    *(a + i) = std::move(*(a + j));
    i = j;
  }
  *(a + i) = std::move(x);
}

template<typename iterator, typename index, typename comparator>
void make_heap(iterator a, index K, index N, comparator less) {
  index L = K;
  while (is_inside(left_child(L), N)) {
    L = left_child(L);
  }
  index R = K;
  while (is_inside(right_child(R), N)) {
    R = right_child(R);
  }
  do {
    L = parent(L);
    R = parent(R);
    index J = R;
    while (true) {
      sift_down(a, J, N, less);
      if (J == L) {
        break;
      }
      J -= 1;
    }
  } while (L != R);
}

template<typename tree, typename offset>
void populate_tournament(tree& t, offset n) {
  offset i = 0;
  for (offset j = parent(n); is_inside(j, n); ++j) {
    t[j] = i;
  }
}
\[ i = i + 1; \]

\[ \}

template <typename tree, typename offset, typename iterator, 
           typename index, typename comparator>
void run_tournament(tree& t, offset n, iterator a, index J, 
                    comparator less) {
  offset j = parent(n - 1);
  while (true) {
    offset k = t[left_child(j)];
    index L = offset_to_index(k, J);
    k = t[right_child(j)];
    index R = offset_to_index(k, J);
    k = left_child(j) + less(*(a + L), *(a + R));
    t[j] = t[k];
    if (j == root(offset)) { 
      break;
    }
    j = j - 1;
  }
}

template <typename tree, typename offset, typename iterator, 
           typename index, typename comparator>
void update(tree& t, offset i, offset n, offset excess, iterator a, 
            index J, comparator less) {
  offset j = i; // root of the subtree considered
  auto top = t[j]; // offset of the old top
  t[j] = excess;
  while (is_inside(left_child(j), n)) {
    j = left_child(j) + (t[right_child(j)] == top);
    t[j] = excess;
  }
  while (j != i) {
    j = parent(j);
    offset k = t[left_child(j)];
    index L = offset_to_index(k, J);
    k = t[right_child(j)];
    index R = offset_to_index(k, J);
    k = left_child(j) + less(*(a + L), *(a + R));
    t[j] = t[k];
  }
}

template <typename tree, typename offset, typename iterator, 
           typename permutation, typename index, typename comparator>
offset handle_base_case(tree& t, offset j, permutation& sigma, 
                        iterator a, index J, comparator less) {
  auto champion = t[j];
  sigma[j + 1] = champion;
  offset k = left_child(j);
offset l = k + (t[k] == champion);
offset w = sibling(l);
auto finalist = t[l];
 sigma[l + 1] = finalist;
k = left_child(l);
offset ll = k + (t[k] == finalist);
offset lw = sibling(ll);
sigma[ll + 1] = t[ll];
k = left_child(lw);
offset lwl = k + (t[lw] == t[k]);
sigma[lwl + 1] = t[lwl];
k = left_child(ll);
offset lll = k + (t[ll] == t[k]);
k = left_child(w);
offset wll = k + (t[k] == champion);
offset wll = t[wll];
k = left_child(wl);
offset wll = t[wll];
sigma[wll + 1] = (line) ? t[lll] : semifinalist;
index K = offset_to_index(semifinalist, J);
index L = offset_to_index(t[lll], J);
bool line = less(*(a + K), *(a + L));
sigma[w + 1] = (line) ? t[lll] : semifinalist;
return excess;
}

template <typename tree, typename offset, typename permutation,
  typename iterator, typename index, typename comparator>
void convert_tournament(tree& t, offset n, permutation& sigma,
  iterator a, index J, comparator less) {

#ifdef IN_PLACE
  cphstl::packed_array stack(ilogb(n));
  stack.resize(ilogb(n));
#else
  std::vector<offset> stack;
  stack.resize(ilogb(n));
#endif

offset s = 0; // stack empty
offset j = root<offset>();
while (true) {
  offset k = t[j];
sigma[j + 1] = k;
  if (general_case(j, n)) {
    offset loser = left_child(j) + (t[j] == t[left_child(j)]);
    offset winner = left_child(j) + (t[j] != t[left_child(j)]);
stack[s] = winner;
++s;
j = loser;
} else {
offset excess = handle_base_case(t, j, sigma, a, J, less);
if (s \neq 0) {
    -- s;
    j = stack[s];
    update(t, j, n, excess, a, J, less);
} else {
    sigma[0] = excess;
    return;
}
}
}
}

