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Aggregation of Information and Beliefs: Asset Pricing Lessons from Prediction Markets*

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Abstract

In a binary prediction market in which risk-neutral traders have heterogeneous prior beliefs and are allowed to invest a limited amount of money, the static rational expectations equilibrium price is demonstrated to underreact to information. This effect is consistent with a favorite-longshot bias, and is more pronounced when prior beliefs are more heterogeneous. Relaxing the assumptions of risk neutrality and bounded budget, underreaction to information also holds in a more general asset market with heterogeneous priors, provided traders have decreasing absolute risk aversion. In a dynamic asset market, the underreaction of the first-period price is followed by momentum.

Keywords: Prediction markets, private information, heterogeneous prior beliefs, limited budget, underreaction.

JEL Classification: D82 (Asymmetric and Private Information), D83 (Search; Learning; Information and Knowledge), D84 (Expectations; Speculations).

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1 Introduction

Prediction markets are incentive-based mechanisms that employ the general ability of asset markets to aggregate population beliefs. It is often argued that forecasts generated by prediction markets are more accurate and less expensive than those obtained through more traditional methods, such as opinion polls or judgement by experts.[1] Partly thanks to their track record as forecasting tools, there is growing interest in using prediction markets to collect information for the purpose of improving decision making in business and public policy contexts.[2]

Given the simplicity of the trading environment and the availability of data on realized (and reasonably exogenous) outcomes, prediction markets offer economists an interesting laboratory to confront theories with evidence on the informational efficiency of markets.[3] This paper explores the theoretical properties of prediction markets as forecasting tools with the objective of drawing implications for asset pricing. The central question we address is how asset prices resulting from a competitive trading process relate to the posterior beliefs of market participants.

Typically, prediction markets target unique events, such as the outcome of a presidential election or the identity of the winner in a sport contest. Given that market participants have limited experience with the underlying events, it is natural to allow individual traders to have heterogeneous *prior beliefs*—these initial opinions are subjective and thus are uncorrelated with the realization of the outcome.[4] Having different prior beliefs, traders gain from trading actively.[5]

While trade in prediction markets may be motivated by traders' heterogeneous prior beliefs, designers of these markets typically are interested in extracting the information possessed by market participants.[6] Therefore, our model allows individual traders to have access to *information* about the outcome—information has an objective nature because

[1] See, for instance, Forsythe et al. (1992) and Berg et al. (2008) on the track record of the Iowa Electronic Markets since 1988.

[2] See Hanson (1999), Wolfers and Zitzewitz (2004), and Hahn and Tetlock (2005).

[3] Indeed, the Iowa Electronic Markets were developed mostly for educational purposes.

[4] For the purpose of our analysis, traders' subjective prior beliefs play the role of exogenous parameters, akin to the role played by preferences.

[5] In the context of more general financial markets, Hong and Stein (2007) survey a large body of evidence that points to heterogeneity of opinions.

[6] As Aumann (1976) notes, "reconciling subjective probabilities makes sense if it is a question of implicitly exchanging information, but not if we are talking about 'innate' differences in priors."

it is correlated with the outcome.[7] Information can be revealed to other traders and the prediction market designer through changes in the price.

In order to clearly distinguish prior belief heterogeneity from information, we make two convenient assumptions. First, we focus on a setting in which the price fully reveals all private information. Without this convenient restriction, the extent of heterogeneity across posterior beliefs would not be constant because the amount of residual private information would depend on the particular realized price.[8] Second, we assume that traders agree on how to interpret information (i.e., that beliefs are concordant). Thus, differences in the posterior beliefs of traders are uniquely due to differences in their prior beliefs.

Our baseline model considers a simple prediction market for a binary event, such as the outcome of a presidential election. Traders can take positions in two Arrow-Debreu contingent assets, each paying one dollar if the corresponding outcome occurs. In our formulation, each trader's initial endowment is constant with respect to the outcome realization. Taking into account a typical institutional feature of prediction markets, we constrain the wealth each trader can invest in the market.[9]

In a fully-revealing rational expectations equilibrium (REE), traders make correct inferences from prices, given common knowledge of the information structure and of the prior beliefs. If the resulting equilibrium price is to behave like a posterior belief of the market based on (all) available information, the implied ex ante belief obtained by inverting Bayes' rule should behave as a prior belief. However, we show that the implied ex ante belief for the market is not independent of the particular information realization, and thus does not behave as a prior belief for the market. Actually, the implied ex ante belief changes systematically in the *opposite* direction of the information realization.

Our main result is that the market price underreacts to information. When the information is more favorable to an event, resulting in a higher market price for the corresponding asset, the implied market ex ante belief is lower. Equivalently, more favorable information

[7] This distinction between prior beliefs and information is standard. An individual updates her belief when learning someone else's information. When exposed instead to the prior belief of another individual, the same individual would not be led to revise her belief.

[8] Because the assumption that the market reaches a fully revealing equilibrium is not warranted for some market rules, ours is the most optimistic scenario for information aggregation. See Plott and Sunder (1982 and 1988) and Forsythe and Lundholm (1990) for experimental investigations of the conditions leading to equilibrium in settings with differential private information.

[9] For example, traders in the Iowa Electronic Markets are allowed to wager up to $500, as also explained in footnote 13. The analysis also applies to general financial markets in which traders can borrow a limited amount of money.

yields a higher market price, but the price increment is smaller than it would be if the price were to respond to information as a Bayesian posterior probability for the event. In this sense, the equilibrium price underreacts to information. This result leads to a new explanation for the favorite-longshot bias, a regularity that is widely documented in the empirical literature on betting markets.

To understand the logic driving this underreaction result, consider a prediction market written on which finalist, Italy or Denmark, will win the World soccer championship. Suppose that the risk-neutral traders are subjectively more optimistic about Italy winning, the further south is their residence. In equilibrium, traders south of a certain threshold latitude spend the maximum amount of money allowed to buy the Italy asset, while, conversely, traders north of the threshold latitude bet all they can on Denmark.

Now what happens when traders overall possess more favorable information about Italy winning? In the fully revealing REE, this information will be revealed and will cause the price for the Italy asset to be higher, while contemporaneously reducing the price for the Denmark asset, compared to the case with less favorable information. As a result, the southern traders (who are optimistic about Italy) are able to buy fewer Italy assets, which are now more expensive—while the northern traders can afford, and end up demanding, more Denmark assets (now cheaper). Hence, there would be an excess supply of the Italy asset and excess demand for the Denmark asset. For the market to equilibrate, some northern traders must turn to the Italian side. In summary, when information more favorable to an outcome is available, the marginal trader who determines the price has a prior belief that is *less* favorable to that outcome. Through this countervailing adjustment, the heterogeneity in priors dampens the effect of information on price.

So far we presented our underreaction result in a setting with risk-neutral traders who are restricted to risk a bounded amount of wealth. As we show, the result also holds under the empirically plausible assumption that traders have decreasing absolute risk aversion, even when there is no exogenous bound on the amount of wealth traders can invest or borrow. In this more general asset exchange model, the equilibrium price can still be meaningfully interpreted as a generalized average of traders' posterior beliefs. But, again, when favorable information is revealed, traders who take long positions on the asset that now becomes more expensive suffer a negative wealth effect, and hence cut back their position because they become more risk averse. Thus, the generalized belief average assigns a lower weight on the beliefs of traders with priors that are in line with the realized information. Again, price underreaction results.

When viewed as special financial markets, a defining feature of prediction markets is that traders have constant endowments across states of the world. Therefore, prediction markets offer an ideal setting to investigate how market prices react to information when traders have heterogeneous beliefs. In a dynamic extension of our model, we find that the initial price underreaction is followed by price momentum. As such our analysis uncovers a novel mechanism that can contribute to the understanding of why prices in financial markets are widely observed to underreact to information and exhibit momentum (or post-announcement drift).[10]

As explained in Section 7, this paper integrates classic results from the literature on belief aggregation (starting with Wilson, 1968, Lintner, 1969, and Rubinstein, 1974) with work on the role of prices as aggregators of information (Grossman, 1976 and Radner, 1979). In the presence of wealth effects, we show that the market price underreacts to information because it allocates an increased weight to traders with beliefs that point in the opposite direction to the realized information. Our model uses the same mix of ingredients (heterogeneous priors, private information, general risk preferences, and REE) as Milgrom and Stokey (1982). To their setting, we add the comparative statics of how the equilibrium resulting in the first round of trade depends on the information available to the market, as well as a characterization of the correlation of price changes over time.[11]

The paper proceeds as follows. Section 2 introduces our baseline model with risk-neutral traders. Section 3 derives the equilibrium and gives a parimutuel re-interpretation. Section 4 demonstrates the general interdependence of information and belief aggregation and the occurrence of the favorite-longshot bias. Section 5 extends the analysis to the case with risk-averse traders. Section 6 shows that momentum arises in a natural dynamic extension of the model. Section 7 details the paper's contribution to the literature. Section 8 concludes. Appendix A collects all proofs. Appendix B analyzes a version of the model with heterogenous endowments but common prior.

[10] Following Jagadeesh and Titman (1993), there is now a large empirical literature documenting momentum effects in asset prices. Hong and Stein (1999) propose a different informational theory of underreaction and momentum. They assume that information diffuses gradually and is not fully understood by all traders, unlike in our fully revealing rational expectations equilibrium.

[11] Milgrom and Stokey (1982) instead focus on the price adjustment that follows the arrival of new information in a second period, after the first round of trade has occurred.

2 Baseline Model

Our model is inspired by the rules of the Iowa Electronic Markets for a binary prediction market, in which traders can take positions on whether an event, E, is realized (e.g., the Democratic candidate wins the 2008 presidential election) or not. There are two Arrow-Debreu assets corresponding to the two possible realizations: one asset pays out 1 currency unit if event E is realized and 0 otherwise, while the other asset pays out 1 currency unit if the complementary event E^c is realized and 0 otherwise.[12]

Traders enter the market by first obtaining an equal number of both assets. Essentially, the designer of the prediction market initially endows each trader i with w_{i0} units of each of the two assets. One important feature of the market is that there is a limit on how much money each trader can invest.[13] After entering the market, traders can exchange their assets with other traders. A second key feature of the market is that traders are not allowed to hold a negative quantity of either asset. As explained below in more detail, these two restrictions (on the amount of money invested and on the number of assets a trader can sell) impose a bound on the number of asset units that each individual trader can purchase and eventually hold.[14]

Markets clear when the aggregate demand for asset 1 precisely equals the aggregate demand for asset 2. We normalize the sum of the two asset prices to one, and focus on the price p of the asset paying in event E.

