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Generalized multicritical one-matrix models

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Abstract

We show that there exists a simple generalization of Kazakov's multicritical one-matrix model, which interpolates between the various multicritical points of the model. The associated multicritical potential takes the form of a power series with a heavy tail, leading to a cut of the potential and its derivative at the real axis, and reduces to a polynomial at Kazakov's multicritical points. From the combinatorial point of view the generalized model allows polygons of arbitrary large degrees (or vertices of arbitrary large degree, when considering the dual graphs), and it is the weight assigned to these large order polygons which brings about the interpolation between the multicritical points in the one-matrix model.

1. Introduction

Matrix models have been among the most important tools when discussing non-critical strings or 2d quantum gravity coupled to conformal field theories with central charge $c < 1$. The main interest in the gravitational aspect came from attempts to non-perturbatively regularize the Polyakov path integral in spacetime dimension different from 26 [1–4]. While the stringy aspect of this program partly failed for physical target space dimensions, the 2d gravity aspect was a very fruitful area of research, initiated in [5,3], and getting full attention after the semi-

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nal paper [6] by Kazakov. The latter used the Hermitian matrix model in the large N limit to describe certain matter fields interacting with 2d quantum gravity. Eventually it was understood that the models in [6] describe 2d quantum gravity coupled to $(2, 2m-1)$ conformal field theories, $m = 2, 3, \dots$ [7] (see e.g. [8] for a review). The susceptibility exponents of these theories were calculated (in a way we will discuss below) to be given by

$$\gamma_s = -\frac{1}{m}. \tag{1}$$

To obtain exponents corresponding to other conformal field theories one had to consider multi-matrix models [9–11]. In this paper we will show that one can in fact obtain the full range of exponents $\gamma_s \in]-\infty, 0[$, in the large-N limit of the standard one-cut Hermitian matrix model by allowing for potentials with "heavy tails". In the range $s \in]3/2, 5/2[$ these matrix models have a combinatorial interpretation in terms of random plane graphs (or random planar maps) with high degree vertices or polygon, which have been of recent interest in the mathematical (physics) literature [12–14].

The rest of this paper is organized as follows. In Sec. 2 we remind the reader of the multicritical matrix model introduced in [6]. In Sec. 3 we generalize the results of Sec. 2, such that any critical exponent $\gamma_s < 0$ can occur. The corresponding potential $V_s(x)$ as well as its derivative $V_s'(x)$ are infinite power series in x with cuts on the real axis. We suggest how one can associate a central charge $c(s)$ to each s. In Sec. 4 we show that the standard way of solving the saddle point equation is still valid. Next we address the question of universality (Sec. 5) and the corresponding continuum limit (Sec. 6). The generalized Kazakov potentials $V_s(x)$ where $s \in]1, \infty]$, allow for a combinatorial interpretation which will be described in Sec. 7 and the relation to $O(n)$ models on random triangulations is outlined in Sec. 8. Finally Sec. 9 summarizes our results.

2. The multicritical matrix model

Let us consider the following $N \times N$ Hermitian matrix model

$$Z = \int dM \, \mathrm{e}^{-N \mathrm{tr} V(M)}, \tag{2}$$

where

$$V(x) = \frac{1}{g} \tilde{V}(x), \quad \tilde{V}(x) = \sum_{n=1}^{m} v_n x^{2n}, \quad v_1 = \frac{1}{2}. \tag{3}$$

In the large-N limit there is a one-cut solution, where the eigenvalues of M condense in an interval $[-a, a]$ and the so-called resolvent (also called the disk amplitude)

$$W(z) = \frac{1}{N} \left\langle \mathrm{tr} \, \frac{1}{z - M} \right\rangle = \int_{-a}^{a} \mathrm{d}x \, \frac{\rho(x)}{z - x} \tag{4}$$

is an analytic function of z outside the cut.

The large-N solution for $W(z)$ is

$$W(z) = \int_0^a \frac{\mathrm{d}x}{\pi} \, \frac{x V'(x)}{(z^2 - x^2)} \, \frac{\sqrt{z^2 - a^2}}{\sqrt{a^2 - x^2}}, \tag{5}$$

where the condition $W(z) \to 1/z$ for $|z| \to \infty$ implies

$$g(a^2) = \int_0^a \frac{\mathrm{d}x}{\pi} \frac{xV'(x)}{\sqrt{a^2 - x^2}} = \sum_{n=1}^{m} \frac{v_n\, a^{2n}}{B(n, \frac{1}{2})}, \qquad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \tag{6}$$

This fixes $g(a^2)$ as a polynomial of a^2 for a polynomial potential.

We can rewrite the integral representation (5) for $W(z)$ in terms of the function $g(a^2)$ instead of the potential $\tilde{V}(x)$. Let us introduce a special notation for this function

$$\tilde{U}(a^2) = \int_0^a \frac{\mathrm{d}x}{\pi} \frac{x\tilde{V}'(x)}{\sqrt{a^2 - x^2}} = \int_0^1 \frac{\mathrm{d}y}{\pi} \frac{G((ay)^2)}{\sqrt{1 - y^2}}, \qquad G(x^2) = x\tilde{V}'(x) \tag{7}$$

so the boundary equation (6) reads

$$\tilde{U}(a^2) = g(a^2). \tag{8}$$

Then one has the following representation of $W(z)$

$$gW(z) = \int_0^{a^2} \mathrm{d}A\, \frac{\tilde{U}'(A)}{\sqrt{z^2 - A}} = \int_0^{g} \frac{\mathrm{d}\tilde{g}}{\sqrt{z^2 - a^2(\tilde{g})}}. \tag{9}$$

The proof is based on the identity

$$\int_0^a \frac{\mathrm{d}x}{\pi} \frac{x^{2n}}{(z^2 - x^2)} \frac{\sqrt{z^2 - a^2}}{\sqrt{a^2 - x^2}} = \frac{1}{2B(n, \frac{1}{2})} \int_0^{a^2} \mathrm{d}A\, \frac{A^{n-1}}{\sqrt{z^2 - A}}. \tag{10}$$

For a general potential defined by a convergent power series we have from (6) the relation

$$\tilde{V}(x) = \sum_{n=1}^{\infty} v_n x^{2n}, \qquad \tilde{U}(A) = \sum_{n=1}^{\infty} u_n A^n, \qquad v_n = u_n\, B(n, \tfrac{1}{2}). \tag{11}$$

Finally note that (8) and (9) lead immediately to the known equation for the disk amplitude with one puncture:

$$\frac{\mathrm{d}\, gW(z)}{\mathrm{d}g} = \frac{1}{\sqrt{z^2 - a^2(g)}}. \tag{12}$$

A so-called mth multicritical point of this matrix model is a point where

$$\left.\frac{\mathrm{d}g(A)}{\mathrm{d}A}\right|_{A=a_c^2} = \cdots = \left.\frac{\mathrm{d}^{m-1}g(A)}{\mathrm{d}A^{m-1}}\right|_{A=a_c^2} = 0, \qquad \left.\frac{\mathrm{d}^m g(A)}{\mathrm{d}A^m}\right|_{A=a_c^2} \neq 0. \tag{13}$$

In order to satisfy this requirement an even potential $\tilde{V}(x)$ has to be at least of order $2m$. If we restrict ourselves to potentials of this order[1] $g(a^2)$ is fixed to be of the form

$$g(a^2) = g_* - c(a_c^2 - a^2)^m, \qquad g_* = c\, a_c^{2m}, \qquad c = \frac{1}{4m a_c^{2m-2}}. \tag{14}$$

The value $a_c^2 > 0$ can be chosen arbitrary, after which the coefficients v_n are completely fixed.

[1] In Sec. 6 we consider more general polynomials.

For convenience we choose $a_c = 1$, i.e.

$$g(a^2) = g_* - \frac{1}{4m}(1-a^2)^m, \quad g_* = \frac{1}{4m}. \tag{15}$$

From (6) and (15) we obtain the coefficients $v_n(m)$ for the mth multicritical Kazakov potential

$$v_n = \frac{(-1)^{n-1}}{4m}\binom{m}{n} B(n, \tfrac{1}{2}) = \frac{1}{4}\,\frac{\Gamma(n-m)\Gamma(\frac{1}{2})}{\Gamma(1-m)\Gamma(n+\frac{1}{2})\,n}, \qquad n \leq m, \tag{16}$$

where the last equality should be understood as the limit where m goes to an integer. For future use we write the mth Kazakov potential as

$$V_s(x) = \frac{1}{g(a^2)}\sum_{n=1}^{m} v_n(s)x^{2n}, \quad v_n(s) = \frac{1}{4}\,\frac{\Gamma(n+\frac{1}{2}-s)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-s)\Gamma(n+\frac{1}{2})\,n}, \quad s \to m + \frac{1}{2}. \tag{17}$$

3. The generalized Kazakov potential

Let us now generalize the potential (17) by simply allowing s in $v_n(s)$ to be a real number larger than 1/2. We thus introduce

$$\begin{aligned}\tilde{V}_s(x) &= \sum_{n=1}^{\infty} v_n(s)\,x^{2n} = {}_3F_2\left(1, 1, \frac{3}{2}-s; 2, \frac{3}{2}; x^2\right)\frac{x^2}{2},\\ v_n(s) &= \frac{1}{4}\,\frac{\Gamma(n+\frac{1}{2}-s)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-s)\Gamma(n+\frac{1}{2})\,n},\end{aligned} \tag{18}$$

where ${}_3F_2$ is the generalized hypergeometric function. Formally, taking $s \to m + 1/2$ the infinite sum is automatically terminated at $n = m$, and the mth multicritical Kazakov potential is reproduced. For $s \neq m + 1/2$ the coefficients behave as $v_n(s) \sim n^{-s-1}$ for $n \to \infty$ and therefore $\tilde{V}_s(x)$ is a power series with radius of convergence equal to one.

