Vectorized Method for Solving the n-queens Problem using Bohrium

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Introduction

On a normal $n \times n$-chessboard, we find that 8 queens can be positioned in
\[
\binom{64}{8} = \frac{64!}{8!(64-8)!} = 4,129,664,688
\]
different legal ways. This is too many to brute force, especially as we go for larger $n$.

For the $8 \times 8$-queens problem, we only have 92 distinct solutions.

Backtracking Algorithm

Since only one queen are allowed in each row and column, we can skip placing another queen in the same column or row in the backtracking algorithm.

![Diagram of backtracking algorithm](image)

The problem is simple, but non-trivial to solve. Using the knowledge stated before, we only have to check
\[
8^3 = 256 = 16 \times 16 \times 16
\]
placements, instead of the roughly four and a half billion legal placements.

Using Permutations

Instead of looking for the right combinations, we can instead compute permutations of rows with a queen in each column. There are
\[
8! = 40,320
\]
permutations of a $8 \times 8$-chessboard’s rows.

Because we construct the rows with a queen in each column, we now know we only have to check the diagonals, to see if the current permutation is a solution.

Matrix Representation

One solution for the four-queens problem can be represented as the following matrix:
\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

To check if this is a solution, we can sum along each column, row, and diagonal, checking if the sum of any of these is greater than 1. If not, then we have a solution.

Gathering all the traces (the sums along the diagonals and offset-diagonals) for the $4 \times 4$ identity board we get
\[
[0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1]
\]

Here we see a 4, which means that the identity board isn’t a solution to the four-queens problem. In fact, the maximum trace must be equal to 4 to be a solution.

Adding a Dimension

Now that we know how to solve one board, we can create all permutations of that board, to solve the entire $n$-queens problem for some $n$.

We can do this by adding a dimension to our matrices above, creating a tensor of $4 \times 4$-chessboards.

Doing so yields
\[
4 \times 4 \times 3 = 288
\]
boards, which all need to be checked. Generalizing for the $n$-queens problem, this would be
\[
4^n \times n!
\]

For the four-queens problem, this yields $4^4 = 256$ traces of which we only need to count the number of 1s. As we have seen already, there is only two solutions, and we only have two max-traces which are equal.

Note that since there is always queens in the diagonals or offset-diagonals, we can never have a max-trace of zero value.

Using NumPy and Bohrium

With our knowledge, we can construct a simple Python program using NumPy or Bohrium to calculate how many distinct solutions are to the $n$-queens problem for a given $n$. In this program we simply list all permutations of the $n \times n$ chessboard, flip it, then trace all diagonals from $-n$ to $n$ and max that matrix into a 10 vector. The result is then found by counting this in vector.

Future Work

Unfortunately we do not currently gain any performance boost using Bohrium, however this is largely due to the creation of the permutation boards.

We are planning to implement a permutation generator into the Bohrium runtime as a streaming generator for the GPU, and will hopefully get a performance speed-up doing so.