We assume that there is a continuum I of risk-neutral traders who aim to maximize their subjective expected wealth.[15] Trader i maximizes $\pi_i w_i(E) + (1-\pi_i) w_i(E^c)$, where π_i denotes the trader's subjective belief. We now turn to the process that determines the trader's subjective belief, π_i.

Initially, trader i has subjective prior belief q_i. Before trading, trader i privately ob-

[12]The state of the world is given exogenously and cannot be affected by the traders. This assumption is realistic in the case of prediction markets on economic statistics, such as non-farm payroll employment. When applied to corporate decision making, prediction market traders might have incentives to manipulate the outcome. Ottaviani and Sørensen (2007) analyze outcome manipulation, disregarding the wealth effect on which we concentrate in this paper.

[13]For example, in the Iowa Electronic Markets each trader cannot invest more than \$500. Exemption from anti-gambling legislation is granted to small stake markets created for educational purposes.

[14]Our main result (Proposition 2) hinges on the property that this bound is endogenous to the model, because the number of assets each trader eventually holds depends on the market-clearing prices.

[15]The results derived in this section immediately extend to the case of risk-loving traders, whose behavior is also to adopt an extreme asset position. We turn to risk-averse traders in Section 5.

serves signal s_i. Conditional on state $\omega \in \{E, E^c\}$, we let $f(s|\omega)$ denote the joint probability density of the signal vector $s = \{s_i\}_{i \in I}$.[16] The likelihood ratio for signal realization s is defined as $L(s) = f(s|E)/f(s|E^c)$. The only constraint imposed on the signal distribution is that there is zero probability of fully state-revealing signals, so $L(s) \in (0, \infty)$ with probability one. If trader i observes the realized signal vector s, then by Bayes' rule the subjective posterior belief π_i satisfies

$$\frac{\pi_i}{1-\pi_i} = \frac{q_i}{1-q_i} L(s). \tag{1}$$

Hence, $L(s)$ is a sufficient statistic for the vector s.

For convenience, we normalize the aggregate endowment of assets to 1. The initial distribution of assets over individuals is described by the cumulative distribution function G. Thus, $G(q) \in [0,1]$ denotes the share of all assets initially held by individuals with subjective prior belief less than or equal to q. We assume that G is continuous, and that G is strictly increasing on the interval where $G \notin \{0,1\}$.[17]

We assume that the model (i.e., preferences, prior beliefs, and signal distributions) and the rationality of all traders are common knowledge. This means that all traders agree on the conditional distributions $f(s|\omega)$, even though they have heterogeneous prior beliefs. In the terminology of Milgrom and Stokey (1982), traders have concordant beliefs.[18] We will assume that the market arrives at a fully revealing rational expectations equilibrium (REE). By definition, the price then varies with the realized signal in such a way that the sufficient statistic L is revealed by the price; the price function $p(L)$ is injective.

Discussion of Assumptions. Before proceeding, we discuss our main assumptions: heterogeneous priors, concordant beliefs, and rational expectations equilibrium. Our results crucially depend on the heterogeneity of prior beliefs across traders.[19] We depart from

[16] We do not assume that the signals are conditionally independent across traders. Indeed, a large number of conditionally independent signals would lead to full revelation of the state. By allowing for conditional dependence, our model encompasses the realistic scenario in which traders overall do not possess full information about the outcome.

[17] The assumption that the priors are continuously distributed is made to simplify the analysis, but is not essential for our underreaction result. See the discussion in footnote 24.

[18] Varian (1989), Harris and Raviv (1993), and Kandel and Pearson (1995) relax this assumption by allowing traders to interpret a publicly observed signal differently. We refer to Morris (1994 and 1995a) for a characterization of the general conditions for no-trade theorems when the interpretation of new information differs across traders.

[19] As in most work on heterogeneous priors, prior beliefs are given exogenously in our model. We refer to Brunnermeier and Parker (2005) and Brunnermeier, Gollier, and Parker (2007) for a model in which

the parsimonious assumption that traders share a common prior mostly on grounds of realism. The common prior assumption is sensible when agents are dealing with objective uncertainty and with commonly experienced events. Prediction markets deal instead with settings in which traders are unlikely to have experienced similar events in the past. Given that in these settings there is no commonly agreed-upon probability from the outset, it is realistic to assume that the traders have heterogeneous prior beliefs.[20]

Second, our assumption that traders have concordant beliefs serves to make our main result particularly striking. Even though each and every individual trader's belief is updated in a Bayes rational way in response to the perfectly revealed information, the market price moves by less than Bayes' rule would predict. Note the strength of the theory that there is no restriction on the distribution of the information, as summarized by L. The only important assumption is that traders and the market designer agree on the interpretation of this information.

A final question is whether it is appropriate to consider the rational expectations equilibrium when priors are heterogeneous. We find it plausible that individuals make some inferences about the state from the realized price in the presence of asymmetric information. Even though these inferences need not be as correct in reality as they are assumed be in a REE, it is a strength of our theory that it works under this narrow, standard assumption.[21]

3 Equilibrium

This section characterizes the fully revealing REE resulting when traders are allowed to exchange assets with other traders in a competitive market.[22] By normalization, the prices of the two assets sum to one, and we focus on the equilibrium determination of the price

heterogeneous prior beliefs arise endogenously.

[20] In addition, note that the common prior assumption is not an implication of rational decision making. An alternative approach would have been to relax the rationality assumption by positing, for example, that traders employ a wrong information model.

[21] Morris (1995b) shows that the REE concept in general relies on strong common knowledge assumptions. In the present setting, we are implicitly assuming that the heterogeneous prior distribution, G, is common knowledge. Such an assumption is no stronger than the usual REE assumption that traders commonly know each others' preferences, for here the subjective prior belief is akin to a parameter characterizing individual preferences.

[22] We focus on the necessary properties of this equilibrium, while Radner (1979) discusses sufficient conditions for its existence.

p for the asset that pays out in event E.

The fully revealing REE price is a sufficient statistic for the likelihood ratio $L(s)$. For every L, trader i's demand solves this trader's maximization problem, given belief $\pi_i(L)$ satisfying (1), and given market price $p(L)$. Market clearing requires the price to be such that aggregate net demand is zero, or that the aggregate holding of each asset equals aggregate wealth (normalized to 1).

Solving the choice problem of the risk-neutral traders is straightforward. Suppose trader i has information with likelihood ratio L resulting in a posterior belief equal to π_i, and suppose that the market price is p. The subjective expected return on the asset that pays out in event E is $\pi_i - p$, while the other asset's expected return is $(1-\pi_1)-(1-p) = p - \pi_i$. With the designer's constraint on asset portfolios, individual demand thus satisfies the following: if $\pi_i > p$, trader i exchanges the entire endowment of the E^c asset into $(1-p)\, w_{i0}/p$ units of the E asset. The final portfolio is then w_{i0}/p units of the E asset and 0 of the E^c asset. Conversely, when $\pi_i < p$, the trader's final portfolio is 0 of the E asset and $w_{i0}/(1-p)$ of the E^c asset. Finally, when $\pi_i = p$, the trader is indifferent between any trade.

Proposition 1 *The fully revealing rational expectations equilibrium price, p, is the unique solution to the equation*

$$p = 1 - G\left(\frac{p}{(1-p)\,L + p}\right) \tag{2}$$

and is a strictly increasing function of the information realization L.

Our analysis permits the different interpretation that a publicly revealed signal with likelihood ratio L is observed before trade takes place. In the fully revealing REE, after L is realized the resulting market price and asset allocation constitute an equilibrium of the perfect information economy.

Parimutuel Interpretation. Given our focus on REE, the equilibrium is invariant with respect to the specific rules used for market trading. To illustrate this point, we now offer a parimutuel reinterpretation of our equilibrium. Suppose that the amount X_ω was bet on outcome $\omega \in \{E, E^c\}$ in a parimutuel betting market. Every unit bet on outcome E returns $1+\rho$ units if E is true, where $X_E(1+\rho) = X_E + X_{E^c} = 1$. The market's implied probability for outcome E is defined as $p = X_E/(X_E + X_{E^c}) = X_E = 1/(1+\rho)$.

This parimutuel market has an REE which is equal to the rational expectations equilibrium above.[23] Suppose that the implied market probability is p. Again solving the utility maximization problem of trader i, we find the following demand: trader i will bet the amount w_{i0} on event E if $\pi_i > p$, and bet the amount w_{i0} on event E^c if instead $\pi_i < p$. Note that if p is a fully revealing REE price, then market clearing implies that

$$X_E = 1 - G\left(\frac{p}{(1-p)\,L + p}\right) = p$$

from which it follows that, indeed, $p = X_E$.

4 Underreaction to Information

Using Bayes' rule (1), we can always interpret the price as the posterior belief of a hypothetical market agent with initial belief $p/\left[(1-p)\,L + p\right]$. This implied ex ante belief for the market then may be interpreted as an aggregate of the heterogeneous subjective prior beliefs of the individual traders. Our main result is that this aggregate market belief depends in a systematic fashion on the information that is initially available to traders. This means that the aggregation of beliefs cannot be separated from the realization of information.

Proposition 2 *If beliefs are truly heterogeneous, i.e. $q_i \neq q_j$ for some pair of traders, then the implied ex ante belief for the market*

$$\frac{p}{(1-p)\,L + p}$$

is strictly decreasing in L.