Given (18) we find

$$x\tilde{V}_s'(x) = {}_2F_1\left(1, \frac{3}{2}-s, \frac{3}{2}, x^2\right)x^2 = {}_2F_1\left(1, s, \frac{3}{2}, \frac{x^2}{x^2-1}\right)\frac{x^2}{1-x^2} \tag{19}$$

and further, from (11):

$$U_s(A) = \frac{1-(1-A)^{s-1/2}}{4(s-1/2)}, \quad U_s'(A) = \frac{1}{4}(1-A)^{s-3/2}, \tag{20}$$

and

$$g_s(a^2) = \frac{1-(1-a^2)^{s-1/2}}{4(s-1/2)}, \quad g_s'(a^2) = \frac{1}{4}(1-a^2)^{s-3/2}, \quad g_s^* = \frac{1}{4(s-1/2)}, \tag{21}$$

which is the most obvious generalization of (14).

If we formally apply (9) we find for the potential (18)

$$gW(z) = \frac{1}{4}\int_0^{a^2} \mathrm{d}A\, \frac{(1-A)^{s-3/2}}{\sqrt{z^2-A}} \tag{22}$$

$$= \frac{{}_2F_1\left(1,\frac{3}{2}-s,\frac{3}{2},z^2\right)z - {}_2F_1\left(1,\frac{3}{2}-s,\frac{3}{2},\frac{z^2-a^2}{1-a^2}\right)(1-a^2)^{s-\frac{3}{2}}\sqrt{z^2-a^2}}{2} \quad (23)$$

$$= \frac{{}_2F_1\left(1,s,\frac{3}{2},\frac{z^2}{z^2-1}\right)z - {}_2F_1\left(1,s,\frac{3}{2},\frac{z^2-a^2}{z^2-1}\right)(1-a^2)^{s-\frac{1}{2}}\sqrt{z^2-a^2}}{2(1-z^2)} \quad (24)$$

$$= \frac{{}_2F_1\left(1,s,\frac{1}{2}+s,\frac{1}{1-z^2}\right)z - {}_2F_1\left(1,s,\frac{1}{2}+s,\frac{1-a^2}{1-z^2}\right)(1-a^2)^{s-\frac{1}{2}}\sqrt{z^2-a^2}}{4(s-\frac{1}{2})(z^2-1)} \quad (25)$$

where the relation between a and g is given by (21), i.e.

$$a^2 = 1 - \left(1 - \frac{g}{g_*}\right)^{\frac{1}{s-\frac{1}{2}}}, \quad g_* = \frac{1}{4(s-1/2)}. \quad (26)$$

All the representations of $W(z)$ given above have their virtues as we will now describe.

A standard representation of $gW(z)$ for an ordinary (even) polynomial $\tilde{V}(z)$ of degree $2n$ is

$$gW(z) = \frac{1}{2}\left[\tilde{V}'(z) - M(z^2-a^2)\sqrt{z^2-a^2}\right], \quad M(x) = \sum_{k=1}^{n} M_k\, x^{k-1}, \quad (27)$$

where $M(x)$ is a polynomial of degree $n-1$, uniquely fixed to cancel $\tilde{V}'(z)$ and to insure that $W(z) \to 1/z$ for $|z| \to \infty$. In our case $\tilde{V}'(z)$ will have a cut along the real axis starting at $z^2 = 1$ as is clear from (19). Correspondingly $M(z^2-a^2)$ should thus have a similar cut and (23) is simply the representation (27) and we have

$$\begin{aligned} &2gW(z) - \tilde{V}'(z) = -M(z^2-a^2)\sqrt{z^2-a^2}, \\ &M(x) = (1-a^2)^{s-\frac{3}{2}}\,{}_2F_1\left(1,\frac{3}{2}-s,\frac{3}{2},\frac{x}{1-a^2}\right), \end{aligned} \quad (28)$$

from which we can read off the coefficients M_k.

The representation (24) is useful because the hypergeometric functions are analytic along the cut $z \in [-a,a]$ of $W(z)$, $0 < a < 1$, and thus the discontinuity across the cut is entirely determined simply by the discontinuity of $\sqrt{z^2-a^2}$. From the very definition (4) of $W(z)$ it follows that the density of eigenvalues, $\rho(x)$, is determined by the discontinuity of $W(z)$ across the cut and we thus obtain:

$$\rho(x) = \frac{\lim_{\epsilon\to 0}(W(x+\mathrm{i}\epsilon) - W(x-\mathrm{i}\epsilon))}{2\pi \mathrm{i}} = \frac{(1-a^2)^{s-\frac{1}{2}}\sqrt{a^2-x^2}\,{}_2F_1\left(1,s,\frac{3}{2},\frac{a^2-x^2}{1-x^2}\right)}{2\pi g(1-x^2)}. \quad (29)$$

This $\rho(x)$ is plotted as $a^2 \to 1$ (or $g \to g_*$) in Fig. 1 for several values of s. Up to normalization these plots correspond to $(1-x^2)^{s-1}$ since we can rewrite $\rho(x)$ as

$$\rho(x) = \frac{\Gamma(s-\frac{1}{2})}{4\sqrt{\pi}\,g\,\Gamma(s)}(1-x^2)^{s-1} - \frac{(g_*-g)}{\pi g}\left[\frac{\sqrt{a^2-x^2}}{1-x^2}\,{}_2F_1\left(1,s,s+\frac{1}{2},\frac{1-a^2}{1-x^2}\right)\right] \quad (30)$$

where the part in brackets is bounded for all $a \in [x,1]$ for fixed $x^2 < 1$. Further, $\rho(x)$ is positive in $x \in]-a,a[$, vanishes at $x = \pm a$ and tends to the delta-function as $s \to \infty$.

Finally the representation (25) shows that $W(z)$ indeed has convergent power expansion in $1/z$ for $|z|$ sufficiently large and using (20) it follows that $W(z) \to 1/z$ for $|z| \to \infty$. For future

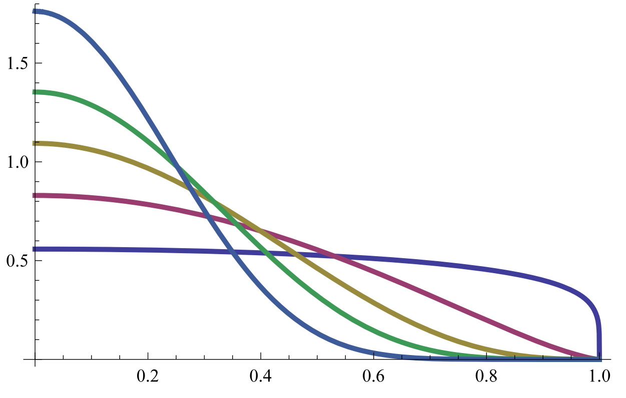

Fig. 1. Plot of $\rho(x)$ versus x for $s = 1.2, 2.4, 4, 6, 10$ from bottom to top.

reference we note that the transformation of the hypergeometric functions from (24) to (25) involves terms not seen in (25). More specifically one has

$$
{}_2F_1\left(1, s, \frac{3}{2}, \frac{z^2}{z^2-1}\right) = {}_2F_1\left(1, s, \frac{1}{2}+s, \frac{1}{1-z^2}\right) \frac{1}{2(\frac{1}{2}-s)} + \\
+\,\mathrm{i}\frac{(1-z^2)^s}{z} \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{2\Gamma(s)} \tag{31}
$$

but the last term on the rhs of eq. (31) cancels against an identical term coming from the other hypergeometric function in (25).

Let us end this section by calculating the susceptibility exponent γ_s associated with the matrix model with potential (18). We define the susceptibility as the second derivative of the free energy of the matrix model with respect to the coupling constant g:

$$
F = \frac{1}{N^2} \log Z, \qquad \chi = \left(g \frac{\mathrm{d}}{\mathrm{d}g}\right)^2 F \tag{32}
$$

and γ_s by

$$
\chi(g) = \chi_a(g) + c(g_* - g)^{-\gamma_s} + \text{less singular}, \tag{33}
$$

where $\chi_a(g)$ is analytic at g_*. Expanding $\mathrm{d}(gW(z))/\mathrm{d}g$ in inverse powers of z, any of the terms $c_n(g)/z^{2n+1}$, $n > 1$, will have $(g_* - g)^{-\gamma_s}$ as the leading non-analytic term. From (12) and (26) it follows immediately that the term is $(g_* - g)^{1/(s-1/2)}$ and therefore

$$
\gamma_s = -\frac{1}{s - \frac{1}{2}}. \tag{34}
$$

For $s \in \,]m - 1/2, m + 1/2[$ with m a positive integer our potential (18) has many of the characteristics of the $s = m + 1/2$ multicritical potential: the first m terms in the power series have oscillating signs, starting out always with $x^2/2$. The signs of terms x^{2n}, $n \geq m$ are the same. At the same time, moving s towards $m + 1/2$, γ_s changes continuously towards the value $-1/m$ of the mth multicritical model. The range $s \in \,]1/2, 3/2]$ is special. It starts out with $s = 3/2$, i.e. $m = 1$ and thus $\tilde{V}(x) = x^2/2$, i.e. a trivial Gaussian potential and we have

$$gW(z) = \frac{1}{2}(z - \sqrt{z^2 - a^2}), \quad g = \frac{1}{4}a^2. \tag{35}$$

a^2 is an analytic function of g, in accordance with the value $\gamma_s = -1$. For $1/2 < s < 3/2$ all coefficients in the power series expansion of $\tilde{V}(x)$ are positive and the derivative $g'(a^2 = 1)$ is infinite rather than zero as for $s > 3/2$. For $s \to 1/2$, $g_* \to \infty$ while $\gamma_s \to -\infty$.

