Vectorized Method for Solving the n-queens Problem using Bohrium
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Publication date:
2016

Document Version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Vectorized Method for Solving the $n$-queens Problem using Bohrium

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Introduction
On a normal $8 \times 8$-chessboard we find that 8 queens can be positioned in

\[
\binom{64}{8} = \frac{64!}{(64-8)!} = 4,426,156,304
\]

different legal ways. This is too many to brute force, especially as we go for larger $n$.

For the $8 \times 8$-queens problem we only have 92 distinct solutions.

Backtracking Algorithm
Since only one queen are allowed in each row and column, we can skip placing another queen in the same column or row in the backtracking algorithm.

The problem is simple, but non-trivial to solve. Using the knowledge stated before, we only have to check

\[
s^8 = 2^{24} = 16,777,216
\]

placements, instead of the roughly four and a half billion legal placements.

Using Permutations
If not, then we have a legal configurations of the board, which can be boiled down to

\[
d^4 = 2^8 = 256
\]

placements where we only check the diagonals, but only $2^8 = 256$ permutations of rows.

The 1 x 4 “identity” chessboard, which is shown in figure 2, can be built similar to figure 1.

For the four-queens problem there are only two solutions. These are shown in figure 3.

Figure 2: The 4 x 4 “identity” chessboard.

For the four-queens problem we only need to check

\[
\text{traces of which we only need to check the number of its. As we have seen already, there are only two solutions, and we only have two max traces which are 1.}
\]

Note that since there is always queens in the diagonals or offset-diagonals, we can never have a max-trace of zero value.

Using NumPy and Bohrium
With our knowledge, we can construct a simple Python program using NumPy or Bohrium to calculate how many distinct solutions are to the $n$-queens problem for a given $n$. In this program we simply list all permutations of the $n \times n$-chessboard, flip it, than trace all diagonals from $-n$ to $n$ and max that matrix into a 10 vector. The result is then found by counting in this vector.

Adding a Dimension
Now that we know how to solve one board, we can create all permutations of that board, to solve the entire $n$-queens problem for some $n$.

We can do this, by adding a dimension to our matrices above, creating a tensor of $4 \times 4$-chessboards.

Doing so yields

\[
\begin{array}{c|c|c}
4 & 4 & 4 \\
4 & 4 & 4 \\
4 & 4 & 4 \\
\end{array}
\]

boards, which all needs to be checked. Generalizing for the $n$-queens problem, this would be

\[
1^2 \cdot n! \quad \text{n-queens problem}
\]

For the four-queens problem, this yields $4 = 24$ traces of which we only need to count the number of its. As we have seen already, there are only two solutions, and we only have two max traces which are 1.

Future Work
Unfortunately we do not currently gain any performance boost using Bohrium, however this is largely due to the creation of the permutation boards.

We are planning to implement a permutation generator into the Bohrium runtime as a streaming generator for the GPU, and will hopefully get a performance speed-up doing so.

---

```python
import numpy as np
import bohrium as np
from itertools import permutations
def nqueens(n):
    # Generate all permutations of n identity boards
    boards = list(permutations(np.eye(n)))
    # Attack the flipped boards as another dimension
    rotation_boards = np.array([boards, np.flip(boards)])
    # Calculate all the traces, from -n to n for all the boards
    n = np.max([np.trace(boards, i, axis1=0, axis2=1) for i in range(-n, n)])
    # Count the number of is in the maximum of the traces
    print(np.sum(n == 1), "solutions for", n, "by", n, "board ")
nqueens(8) # => 92 solutions for 8 by 8 board.
```