template <typename permutation, typename offset, typename iterator, 
        typename index, typename bit_vector>
void permute_in_place(permutation& sigma, offset n, iterator a, 
        index J, bit_vector& b) {
    using element = typename std::iterator_traits<iterator>::
            value_type;
    offset i = 0;
    while (i < n) {
        b[i] = false;
        offset k = sigma[i];
        if (k == i) {
            b[i] = true;
        }
        ++i;
    }
    i = 0;
    while (i < n) {
        if (b[i] == false) { // process next cycle
            b[i] = true;
            index H = offset_to_index(i, J); // H as in hole
            element temporary = std::move(*(a + H));
            offset k = sigma[i];
            while (k \neq i) {
                b[k] = true;
                index K = offset_to_index(k, J);
                *(a + H) = std::move(*(a + K));
                H = K;
                k = sigma[k];
            }
            *(a + H) = std::move(temporary);
        }
        ++i;
    }
template <typename iterator, typename comparator>
void make_heap(iterator a, iterator past, comparator less) {
using index = std::size_t;
index n = past - a;
if (n < 2) {
    return;
}
index lg_n = ilogb(n);
index lg/lg_n = ilogb(std::max(index(2), lg_n));
index s = 8;
index h = 3;
while (s < FACTOR * lg_n / lg/lg_n) {
    s = 2 * s;
    h = h + 1;
}
if (n < 2 * s) {
    make_heap(a, root(), n, less);
    return;
}

#ifdef IN_PLACE
    cphstl::packed_array sigma(h + 1);
sigma.resize(2 * s);
cphstl::packed_array t(h + 1);
t.resize(4 * s);
cphstl::packed_array b(1);
b.resize(2 * s);
#else
    std::vector<int> sigma;
sigma.resize(2 * s);
    std::vector<int> t;
t.resize(4 * s);
    std::vector<unsigned char> b;
b.resize(2 * s);
#endif

    index const R = (1 << lg_n) - 1;
    index K = ancestor(R - 1, h - 1);
    index J = ancestor(n - 1, h - 1);
    index I = parent(K + 1);
    if (I < J) {
        J = parent(J);
    }
    while (K > J) {
        populate_tournament(t, 2 * s - 1);
        run_tournament(t, 2 * s - 1, a, K, less);
        convert_tournament(t, 2 * s - 1, sigma, a, K, less);
        permute_in_place(sigma, s, a, K, b);
        for (index Z = K; (Z bitand 1) == 1; ) {
            Z = parent(Z);
sift_down(a, Z, n, less);
}  -- K;
}
make_heap(a, J, n, less);
for (index Z = J; (Z bitand 1) == 1; ) {
    Z = parent(Z);
    sift_down(a, Z, n, less);
}
while (K > I) {
    -- K;
    populate_tournament(t, 4 * s - 1);
    run_tournament(t, 4 * s - 1, a, K, less);
    convert_tournament(t, 4 * s - 1, sigma, a, K, less);
    permute_in_place(sigma, 2 * s, a, K, b);
    for (index Z = K; (Z bitand 1) == 1; ) {
        Z = parent(Z);
        sift_down(a, Z, n, less);
    }
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}

template<typename index>
index left_child(index i) {
    return 2 * i + 1;
}

template<typename index>
index right_child(index i) {
    return 2 * i + 2;
}

template<typename index>
index odd(index i) {
    return (i & 1) == 1;
}

template<typename index>
index even(index i) {
    return (i & 1) == 0;
}

template<typename index>
index sibling(index i) {
    return i + odd(i) - even(i);
}

template<typename index, index n>
bool is_inside(index i, index n) {
    return i < n;
}

template<typename integer>
bool is_power_of_2(integer n) {
    return __builtin_popcount(n) == 1;
}

template<typename iterator, typename index, typename comparator>
void sift_down(iterator a, index i, index n, comparator less) {
    using element = typename std::iterator_traits<iterator>::
        value_type;
    element x = std::move(*(a + i));
    index j = left_child(i);
    while (is_inside(j, n)) {
        if (is_inside(j + 1, n)) {
            j = j + less(*(a + j), *(a + (j + 1)));
        } else {
            break;
        }
    }
    *(a + i) = std::move(*(a + j));
    i = j;
    j = left_child(i);
template<typename iterator, typename index, typename comparator>
void make_heap(iterator a, index K, index N, comparator less) {
    if (not is_inside(left_child(K), N)) {
        return;
    }
    index L = K;
    while (is_inside(left_child(L), N)) {
        L = left_child(L);
    }
    index R = K;
    while (is_inside(right_child(R), N)) {
        R = right_child(R);
    }
    if (R < L) {
        R = right_child(R);
    }
    do {
        L = parent(L);
        R = parent(R);
        index J = R;
        while (true) {
            sift_down(a, J, N, less);
            if (J == L) {
                break;
            }
            J -= 1;
        }
    } while (L != R);
}

template<typename input, typename index, typename output>
void move_heap(input& a, index j, index n, output& o) {
    index i = j;
    index k = j;
    index p = root();
    while (i < n) {
        j = i;
        while (j ≤ k) {
            *(o + p) = std::move(*(a + j));
            ++j;
            ++p;
        }
        i = left_child(i);
        k = right_child(k);
    }
}