The arrival of more favorable pre-trade information yields a higher market price that nevertheless underreacts to the information. Consider the inference of an observer (such as the market designer, any of the traders, or a truly outside observer) who desires to extract information from the market price. Given common knowledge of the model and

[23] As argued by Ottaviani and Sørensen (forthcoming a) and (forthcoming b), the assumption that the market reaches an REE might be unrealistic if traders take simultaneous positions in a parimutuel market. Instead, here we consider the most optimistic scenario for information aggregation by allowing traders to observe the equilibrium price. See also the discussion in Section 7.

the observation of a price that reveals information L, this observer's posterior probability, $\pi(L)$, for the event E derived from any fixed prior belief q satisfies (1), or

$$\log\left(\frac{\pi(L)}{1-\pi(L)}\right)=\log\left(\frac{q}{1-q}\right)+\log L. \tag{3}$$

The expression on the left-hand side is the posterior log-likelihood ratio for event E, which clearly moves one-to-one with changes in $\log L$. Proposition 2 implies that this observer's belief reacts more than the price:

Proposition 3 *If beliefs are truly heterogeneous, then for any two different information realizations $L > L'$,*

$$\log\left(\frac{\pi(L)}{1-\pi(L)}\right)-\log\left(\frac{\pi(L')}{1-\pi(L')}\right)>\log\left(\frac{p(L)}{1-p(L)}\right)-\log\left(\frac{p(L')}{1-p(L')}\right)>0. \tag{4}$$

To understand the intuition for this underreaction result, consider what happens when traders have information more favorable to event E (corresponding, say, to the Democratic candidate winning the election), i.e., when L is higher. According to (2), the price of the E asset, p, is clearly higher when L is higher. Now, this means that traders who are optimistic about a Democratic victory can buy fewer units of asset E, because the bound w_{i0}/p is decreasing in p. In addition, traders who are pessimistic about a Democratic victory can buy more units of asset E^c, which they want to buy. If all the traders who were purchasing E before the increase in L were still purchasing E at the higher price that results with higher L, there would be insufficient demand for E. Similarly, there would also be excess demand for E^c. To balance the market it is necessary that some traders who were betting on the Republican candidate before now change sides and put their money on the Democratic candidate. In the new equilibrium, the price must change to move traders from the pessimistic to the optimistic side. Thus, the indifferent trader who determines the equilibrium price at the margin holds a more pessimistic prior belief about Democratic victory, the more favorable to Democratic candidate (i.e., the higher) the information, L, is. Hence, although the price, p, rises with the information, L, it rises more slowly than a posterior belief, because of this negative effect on the prior belief of the marginal trader.[24]

[24] Our assumption that prior beliefs are distributed continuously in the population is not essential for the result. If the population is finite, or there are gaps in the distribution, then there can be ranges of information, L, over which the equilibrium price is constant. In that case, the rational expectations equilibrium cannot fully reveal L, but can reveal the information needed for the proper allocation of the assets. Underreaction would still hold, but only relative to the limited information revealed by the equilibrium price.

The underreaction result derived in this section is driven by the restriction on the amount of money invested (see footnote 13) and, therefore, on the number of assets a trader can sell. In turn, this restriction imposes a bound on the number of assets that each individual can purchase and eventually hold. The result hinges on the fact that this bound (equal to w_{i0}/p) is inversely related to the equilibrium price.[25]

Proposition 3 offers a new informational explanation of the favorite-longshot bias, a widely-documented fact in the empirical literature on betting markets (see Thaler and Ziemba, 1988, and Snowberg and Wolfers, 2005).[26] The bias says that outcomes favored by the market occur more often than if the market price is interpreted as a probability—and, conversely, longshots win less frequently than suggested by the market price. To see how this effect arises in our context, compare the market price, $p(L)$, with the posterior belief, $\pi(L)$, held by an outside observer with (fixed) prior belief q:

Proposition 4 *If beliefs are truly heterogeneous, there exists a market price* $p^* \in [0,1]$ *with the property that* $p(L) > p^*$ *implies* $\pi(L) > p(L)$*, and* $p(L) < p^*$ *implies* $\pi(L) < p(L)$*.*

There is a threshold level, p^*, for the realized market price, such that a market price below p^* classifies event E as a longshot and a market price above p^* makes E a favorite. The observer expects longshots to occur less often than indicated by the market price, and favorites to occur more often.

Comparative Statics. We are now ready for the key comparative statics result of the paper. We show that underreaction is more pronounced if traders' heterogeneous beliefs are more dispersed. In analogy with Rothschild and Stiglitz's (1970) definition of mean preserving spread, define distribution G' to be a *median-preserving spread* of distribution G if G and G' have the same median m and satisfy $G'(q) \geq G(q)$ for all $q \leq m$ and $G'(q) \leq G(q)$ for all $q \geq m$.

[25] Suppose, instead, that the market designer were to impose a direct cap on the number of assets that each trader can buy, rather than on the budget each trader can invest. Then, for a large range of information realizations, the same fixed set of optimists (or pessimists) would buy the full allowance of the E (or E^c) asset. Since the sets of optimists and pessimists are constant, there would be no underreaction. However, typically in prediction markets there is an upper bound on the traders' budget, rather than on the number of nominal positions they can take.

[26] Wolfers and Zitzewitz (2004) report that the favorite-longshot bias has been observed in a prediction market for Standard and Poor's 500 index.

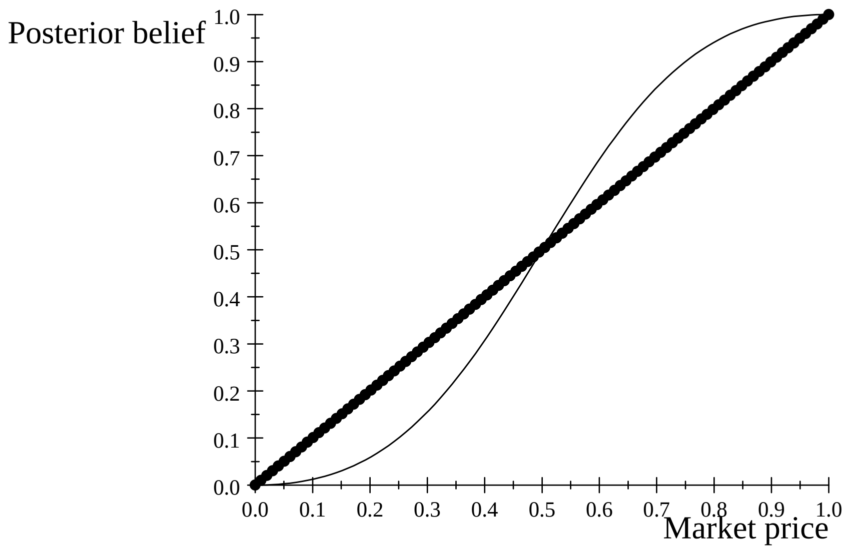

Figure 1: This plot shows the posterior probability for event E as a function of the market price p for the E asset, when the prior beliefs of the risk-neutral traders are uniformly distributed ($\beta = 1$ in the example). The dotted line is the diagonal.

Proposition 5 *Suppose that G' is a median-preserving spread of G, denoting the common median by m. Then, more underreaction results under G' than under G: $L > (1-m)/m$ implies $\pi(L) > p(L) > p'(L) > 1/2$, and $L < (1-m)/m$ implies $\pi(L) < p(L) < p'(L) < 1/2$.*

This result is in line with empirical evidence by Verardo (2007) that momentum profits are significantly larger for portfolios characterized by higher heterogeneity of beliefs.

Example. To illustrate our results, suppose that the distribution of subjective prior beliefs over the interval $[0,1]$ is $G(q) = q^\beta / \left[q^\beta + (1-q)^\beta\right]$, where $\beta > 0$ is a parameter that measures the concentration of beliefs. The greater is β, the less spread is this symmetric belief distribution around the average belief $q = 1/2$. For $\beta = 1$ beliefs are uniformly distributed, as $\beta \to \infty$ beliefs become concentrated near $1/2$, and as $\beta \to 0$ beliefs are maximally dispersed around the extremes of $[0,1]$. The equilibrium condition (2) becomes

$$\log\left(\frac{p}{1-p}\right) = \log\left(\frac{1 - G\left(\frac{p}{(1-p)L+p}\right)}{G\left(\frac{p}{(1-p)L+p}\right)}\right) = \beta \log\left(\frac{(1-p)L}{p}\right),$$

so that the market price $p(L)$ satisfies the linear relation

$$\log\left(\frac{p(L)}{1-p(L)}\right)=\frac{\beta}{1+\beta}\log L.$$

Hence, $\beta/(1+\beta)\in(0,1)$ measures the extent to which the price reacts to information. Price underreaction is minimal when β is very large, corresponding to the case with nearly homogeneous beliefs. Conversely, there is an arbitrarily large degree of underreaction when beliefs are maximally heterogeneous (i.e. β is close to zero).

Assume that a market observer's prior is $q=1/2$ for event E, consistent with a symmetric market price of $p(1)=1/2$ in the absence of additional information. The posterior belief associated with price p then satisfies

$$\log\left(\frac{\pi(L)}{1-\pi(L)}\right)=\log L=\frac{1+\beta}{\beta}\log\left(\frac{p(L)}{1-p(L)}\right).$$

As illustrated in Figure 1 for the case with uniform beliefs ($\beta=1$), the market price overstates the winning chance of a longshot and understates the winning chance of a favorite by a factor of two.

5 Risk Aversion

So far we have assumed that each individual trader is risk neutral, and thus ends up taking as extreme a position as possible on either side of the market. Now, we show that our result extends nicely to risk-averse individuals, under the plausible assumption that traders' absolute risk aversion is decreasing in wealth, even when no exogenous bounds are imposed on the amounts that traders can invest.

Model. Realistically, suppose that each trader is endowed with the same number w_0 of each asset. To properly capture the effect of risk aversion, we suppose that each trader i is also characterized by an initial, state-independent level W_i of additional wealth.[27] Trader i maximizes subjective expected utility of final wealth, $\pi_i u_i(w_i(E))+(1-\pi_i)\,u_i(w_i(E^c))$, where π_i is the trader's subjective belief. We suppose that u_i is twice differentiable with $u_i'>0$ and $u_i''<0$, and satisfies the DARA assumption that the de Finetti-Arrow-Pratt coefficient of absolute risk aversion, $-u_i''/u_i'$, is weakly decreasing in wealth, w_i. The

[27] See Musto and Yilmaz (2003) for a model in which, instead, traders are subject to wealth risk, because they are differentially affected by the redistribution associated with different electoral outcomes.

cumulative distribution function G again describes the distribution of subjective beliefs, and it is assumed to satisfy the same properties as before. Private information is distributed as before.

Let Δx_i denote the choice variable of trader i, such that $p\Delta x_i$ units of the E^c asset are exchanged for $(1-p)\,\Delta x_i$ units of the E asset. Note that this is a zero net value trade, since the asset sale generates $(1-p)\,p\,|\Delta x_i|$ of cash that is spent on buying the other asset. The final wealth levels in the two states are

$$w_i\,(E) = W_i + w_0 + (1-p)\,\Delta x_i \tag{5}$$

and

$$w_i\,(E^c) = W_i + w_0 - p\Delta x_i. \tag{6}$$

To properly connect with the previous model, we maintain the trading constraints that $\Delta x_i \in [-w_0/\,(1-p)\,, w_0/p]$. We stress that our main results hold regardless of whether these constraints are binding.