For the mth multicritical potential it is well known that $\gamma = -1/m$ does not correspond to the KPZ area susceptibility exponent γ_A [7]. Rather, it is related to insertions of the primary operator with the most negative scaling dimension, which in non-unitary conformal theories coupled to 2d gravity need not be the cosmological constant. In the multicritical models one obtains the KPZ exponent by identifying the cosmological constant via the length of the boundary of the disk. One thus looks at

$$\langle W(2\ell)\rangle := \frac{1}{N}\left\langle \mathrm{tr}\, M^{2\ell}\right\rangle = 2\int_0^a \mathrm{d}x\, \rho(x)\, x^{2\ell} \to a^{2l}\left[\frac{(1-a^2)^{s-\frac{3}{2}}a^2\Gamma(\ell+\frac{1}{2})}{4\sqrt{\pi}\, g(a^2)\Gamma(\ell+2)}\right] \tag{36}$$

where the average $\langle\cdot\rangle$ is with respect to the partition function (2). We are interested in the limit $\ell \to \infty$ where the integral will be dominated by x close to the boundary a. One obtains the leading ℓ behavior

$$\langle W(2\ell)\rangle \sim \exp(2\ell \log a + O(\log \ell)) = \exp\left(-\left(1-\frac{g}{g_*}\right)^{\frac{1}{s-1/2}}\ell + O(\log \ell)\right), \tag{37}$$

where we have used (26). Thus we identify the dimensionless boundary cosmological constant μ_B and we introduce the dimensionless bulk cosmological constant $\mu \sim \mu_B^2$ as follows

$$\mu_B \sim \left(1-\frac{g}{g_*}\right)^{\frac{1}{s-1/2}} = \left(1-\frac{g}{g_*}\right)^{-\gamma_s}, \qquad \mu \sim \left(1-\frac{g}{g_*}\right)^{-2\gamma_s}. \tag{38}$$

From the definition (32) we have

$$F(g)\Big|_{\text{singular}} \sim (g_* - g)^{2-\gamma_s} \sim \mu^{2-\gamma_A} \tag{39}$$

and we conclude that

$$\gamma_A = \frac{3}{2} + \frac{1}{\gamma_s} = 2 - s. \tag{40}$$

If we assume that γ_A is related to an underlying conformal field theory coupled to 2d quantum gravity, as is the case for the multicritical points where $s = m + 1/2$, we have from the standard KPZ relation that the central charge of the matter fields related to s is

$$c(s) = 1 - 6\frac{\gamma_A^2}{\gamma_A - 1} = 1 - 6\frac{(s-2)^2}{s-1}. \tag{41}$$

The same $c(s)$ corresponds to two different γ_A's, related by

$$\gamma_A \to \gamma_A' = -\frac{\gamma_A}{1-\gamma_A}, \quad \text{i.e.} \quad s \to s' = \frac{s}{s-1}. \tag{42}$$

The two γ_A's correspond to the two different solutions to the KPZ relation (41). Usually the conformal field theory associated with a given central charge c is assigned a $\gamma(c)$ from the branch where $\gamma(c) \to -\infty$ for $c \to -\infty$. However, the other branch also has an interpretation in terms of random surfaces and 2d quantum gravity [17–19].

If we follow the above conjectures we are led to the following picture: $s = 2$ corresponds to $c = 1$ ($\gamma_A = 0$) where the two branches meet. The region $s \in]2, \infty[$ corresponds to the "physical" branch of the KPZ equation where γ_A changes from 0 to $-\infty$. The other branch corresponds to $s' \in]1, 2[$ and $\gamma'_A > 0$, approaching 1 for $s' \to 1^+$. An interesting example is $s' = 3/2$ considered above. Formally it corresponds to the $m = 1$ "multicritical" matrix model which is just the Gaussian matrix model with $W(z)$ given by (35). In KPZ context it can be viewed as the (2,1) conformal field theory coupled to 2d gravity in the series of $(2, 2m - 1)$ conformal field theories corresponding to the multicritical models, although it, contrary to the larger m theories, is not a standard *minimal* conformal field theory. The KPZ assignment of central charge to this theory is $c = -2$ and the corresponding $\gamma_A = -1$. In fact we found $\gamma_s = -1$ above, but according to (40) the corresponding $\gamma'_A = 1/2$, in agreement with the fact that $W(z)$ in (35) *is* the partition function for branched polymers which is known to have $\gamma = 1/2$. That branched polymers play an important role in the interpretation of γ'_A is the essence of the work [17–19]. It also follows from (42) that $s' = 3/2 \to s = 3$ and $s = 3$ indeed gives $c = -2$ and $\gamma_A = -1$. In Sec. 7 we will see it is possible to give a combinatorial explanation of the relation between s and s' which is in agreement with the picture developed in [17–19].

Clearly $s = 1$ is special, being the limit where the assumed central charge $c(s) \to -\infty$ and $\gamma_{A'} \to 1$. The potential (18) is in this case

$$\tilde{V}'_{s=1}(x) = \log\left(\frac{1+x}{1-x}\right) = 2\,\mathrm{arctanh}\,x, \tag{43}$$

and the corresponding disk function from (25)

$$W(z) = \frac{\mathrm{arcsinh}\sqrt{\frac{1}{z^2-1}} - \mathrm{arctanh}\sqrt{\frac{1-a^2}{z^2-a^2}}}{1-\sqrt{1-a^2}}. \tag{44}$$

It is interesting that all potentials corresponding to integer $s > 1$, i.e. non-negative integer γ_A, are simple modifications of (43). Similarly the corresponding Ws are simple modifications of (44). These statements follow from Gauss' recursion relations for hypergeometic functions.

4. The Riemann–Hilbert method at work

Above we assumed that one can use the standard large N one-matrix model formula to obtain the disk function. Let us briefly discuss why the formula is still valid in certain cases where $V'(x)$ has cuts and poles at the real axis. It represents a simple generalization of the usual case of the one-matrix model with polynomial $V'(x)$ which can still be treated by the Riemann–Hilbert method.

The large N saddelpoint of the matrix model is the principle value integral

$$V'(x) = 2 \fint \mathrm{d}y \, \frac{\rho(y)}{x-y}, \tag{45}$$

which is valid when x belongs to the support of the eigenvalue density ρ which is assumed to avoid possible cuts and poles of V'. We proceed in the usual way by introducing the analytic function

$$W(z) = \int \mathrm{d}y \, \frac{\rho(y)}{z-y}, \tag{46}$$

and rewriting eq. (45) at the real axis as

$$\Im\left(W^2 - V'W\right) + \Im V' \Re W = 0. \tag{47}$$

Usually, the term with $\Im V'$ is missing since V' is real at the real axis, but we now have to include it since V' can have cuts located on the real axis.

Equation (47) on the real axis implies the following equation in the whole complex plane:

$$W^2(z) - V'(z)W(z) + \int_{C_2} \frac{d\omega}{2\pi i} \frac{V'(\omega)W(\omega)}{(z-\omega)} = Q(z), \tag{48}$$

where the contour C_2 encircles possible cuts and poles of $V'(\omega)$ on the real axis, but not z and not the cut(s) of $W(\omega)$. $Q(z)$ is an entire function (a polynomial if V' is itself a polynomial) and its role is to compensate nonnegative powers of z in the product $V'(z)W(z)$. The third term on the left-hand side of eq. (48) plays thus no role in determining $Q(z)$.

We can rewrite eq. (48) as

$$W^2(z) - \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)W(\omega)}{(z-\omega)} = 0, \tag{49}$$

where the contour C_1 encircles (anti-clockwise) the cut(s) of $W(\omega)$, but not z and possible cuts and poles of $V'(\omega)$. We can prove the equivalence of Eqs. (48) and (49) by deforming the contour C_1 in eq. (49) to C_2, which will give the third term on the left-hand side of eq. (48). We get in addition the residual at $\omega = z$, which accounts for the second term on the left-hand side of eq. (48), and finally we get the contribution from $\omega = \infty$, which is equal $Q(z)$.

Equation (49) is the usual loop equation of the one-matrix model at $N = \infty$ with the potential $\operatorname{tr} V(M)$. Its standard derivation by an infinitesimal shift of M apparently works for all potentials, including the ones with cuts on the real axis. Correspondingly, eq. (49) results in the usual formula for the one-cut solution

$$W(z) = \int_a^b \frac{dx}{2\pi} \frac{V'(x)}{(z-x)} \frac{\sqrt{(z-a)(z-b)}}{\sqrt{(x-a)(b-x)}}, \tag{50}$$

where the cut is from a to b. For an even potential $V(x) = V(-x)$, when the cut is from $-a$ to $+a$, it simplifies to (5). The values of a and b are determined from the condition $W(z) \to 1$ as $z \to \infty$, which for an even potential reduces to (6). Explicit formulas for a simplest non-even logarithmic potential are presented in Appendix A.

5. Universality

Let us recall the universality situation when the potential $V(x)$ is (an even) polynomial. Using a Wilsonian wording we have an infinite dimensional space of coupling constants, the coefficients in all polynomials $V(x)$ and the mth critical surface is characterized by the condition (13). It has finite co-dimension $m-1$ and one can approach the surface such that $m-1$ parameters survive in the "continuum" limit (see [20] for a review). The Kazakov potential (17) is a particular simple choice of polynomial which only depends on one parameter, g. We would like to understand the universality situation for the new critical points defined by the generalized Kazakov potentials $V_s(x) = \frac{1}{g}\tilde{V}_s(x)$.

Clearly the new critical behavior is related to the tail $v_n \sim n^{-1-s}$ in $\tilde{V}(x)$. Let us choose another potential with the same tail but depending on two parameters, g and c, rather than the single g in $V_s(x)$,

$$\hat{V}(x) = \frac{1}{g}\left[\frac{x^2}{2}(1+c) - \frac{c}{2}\mathrm{Li}_{1+s}\left(x^2\right)\right] = \frac{1}{g}\left[\frac{x^2}{2} - \frac{c}{2}\sum_{n=2}^{\infty}\frac{x^{2n}}{n^{1+s}}\right], \tag{51}$$

where Li_{1+s} is the polylogarithm. This potential is rather general. In particular, we can get a quartic potential from (51) in the limit $c \to \infty$, $g \sim 1/c$.