template<typename tournament, typename index>
```cpp
void populate_tournament(tournament& t, index n, index J, index E)
{
    index i = parent(n);
    t[i] = E;
    ++i;
    index I = J;
    index K = J;
    while (i < n) {
        J = I;
        while (J <= K) {
            t[i] = J;
            ++i;
            ++J;
        }
        I = left_child(I);
        K = right_child(K);
    }
}

// template <typename tournament, typename index, typename input, typename comparator>
void run_tournament(tournament& t, index n, input const& a, comparator less)
{
    index i = parent(n - 1);
    while (true) {
        index C = t[left_child(i)];
        index D = t[right_child(i)];
        t[i] = less(*(a + C), *(a + D)) ? D : C;
        if (i == root()) {
            break;
        }
        -- i;
    }
}

// template <typename tournament, typename index, typename input, typename comparator>
void update(tournament& t, index j, index h, index E, input const& a, comparator less)
{
    index M = t[j]; // location of the old champion
    index k = j;
    do {
        k = left_child(k);
        k = k + (t[k] != M);
        -- h;
    } while (h != 0);
    t[k] = E;
    do {
        k = parent(k);
        index C = t[left_child(k)];
        index D = t[right_child(k)];
        t[k] = less(*(a + C), *(a + D)) ? D : C;
    } while (k != 0);
}
```

template <typename tournament, typename index, typename input,
         typename output, typename comparator>
index handle_base_case(tournament& t, index j, input& a, output& o,
        index J, comparator less) {
    index champion = t[j];
    *(o + J) = std::move(*(a + champion));
    index k = left_child(j);
    index l = k + (t[k] == champion);
    index w = sibling(l);
    index L = left_child(J);
    index W = right_child(J);
    index second = t[l];
    *(o + L) = std::move(*(a + second));
    k = left_child(l);
    index ll = k + (t[k] == second);
    index lw = sibling(ll);
    *(o + left_child(L)) = std::move(*(a + t[ll]));
    k = left_child(ll);
    index lw1 = k + (t[k] == second);
    *(o + right_child(L)) = std::move(*(a + t[lw1]));
    k = left_child(ll);
    index lll = k + (t[k] == ll);
    k = left_child(w);
    index wll = k + (t[k] == champion);
    index ww = sibling(wll);
    k = left_child(wll);
    index wll1 = k + (t[k] == wll);
    *(o + left_child(W)) = std::move(*(a + t[lll]));
    index U = t[lll];
    index V = t[lll];
    if (less(* (a + U), *(a + V))) {
        *(o + W) = std::move(*(a + V));
        *(o + right_child(W)) = std::move(*(a + U));
    }
    else {
        *(o + W) = std::move(*(a + U));
        *(o + right_child(W)) = std::move(*(a + V));
    }
    return excess;
}

template <typename tournament, typename index, typename input,
         typename output, typename comparator>
index convert_tournament(tournament& t, index j, index h, input& a,
                        output& o, index P, comparator less) {
    // P: current location in the output heap
// j: current root in the tournament tree
if (h == 3) {
    return handle_base_case(t, j, a, o, P, less);
}

index champion = t[j];
*(o + P) = std::move(*a + champion));
index k = left_child(j);
k = k + (t[k] == champion);
index excess = convert_tournament(t, k, h - 1, a, o, left_child(P), less);
k = sibling(k);
update(t, k, h - 1, excess, a, less);
return convert_tournament(t, k, h - 1, a, o, right_child(P), less);
}

template<typename input, typename index, typename tournament, 
        typename output, typename comparator>
index make_heap(input& a, index K, index N, index E, 
                tournament& t, output& o, index P, comparator less) {
    index height = 0;
    index L = K;
    while (is_inside(left_child(L), N)) {
        ++height;
        L = left_child(L);
    }
    index n = 1 << (height + 1);
populate_tournament(t, 2 * n - 1, K, E);
run_tournament(t, 2 * n - 1, a, less);
E = convert_tournament(t, root(), height + 1, a, o, P, less);
return E;
}

template<typename iterator, typename comparator>
void make_heap(iterator a, iterator past, comparator less) {
    using index = std::size_t;
    using element = typename std::iterator_traits<iterator>::
                  value_type;
    index n = past - a;
    index lg_n = ilogb(std::max(index(2), n));
    index s = 8;
    index h = 3;
    while (s < FACTOR * lg_n) {
        s = 2 * s;
        h = h + 1;
    }
    if (n < 2 * s) {
        make_heap(a, root(), n, less);
        return;
    }
    std::vector<index> t;
t.resize(4 * s);
element* o = (element*) malloc(n * sizeof(element)); // no