Equilibrium. Given price p and posterior belief π_i, trader i chooses Δx_i to maximize posterior expected utility $\pi_i u_i\,(w_i\,(E)) + (1-\pi_i)\,u_i\,(w_i\,(E^c))$. If the trading constraints are not binding, then the choice satisfies the necessary first-order condition

$$\frac{\pi_i}{1-\pi_i}\frac{u_i'\,(w_i\,(E))}{u_i'\,(w_i\,(E^c))} = \frac{p}{1-p}. \tag{7}$$

The endogenously determined net trade Δx_i then satisfies the standard condition that the marginal rate of substitution equals the price ratio. In particular, a trader with $\pi_i = p$ will choose $w_i\,(E) = w_i\,(E^c)$, corresponding to $\Delta x_i = 0$. A trader with a posterior belief π_i above the price p will choose $\Delta x_i > 0$. Note that the trading constraint does not bind, unless the trader is nearly risk neutral or the difference between π_i and p is sufficiently large. In analogy with Proposition 1 we have:[28]

Proposition 6 *There exists a unique fully-revealing rational expectations equilibrium. The price, p, is a strictly increasing function of the information realization L.*

[28] The DARA assumption implies a monotonicity of the demand function, which plays an important role in the proof of this result. With Increasing Absolute Risk Aversion (IARA) preferences, monotonicity may fail. Existence of a unique fully revealing equilibrium REE also holds for IARA preferences for which monotonicity hold—in which case overreaction rather than underreaction results. We focus on the DARA case because it is empirically more plausible.

Belief Aggregation with CARA Preferences. Suppose first that the traders have constant absolute risk aversion (CARA) utility functions, with heterogeneous degrees of risk aversion, such that $u_i(w) = -\exp(-w/t_i)$, where $t_i > 0$ is the constant coefficient of risk tolerance, the inverse of the coefficient of absolute risk aversion. Denoting the relative risk tolerance of trader i in the population by $\tau_i = t_i / \int_0^1 t_j dG(q_j)$, we have:

Proposition 7 *Suppose that the traders have CARA preferences and heterogeneous beliefs. Define an average prior belief q by*

$$\log \frac{q}{1-q} = \int_0^1 \tau_i \log \frac{q_i}{1-q_i} dG(q_i),$$

and for each individual let

$$d_i^* = t_i \log \left(\frac{q_i - qq_i}{q - qq_i} \right). \tag{8}$$

Suppose that $1 + w_0 / \inf_i d_i^* < w_0 / \sup_i d_i^*$. When the realized information L is in the range satisfying

$$1 + \frac{w_0}{\inf_i d_i^*} \leq \frac{qL}{qL + 1 - q} \leq \frac{w_0}{\sup_i d_i^*} \tag{9}$$

then the fully-revealing REE price satisfies Bayes' rule with market prior q, i.e., $p(L) = qL/(qL + 1 - q)$. When L falls outside this range, the price underreacts to changes in L compared to Bayes' rule.

Risk aversion allows for the possibility that no trader meets the constraint. This is more likely to happen when w_0 is large, as also suggested by condition (9). With CARA preferences and when no trader is constrained, trader i chooses net demand d_i^* in equilibrium. The market price thus behaves as a posterior belief and there is no bias.

This result for the case with CARA preferences and unconstrained positions is consistent with Varian's (1989) analysis. As shown by Wilson (1968), under CARA preferences the traders' heterogeneous priors can be aggregated. Once private information is added, the price then behaves as a posterior belief.[29]

[29] Note that in the very special case in which all traders share the same common prior, $q_i = q$, they have all the same posterior, $\pi = \pi_i$, so that by (7) there is no trade and the price satisfies Bayes' rule as well. Note that this result follows regardless of the degree of risk aversion of individual traders. Thus, provided one maintains Grossman's (1976) second assumption that traders share a common prior, the equilibrium price behaves like a posterior belief regardless of his CARA assumption. Our underreaction result below obtains by relaxing simultaneously *both* of these two assumptions made by Grossman (1976).

Underreaction with DARA Preferences. We have seen that CARA preferences lead to an unbiased price reaction to information when the trading constraints are not binding in equilibrium. Now we verify that, for strict DARA preferences, a bias arises in the price, whether traders are constrained or not. When L rises, the rising equilibrium price yields a negative wealth effect on any optimistic individual (with $\pi_i > p$) who is a net demander ($\Delta x_i > 0$). Conversely, pessimistic traders benefit from the price increase. With DARA preferences, the wealth effect implies that optimists become more risk averse while pessimists become less risk averse. An increase in L thus must shift weight from optimists to pessimists when the market price is calculated as a belief average. Hence, although the price rises with L, it rises less fast than a posterior belief, because the weight is shifted more to pessimists when information is more favorable.

Proposition 8 *Suppose that beliefs are truly heterogeneous and that all individuals have strict DARA preferences. Then the market price underreacts to information, as*

$$\left|\log\left(\frac{\pi(L)}{1-\pi(L)}\right)-\log\left(\frac{\pi(L')}{1-\pi(L')}\right)\right| > \left|\log\left(\frac{p(L)}{1-p(L)}\right)-\log\left(\frac{p(L')}{1-p(L')}\right)\right|$$

for any $L \neq L'$.

Example with Logarithmic Preferences. Suppose that prior beliefs are uniformly distributed over the interval $[0,1]$, with $G(q) = q$, and traders have logarithmic preferences, $u_i(w) = \log w$, satisfying the DARA assumption. In order to highlight the difference between Propositions 3 and 8, namely the inclusion of individuals with an interior solution to their maximization problem, we again remove completely the exogenous bound on the wealth invested. The individual demand function solving (7) is $\Delta x_i = (W_i + w_0)(\pi_i - p) / [p(1-p)]$. The equilibrium price is then an average of the posterior beliefs,

$$p(L) = \int_0^1 \pi(L)\, dq = \int_0^1 \frac{qL}{qL + (1-q)}\, dq. \tag{10}$$

For $L \neq 1$, integration by parts of (10) yields $p(L) = L(L-1-\log L)/(L-1)^2$. If we keep fixed $p(1) = \int_0^1 q\, dq = 1/2$ as the prior belief of the outside observer, the favorite-longshot bias can be illustrated in a graph with the same qualitative as Figure 1.

Edgeworth Box Illustration. We now present a graphical illustration of the logic of Proposition 8 for an economy with two types of traders (with prior beliefs $q_1 < q_2$)

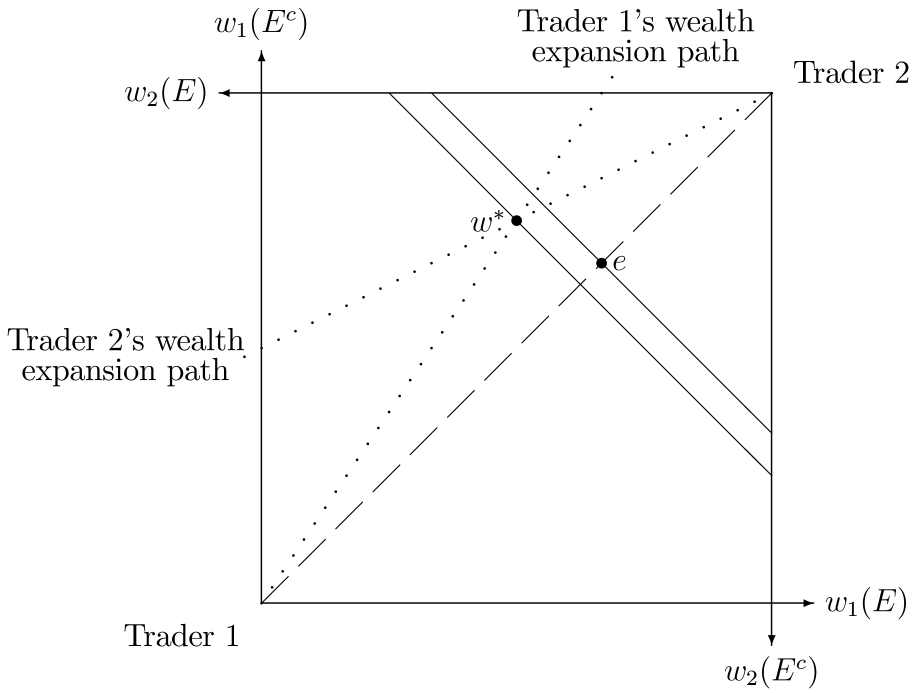

Figure 2: Edgeworth box representation of the underreaction result for the case with DARA preferences and interior solution. The wealth expansion paths are linear for the special case with logarithmic preferences.

and no exogenous bounds on the wealth traders can invest. As represented in Figure 2, the Edgeworth box is a square because there is no aggregate uncertainty. The initial endowment, e, lies on the diagonal, being the same in the two events: $w_i\left(E\right) = W_i + w_0 = w_i\left(E^c\right)$. If traders are strictly risk averse, they have strictly convex indifference curves, which are not drawn to avoid cluttering the picture. The slope of the indifference curves at any safe allocation is $-\pi_i/\left(1-\pi_i\right) = -q_iL/\left(1-q_i\right)$. Thus, for any allocation along the diagonal, the indifference curve for trader 2 (optimist) is steeper than for trader 1 (pessimist).

We focus on interior equilibria in which the exogenous trading constraints are not binding. When information L is available and revealed, the marginal rates of substitution of the two traders are equalized at the equilibrium allocation, w^*. Thus, the equilibrium allocation lies above the diagonal, and the optimistic trader 2 is a net buyer for asset E.

How is the equilibrium affected by an exogenous change in information to $L' > L$? As a result of this change in information, indifference curves become steeper by a factor of L'/L.

For the sake of argument, suppose that the price were to change without any underreaction from the original $p(L)$ to the Bayes-updated $p' = p(L) L' / (p(L) L' + (1 - p(L)) L)$. At allocation w^*, the marginal rates of substitutions are still equalized, and the straight line dividing the two traders' preferred sets has slope $-p'/(1-p')$. However, since $p' > p$, this straight line passes through the diagonal below the initial endowment point, proving that w^* cannot be the new equilibrium allocation. Maintaining price p', the new budget line must pass through the initial endowment. As this new budget line is further up on the diagonal compared to the one passing through w^*, we see illustrated here the positive wealth effect for the pessimistic trader 1 and the negative wealth effect for the optimistic trader 2.