The boundary equation (8) now reads

$$g(a^2) = \frac{1}{4}\left[a^2(1+c) - cF_s(a^2)\right], \tag{52}$$

where the function $F_s(A)$ (trivially related to $\tilde{U}(A)$) is defined by

$$F_s(A) = \int_0^{\sqrt{A}} \frac{\mathrm{d}x}{\pi}\frac{\mathrm{Li}_s(x^2)}{\sqrt{A-x^2}} = \frac{2}{\Gamma(s)}\int_0^{\infty}\mathrm{d}\tau\,\tau^{s-1}\left(\frac{1}{\sqrt{1-A\,\mathrm{e}^{-\tau}}}-1\right). \tag{53}$$

It has the following expansion (see also (11))

$$F_s(A) = \sum_{n=1}^{\infty}\frac{2A^n}{B(\frac{1}{2},n)\,n^{s+1}} = A + \frac{3}{2^{2+s}}A^2 + \ldots. \tag{54}$$

Using the properties of $F_s(A)$ listed in Appendix B, one can analyze the function $g(a^2)$. It is an analytic function of a^2 for $0 \le a < 1$ and the behavior close to $a^2 = 1$ is as follows:

$$g(a^2) = f(1-a^2) - \frac{2c}{\sqrt{\pi}}(1-a^2)^{s-1/2}\Big(1 + O((1-a^2)\Big), \quad s > 3/2, \tag{55}$$

where $f(x)$ can be expanded to order $[s-1/2]$:

$$f(x) = g_s^* + g'(1)\,x + O(x^2), \quad g_s^* = \frac{1}{4}(1+c-cF_s(1)), \quad g'(1) = -\frac{1}{4}(1+c-cF_{s-1}(1)). \tag{56}$$

The function $g(a^2)$ starts out as an increasing function of a^2. By increasing a, eventually a might become a non-analytic function of g. This happens either at the first a where $g'(a) = 0$ or, if $g'(a) > 0$ for all a, at $a = 1$, the radius of convergence for $g(a^2)$. In the former case we have an $a_c < 1$ where $g'(a_c) = 0$ and a corresponding critical value of g, $g_c = g(a_c)$. For our choice of the potential (depending only on g, c) one can show that $g''(a) \neq 0$ for all values of $a < 1$.

In a neighborhood of a_c we can therefore write

$$g(a^2) = g(a_c^2) - k^2(a_c^2 - a^2)^2. \tag{57}$$

We thus conclude that the leading non-analytic behavior of a as a function of g is $(a_c^2 - a^2)^{1/2}$, i.e. we have the standard situation with $\gamma_s = -1/2$, corresponding to the $m = 2$ Kazakov potential. Whether or not this situation is realized depends on the value of c. We have

$$a^2\frac{\mathrm{d}}{\mathrm{d}a^2}g = \frac{1}{4}\left[a^2(1+c) - cF_{s-1}(a^2)\right], \tag{58}$$

and thus the following equation for the value $c_*(s)$ separating the two situations:

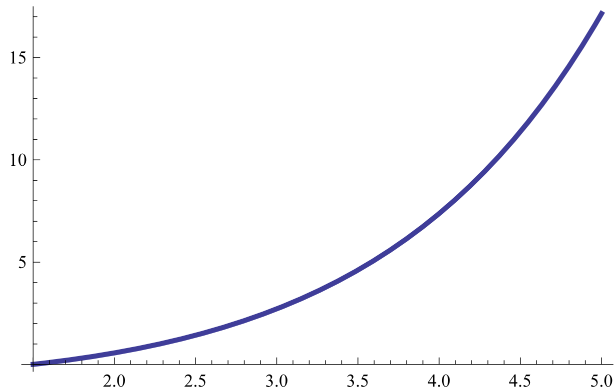

Fig. 2. Value of c_* versus s above which the usual 2d gravity scaling limit is realized for the polylog potential (51).

$$a^2 \frac{\mathrm{d}}{\mathrm{d}a^2} g\Big|_{a=1} = 0, \quad \text{i.e.} \quad c_*(s) = \frac{1}{F_{s-1}(1) - 1} \quad (\geq 0). \tag{59}$$

This $c_*(s)$ is positive for $s > 3/2$, because then $1 < F_{s-1}(1) < \infty$ and increases rapidly with s as is depicted in Fig. 2.

Let us first discuss the situation for $s \in]3/2, 5/2[$. For a given s in this interval and a given $c \leq c_*(s)$ the critical point is thus g_s^* corresponding to $a_c = 1$ and the relation between a and g close to a_c is determined by (55) and (56). For fixed $c < c_*(s)$ the analytic term from $f(1-a^2)$ will dominate over the non-analytic term $(1-a^2)^{s-1/2}$ and we have formally the situation corresponding to $\gamma = -1$. However, precisely for $c = c_*(s)$ this term will by definition vanish and we obtain from (55)

$$g(a^2) = g_s^* - \frac{2c_*(s)}{\sqrt{\pi}}(1-a^2)^{s-1/2}, \quad s \in]3/2, 5/2[, \tag{60}$$

i.e. precisely the same scaling relation as for the generalized Kazakov potential, and thus also $\gamma_s = 1/(1/2 - s)$. If $c > c_*(s)$ we have $\gamma_s = -1/2$, but for $c \to c_*(s)$ (60) will take over since the term non-analytic in $(1-a^2)$ will dominate over the contribution (57) when $a_c \to 1$. In the limit $s \to 5/2$ they will agree and give $\gamma_{5/2} = -1/2$.

If we consider $s \in]5/2, 7/2[$ we still have the same the curve $c_*(s)$, and results identical to those for $s \in]3/2, 5/2[$ if $c \neq c_*(s)$. For $c = c_*(s)$ the term in $f(1-a^2)$ linear in $(1-a^2)$ will still cancel, but the term proportional to $(1-a^2)^2$ will be dominant compared to $(1-a^2)^{s-1/2}$. Only if we can cancel the analytic $(1-a^2)^2$ term will we obtain a scaling like (60) also for $s \in]5/2, 7/2[$. To obtain such a cancellation we need one further adjustable coupling constant apart from g and c.

There are many ways to introduce such a coupling constant, but maybe the simplest is to add a term $v_2 x^4$ to the potential (51). With this new coupling constant at our disposal we can always find a point $a_c < 1$ such that $g'(a_c^2) = 0$. We can also try to find a point a_c where not only $g'(a_c^2) = 0$ but also $g''(a_c^2) = 0$, precisely as for the $m = 3$ multicritical matrix model. In Fig. 3 we show such a situation. Whether or not this is possible depends again on c and corresponding to eq. (59) one obtains

$$\left(a^2 \frac{\mathrm{d}}{\mathrm{d}a^2}\right)^n g\Big|_{a=1} = 0, \quad n = 1, 2, \quad \text{i.e.} \quad c_*(s) = \frac{1}{2F_{s-1}(1) - F_{s-2}(1) - 1} \quad (\leq 0) \tag{61}$$

and the corresponding value of $v_2(s)$ is

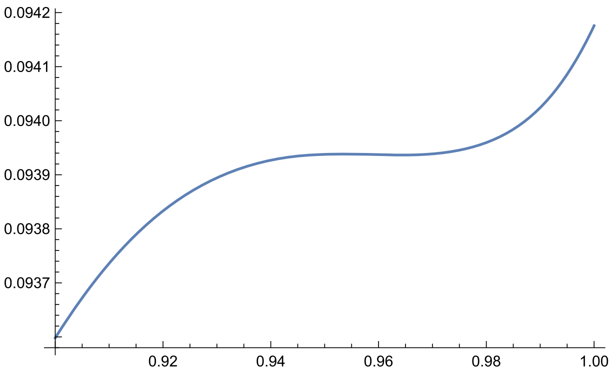

Fig. 3. Plot of $g(a^2)$ for $s=3$, $c=-3$ and $v_2=-0.3433$ where $g'(a_c)=g''(a_c)=0$ for an $a_c<1$.

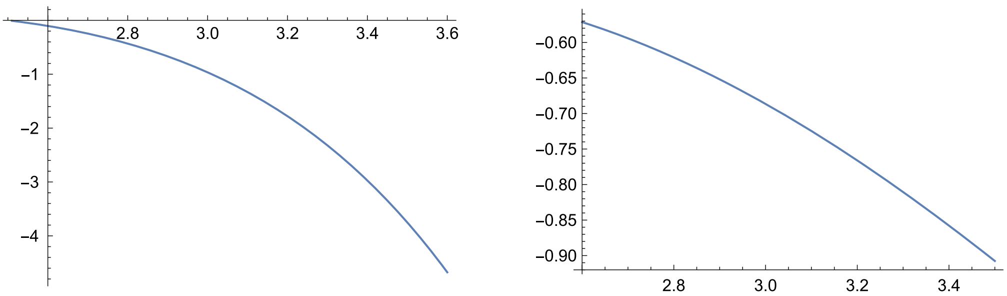

Fig. 4. The left figure plots $c_*(s)$ from (61) for $s>5/2$. $c_*(s)\to 0$ for $s\to 5/2$ since $F_{s-2}(1)\to\infty$ for $s\to 5/2$. The right figure plots $v_2(s)$ given by eq. (62).

$$v_2(s)=\frac{4}{3}u_2(s)=\frac{2c_*(s)}{3}(F_{s-2}(1)-F_{s-1}(1)) \quad (<0). \tag{62}$$

We show $c_*(s)$ and $v_2(s)$ in Fig. 4. Note that they are both negative. For $c<c_*(s)$ we can approach $c_*(s)$ by changing c while satisfying $g'(a_c)=g''(a_c)=0$, where $a_c(c)<1$ and $a_c(c)\to 1$ for $c\to c_*(s)$. The condition $g'(a_c)=g''(a_c)=0$ determines v_2 uniquely for fixed c. For $c>c_*(s)$ one can approach $c_*(s)$ in such a way that $g'(a_c)=0$. This does not fix v_2 and the corresponding a_c, but by demanding that $v_2\to v_2(s)$ given by (62) we have by construction that $a_c\to 1$ and $g''(a_c)\to 0$ for $c\to c_*(s)$. For $s\in\,]5/2,7/2[$ we thus have a situation completely analogous to $s\in\,]3/2,5/2[$, except that the multicriticality while approaching $c_*(s)$ has changed from $m=1$ and 2 to $m=2$ and 3. At $c_*(s)$ we have $\gamma_s=-1/(s-1/2)$ and the potential $\hat{V}(x)$ is qualitatively the same as the generalized Kazakov potential $V_s(x)$ in the same range of s.