    element moves

index const R = (1 << lg_n) - 1;
index K = ancestor(R - 1, h - 1);
index J = ancestor(n - 1, h - 1);
index I = parent(K + 1);
if (I < J) {
    J = parent(J);
}
index excess = root();
if (K == J) {
}
else if (K == J + 1) {
    make_heap(a, K, n, less);
}
else {
    move_heap(a, K, n, o);
    --K;
    while (K > J) {
        excess = root();
        excess = make_heap(a, K, n, excess, t, a, K + 1, less);
        *(a + root()) = std::move(*(a + excess));
        for (index Z = K + 1; (Z bitand 1) == 1; ) {
            Z = parent(Z);
            sift_down(a, Z, n, less);
        }
        --K;
    }
    excess = root();
    *(o + (s - 1)) = std::move(*(a + excess));
    excess = make_heap(o, root(), s - 1, s - 1, t, a, K + 1, less);
    *(a + root()) = std::move(*(o + excess));
    for (index Z = K + 1; (Z bitand 1) == 1; ) {
        Z = parent(Z);
        sift_down(a, Z, n, less);
    }
}
else {
    K = J - 1;
    move_heap(a, K, n, o);
    -- K;
    while (K ≥ I) {
        excess = root();
        excess = make_heap(a, K, n, excess, t, a, K + 1, less);
        *(a + root()) = std::move(*(a + excess));
        for (index Z = K + 1; (Z bitand 1) == 1; ) {
            Z = parent(Z);
            sift_down(a, Z, n, less);
        }
        -- K;
    }
    excess = root();
    *(o + (2 * s - 1)) = std::move(*(o + excess));
    excess = make_heap(o, root(), 2 * s - 1, 2 * s - 1, t, a, I, less);
    *(a + root()) = std::move(*(o + excess));
    for (index Z = I; (Z bitand 1) == 1; ) {
        Z = parent(Z);
        sift_down(a, Z, n, less);
    }
    free(o);
}
return 0;
}

template <typename index>
index parent(index i) {
    return (i - 1) / 2;
}

template <typename index>
index left_child(index i) {
    return (i << 1) + 1;
}

template <typename index>
index right_child(index i) {
    return (i << 1) + 2;
}

template <typename index, typename height>
index ancestor(index i, height h) {
    return ((i + 1) >> h) - 1;
}

template <typename index>
bool is_inside(index i, index n) {
    return i < n;
}

template <typename iterator, typename index, typename comparator>
void sift_up(iterator a, index j, comparator less) {
    using element = typename std::iterator_traits<iterator>::value_type;
    element in = std::move(*(a + j));
    while (j > root()) {
        index i = parent(j);
        if (less(*(a + i), in)) {
            *(a + j) = std::move(*(a + i));
            j = i;
        } else {
            break;
        }
    }
    *(a + j) = std::move(in);
}

template <typename iterator, typename index, typename comparator>
void sift_down(iterator a, index i, index n, comparator less) {
    using element = typename std::iterator_traits<iterator>::value_type;
    element x = std::move(*(a + i));
    index j = left_child(i);
while (is_inside(j, n)) {
    j = j + less(*(a + j), *(a + (j + 1)));
    if (not less(x, *(a + j))) {
        break;
    }
    *(a + i) = std::move(*(a + j));
    i = j;
    j = left_child(i);
} *(a + i) = std::move(x);

enum {LEFT = 0, RIGHT = 1, UNDEFINED = 2};

template <typename iterator, typename index, typename comparator, 
           typename bit_vector>
void make_heap(iterator a, index K, index N, comparator less, 
               bit_vector & b) {
    using element = typename std::iterator_traits<iterator>::
        value_type;
    index L = K;
    index R = K;
    index n = root();
    while (is_inside(L, N)) {
        L = left_child(L);
        n = left_child(n);
        R = right_child(R);
    }
    for (index i = root(); is_inside(i, n); ++i) {
        b[i] = UNDEFINED;
    }
    index m = n;
    do {
        L = parent(L);
        R = parent(R);
        for (index M = R + 1; M > L;) {
            -- M;
            -- m;
            index J = M;
            index j = m;
            index d = 0;
            while (is_inside(left_child(J), N)) {
                ++ d;
                if (b[j] != UNDEFINED) {
                    index flag = b[j];
                    J = left_child(J) + flag;
                    j = left_child(j) + flag;
                } else {
                    index flag = less(*(a + left_child(J)), *(a + right_child (J)));
                }
b[j] = flag;
J = left_child(J) + flag;
} while (J > M and less(*(a + J), *(a + M))) {
    J = parent(J);
    j = parent(j);
    -- d;
} if (J != M) {
    element temp = std::move(*(a + M));
    while (d > 0) {
        *(a + ancestor(J, d)) = std::move(*(a + ancestor(J, d - 1)));
        index i = ancestor(j, d);
        b[i] = UNDEFINED;
        -- d;
    } *(a + J) = std::move(temp);
} while (L != K);
}