We now turn to the implication of this wealth effect. Given price p', the choice w_1^* would be optimal for trader 1, if the true budget line were passing through w^*. This point lies above the diagonal in the Edgeworth box. Now, as it is well known since Arrow (1965), DARA implies that the wealth expansion paths diverge from the diagonal. The richer trader 1 demands a riskier bundle (further away from the diagonal) by increasing $w_1(E^c) - w_1(E)$, whereas the poorer trader 2 demands a safer bundle (closer to the diagonal) by decreasing $w_2(E) - w_2(E^c)$.[30] Therefore p' cannot be an equilibrium after the exogenous information change. To reach an equilibrium both traders must shift their portfolio towards lower $w_i(E^c) - w_i(E)$ so as to eliminate the excess demand for asset E^c as well as the excess supply for asset E. This is achieved by a relative reduction in the price for asset E, so that $p(L') < p'$. Thus, prices must underreact to information.

6 Dynamic Extension: Momentum

In this section we extend our model to a dynamic setting in which information arrives sequentially to the market after the initial round of trade. We verify that there exists an equilibrium where the initial round of trade is captured by our baseline model, and where there is no trade in subsequent periods, consistent with Milgrom and Stokey's (1982) no trade theorem. We then show that the initial under-reaction of the price to information implies momentum of the price process in subsequent periods—if the initial price movement

[30] The trader's choice problem can be reformulated with a safe asset, always paying 1, and a risky asset paying 1 in E^c and -1 in E. The richer trader demands more units of the risky asset, and hence $|w(E^c) - w(E)|$ rises.

is upwards, prices subsequently move up on average. Intuitively, first-round information is swamped by the information revealed in subsequent rounds, and hence over time the price comes to approximate the correctly updated prior belief.

In this model, the constant set of traders I is initially in the same situation as in our baseline model. Each trader is allowed to trade at every time date $t \in \{1, \ldots, T\}$ at the competitive price p_t. The joint information received by traders up until period t has likelihood ratio L_t. The asset position of trader i after trade at period t is summarized by Δx_{it}. At time $T+1$ the true event is revealed, and the asset pays out. Each trader aims to maximize the expected utility of period $T+1$ wealth, which consists of other wealth W_i and prediction market wealth w_{iT}.

Before we have characterized the static equilibrium that corresponds to the case $T = 1$. When $T > 1$, a fully revealing rational expectations equilibrium is defined as follows. First, for every $t = 1, \ldots, T$ there is an injective price function $p_t(L_t)$. By convention, $p_{T+1} = 1$ when E is true, and $p_{T+1} = 0$ when E^c is true. Second, given these price functions, every trader i chooses a strategy of net positions $\Delta x_{it}(L_t)$ in order to maximize expected utility of final wealth. By convention, $\Delta x_{i0} = 0$. The trader's prediction market wealth evolves randomly over time as $w_{it}(L_t) = w_{it-1}(L_{t-1}) + (p_t(L_t) - p_{t-1}(L_{t-1}))\,\Delta x_{it-1}(L_{t-1})$ for $t = 1, \ldots, T+1$, with $w_{i0} > 0$ specified by the market designer as before. If constrained, the trader's prediction market wealth must always stay non-negative, meaning that the net position choice at $t-1$ is constrained by $\Delta x_{it-1}(L_{t-1}) \in [-w_{it-1}(L_{t-1}) / (1 - p_{t-1}(L_{t-1})), w_{it-1}(L_{t-1}) / p_{t-1}(L_{t-1})]$. Finally, in every period t at any information L_t realization the market clears, $\int_i \Delta x_{it}(L_t)\, dG(q_i) = 0$.

Proposition 9 *There exists a fully revealing rational expectations equilibrium of the dynamic trade model with the following properties. In the first round of trade, the price* $p_1(L_1)$ *is equal to the static equilibrium price* $p(L_1)$ *characterized before. In all subsequent periods there is no trade, so* $\Delta x_{it}(L_t) = \Delta x_i$ *of the static equilibrium, and the price satisfies Bayes' updating rule,*

$$\frac{p_t(L_t)}{1 - p_t(L_t)} = \frac{L_t}{L_1} \frac{p_1(L_1)}{1 - p_1(L_1)}. \tag{11}$$

The marginal trader, who holds belief $\pi_i(L_1) = p(L_1)$ after the first round of trading, remains the marginal trader in future rounds. The market price in future periods is the

Bayesian posterior of this belief updated with the newly arriving information. From this trader's point of view prices follow a martingale, i.e., $E\left[p_t\left(L_t\right)|L_{t'}\right] = p_{t'}\left(L_{t'}\right)$ for all $t' < t$.

Every trader who is initially more optimistic than this marginal trader, and hence has first-round posterior $\pi_i\left(L_1\right) > p\left(L_1\right)$ and has chosen $\Delta x_i > 0$, believes that the price is a sub-martingale (trending upwards). Despite this belief, the no-trade theorem establishes that such a trader does not wish to alter the position away from the initial Δx_i. The position already reflects a wealth- or risk-constrained position on the asset eventually rising in price, and there is no desire to further speculate on the upward trend in future asset prices.

The underreaction to the first-period information implies momentum in returns, consistent with the findings of Jagadeesh and Titman (1993). If the information revealed in the first period is favorable, i.e. if the realized likelihood ratio satisfies $L_1 > 1$, then the trader with neutral prior belief $q = p\left(1\right)$ joins the group of optimists taking long positions on E. From the natural point of view of an observer with a prior belief equal to this neutral prior, how are prices expected to behave? We show that prices trend upwards (downwards) in expectation following the initial realization of favorable (unfavorable) information.

Proposition 10 *Suppose that beliefs are truly heterogeneous and that all individuals are constrained or have strictly decreasing absolute risk aversion (DARA). Fix the market prior at the natural level* $q = p\left(1\right)$*. When non-degenerate new information* L_t *arrives to the market in later trading periods, then prices exhibit momentum, in the sense that changes in prices in later periods are positively correlated with early changes in prices. For any* $t_1 > t_2 > 0$,

$$E\left[\left(p_{t_1}\left(L_{t_1}\right) - p_{t_2}\left(L_{t_2}\right)\right)\left(p_{t_2}\left(L_{t_2}\right) - p\left(1\right)\right)\right] > 0. \tag{12}$$

Price changes are positively correlated with the opening price; in a regression of price changes on earlier price changes there should be a positive coefficient. Thus, the initial underreaction must be followed by a correcting price momentum. Although the present analysis focuses on a period of trade opening, and immediately subsequent periods, our results apply more broadly to trading environments in which the arrival of new information coincides with trade—either because of added liquidity reasons or differential interpretation of information, from which the present analysis abstracts.

Proposition 10 is also consistent with the seemingly conflicting findings on price drift recently documented by Levitt and Gil (2007) and Croxson and Reade (2008) in the context of sport betting markets. On the one hand, Levitt and Gil (2007) find that the immediate price reaction to goals scored in the 2002 World Cup games is sizeable but incomplete and that price changes tend to be positively correlated, as predicted by our model. On the other hand, Croxson and Reade (2008) find no drift during the half-time break, thus challenging the view that the positive correlation of price changes during play time indicates slow incorporation of information. Consistent with this second bit of evidence, our model predicts the absence of drift when no new information arrives to the market, as it is realistic to assume during the break when the game is not played. This results follows immediately from Proposition 10 when $L_t = L_{t'} = L_{t''}$ for all periods t in the break $\{t', ..., t''\}$.

7 Contribution to Literature

This paper bridges the gap between the literature on trade with heterogeneous beliefs and the REE literature on information aggregation through prices. In an important paper on betting, Ali (1977) formulates a static model in which risk-neutral bettors with limited wealth have heterogeneous beliefs about a binary outcome. Ali shows that if the bettor with the median belief thinks that one outcome (defined to be the favorite) is more likely, then the equilibrium fraction of parimutuel bets on this outcome is lower than the belief of the median bettor. By identifying the belief of the median bettor as the correct benchmark for the empirical probability, Ali concludes that the favorite is underbet as compared to the longshot. Following Ali, the fledgling literature on belief aggregation in prediction markets (Manski, 2006, Gjerstad, 2005, and Wolfers and Zitzewitz, 2005) analyzes the relation between the equilibrium price and the average belief of traders, depending on the traders' preferences for risk. In this literature, traders do not make any inference from market prices.

But if the traders' beliefs really have information content (about the empirical probability of the state), their positions should depend on the information about these beliefs that is contained in the market price. This tension underlies the rational expectations critique of the Walrasian approach to price formation with heterogeneous beliefs (see, e.g., the discussion in Chapter 1 of Grossman, 1989). Unlike the prediction market literature, we remain agnostic about the relationship between (the distribution of prior) beliefs and the empirical chance of the outcome. Instead, we conduct a comparative statics exercise with respect to information realizations. Ours is the appropriate theoretical benchmark for empirical testing, because the different opinions underlying the heterogeneous priors of traders have no information content and, thus, should have no bearing on the empirical probabilities. We show that changes in the realized information translate in less than one-for-one changes in the market price—the variation in information is dampened by the market.

Following Miller (1977), Harrison and Kreps (1978), and Morris (1996) there is also an extensive literature in which active trade results from heterogeneous prior beliefs, in the absence of information. Our model departs from this finance literature (surveyed by Scheinkman and Xiong, 2004) in two key ways. First, risk-neutral traders are not wealth constrained in Harrison and Kreps (1978), but the market is incomplete, so that traders can take positions on only one side of the market. Thus in their model, the entire net supply of the asset is held by the most optimistic trader, whose belief determines the market price. This trader's belief then can be taken as the market belief, so there is no informational favorite-longshot bias. In our model, instead, markets are complete, but the wealth that can be invested is constrained either exogenously or endogenously through risk aversion—thus our model is perfectly symmetric by imposing a limit on the amount of wealth that can be invested (either for long buying or for short selling). Second, we allow traders with heterogeneous priors to also have private information, unlike in most of this literature with the exception of some of the models discussed below.

On the other hand, the REE literature typically assumes that traders have a common prior belief, so that differences in beliefs across agents can be attributed only to private information (Grossman, 1976). Under the common prior assumption, the price reacts one-for-one to information, regardless of risk attitudes (see footnote 29 above). Under CARA, heterogeneous beliefs can be aggregated (Wilson, 1968, Lintner, 1969, and Rubinstein, 1974), and thus the price again behaves as a posterior belief (see Proposition 7 above).[31] Our underreaction result thus holds once we depart from Grossman's (1976) characterization of REE by allowing for *both* heterogeneous priors *and* wealth effects. As compared to Varian's (1989) generalization of Grossman (1976) to heterogeneous priors, here we further introduce the possibility of wealth effects by imposing a limit on the amount that traders

[31] On the optimal allocation of risk with heterogeneous prior beliefs and risk preferences but without private information, see also the recent developments by Gollier (2006).

can invest (in Section 2's baseline model) *or*, more generally, by relaxing the assumption of constant absolute risk aversion (in Section 5's general model).[32]

The positive characterization we initially provide focuses on the REE that results when privately informed traders with heterogeneous priors are asked to trade once.[33] Thus, we contribute to the REE literature the characterization of the interaction of heterogeneous beliefs and information in the *first* round of trade. Our analysis complements Milgrom and Stokey (1982), who focus instead on the subsequent rounds of (no) trade that follows the arrival of additional information (with concordant beliefs). In the dynamic version of the model, we also characterize the co-variation of price changes over time and obtain a simple and novel mechanism for momentum.