The generalization to higher values of s is straight forward. For $s\in\,]m-1/2,m+1/2[$ we allow deformations of $\hat{V}$ involving $v_2,\ldots,v_{m-1}$. We can define a critical $c_*(s)$ and approach it from the two sides via $m-1$ and m critical points by changing c, and the potential $\hat{V}(x)$ at $c=c_*(s)$ will be qualitatively the same as $V_s(x)$. We have thus seen that the new scaling limits for $s>3/2$ are universal in the same way as the standard multicritical points of the one-matrix model which correspond to $s=m+1/2$.

Let us finally consider the region $s\in\,]1/2,3/2[$. For s in this region we have

$$g(a^2)=g_s^*-\frac{2c}{\sqrt{\pi}}(1-a^2)^{s-1/2}+O(1-a^2). \tag{63}$$

Thus $a_c = 1$ and $\gamma_s = -1/(s-1/2)$ and no fine tuning of c is needed (except if one insists on g_s^* positive one has to choose c negative). The range of γ_s is from -1 to $-\infty$, i.e. outside the range of the original Kazakov range of $\gamma(m) = -1/m$ with integer m.

6. The continuum limit

For the (even) matrix models the scaling limit is usually performed by the following assignment

$$a^2 \to a_c^2 - \sqrt{\Lambda}\epsilon, \quad z^2 = a_c^2 + P\epsilon. \tag{64}$$

In our case $a_c = 1$. For most of the "observables" considered for matrix models, this scaling is straight forward and unproblematic. As examples we have for the disk amplitude with one puncture, $\mathrm{d}(gW(z))/\mathrm{d}g$, that

$$\frac{\mathrm{d}\, gW(z)}{\mathrm{d}g} = \frac{1}{\sqrt{z^2 - a^2}} \to \frac{1}{\sqrt{\epsilon}} \frac{1}{\sqrt{P + \sqrt{\Lambda}}} \tag{65}$$

and for the universal two-loop function (which can be derived for our more general potentials precisely as for the ordinary polynomial potentials [21,22]):

$$W(z_1, z_2) = \frac{a^4}{2(z_2\sqrt{z_1^2 - a^2} + z_1\sqrt{z_2^2 - a^2})^2} \frac{1}{\sqrt{z_1^2 - a^2}\sqrt{z_2^2 - a^2}} \tag{66}$$

$$\to \frac{1}{\epsilon^2} \frac{1}{\left(\sqrt{P_1 + \sqrt{\Lambda}} + \sqrt{P_2 + \sqrt{\Lambda}}\right)^2} \frac{1}{\sqrt{P_1 + \sqrt{\Lambda}}\sqrt{P_2 + \sqrt{\Lambda}}}. \tag{67}$$

The same is true for any higher loop functions. Approaching the mth multicritical point for ordinary matrix models one obtains

$$W(z_1, \dots, z_b) \to \frac{1}{\epsilon^{(b-2)m + \frac{3}{2}b - 1}} W^{\mathrm{cont}}(P_1, \dots, P_b; \Lambda), \quad b > 2, \tag{68}$$

where $W^{\mathrm{cont}}(P_1, \dots, P_b; \Lambda)$ denotes the *continuum* b-loop function.[2] The one (natural) difference in our more general case will be that in the divergent pre-factor m is replaced by $s - 1/2$.

The so-called continuum limit of the disk amplitude requires a more detailed discussion since it contains a non-scaling part. If we use the representation (27) the potential term $\tilde{V}'(z)$ will not scale when using the prescription (64). On the other hand the rest of the expression will scale, as is clear from (27) for a polynomial potential and from (28) for the generalized Kazakov potential. However the rhs of (28) does not fall off as a function of the continuum P for $|P| \to \infty$ the way one requires for the continuum disk-amplitude $W(P)$. One cures this by introducing a "continuum" potential $V_{\mathrm{cont}}(P)$ which is determined by the requirement[3] that $W_{\mathrm{cont}}(P)$ has a power expansion in $P^{-n-\frac{1}{2}}$, $n \geq 0$, for $P \to \infty$. We thus write[4]

---

[2] For the continuum two-loop function defined by eq. (66) one often makes a subtraction which is irrelevant for our discussion, see e.g. [20].

[3] It is often required that the power series of $W_{\mathrm{cont}}(P)$ starts with the term $P^{-3/2}$, i.e. one includes the first term $1/\sqrt{P}$ in $V'_{\mathrm{cont}}(P)$.

[4] Sometimes a factor of 2 is inserted on the rhs of this formula to emphasize a doubling of continuum degrees of freedom for an even potential owing to the symmetry $z \to -z$.

$$\left(gW(z)-\frac{\tilde{V}'(z)}{2}\right)=\epsilon^{s-1}\left(W_{\rm cont}(P)-\frac{\tilde{V}'_{\rm cont}(P)}{2}\right). \tag{69}$$

That the scaling factor is ϵ^{s-1} follows immediately from (28). When $s=m+1/2$ it reduces to the ordinary scaling factor for the ordinary mth multicritical matrix model. Equations (25), (28) and (31) and the remarks surrounding (31) allow us immediately to substitute the continuum limit (64) and we obtain

$$W_{\rm cont}(P)=-g_s^*\,(\sqrt{\Lambda})^{s-\frac{1}{2}}\,\frac{\sqrt{P+\sqrt{\Lambda}}}{P}\,{}_2F_1\Big(1,s,\frac{1}{2}+s;-\frac{\sqrt{\Lambda}}{P}\Big), \tag{70}$$

$$\tilde{V}'_{\rm cont}(P)=\mathrm{i}\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{2\Gamma(s)}\,(-P)^{s-1}. \tag{71}$$

This $W_{\rm cont}(P)$ can indeed be expanded in powers $1/P^{n+\frac{1}{2}}$ and the series is absolutely convergent for $|P|>\sqrt{\Lambda}$ and from the integral representation of hypergeometric functions it follows that it is analytic for positive P. It has a cut for negative P starting at $P=-\sqrt{\Lambda}$, coming from $\sqrt{P+\sqrt{\Lambda}}$. Like for the ordinary matrix models, this cut is the scaled version of the original cut $[-a,a]$ in z. The potential $\tilde{V}'_{\rm cont}(P)$ in (71) has a cut along the positive P axis. This is the scaled version of the original cut of $\tilde{V}'(z)$ starting at $z=1$.

If $s=m+1/2$ it is instructive to rederive the standard continuum results for the mth model directly from (28). Using (28) we obtain

$$\left(gW(z)-\frac{\tilde{V}'(z)}{2}\right)=-\frac{1}{2}\epsilon^{m-1/2}\Big[(\sqrt{\Lambda})^{m-1}{}_2F_1\Big(1,1-m,\frac{3}{2};1+\frac{P}{\sqrt{\Lambda}}\Big)\Big]\sqrt{P+\sqrt{\Lambda}}, \tag{72}$$

where the expression part in square brackets is a polynomial in P of order $m-1$, which can be written as

$$-g^*P^{m-1}\sum_{k=0}^{m-1}(-1)^{m-k}\frac{c_k}{c_m}\Big(\frac{\sqrt{\Lambda}}{P}\Big)^k,\qquad c_k=\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k+1)}, \tag{73}$$

c_k being the coefficients in the Taylor expansion of $1/\sqrt{1-x}$. This implies that except for $P^{m-1/2}$ all positive powers of P will cancel on the rhs of eq. (72) and we obtain

$$\left(gW(z)-\frac{\tilde{V}'(z)}{2}\right)=\frac{(-1)^m\Gamma(\frac{1}{2})\Gamma(m)}{4\Gamma(m+\frac{1}{2})}P^{m-\frac{1}{2}}-g^*\frac{\Lambda^{m/2}}{\sqrt{P}}+O(P^{-3/2}), \tag{74}$$

i.e. precisely the representation (69)–(71).

Let us briefly discuss the perturbation away from one of the generalized multicritical points. One convenient way to characterize the deformation away from the ordinary mth multicritical point is to use the so-called moments M_k [21–23]. They are defined by[5]

$$M_k(a^2,v_n)=\frac{2}{k!c_k}\left(\frac{\partial}{\partial a^2}\right)^k\tilde{U}(a^2,v_n). \tag{75}$$

[5] An equivalent definition is

$$M_k=\oint_C\frac{\mathrm{d}x}{2\pi i}\,\frac{x\tilde{V}(x)}{(x^2-a^2)^{k+1/2}},$$

where the contour C encircles to cut of $W(z)$ but not any poles or cuts of $V'(x)$.

In (75) we view $\tilde{U}$ and M_k as functions of a^2 and the coupling constants v_n. For a given choice of coupling constants v_n and g the position or the cut, i.e. the determination of a as a function of v_n and g will then finally be determined by (8). The coupling constants v_n^c and g_* correspond to an mth multicritical point if the corresponding value $a = a_c$ is such that $M_k(a_c, v_n^c) = 0$, $k = 1, \ldots, m-1$, $M_m(a_c, v_n^c) \neq 0$. In the case of the Kazakov potential we have chosen a particular simple way to move away from the critical point, namely by keeping the $v_n = v_n^c$ and only changing g and that case we had explicitly

$$M_k(a^2) \propto (1-a^2)^{m-k}, \quad 0 < k \leq m, \quad M_k = 0, \quad k > m. \tag{76}$$

For the generalized Kazakov potential this is changed to

$$M_k(a^2) \propto (1-a^2)^{s-\frac{1}{2}-k}, \quad k > 0, \tag{77}$$

the difference being that now infinitely many moments are different from zero.