template <typename iterator, typename comparator>
void make_heap(iterator first, iterator past, comparator less) {
    using index = std::size_t;
    index const n = past - first;
    if (n < 2) {
        return;
    }
    index const m = (n bitand 1) ? n - 1;
    index lg_m = ilogb(m);
    index s = 2;
    index h = 1;
    while (s < FACTOR * lg_m) {
        s = 2 * s;
        h = h + 1;
    }
    if (n <= s) {
        std::make_heap(first, past, less);
        return;
    }
    #ifdef IN_PLACE
    cphstl::packed_array extra_space(2);
    extra_space.resize(2 * s);
    #else
    std::vector<unsigned char> extra_space;
    extra_space.resize(2 * s);
    #endif
}
index const last_leaf = (1 << lg_m) - 2;
index K = ancestor(last_leaf, h - 1);
index I = parent(K + 1);
while (true) {
    make_heap(first, K, m, less, extra_space);
    for (index Z = K; (Z bitand 1) == 1; ) {
        Z = parent(Z);
    }
    if (K == I) {
        break;
    } - - K;
    sift_down(first, Z, m, less);
}
if (K == I) {
    break;
} - - K;
sift_up(first, n - 1, less);
}

Helper

packed_array.h++

/*
A packed array stores a sequence a small integers of equal length
  compactly in memory
*/
#
#include <climits> // CHAR_BIT
#include <cstdint> // std::size_t
#include <vector> // std::vector
namespace cphstl {
    class packed_array {
    public:
        using value_type = std::size_t;
        using size_type = std::size_t;
        static word_size = CHAR_BIT * sizeof(value_type);
    class reference {
        private:
            friend class packed_array;
            value_type* word_pointer;
            value_type offset;
            value_type mask;
    }
reference();

public:

reference(packed_array& a, std::size_t position)
  : word_pointer(a.data[0] + (position << a.shift) / a.
    word_size),
    offset((position << a.shift) bitand (a.word_size - 1)),
    mask(a.mask) {
}

~reference() {
}

// For a[i] = x;
template <typename integer>
reference& operator=(integer x) {
  value_type y = x;
  y = y bitand mask;
  y = y << offset;
  value_type z = mask << offset;
  *word_pointer &= ~z;
  *word_pointer |= y;
  return *this;
}

// For a[i] = a[j];
reference& operator=(reference const& r) {
  value_type x = *r.word_pointer;
  x = x >> r.offset;
  x = x bitand r.mask;
  value_type y = x bitand mask;
  y = y << offset;
  value_type z = mask << offset;
  *word_pointer &= ~z;
  *word_pointer |= y;
  return *this;
}

// For x = a[i];
operator value_type() const {
  value_type x = *word_pointer;
  x = x >> offset;
  x = x bitand mask;
  return x;
}

bool operator==(reference const& r) const {
  value_type x = (value_type) r;
  value_type y = (value_type) *this;
  return x == y;
}
bool operator!=(reference const& r) const {
    return not (*this == r);
}

template<typename integer>
bool operator==(integer x) const {
    value_type y = (value_type)*this;
    return value_type(x) == y;
}

template<typename integer>
bool operator!=(integer x) const {
    return not (*this == x);
};

friend class reference;

template<typename integer>
explicit packed_array(integer min_width_in_bits = 1) {
    n = 0;
    value_type delta = 1;
    shift = 0;
    while (delta < min_width_in_bits) {
        delta = 2 * delta;
        shift = shift + 1;
    }
    mask = (1 << delta) - 1;
}

size_type size() const {
    return n;
}

template<typename integer>
void resize(integer new_size) {
    n = new_size;
    data.resize(((n << shift) + word_size - 1) / word_size);
}

template<typename integer>
reference const operator[](integer position) const {
    return reference(*this, position);
}

template<typename integer>
reference operator[](integer position) {
    return reference(*this, position);
}