We also share our focus on the interaction of private information with heterogeneous priors with a handful of papers on betting. Notably, Shin (1991 and 1992) considers asset pricing by an uninformed monopolist bookmaker in a market in which some traders take positions on the basis of their beliefs while others are perfectly informed about the outcome. Morris (1997) characterizes the equilibrium in a game-theoretic model of bilateral betting with asymmetric information and heterogeneous priors. As in these contributions, in our model heterogeneous beliefs coexist with private information, but we focus here on the competitive equilibrium.

Allen, Morris, and Shin (2006) analyze a dynamic noisy REE model with overlapping generations of traders. In their model, traders with short horizons are subject to noise, which makes prices partially revealing. Because of their short-term bias, traders must forecast the next period average forecasts and so end up overweighting the common public information.[34] Our mechanism behind the underreaction to information is instead driven by income effects that are absent in Allen, Morris, and Shin's (2006) model with CARA preferences. Banerjee, Kaniel, and Kremer (forthcoming) further investigate the conditions for momentum with CARA preferences. Deviating from the REE framework, momentum results when traders react to their private information, but are assumed not to react the

[32] Once we allow for heterogeneous priors, we stress the wealth effect introduced by *either one* of these two channels is sufficient to result in underreaction in our setting.

[33] This market-opening scenario is particularly relevant for prediction markets (which are often created with the express purpose of aggregating information and beliefs already held by traders), but is also valid for financial markets with constant inflows of new traders and changes in the state of the world.

[34] Kyle (1989) notes that prices may overreact in a noisy REE, when risk-averse traders require a risk-premium for accommodating noisy demand, positively correlated with the price.

information contained in the equilibrium price because they are unable to recognize the information of other traders. Instead, we analyze the implications of belief heterogeneity within the REE setting with concordant beliefs, where traders fully incorporate the information available to them and to other traders.[35]

Our analysis could be extended to a partially revealing REE. However, the residual private information not revealed by the market then generates private belief heterogeneity that may, in general, depend on the realized L. In principle, there could then be a systematic relationship between favorable information and the extent of belief heterogeneity. This additional mechanism is reminiscent of the effect of short-selling constraints in a Glosten and Milgrom (1985) setting, as explored by Diamond and Verrecchia (1987). When the realized news is negative, short-sale constrained informed traders are less frequently encountered in the market, and it takes longer for the market to learn negative news. Diamond and Verrecchia conclude, however, that the market maker continues to quote unbiased prices given the available information.

Finally, our explanation for underreaction is distinct from the one proposed by Ottaviani and Sørensen (forthcoming a) and (forthcoming b). In their setting, individual traders have common prior beliefs and they are unable to condition their behavior on the information that is available to the other traders. In the model formulated here, instead, we allow traders to perfectly share information through the trading process and offer a complementary explanation that crucially relies on the heterogeneity of prior beliefs. An important advantage of the explanation proposed here is that it does not depend on particular specifications of the market structure.

8 Conclusion

Prediction markets are special financial markets in which traders' endowments are constant with respect to different outcome realizations. Our model of these markets has three key ingredients: heterogeneous priors, private information, and limited positions. First, market participants do not share a common prior belief because there is genuine uncertainty about the underlying event. Second, the market designer typically is interested in aggregating

[35] We assume that traders extract information from the market price. To further appreciate the fundamental difference between the two settings, note that in Banerjee, Kaniel, and Kremer's (forthcoming) static model with unconstrained traders, underreaction immediately results even with CARA preferences, in contrast to our Proposition 7.

the private information of participants—so it is natural to allow for the presence of this information. Third, prediction market traders are allowed to wager only a limited amount of wealth, so their positions are bounded. While these three ingredients are inspired by the special features of prediction markets, they are also relevant for more general financial markets (see Hong and Stein, 2007).

In this setting, the REE price reveals all the traders' private information, but underreacts to information. This result is driven by a wealth effect arising because traders with heterogeneous beliefs take speculative net positions. Underreaction also results when the risk-neutral traders are allowed to invest any amount they wish, provided that they have finite wealth and that they can borrow a finite amount of money—as it is the case in financial markets. Even in the absence of any exogenous bound on positions or without borrowing constraints, underreaction holds under the realistic assumption that traders become less risk averse when their wealth increases. The general lesson is that the incorporation of information into the market price is intertwined with (and cannot easily be separated from) the aggregation of subjective priors whenever there are wealth effects.

Is it natural to wonder whether income effects would lead to underreaction to information in a more traditional setting with common priors, but heterogenous endowments. Appendix B investigates how prices react to information in the presence of wealth effects in a financial market in which gains from trade are generated by idiosyncratic uncertainty. Extending Rubinstein's (1974) results to allow for private information, we show that there is no underreaction bias when traders have common prior but heterogeneous endowments, provided they have Hyperbolic Absolute Risk Aversion (HARA) with common cautiousness parameters. For example, if all traders have logarithmic preferences (our leading DARA example analyzed in Section 5) the price behaves as a posterior belief when traders have heterogeneous endowments, but common prior. Intuitively, all liquidity-motivated traders take more (or less) extreme positions, as the available information varies. With heterogeneous beliefs, instead, optimists (who buy) buy less on favorable information, while pessimists (who sell) sell more, so that the price must equilibrate in a direction contrary to information. Thus, heterogeneity of priors is important for underreaction and cannot be replaced by heterogeneity in endowments across individuals.

We see our analysis as a first step toward understanding information aggregation in markets with wealth effects. Beyond the setting with two states on which we concentrate in this paper, the wealth effect underlying our results introduces an additional channel through which information affects prices: information about the likelihood of a state relative to a second state can impact the price of the asset for a third state. The adjustment related by this contagion effect can induce overreaction to information in the relative prices of the assets for the first two states, as can be shown through simple examples. A general analysis of how prices react to information in the presence of wealth effects and heterogeneous priors is a challenging but promising problem for future research.[36]

[36] With the notable exception of Grossman (1978), most of the REE literature obtaining positive characterizations of equilibrium prices follows Grossman (1976) in restricting attention to CARA preferences and normally distributed returns. In our two-state model, instead, we can be fully general about risk preferences and information, as well as allowing for heterogeneous priors.

Appendix A: Proofs

Proof of Proposition 1. For a given likelihood ratio L, the prior of an individual with posterior belief π_i is, using (1), $q_i = \pi_i / [(1-\pi_i) L + \pi_i]$. The E^c asset is demanded in amount $w_{i0}/(1-p)$ by every individual with $\pi_i < p$, or equivalently $q_i < p/[(1-p)L + p]$. The aggregate demand for this asset is then $G(p/[(1-p)L+p])/(1-p)$. In equilibrium, this aggregate demand is equalized to the aggregate supply, equal to 1, resulting in equation (2).

We complete the proof by noting that the price defined by (2) reveals L, because it is a strictly increasing function of L. The left-hand side of (2) is a strictly increasing continuous function of p, which is 0 when $p=0$ and 1 when $p=1$. For any $L \in (0,\infty)$, the right-hand side is a weakly decreasing continuous function of p, for the cumulative distribution function G is non-decreasing. The right-hand side is equal to 1 at $p=0$, while it is 0 at $p=1$. Thus, there exists a unique solution, such that $G \notin \{0,1\}$. When L rises, the left-hand side is unaffected, while the right-hand side rises for any p, strictly so near the solution to (2) by the assumptions on G. Hence, the solution p must be increasing with L. □

Proof of Proposition 2. The market price p is the posterior belief given information L and market prior belief $p/[(1-p)L+p]$. When L increases, so does p. By equation (2), when p increases, $p/[(1-p)L+p]$ must fall, because the cumulative distribution function G is non-decreasing. □

Proof of Proposition 3. By Proposition 1, $p(L) > p(L')$. By (3), (4) is equivalent to

$$\log\left(\frac{p(L)}{1-p(L)}\right) - \log\left(\frac{p(L')}{1-p(L')}\right) < \log L - \log L',$$

or

$$\frac{p(L)}{1-p(L)}\frac{1}{L} < \frac{p(L')}{1-p(L')}\frac{1}{L'}.$$

Using the strictly increasing transformation $z \to z/(1+z)$ on both sides of this inequality, it is equivalent to

$$\frac{p(L)}{[1-p(L)]L + p(L)} < \frac{p(L')}{[1-p(L')]L' + p(L')},$$

which is true by Proposition 2. □

Proof of Proposition 4. By Proposition 3, the function

$$\Psi(L) = \log\left(\frac{\pi(L)}{1-\pi(L)}\right) - \log\left(\frac{p(L)}{1-p(L)}\right)$$

is strictly increasing in L. Hence, one of the following three cases will hold. In the first case, there exists an $L^* \in (0,\infty)$ such that $\Psi(L)$ is negative for $L < L^*$ and positive for $L > L^*$ — in this case, the result follows with $p^* = p(L^*)$. In the second case, $\Psi(L)$ is negative for all L, and the result holds for $p^* = 1$. In the third case, $\Psi(L)$ is positive for all L, and the result is true with $p^* = 0$. □

Proof of Proposition 5. Note first that $p((1-m)/m) = 1/2$ by (2). Consider now $L > (1-m)/m$ such that the equilibrium prices satisfy $\pi(L) > p(L), p'(L) > 1/2$. If, contrary to the claim, $p(L) < p'(L)$, then (2) implies that

$$G\left(\frac{p(L)}{(1-p(L))L + p(L)}\right) = 1 - p(L) > 1 - p'(L) = G'\left(\frac{p'(L)}{(1-p'(L))L + p'(L)}\right).$$

Further,

$$\frac{p'(L)}{(1-p'(L))L + p'(L)} > \frac{p(L)}{(1-p(L))L + p(L)},$$

while $p'(L) > 1/2$ in equilibrium implies

$$\frac{p'(L)}{(1-p'(L))L + p'(L)} < m.$$

Thus, the median preserving spread property implies the contradiction,

$$G\left(\frac{p(L)}{(1-p(L))L + p(L)}\right) < G'\left(\frac{p'(L)}{(1-p'(L))L + p'(L)}\right).$$

A similar argument applies when $L < (1-m)/m$. □

Proof of Proposition 6. The individual trader solves the problem

$$\max_{\Delta x_i \in [-w_0/(1-p), w_0/p]} \pi_i u_i(w_i(E)) + (1-\pi_i) u_i(w_i(E^c)).$$

Strict concavity of u_i ensures that the maximand Δx_i is unique. By the Theorem of the Maximum, Δx_i is a continuous function of π_i and p. We first show that the optimizer Δx_i is strictly decreasing in p and weakly increasing in π_i, strictly so when $\Delta x_i \in (-w_0/(1-p), w_0/p)$.