For the mth multicritical model a general deformation away from the multicritical point could be described as a change of coupling constants away from the critical values such that

$$M_k = \mu_k \epsilon^{m-k}, \quad 1 \leq k \leq m, \quad a^2 = a_c^2 - \sqrt{\Lambda}\epsilon, \tag{78}$$

where μ_k and $\sqrt{\Lambda}$ are kept fixed when the coupling constants change towards their critical values. As shown in [23] all multiloop functions can in the continuum limit be expressed as functions of μ_k's, $\sqrt{\Lambda}$ and the variables $P_1, \ldots, P_b$. The obvious generalization to a deformation around the generalized Kazakov potential is to assume that

$$M_k = \mu_k \epsilon^{s-1/2-k}, \quad 1 \leq k < \infty, \quad a^2 = a_c^2 - \sqrt{\Lambda}\epsilon \tag{79}$$

and that the μ_k's and Λ are kept fixed when the coupling constants flow towards their critical values. With such a behavior all formulas for multiloop functions derived for the deformation around an arbitrary mth model will remain valid of any choice of s. For an arbitrary $s > 1/2$ it is possible to define so-called continuous times T_k, related to the μ_k's, and to study the so-called *KdV* flow equations in terms of the T_k's. Finally we should mention that the loop equations connecting loop functions of different genera remain valid for the general potentials considered here. When taking the formal double scaling limit one keeps $N\epsilon^s$ fixed. This is clearly the natural generalization of keeping $N\epsilon^{m+1/2}$ fixed for the mth multicritical model. Details of this will appear in a forthcoming paper [24].

7. Combinatorial interpretation

To better understand the duality $s \to \frac{s}{s-1}$ discussed in Sec. 3 let us have a look at the combinatorial interpretation of the matrix model in terms of planar maps, i.e. graphs embedded in the plane modulo orientation-preserving homeomorphisms. The boundary of a planar map m is the contour of its "outer face", and we assume that m has a distinguished oriented edge on the boundary, which is called the root edge. We denote by $\mathcal{M}^{(l)}$, $l \geq 1$, the set of all such rooted planar maps that are bipartite, i.e. having all faces of even degree, and have boundary length $2l$. By convention we let $\mathcal{M}^{(0)}$ contain a single map consisting of just a vertex. If we write $q_n := \delta_{n,1} - 2n\, v_n$ for $n \geq 1$ then the disk amplitude $W(z)$ for $z^2 \geq a^2(g)$ and $g \leq g_*$ can be expressed as the convergent sum

$$W(z) = \sum_{l=0}^{\infty} z^{-2l-1} \sum_{m \in \mathcal{M}^{(l)}} g^{\#\mathrm{Vertices}(m)-1} \prod_{f \in \mathrm{Faces}(m)} q_{\deg(f)/2}. \tag{80}$$

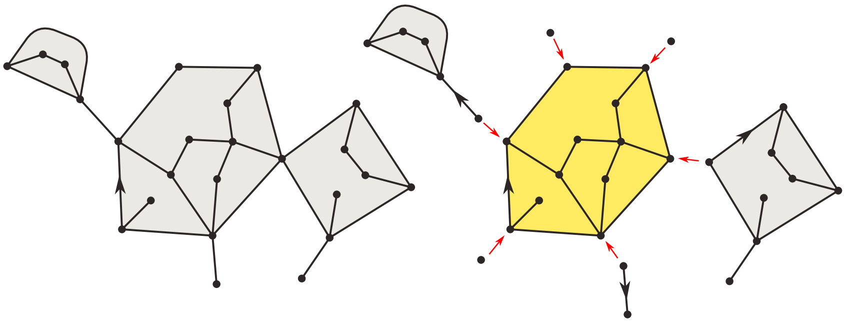

Fig. 5. A general rooted planar map with non-simple boundary (left) can be obtained uniquely by gluing rooted planar maps to each boundary vertex of a planar map with simple boundary (right).

One should notice that $\mathcal{M}^{(l)}$ contains planar maps with a boundary of the most general "non-simple" form, meaning that it may have pinch points in the sense that vertices appear multiple times in the boundary contour (see Fig. 5 for an example). As we will see shortly, if $s \le 2$ dropping the contribution of planar maps with non-simple boundaries from (80) has a non-trivial effect on the scaling properties of the disk amplitude.

We denote by $\hat{\mathcal{M}}^{(l)} \subset \mathcal{M}^{(l)}$ the planar maps with a "simple" boundary, meaning that all vertices in the boundary contour are unique, and define the simple disk amplitude $\hat{W}(x)$ for x^2 sufficiently small by

$$\hat{W}(x) := \sum_{l=0}^{\infty} x^{2l} \sum_{m\in\hat{\mathcal{M}}^{(l)}} g^{\#\mathrm{Vertices}(m)} \prod_{f\in\mathrm{Faces}(m)} q_{\deg(f)/2}. \tag{81}$$

From the dual point of view $W(z)$ and $\hat{W}(x)$ can be interpreted respectively as the disconnected and connected planar Green functions, and it has long been recognized that they satisfy a simple relation [15,16]. Indeed, since a planar map m with non-simple boundary contains a unique submap with simple boundary sharing the same root edge (see Fig. 5), one easily observes that

$$W(z) = \frac{1}{gz}\sum_{l=0}^{\infty}(W(z))^{2l} \sum_{m\in\hat{\mathcal{M}}^{(l)}} g^{\#\mathrm{Vertices}(m)} \prod_{f\in\mathrm{Faces}(m)} q_{\deg(f)/2} = \frac{1}{gz}\hat{W}(W(z)). \tag{82}$$

This implies that

$$\hat{W}(x) = gx\, W^{-1}(x) \qquad \text{when } |x| \le W(a(g)), \tag{83}$$

where $W^{-1}(\cdot)$ is the functional inverse of $z \to W(z)$. Notice that the position of the cut in this simple disk amplitude is now determined by $W(a)$ which when $a \to 1$ scales as

$$gW(a(g)) = {}_2F_1(1, 3/2-s, 3/2, a^2)\frac{a}{2} \tag{84}$$

$$= \frac{1}{4(s-1)} - \frac{1-a^2}{2(2-s)} + \frac{\sqrt{\pi}\,\Gamma(1-s)}{4\Gamma(3/2-s)}(1-a^2)^{s-1} + \dots \tag{85}$$

$$= \frac{1}{4(s-1)} - \frac{(1-g^2/g_*^2)^{\frac{1}{s-1/2}}}{2(2-s)} + \frac{\sqrt{\pi}\,\Gamma(1-s)}{4\Gamma(3/2-s)}(1-g^2/g_*^2)^{\frac{s-1}{s-1/2}} + \dots \tag{86}$$

Which of the two last terms dominates depends on whether $s > 2$ or $s < 2$ (we will not discuss integer s). In particular, if one identifies the "simple" boundary cosmological constant $\hat{\mu}_B$ in analogy with the discussion above (38) one obtains

$$\hat{\mu}_B \sim \begin{cases} (1-g/g_*)^{\frac{1}{s-1/2}} & \text{for } s > 2 \\ (1-g/g_*)^{\frac{s-1}{s-1/2}} & \text{for } s < 2. \end{cases} \tag{87}$$

Defining a corresponding bulk cosmological constant $\hat{\mu} \sim \hat{\mu}_B^2$ and requiring $F(g)\Big|_{\text{singular}} \sim \hat{\mu}^{2-\hat{\gamma}_A}$, one gets exactly

$$\hat{\gamma}_A = \begin{cases} 2-s & \text{for } s > 2 \\ (s-2)/(s-1) & \text{for } s < 2, \end{cases} \tag{88}$$

which is invariant under $s \to s/(s-1)$ and corresponds to the "right" branch of (41).

Let us now have a look at the continuum limit of the simple disk amplitude using (69). Based on (85) one expects that one should scale $x^2 \to x_c^2(1 - X\epsilon^\beta)$ with $x_c = (s-1/2)/(s-1)$ and $\beta = 1$ for $s > 2$ and $\beta = s-1$ for $s < 2$, in addition to $a^2 \to 1 - \sqrt{\Lambda}\epsilon$. If we denote the leading order of $W^{-1}(x)$ in ϵ by $W^{-1}(x) \sim 1 + P\epsilon/2$ then for $P > 0$

$$x = W(1 + P\epsilon/2) = x_c - \epsilon P\frac{s-1/2}{2s-4} + \text{analytic} + \epsilon^{s-1} W_\Lambda(P) + \dots \tag{89}$$

with[6]

$$W_\Lambda(P) := \frac{\Gamma(1-s)\Gamma(s+1/2)}{\sqrt{\pi}} P^{s-1} - (s-\frac{1}{2})\sqrt{\Lambda}^{s-1/2}\frac{\sqrt{P+\sqrt{\Lambda}}}{P}\,{}_2F_1\Big(1, s; s+\frac{1}{2}; -\frac{\sqrt{\Lambda}}{P}\Big) \tag{90}$$

It follows that for $s > 2$ we have $P = (s-2)/(s-1)X + \epsilon^{s-2}\frac{2s-4}{s-1/2}W_\Lambda\left(\frac{s-2}{s-1}X\right) + \dots$ and therefore

$$\frac{1}{gx}\hat{W}(x) = 1 + \text{analytic} + \epsilon^{s-1}\frac{s-2}{s-1/2}W_\Lambda\left(\frac{s-2}{s-1}X\right) + \dots, \tag{91}$$

which has the same form (up to rescaling) as the continuum limit of the non-simple disk function $W(z)$. On the other hand, when $s < 2$ one may check that $W_\Lambda(P)$ is monotonically decreasing on $P \in [-\sqrt{\Lambda}, \infty[$ and therefore we identify $P = W_\Lambda^{-1}(-x_c X/2) + \mathcal{O}(\epsilon^{2-s})$. This implies that

$$\frac{1}{gx}\hat{W}(x) = 1 + \epsilon W_\Lambda^{-1}(-x_c X/2) + \mathcal{O}(\epsilon^{3-s}). \tag{92}$$

Since we took $x^2 \to x_c^2(1 - X\epsilon^{s-1})$, the linear term in (92) is in fact the dominant singular part and we conclude that it is really the functional inverse of (90) that provides the continuum limit of the simple disk amplitude.