private:
Drivers

driver.c++

```cpp
# if not defined(MAXSIZE)
#define MAXSIZE 64 * 1024 * 1024 // 715827882
#endif

#include <algorithm> // std::random_shuffle std::make_heap std::sort
#include <ctime> // std::clock_t std::clock CLOCKS_PER_SEC
#include <functional> // std::less
#include <iostream> // std::cout std::cerr
#include <iterator> // std::iterator_traits
#include <utility> // std::move

template <typename iterator>
bool is_permutation(iterator first, iterator past) {
    using element = typename std::iterator_traits<iterator>::value_type;
    std::sort(first, past);
    for (iterator q = first; q != past; ++q) {
        element i = element(q - first);
        if (*q != i) {
            std::cerr << i << ": element missing " << *q << " instead" << std::endl;
            return false;
        }
    }
    return true;
}

template <typename iterator, typename comparator>
bool in_heap_order(iterator first, iterator past, comparator less) {
    using index = typename std::iterator_traits<iterator>::
        difference_type;
    const iterator a = first - 1;
    const index N = past - first;
    bool violated = false;
    for (index i = N; i > 1; i--) {
        if (less(a[i / 2], a[i])) {
            std::cerr << i << ": parent=" << a[i / 2] << " me=" << a[i] << std::endl;
            violated = true;
        }
    }
    return !violated;
}
```
template <typename iterator, typename index, typename comparator>
bool in_heap_order(iterator a, index j, index n, comparator less) {
    if (j > n) {
        return true;
    }
    bool ok = true;
    index left = 2 * j;
    index right = left + 1;
    if (left <= n and less(a[j], a[left])) {
        std::cerr << a[j] << std::endl;
        std::cerr << a[left] << " " << a[right] << std::endl;
        if (2 * left < n) {
            std::cerr << a[2 * left] << " " << a[2 * left + 1] << " ";
            std::cerr << a[2 * right] << " " << a[2 * right + 1] << std::endl;
        }
        ok = false;
    }
    if (right <= n and less(a[j], a[right])) {
        std::cerr << a[j] << std::endl;
        std::cerr << a[left] << " " << a[right] << std::endl;
        if (2 * left < n) {
            std::cerr << a[2 * left] << " " << a[2 * left + 1] << " ";
            std::cerr << a[2 * right] << " " << a[2 * right + 1] << std::endl;
        }
        ok = false;
    }
    ok &= in_heap_order(a, 2 * j, n, less);
    ok &= in_heap_order(a, 2 * j + 1, n, less);
    return ok;
}

#include "algorithm.h++"
#ifdef MEASURE_COMPARISONS
long long volatile comparisons = 0;
#endif

template <typename T>
class counting_comparator {
public:
    using first_argument_type = T;
    using second_argument_type = T;
    using result_type = bool;

    bool operator()(T const& a, T const& b) const {
}
++ comparisons;

return a < b;

};
#endif

#ifdef MEASURE_MOVES
long long volatile moves = 0;
#endif

template<typename T>
class move_counter {
private:

T datum;

move_counter(move_counter const&) = delete;
move_counter& operator=(move_counter const&) = delete;

public:

explicit move_counter()
: datum(0) {
    moves += 1;
}

template<typename number>
explicit move_counter(number x = 0)
: datum(x) {
    moves += 1;
}

move_counter(move_counter& other) {
    datum = std::move(other.datum);
    moves += 1;
}

move_counter& operator=(move_counter& other) {
    datum = std::move(other.datum);
    moves += 1;
    return *this;
}

operator T() const {
    return datum;
}

template<typename U>
friend bool operator<(move_counter<U> const&, move_counter<U> const &);

template<typename U>
friend bool operator==(move_counter<int> const& x, move_counter<int> const& y) {
    return x.datum == y.datum;
}

template<
    typename T>
bool operator<(move_counter<T> const x, move_counter<T> const& y) {
    return x.datum < y.datum;
}

template<
    typename T>
bool operator==(move_counter<T> const& x, move_counter<T> const& y) {
    return x.datum == y.datum;
}

#endif

template<
    typename iterator>
void generate(iterator p, iterator r, char z) {
    using element = typename std::iterator_traits<iterator>::value_type;
    switch (z) {
    case 'd':
        for (iterator q = p; q < r; ++q)
            *q = element((r - 1) - q);
        break;
    case 'i':
        for (iterator q = p; q < r; ++q)
            *q = element(q - p);
        break;
    case 'r':
        for (iterator q = p; q < r; ++q)
            *q = element(q - p);
        std::random_shuffle(p, r);
        break;
    case 'z':
        bool t = false;
        for (iterator q = p; q < r; ++q) {
            *q = element(t);
            t = not t;
        }
    break;
    }
}

void usage(char const* program) {
    std::cerr << "Usage: 
    " << program
    << " <N> <'i'ncreasing | 'd'ecreasing | 'r'andom | 'b'ool>";
    << std::endl;
    exit(1);
}

int main(int argc, char** argv) {
#ifndef MEASURE_MOVES
    using element = move_counter<int>;
#else
    using element = int;
#endif

#ifndef MEASURE_COMPARISONS
    using C = counting_comparator<element>;
#else
    using C = std::less<element>;
#endif