The constraint set $[-w_0/(1-p), w_0/p]$ does not depend on π_i and falls in Veinott's set order when p rises. The trader's objective function $\pi_i u_i (W_i + w_0 + (1-p)\Delta x_i) + (1-\pi_i) u_i (W_i + w_0 - p\Delta x_i)$ has first derivative

$$\pi_i (1-p) u_i' (w_i (E)) - (1-\pi_i) p u_i' (w_i (E^c))$$

with respect to Δx_i. Since $u_i' > 0$, the cross-partial of the objective with respect to the choice variable Δx_i and the exogenous π_i is strictly positive, and hence Δx_i is weakly increasing in π_i, strictly so when the unique Δx_i optimizer satisfies (7). A sufficient condition for a strictly negative cross-partial with respect to Δx_i and p is

$$\Delta x_i [\pi_i (1-p) u_i'' (w_i (E)) - (1-\pi_i) p u_i'' (w_i (E^c))] > 0. \tag{13}$$

Using the first order condition for optimality, the second factor of (13) is positive if and only if

$$-\frac{u_i'' (w_i (E^c))}{u_i' (w_i (E^c))} > -\frac{u_i'' (w_i (E))}{u_i' (w_i (E))}.$$

By the DARA assumption, this inequality holds if and only if $w_i (E) > w_i (E^c)$, i.e., $\Delta x_i > 0$. Hence the cross-partial is strictly negative, and it follows that Δx_i is strictly decreasing in p.

Equilibrium is characterized by the requirement that the aggregate purchase of asset E must be zero, i.e., $\int_0^1 \Delta x_i (p, q_i, L) \, dG (q_i) = 0$. When $p = 0$, every trader has $\pi_i > p$ and hence $\Delta x_i > 0$, while the opposite relation holds when $p = 1$. Individual demands are continuous and strictly decreasing in p, so there exists a unique equilibrium price in $(0, 1)$. When L is increased, $\pi_i (L)$ rises, and hence Δx_i rises for every trader. The price must then be strictly increased, in order to restore equilibrium. Finally, since the equilibrating price p is thus a strictly increasing function of L, the equilibrium price schedule is fully revealing. □

Proof of Proposition 7. Suppose for a moment that no trader is constrained in equilibrium. The necessary and sufficient first order condition (7) for the unconstrained optimum is solved by

$$\Delta x_i = t_i \log \left(\frac{1 - p(L)}{p(L)} \frac{\pi_i (L)}{1 - \pi_i (L)} \right). \tag{14}$$

Market clearing occurs when $\int_0^1 \Delta x_i dG (q_i) = 0$. By (14) and using $\pi_i (L) / (1 - \pi_i (L)) = q_i L / (1 - q_i)$ this is solved by $p (L) = qL / (qL + 1 - q)$. Inserting this market price in the individual demand (14), the resulting equilibrium demand is d_i^*, as given in (8). This analysis describes the equilibrium, provided no individual is constrained. The lower bound constrains no individual when $0 > \inf_i d_i^* \geq -w_0/(1-p(L))$, or equivalently $p(L) \geq 1 + w_0/\inf_i d_i^*$. Likewise, the upper bound is equivalent to $p(L) \leq w_0/\sup_i d_i^*$.

When a positive mass of traders are constrained, the bias follows from the argument of Proposition 8 reported below. $\square$

Proof of Proposition 8. The result follows as in the proof of Proposition 3, once we establish that $\log[p(L)/(1-p(L))] - \log(L)$ is strictly decreasing in L. Suppose, for a contradiction, that $\log[p(L)/(1-p(L))] - \log(L)$ is non-decreasing near some L. Traders at the boundary $\Delta x_i = -w_0/(1-p)$ have their demand decreasing in p, and hence $d\Delta x_i/dL < 0$. Likewise, $d\Delta x_i/dL < 0$ at the other boundary $\Delta x_i = w_0/p$. We will argue in the next paragraph that the same effect holds for traders satisfying (7). Since market clearing $\int_0^1 \Delta x_i(p(L), q_i, L)\, dG(q_i) = 0$ implies $\int_0^1 [d\Delta x_i(p, q_i, L)/dL]\, dG(q_i) = 0$, we will then obtain a contradiction establishing the claim.

Since $\log[\pi_i(L)/(1-\pi_i(L))] - \log(L)$ is constant, (7) implies that $u_i'(w_i(E))/u_i'(w_i(E^c))$ is non-decreasing in L. Using the expressions for the final wealth levels (5) and (6), non-negativity of the derivative of $u_i'(w_i(E))/u_i'(w_i(E^c))$ implies that

$$u_i''(w_i(E))\, u_i'(w_i(E^c)) \left[(1-p)\frac{d\Delta x_i}{dL} - \Delta x_i \frac{dp}{dL}\right] \geq -u_i''(w_i(E^c))\, u_i'(w_i(E)) \left[p\frac{d\Delta x_i}{dL} + \Delta x_i \frac{dp}{dL}\right].$$

The second derivative of the utility function is negative, so this implies

$$\frac{d\Delta x_i}{dL} \leq \Delta x_i \frac{dp}{dL} \frac{u_i''(w_i(E))\, u_i'(w_i(E^c)) - u_i''(w_i(E^c))\, u_i'(w_i(E))}{(1-p)\, u_i''(w_i(E))\, u_i'(w_i(E^c)) + p u_i''(w_i(E^c))\, u_i'(w_i(E))}. \tag{15}$$

On the right-hand side of (15), $dp/dL > 0$ by Proposition 6, and the denominator is negative. Recall that $\Delta x_i > 0$ if and only if $w_i(E) > w_i(E^c)$. By DARA, this implies that

$$-\frac{u_i''(w_i(E))}{u_i'(w_i(E))} < -\frac{u_i''(w_i(E^c))}{u_i'(w_i(E^c))}$$

or that the numerator is positive. Likewise, when $\Delta x_i < 0$, the numerator is negative. In either case, the right-hand side of (15) is strictly negative. Hence, $d\Delta x_i/dL < 0$ for every trader who satisfies the first-order condition (7). $\square$

Proof of Proposition 9. We verify that the described outcome is an equilibrium. First, we know from before that $p_1(L_1)$ is injective. From (11), also $p_t(L_t)$ is injective. For the final equilibrium condition, note that the market will clear because trader positions are the same as in the static REE. The remainder of the proof verifies that this constant position is indeed optimal in the individual optimization problem.

Suppose at period t, information L_t has been realized. To save notation, write p_t for the realization of $p_t(L_t)$ and w_{it} for the realization of $w_{it}(L_t)$. Two observations are essential. First, Δx_{it} is at the upper bound (interior, lower bound) of the constraint set $[-w_{it}/(1-p_t), w_{it}/p_t]$ if and only if, for all L_{+1}, Δx_{it} is on the upper bound (interior, lower bound) of the constraint set $[-w_{it+1}(L_{t+1})/(1-p_{it+1}(L_{t+1})), w_{it+1}(L_{t+1})/p_{it+1}(L_{t+1})]$. Second, for all realizations of the string $(L_{t+1}, \ldots, L_T)$, the feasible choice $\Delta x_{iT}(L_T) = \ldots = \Delta x_{it+1}(L_{t+1}) = \Delta x_{it}$ implies

$$\frac{u_i'(W_i + w_{iT}(E))}{u_i'(W_i + w_{iT}(E^c))} = \frac{u_i'(W_i + w_{it} + (1-p_t)\Delta x_{it})}{u_i'(W_i + w_{it} - p_t \Delta x_{it})}.$$

Both observations follow from the wealth evolution equation $w_{i\tau}(L_t) = w_{i\tau-1}(L_{\tau-1}) + (p_\tau(L_\tau) - p_{\tau-1}(L_{\tau-1}))\Delta x_{i\tau-1}(L_{\tau-1})$ for periods $\tau = t+1, \ldots, T$.

To prove our claim that the trader in every period selects the same position $\Delta x_{it} = \Delta x_{i1}(L_1)$ as in the static model given price $p_1(L_1)$, we proceed by backwards induction. The induction hypothesis t states that the agent in period t given price $p_t(L_t)$ (i) chooses Δx_{it} to satisfy the static first-order condition

$$\frac{p_t(L_t)}{1-p_t(L_t)} = \frac{\pi_i(L_t)}{1-\pi_i(L_t)} \frac{u_i'(W_i + w_{it}(L_t) + (1-p_t(L_t))\Delta x_{it})}{u_i'(W_i + w_{it}(L_t) - p_t(L_t)\Delta x_{it})}$$

if feasible, or (ii) chooses $\Delta x_{it} = w_{it}(L_t)/p_t(L_t)$ if the left hand side of this static condition is below the right hand side at this choice, and (iii) chooses $\Delta x_{it} = -w_{it}(L_t)/(1-p_t(L_t))$ if the left hand side of this static condition exceeds the right hand side at this choice. Note from the previous two essential observations, that once we have proved the induction hypothesis for all t, we have $\Delta x_{iT}(L_T) = \ldots = \Delta x_{i1}(L_1)$, and $\Delta x_{i1}(L_1)$ is the solution to the individual problem in Proposition 6.