---

[6] $W_\Lambda(P)$ differs slightly from both $W_{\text{cont}}(P)$ and $W_{\text{cont}}(P) - \tilde{V}_{\text{cont}}(P)/2$ appearing on the rhs of (69) because (1) it is defined without the factor g multiplying $W(z)$ on the lhs of (69) and (2) it is defined as the part of $W(z)$ which scales as ϵ^{s-1}. When making the substitution $z = 1 + \epsilon P$ in $\tilde{V}(z)$ in the lhs of (69) we obtain such a term which together with the appropriately normalized rhs of (69) constitute $W_\Lambda(P)$.

8. Relation to the multicritical $O(n)$ loop models

Above we have argued that at criticality the generalized Kazakov model with parameter s allows a continuum interpretation as a conformal field theory with central charge $c(s)$ given by (41), for $s \in]1, \infty[$ and coupled to 2d quantum gravity. We have also argued that by universality one would expect $c(s)$ to be the same for all potentials $V_s(x)$ where the coefficients $v_n \sim 1/n^{s+1}$ for large n and were one can adjust at least $m-1$ of the coefficients v_n, $m - 1/2 < s < m+1$, such that $g(a)$ behaves like in the case of the generalized Kazakov model. It is of interest to check this behavior in an explicit model where matter with central charge $c(s)$ is coupled to random graphs. This can indeed be done for $s \in]3/2, 5/2[$. If we define $n = 2\cos \pi s$ it is known that the corresponding $O(n)$ model (which can formally be defined on a regular lattice for non-integer n via the strong coupling expansion), at criticality corresponds to a conformal field theory with central charge $c(s)$. Now, defining the $O(n)$ model on an suitable ensemble of planar random graphs, we have a model which at criticality should correspond to a conformal field theory with central charge $c(s)$ coupled to 2d quantum gravity, in the same way as the Ising model on planar random graphs at criticality corresponds to a conformal field theory with central charge $1/2$ coupled to 2d quantum gravity. On the other hand it has been observed in [12,13] that at criticality there exists a natural relation between $O(n)$ models on random planar graphs and random planar maps with non-trivial weights on the faces. These random planar maps with non-trivial weights are described by one-matrix models with potentials $V_s(x)$ where the coefficients v_n for large n behave as $1/n^{s+1}$ and where all weights given to the faces are positive. Thus, for a given $s \in]3/2, 5/2[$ we would expect such a potential to be in the same universality class as the generalized Kazakov potential with the same s. Let us briefly describe this connection between $O(n)$ models on random graphs and random planar maps with additional weights on the faces.

The $O(n)$ matrix model for positive integer n is defined by

$$Z = \int dM \prod_{i=1}^{n} d\Phi_i \exp\left(-N\mathrm{tr}\left[\bar{V}(M) + \frac{1}{2}\sum_{i=1}^{n} \Phi_i^2 - \frac{1}{2z_*}\sum_{i=1}^{n} \Phi_i^2 M\right]\right), \tag{93}$$

where $\bar{V}(M) = \frac{1}{2}M^2 - \sum_{k=2}^{m+1} \frac{1}{k}\bar{q}_k M^k$ is some polynomial potential and z_* is an independent coupling constant (the definition of the constants $\bar{q}_k$ in $\bar{V}$ differs from the v_k used elsewhere, like in (11), by a factor k). We first assume that the couplings $\bar{q}_k$ are positive, such that the matrix model in the large N limit generates loop-decorated planar graphs constructed from faces up to order m, as shown in Fig. 6 (left). Each such loop-decorated planar map comes with a weight

$$n^{\#\text{loops}}\,(2z_*)^{-\#\text{loop-decorated faces}} \prod_{\text{non-loop faces } f} \bar{q}_{\deg(f)}, \tag{94}$$

which makes sense for any real positive value of n. For $s \in]3/2, 5/2[$ one may tune the parameter z_* and some of the $\bar{q}_k$'s such that the singular part of the disk amplitude $\bar{W}(z)$ takes the form [11]

$$\bar{W}(z)|_{\text{sing.}} \sim (z - z_*)^{s-1}. \tag{95}$$

One can define a one-matrix model which reproduces the weight (94) [13]. For this purpose it is convenient to introduce the *gasket* $\mathcal{G}(m)$ of a loop-decorated planar map m with a boundary to be the planar map obtained by removing all faces intersected or surrounded by loops (see Fig. 6). Given a planar map m' with boundary (and no loops), one may ask for the total weight in the sense of (94) of all loop-decorated planar maps m that have $m' = \mathcal{G}(m)$ as their gasket. Since

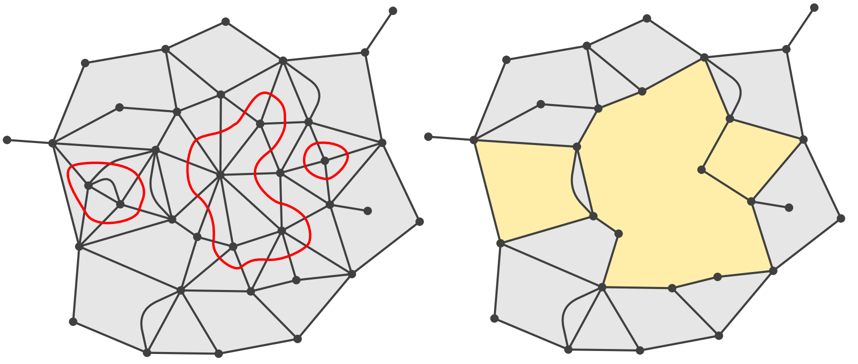

Fig. 6. An example of a loop-decorated planar map with a boundary (left) and its gasket (right).

each face of m' corresponds to either a face or a loop of m, we easily find that this total weight factorizes as $\prod_{f \in \mathrm{Faces}(m')} q_{\deg(f)}$ where q_k is given by

$$q_k := \bar{q}_k + n \sum_{l=0}^{\infty} \binom{l+k}{l} (2z_*)^{-l-k} \bar{W}^{(l)}. \tag{96}$$

This is precisely the weight associated to m' in a one-matrix model with "effective" potential

$$V'_{\mathrm{eff}}(z) := z - \sum_{k=1}^{\infty} q_k z^{k-1} \tag{97}$$

$$= \bar{V}'(z) - n \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} z^{k-1} \binom{l+k}{l} (2z_*)^{-l-k} \bar{W}^{(l)} \tag{98}$$

$$= \bar{V}'(z) - n \frac{2z_*}{z} (\bar{W}(2z_* - z) - \bar{W}(2z_*)). \tag{99}$$

Consequently $\bar{W}(z)$ may be identified as the disk amplitude of the one-matrix model corresponding to this potential.

From (95) it follows that the singular behavior of the effective potential is as follows: $V'_{\mathrm{eff}}(z)|_{\mathrm{sing.}} \sim (z_* - z)^{s-1}$. This agrees with the singular behavior that the generalized Kazakov potential (18). The singular behavior fixes the asymptotic behavior of the coefficients to be $v_n \sim 1/n^{s+1}$, i.e. it is the same as for the generalized Kazakov potential. Correspondingly the scaling of the generalized Kazakov disk amplitude $W(z)$ at the critical value $g = g_*$ agrees precisely with the singular behavior (95) of the disk amplitude $\bar{W}(z)$, since at $g = g_*$

$$W(z) = \frac{1}{z} {}_2F_1(1/2, 1; 1/2 + s; z^{-2}), \quad W(z)|_{\mathrm{sing.}} \sim (z-1)^{s-1}, \tag{100}$$

provided $s \neq m + 1/2$.

In [11] these considerations were generalized in the following way: rather than considering a potential $\bar{V}(M)$ with positive coefficients, which generate graphs related to 2d gravity, the authors of [11] considered potentials which generated graphs with both positive and negative weights like in the case of the mth multicritical point. By properly adjusting the coefficients $\bar{v}_k$ and z_* it was shown that one could obtain the critical behavior (95) for the disk amplitude $\bar{W}(z)$, with $s \in]m - 1/2, m + 1/2[$. The eqs. (96) and (97) are still valid with s in the range

$]m-1/2, m+1/2[$, and we conclude again that the $O(n)$ models coupled to the class of random graphs defined by the $\bar{V}(M)$ leading to this new critical behavior has an effective one-matrix potential with coefficients $v_n \sim 1/n^{s+1}$, $s \in]m-1/2, m+1/2[$. Unfortunately, the relation to matter models coupled to 2d gravity is not so clear if $m > 2$, as already discussed in [11]. The problem is that the multicritical matrix models for $m > 2$ already themselves at criticality represent conformal field theories coupled to 2d gravity, the simplest example being the representation of the $m = 3$ model as a dimer model coupled to random graphs [7]. However, the additional coupling to an $O(n)$ model will lead to an interacting model of dimers and $O(n)$ loops, even on a flat lattice. Since we cannot solve this interacting model on a flat lattice we do not know the corresponding central charge c at criticality and we cannot really claim that the corresponding random matrix model (93) with associated central charge $c(s)$ represents such a matter model coupled to 2d gravity, although there is no reason to doubt that it is the case.

The precise connection with the multi-critical $O(n)$ model only holds at criticality, i.e. $g = g_*$. This explains why the continuum limit (70) of our disk amplitude for $\Lambda \neq 0$ is different from the standard one of the $O(n)$ [11]. The reason is that the deformation away from criticality is not the same in the two situations. While Λ in the $O(n)$ model refers to the cosmological constant which couples to the “area” (i.e. all polygons) of the complete random graph, the Λ in the corresponding one-matrix model (and in the generalized Kazakov model) refers to the “area” of the gasket, as illustrated in Fig. 6. It is somewhat unclear how to relate the two and how to explain the difference in terms of deformations away from criticality, since we in general are talking about irrational conformal field theories.