unsigned long N = 15;
char method = 'i';
if (argc == 2) {
    N = atoi(argv[1]);
    method = 'i';
}
else if (argc != 3) {
    usage(argv[0]);
}
else {
    N = atoi(argv[1]);
    method = argv[2];
}
if (N > MAXSIZE) {
    std::cerr << "N out of bounds [0.." << MAXSIZE << "]" << std::endl;
    usage(argv[0]);
}
switch (method) {
    case 'd':
    case 'i':
    case 'r':
    case 'b':
        break;
    default:
        std::cerr << "Method not in ["d","i","r","b"]" << std::endl;
        usage(argv[0]);
}
element* a = new element[MAXSIZE];
element* b = a;
for (volatile unsigned long t = MAXSIZE / N; t > 0; t--) {
    generate(b, b + N, method);
    b = b + N;
}
#else defined(MEASURE_MOVES)
moves = 0;
#elif defined(MEASURE_COMPARISONS)
comparisons = 0;
#endif
#if defined(REPETITIONS)
unsigned long const repetitions = REPETITIONS;
#else
unsigned long const repetitions = MAXSIZE / N;
#endif
b = a;
#if not defined(MEASURE_COMPARISONS) and not defined(MEASURE_MOVES)
std::clock_t start = std::clock();
#endif
for (volatile unsigned long t = 0; t < repetitions; ++t) {
   NAME::make_heap(b, b + N, C());
   b = b + N;
}
#if not defined(MEASURE_COMPARISONS) and not defined(MEASURE_MOVES)
std::clock_t stop = std::clock();
#endif
#if not defined(NDEBUG)
    b = a;
    for (volatile unsigned long t = 0; t < repetitions; ++t) {
        bool ok = in_heap_order(b, b + N, std::less<element>());
        if (!ok) {
            return 1;
        }
        if (method == 'd' or method == 'i' or method == 'r') {
            ok = is_permutation(b, b + N);
            if (!ok) {
                return 2;
            }
        }
        b = b + N;
    }
#endif
double t = double(repetitions) * double(N);
#if defined(MEASURE_COMPARISONS)
std::cout.precision(3);
std::cout << N << " \t " << double(comparisons) / t << std::endl;
#elif defined(MEASURE_MOVES)
std::cout.precision(3);
std::cout << N << " \t " << double(moves) / t << std::endl;
#else
double ns = 1000000000.0 * double(stop - start) / double(CLOCKS_PER_SEC);
std::cout.precision(4);
std::cout << N << ' ' << ns / t << std::endl;
#endif

delete[] a;
return 0;
}

test-driver.cpp

#include <algorithm>  // std::random_shuffle std::make_heap std::sort
#include <cassert>   // assert macro
#include <functional> // std::less
#include <iostream>  // std::cout std::cerr
#include <vector>    // std::vector

template <typename iterator>
void show(iterator a, iterator z) {
    while (a != z) {
        std::cout << long(*a) << ' ';
        ++a;
    }
    std::cout << std::endl;
}

template <typename index>
index left_heap_child(index i) {
    return 2 * i + 1;
}

template <typename index>
index right_heap_child(index i) {
    return 2 * i + 2;
}

template <typename iterator, typename index>
void print(iterator a, index j, index N) {
    if (j < N) {
        std::cerr << a[j];
    }
    std::cerr << ' ';
}

template <typename iterator, typename index, typename comparator>
bool in_heap_order(iterator a, index j, index n, comparator less) {
    if (j >= n) {
        return true;
    }
    bool ok = true;
    index left = left_heap_child(j);
index right = right_heap_child(j);
if (left < n and less(a[j], a[left])) {
    std::cerr << "left child not in heap order " << j << std::endl;
    print(a, j, n);
    std::cerr << std::endl;
    print(a, left, n);
    print(a, right, n);
    std::cerr << std::endl;
    print(a, left_heap_child(left), n);
    print(a, right_heap_child(left), n);
    print(a, left_heap_child(right), n);
    print(a, right_heap_child(right), n);
    std::cerr << std::endl;
    ok = false;
}
if (right < n and less(a[j], a[right])) {
    std::cerr << "right child not in heap order " << j << std::endl;
    print(a, j, n);
    std::cerr << std::endl;
    print(a, left, n);
    print(a, right, n);
    std::cerr << std::endl;
    print(a, left_heap_child(left), n);
    print(a, right_heap_child(left), n);
    print(a, left_heap_child(right), n);
    print(a, right_heap_child(right), n);
    std::cerr << std::endl;
    ok = false;
}
ok &= in_heap_order(a, left_heap_child(j), n, less);
ok &= in_heap_order(a, right_heap_child(j), n, less);
return ok;
}

template <typename iterator>
bool is_permutation(iterator first, iterator past) {
    using integer = typename std::iterator_traits<iterator>::value_type;
    std::vector<integer> v;
v.resize(past - first);
    std::copy(first, past, v.begin());
    std::sort(v.begin(), v.end());
    for (auto q = v.begin(); q != v.end(); ++q) {
        integer i = integer(q - v.begin());
        if (*q != i) {
            std::cerr << i << " missing " << *q << " instead" << std::endl;
            return false;
        }
    }
    return true;
}
template<typename iterator>
void generate(iterator p, iterator r) {
  using element = typename std::iterator_traits<iterator>::value_type;
  for (iterator q = p; q < r; ++q) {
    *q = element(q - p);
  }
  std::random_shuffle(p, r);
}