9. Conclusions

We have shown that standard matrix model calculations extend to potentials of the form

$$V(x) = \frac{1}{g}\sum_n v_n x^{2n}, \quad v_n \sim \frac{1}{n^{s+1}} \quad \text{for } n \to \infty. \tag{101}$$

Both the potential and their derivatives have cuts on the real axis. Nevertheless one can find 1-cut solutions $W(z)$ to the disk amplitude which are natural generalizations of the standard multicritical disk amplitudes and in this way the generalized Kazakov potentials $V_s(x)$ serve as generalized multicritical points interpolating between the standard multicritical points. In particular the b-loop functions are universal functions when expressed in terms of $z_j^2 - a^2$, $j = 1, \dots, b$ and the $b-2$ first moments M_k, $k = 1, \dots, b-2$, even if $W(z)$ itself depends on infinite many M_k's. Also, for the multiloop functions the continuum limit is obtained in a straigth forward manner.

To each $s > 1$ one can formally associate a central change $c(s)$ given by (41) and conversely to each central charge $c < 1$ one can associate two values $s(c) > 2$ and $s'(c) < 2$ related by $s' = s/(s-1)$ corresponding the KPZ exponents $\gamma_A(s) = 2 - s$ and $\gamma'_A(s') = 2 - s'$, related by (42) and corresponding to the two solutions of the KPZ equation (41). The “wrong” solution of the KPZ equation where $\gamma'_A(s') > 0$ has been associated with so-called touching interactions where one in matrix model context has added terms like $g_t(\mathrm{tr}\phi^2)^2$ to the ordinary matrix potential. By fine-tuning the touching coupling like g_t one could obtain certain critical exponents $\gamma > 0$. We have here seen very explicitly in Sec. 7 that for potentials with the most heavy tail, namely $1 < s < 2$ the “touching” picture appears automatically, without adding any explicit touching interaction, and that the whole range $0 < \gamma'_A < 1$ is spanned.

A number of interesting questions remain to be answered. Is there any conformal field theory interpretation of the region $1/2 < s < 1$? How do the perturbations away from the generalized Kazakov point relate to the corresponding conformal field theory, which in general will be irrational? What *is* the most natural way to perturb away from the generalized Kazakov point and how does it relate to the standard KdV flow equations valid for any standard multicritical model? These questions deserve further considerations.

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Appendix A. Simplest example of logarithmic potential

We illustrate in this Appendix how general formulas of Sect. 4 work for potentials which are not even, i.e. $V(x) \neq V(-x)$. A simplest such a potential for which V' has a cut at the real axis is the logarithmic potential

$$V(x) = \frac{1}{g}\left[(1-x)\log(1-x) + x\right] = \frac{1}{g}\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}, \tag{102}$$

so that both $V(x)$ and

$$V'(x) = -\frac{1}{g}\log(1-x) = \frac{1}{g}\sum_{n=1}^{\infty}\frac{x^n}{n} \tag{103}$$

have a cut from 1 to ∞.

From eq. (50) we find the solution

$$W(z) = \frac{1}{g}\left[\operatorname{arctanh}\sqrt{\frac{(z-b)}{(z-a)}} - \operatorname{arctanh}\sqrt{\frac{(1-a)(z-b)}{(1-b)(z-a)}} - \frac{1}{2}\log(1-z)\right], \tag{104}$$

where

$$a = b - 4\left(1 - \sqrt{1-b}\right) \tag{105}$$

and

$$g = \frac{(b-a)^2}{16} = \left(1-\sqrt{1-b}\right)^2. \tag{106}$$

The cut $[a, b]$ is non-symmetric.

The solution (104) has all required properties: it is analytic outside of $[a, b]$, reproduces Wigner's law as $g \to 0$ etc. The discontinuity across the cut determines the (normalized) spectral density

$$\rho(x) = \frac{1}{\pi g}\left[\arctan\sqrt{\frac{(1-a)(b-x)}{(1-b)(x-a)}} - \arctan\sqrt{\frac{(b-x)}{(x-a)}}\right], \tag{107}$$

which indeed obeys eq. (45) with the potential (103) as can be explicitly checked. The spectral density (107) is positive for $b < 1$, vanishes at the ends of the cut, but looks pretty different from the previously known cases, where V' has no cut at the real axis. In those usual cases ρ has a square-root singularity, which is now hidden under the arctan.

A critical behavior is now reached as $b \to 1$, when

$$g \to g_* - 2\sqrt{1-b}, \qquad g_* = 1 \tag{108}$$

from eq. (106). Expanding near the critical point similarly to (64),

$$b = 1 - \epsilon\sqrt{\Lambda}, \qquad z = 1 + \epsilon P, \tag{109}$$

we find from eq. (104)

$$W - \frac{V'}{2} \propto \operatorname{arctanh}\sqrt{\frac{P}{\sqrt{\Lambda}} + 1} \tag{110}$$

which has a cut along the real axis for $p < -\sqrt{\Lambda}$ and

$$\rho_{\text{cont.}}(p) = \frac{1}{\pi}\operatorname{arctanh}\sqrt{-1 - p/\sqrt{\Lambda}}. \tag{111}$$

Notice that ϵ has canceled on the right-hand side of eq. (110). This might imply that the double scaling limit does not exist for this matrix model, but it rather corresponds to a certain continuum combinatorial problem like the Kontsevich matrix model. A similar behavior occurs for the potential (51) for $s = 1$. The potentials (102) and (51) with $s = 1$ thus belong to the same universality class.

Appendix B. An extension of the polylogarithm

The polylogarithm has the integral representation

$$\mathrm{Li}_s(A) = \sum_{n=1}^{\infty}\frac{A^n}{n^s} = \frac{1}{\Gamma(s)}\int_0^{\infty} \mathrm{d}\tau\, \tau^{s-1}\left(\frac{1}{1 - A\,\mathrm{e}^{-\tau}} - 1\right), \tag{112}$$

where the integral is convergent at small t for $s > 0$ and $|A| < 1$ and $s > 1$ for $A = 1$. The asymptotic behavior of $\mathrm{Li}_s(A)$ as $A \to 1$ depends on the value of s. For $0 < s < 2$ we have from eq. (112)

$$\mathrm{Li}_s(A) \to \zeta(s) + \Gamma(1-s)\,(1-A)^{s-1} \qquad \text{for } 0 < s < 2 \tag{113}$$

and

$$\mathrm{Li}_s(A) \to \zeta(s) + \zeta(s-1)\,(1-A) \qquad \text{for } s > 2. \tag{114}$$

Let us define the function

$$F_s(\alpha, A) = \sum_{n=1}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+1)\,n!}\frac{A^n}{n^s} = \frac{1}{\alpha\Gamma(s)}\int_0^{\infty}\mathrm{d}\tau\,\tau^{s-1}\left[\frac{1}{\left(1 - A\,\mathrm{e}^{-\tau}\right)^{\alpha}} - 1\right] \tag{115}$$

which for $\alpha = 1$ reduces to the polylogarithm and for $\alpha = 1/2$ reproduces the function (53). The integral in eq. (115) is convergent for $s > 0$ if $|A| < 1$ and $s > \alpha$ if $A = 1$. The derivative of (115) reads

$$\frac{\mathrm{d}}{\mathrm{d}A} F_s(\alpha, A) = \frac{1}{A} F_{s-1}(\alpha, A)\,. \tag{116}$$

The asymptotic behavior near $A = 1$ can be found from the difference

$$F_s(\alpha, A) - F_s(\alpha, 1) = \frac{1}{\alpha\Gamma(s)} \int_0^\infty \mathrm{d}\tau\, \tau^{s-1} \left[\frac{1}{\left(1 - A\,\mathrm{e}^{-\tau}\right)^\alpha} - \frac{1}{\left(1 - \mathrm{e}^{-\tau}\right)^\alpha} \right]. \tag{117}$$

If we expand the difference in $(1 - A)$, we find

$$F_s(\alpha, A) - F_s(\alpha, 1) = -\frac{(1-A)}{\Gamma(s)} \int_0^\infty \mathrm{d}\tau\, \tau^{s-1} \left[\frac{\mathrm{e}^{-\tau}}{\left(1 - \mathrm{e}^{-\tau}\right)^{\alpha+1}} \right], \tag{118}$$

where the integral converges for $s > 1 + \alpha$. We then find

$$F_s(\alpha, A) - F_s(\alpha, 1) = -(1 - A)\, F_{s-1}(\alpha, 1) \qquad \text{for} \;\; s > 1 + \alpha. \tag{119}$$

If $s < 1 + \alpha$, the integral in eq. (118) diverges as $\tau \to 0$ and we cannot expand in $(1 - A)$. Then for $\alpha < s < 1 + \alpha$ the right-hand side of eq. (117) is dominated by small $\tau \sim (1 - A)$ and we write

$$\begin{aligned} F_s(\alpha, A) - F_s(\alpha, 1) &= \frac{1}{\alpha\Gamma(s)} \int_0^{\sim 1} \mathrm{d}\tau\, \tau^{s-1} \left[\frac{1}{(1 - A + \tau)^\alpha} - \frac{1}{\tau^\alpha} \right] \\ &= (1 - A)^{s-\alpha} \frac{\Gamma(\alpha - s)}{\Gamma(1 + \alpha)} \qquad \text{for} \;\; \alpha < s < 1 + \alpha. \end{aligned} \tag{120}$$

For $\alpha < s < 1 + \alpha$ this is larger than the contribution from the domain of large τ, where

$$\int_{\sim 1}^\infty \mathrm{d}\tau\, \tau^{s-1} \left[\frac{1}{\left(1 - A\,\mathrm{e}^{-\tau}\right)^\alpha} - \frac{1}{\left(1 - \mathrm{e}^{-\tau}\right)^\alpha} \right] \propto -(1 - A)\,. \tag{121}$$

Thus the asymptote (120) holds for $\alpha < s < 1 + \alpha$ and the asymptote (119) holds for $s > 1 + \alpha$.

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