#include "algorithm.h++" // NAME

int main() {
  int const magic = 20;
  int N = (1 << magic);
  int a[N];
  std::vector<int> testcase;
  for (int n = 0; n ≤ 20000; ++n) { // 8448
    testcase.push_back(n);
  }
  for (int h = 3; h ≠ magic; ++h) {
    for (int delta = -5; delta < 6; ++delta) {
      testcase.push_back((1 << h) + delta);
    }
  }
  testcase.push_back(N);
  for (int n: testcase) {
    std::cout << "n: " << n << std::endl;
  }
  NAME::make_heap(a + 0, a + n, std::less<int>());
  assert(NAME::is_heap_order(a, 0, n, std::less<int>()));
  assert(NAME::is_permutation(a + 0, a + n));
}

Makefile

CXX=g++
CXXFLAGS=-O3 -std=c++11 -x c++ -Wall -Wextra -DNDEBUG # -DIN_PLACE

header-files:=$(wildcard *.h++)
versions:=$(basename $(header-files))
time-tests:=$(addsuffix .time, $(versions))
log-files:=$(addsuffix .log, $(versions))
comp-tests:=$(addsuffix .comp, $(versions))
move-tests:=$(addsuffix .move, $(versions))
branch-tests:=$(addsuffix .branch, $(versions))
cache-tests:=$(addsuffix .cache, $(versions))
instruction-tests:=$(addsuffix .count, $(versions))
unittests := $(addsuffix .test, $(versions))
portability-tests := $(addsuffix .port, $(versions))
profilings := $(addsuffix .prof, $(versions))

N = 1023 32767 1048575 33554431
data = r i d b

$(time-tests): %.time : %.h++
  @cp $*.h++ algorithm.h++
  $(CXX) $(CXXFLAGS) -DNAME=*$ driver.c++
  @for n in $(N) ; do \
    ./a.out $n r : \n  done;
  rm -f algorithm.h++ ./a.out

$(log-files): %.log : %.h++
  @cp $*.h++ algorithm.h++
  $(CXX) $(CXXFLAGS) -DNAME=*$ driver.c++
  @for n in `cat n.txt` ; do \
    ./a.out $n r >> $*.log ; \
  done;
  rm -f algorithm.h++ ./a.out

$(move-tests): %.move : %.h++
  @cp $*.h++ algorithm.h++
  $(CXX) $(CXXFLAGS) -DMEASURE_MOVES -DNAME=*$ driver.c++
  @for d in $(data) ; do \
    echo $$d ; \
    for n in $(N) ; do \
      ./a.out $n $d ; \
    done \
  done;
  rm -f algorithm.h++ ./a.out

$(comp-tests): %.comp : %.h++
  @cp $*.h++ algorithm.h++
  $(CXX) $(CXXFLAGS) -DMEASURE_COMPARISONS -DNAME=*$ driver.c++
  @for d in $(data) ; do \
    echo $$d ; \
    for n in $(N) ; do \
      ./a.out $n $d ; \
    done \
  done;
  rm -f algorithm.h++ ./a.out

$(instruction-tests): %.count : %.h++
  @cp $*.h++ algorithm.h++
  @for n in $(N) ; do \
    python instruction_count.py $* $n ; \
    rm -f ./a.out ; \
    rm -f ./cachegrind.out.* ; \
  done
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$(cache-tests): %.cache : %.h++
@for n in $(N) ; do 
  python cache_misses.py driver.c++ $* $$n ; 
  rm -f ./a.out ; 
  rm -f ./cachegrind.out.* ; 
done

$(branch-tests): %.branch : %.h++
@for n in $(N) ; do 
  python branch_mispredictions.py $* $$n ; 
  rm -f ./a.out ; 
  rm -f ./cachegrind.out.* ; 
done

TESTFLAGS=-O3 -std=c++11 -Wall -Wextra -x c++ -g -DDDEBUG

$(unittests): %.test : %.h++
@cp $*.h++ algorithm.h++
$(CXX) $(TESTFLAGS) -DNAME=$* test-driver.c++
   ./a.out
  rm -f algorithm.h++ ./a.out

$(portability-tests): %.port : %.h++
@cp $*.h++ algorithm.h++
$(CXX) $(TESTFLAGS) -DNAME=$* portability-driver.c++
   ./a.out
  rm -f algorithm.h++ ./a.out

PROFILERFLAGS = -DDEBUG -Wall -std=c++11 -pedantic -x c++ -g

$(profilings): %.prof : %.h++
@cp $*.h++ algorithm.h++
$(CXX) $(PROFILERFLAGS) -DNAME=$* -DMAXSIZE=4194300 driver.c++
  valgrind --tool=callgrind --dump-instr=yes --collect-jumps=yes 
     --callgrind-out-file=$*.callgrind.out ./a.out 1048575
  rm -f algorithm.h++

# Other tools

clean:
  - rm -f a.out temp algorithm.h++ 2>/dev/null

everclean: clean
  - rm *~ */*~ 2>/dev/null

find:
  find . -type f -exec grep $(word) {} \; | less