The induction hypothesis T is satisfied because the static first-order condition characterizes the solution to the remaining one-period problem. We now assume that the induction hypothesis is true at $t+1, \ldots, T$, and will prove that induction hypothesis $t < T$ is true. Suppose at period t, information L_t has been realized. Final wealth will be

$$w_{iT}(E) = W_i + w_{it} + (p_{t+1}(L_{t+1}) - p_t)\Delta x_{it} + (1 - p_{t+1}(L_{t+1}))\Delta x_{it+1}(L_{t+1})$$

and

$$w_{iT}(E^c) = W_i + w_{it} + (p_{t+1}(L_{t+1}) - p_t)\,\Delta x_{it} - p_{t+1}(L_{t+1})\,\Delta x_{it+1}(L_{t+1})$$

where $\Delta x_{it+1}(L_{t+1})$ is the reaction prescribed by induction hypothesis $t+1$. The time t problem is

$$\max_{\Delta x_{it} \in [-w_{it}/(1-p_t), w_{it}/p_t]} \pi_i(L_t)\, E[u_i(w_{iT}(E))\,|E] + (1-\pi_i(L_t))\, E[u_i(w_{iT}(E^c))\,|E^c]$$

where the expectations are taken over the realization of L_{t+1}. In case (i), the static first-order condition can be satisfied with an interior choice of Δx_t. Evaluated at this choice, the derivative of the time t objective function is, by the envelope theorem,

$$\begin{aligned}
& \pi_i(L_t)\, E[(p_{t+1}(L_{t+1}) - p_t)\, u_i'(w_{iT}(E))\,|E] \\
& + (1-\pi_i(L_t))\, E[(p_{t+1}(L_{t+1}) - p_t)\, u_i'(w_{iT}(E^c))\,|E^c] \\
= \; & p_t E\left[\tfrac{\pi_i(L_t) u_i'(w_{iT}(E))}{p_t}(p_{t+1}(L_{t+1}) - p_t)\,|E\right] \\
& + (1-p_t)\, E\left[\tfrac{(1-\pi_i(L_t)) u_i'(w_{iT}(E^c))}{1-p_t}(p_{t+1}(L_{t+1}) - p_t)\,|E^c\right].
\end{aligned}$$

Here $w_{iT}(E)$ and $w_{iT}(E^c)$ are constant across realizations of L_{t+1}. The static first-order condition then allows us to rewrite the derivative with respect to the control variable as

$$\frac{\pi_i(L_t)\, u_i'(w_{iT}(E))}{p_t} \left\{ p_t E[p_{t+1}(L_{t+1}) - p_t | E] + (1-p_t)\, E[(p_{t+1}(L_{t+1}) - p_t)\,|E^c] \right\}.$$

By the martingale property of Bayes-updated prices at market belief p_t, we have

$$p_t E[p_{t+1}(L_{t+1}) - p_t | E] + (1-p_t)\, E[(p_{t+1}(L_{t+1}) - p_t)\,|E^c] = 0.$$

Thus, the first order condition for optimality of Δx_{it} is satisfied at the choice resulting from the static model. The two other cases (with constrained choices) follow likewise.

Proof of Proposition 10. For a given realization of L_{t_2}, we denote for simplicity the resulting price by $p = p_{t_2}(L_{t_2})$ and the natural posterior by $\pi = qL_{t_2}/(qL_{t_2} + 1 - q)$. Under the natural posterior, we have

$$\begin{aligned}
E[p_{t_1}(L_{t_1}) - p | L_{t_2}] &= \pi E[p_{t_1}(L_{t_1}) - p | E] + (1-\pi)\, E[p_{t_1}(L_{t_1}) - p | E^c] \\
&= (\pi - p)\left\{ E[p_{t_1}(L_{t_1}) - p | E] - E[p_{t_1}(L_{t_1}) - p | E^c] \right\},
\end{aligned}$$

using the martingale property of prices at the market belief p. Next,

$$p_{t_1}(L_{t_1}) = \frac{pL_{t_1}}{pL_{t_1} + (1-p)\, L_{t_2}} = p + \frac{(1-p)\, p\, (L_{t_1} - L_{t_2})}{pL_{t_1} + (1-p)\, L_{t_2}}$$

follows from equation (11). At t_2, there is uncertainty about the realization of the future L_{t_1}. Bayes' rule implies $L_{t_1}/L_{t_2} = f_{t_2}(L_{t_1}|E)/f_{t_2}(L_{t_1}|E^c)$ where f_{t_2} denotes the p.d.f. for L_{t_1}. Collecting these pieces, we obtain

$$E[p_{t_1}(L_{t_1}) - p|L_{t_2}] = (\pi - p)\int_0^\infty \frac{(1-p)\,p\,(L_{t_1} - L_{t_2})}{pL_{t_1} + (1-p)\,L_{t_2}}(L_{t_1} - L_{t_2})\,f_{t_2}(L_{t_1}|E^c)\,dL_{t_1}.$$

Now, averaging over realizations of L_{t_2}, we find

$$\begin{aligned}
& E\left[(p_{t_1}(L_{t_1}) - p_{t_2}(L_{t_2}))\,(p_{t_2}(L_{t_2}) - p(1))\right] \\
= \;& E\left[E\left[(p_{t_1}(L_{t_1}) - p_{t_2}(L_{t_2}))\,(p_{t_2}(L_{t_2}) - q)\,|L_{t_2}\right]\right] \\
= \;& E\left[(p_{t_2}(L_{t_2}) - q)\,(\pi(L_{t_2}) - p_{t_2}(L_{t_2}))\int_0^\infty \frac{(1-p_{t_2}(L_{t_2}))p_{t_2}(L_{t_2})(L_{t_1}-L_{t_2})^2}{p_{t_2}(L_{t_2})L_{t_1}+(1-p_{t_2}(L_{t_2}))L_{t_2}} f_{t_2}(L_{t_1}|E^c)\,dL_{t_1}\right].
\end{aligned}$$

Underreaction states that $(p_{t_2}(L_{t_2}) - q)\,(\pi(L_{t_2}) - p_{t_2}(L_{t_2})) > 0$ for all $L_{t_2} \neq 1$. Since all terms in the expectation are positive, we have proved (12). $\square$

Appendix B: Heterogeneous Endowments and Common Prior

In Proposition 8, traders are motivated to actively trade from the initial allocation because they have heterogeneous prior beliefs. It is natural to wonder how essential is the heterogeneous prior assumption for the result that the price does not react one-for-one to information. In particular, would wealth effects lead to underreaction more generally when traders are motivated to trade because of traditional liquidity motives (e.g., risk-sharing when individual endowments depend on the state), even when they share a common prior belief? This appendix shows that the price reacts one-for-one to information for a broad class of preferences, which include our leading logarithmic example from Section 5.

We modify the model from that section by assuming that all traders share the common prior q, but allowing trader's i initial endowment, $(w_{i0}(E), w_{i0}(E^c))$, to vary across states, $w_{i0}(E) \neq w_{i0}(E^c)$. For simplicity, we do not constrain the positions traders can take.

Suppose that there exists constants α_i and β such that trader i has Hyperbolic Absolute Risk Aversion (HARA), $-u_i''(w)/u_i'(w) = 1/(\alpha_i + \beta w)$. The fact that β is constant across traders means that traders are equally cautious.[37]

Proposition 11 *If all agents have HARA preferences with common cautiousness parameter, then the market price reacts as a Bayesian posterior belief to information.*

The result follows from Rubinstein's (1974) observation that in this case there exists a representative trader. Denoting the utility function of this representative trader by U, the equilibrium price $p(L)$ must then satisfy the equivalent of (7),

$$\frac{qL}{1-q}\frac{U'(w_0(E))}{U'(w_0(E^c))} = \frac{p(L)}{1-p(L)},$$

where $w_0(E) = \int_{i\in I} w_{i0}(E)\,di$ and $w_0(E^c) = \int_{i\in I} w_{i0}(E^c)\,di$ denote the aggregate endowments in states E and E^c. In the special case without aggregate uncertainty ($w_0(E) = w_0(E^c)$), the market price is equal to the common posterior. More generally, in the presence of aggregate uncertainty the price still reacts one-for-one to information.

To understand more generally why the logic of Section 5 does not carry over to this model, reconsider its Edgeworth box illustration. The box is no longer a square when $w_0(E) \neq w_0(E^c)$. Without loss of generality, suppose $w_0(E) > w_0(E^c)$. Unlike in

[37] In the special case with $\beta \geq 0$, the absolute risk aversion $1/T$ is decreasing in w. In this case, these preferences are a special case of DARA preferences. CARA results when $\beta = 0$.

Figure 2, the two risk-averse traders with common prior have an equilibrium bundle on the same side of their respective diagonal (i.e., $w_1(E) > w_1(E^c)$ and $w_2(E) > w_2(E^c)$). The DARA income expansion paths no longer contradict the possibility of staying in equilibrium when the price is Bayes updated to information. With general risk-averse preferences, underreaction or overreaction to information depends on demand elasticity.

To further illustrate the difference of heterogeneous endowments to heterogeneous priors, reconsider the model with a continuum of traders with logarithmic preferences. Logarithmic preferences correspond to our leading DARA example from Section 5 and belong to the HARA class with $\alpha_i = 0$ and common cautiousness parameter $\beta = 1$. Denote by π the common posterior belief given information L. The first-order condition for Δx_i is

$$\frac{\pi}{1-\pi}\frac{w_{i0}(E^c) - p\Delta x_i}{w_{i0}(E) + (1-p)\Delta x_i} = \frac{p}{1-p}. \tag{16}$$

Solving for equilibrium using (16) and market clearing, $\int_{i\in I} \Delta x_i di = 0$, we obtain

$$\frac{\pi}{1-\pi}\frac{w_0(E^c)}{w_0(E)} = \frac{p(L)}{1-p(L)}. \tag{17}$$

Combining (16) with (17), trader i's net asset position is

$$\Delta x_i = \frac{\pi}{p}\left(w_{i0}(E^c) - \frac{w_0(E^c)}{w_0(E)} w_{i0}(E)\right).$$

Thus, in this risk-sharing model, trader i has a long (or short) equilibrium asset position, $\Delta x_i > 0$ (or $\Delta x_i < 0$), when i is initially endowed with less (or more) of the E asset than the aggregate market. The size of the asset position, $|\Delta x_i|$, is increasing in π/p. Now note that the expected return from the first asset,

$$\frac{\pi}{p} = (1-\pi)\frac{w_0(E)}{w_0(E^c)} + \pi,$$

is strictly monotone in L when $w_0(E) \neq w_0(E^c)$. If $w_0(E) > w_0(E^c)$ then *all* traders take more extreme positions when L increases—and, conversely, they all take less extreme positions if instead $w_0(E) < w_0(E^c)$. In this setting, when the price responds to information as a posterior belief, buyers buy more aggressively when sellers sell more aggressively. Thus, there is no reweighing across traders and no countervailing adjustment in the price depending on the realized information, L. With heterogeneous beliefs, instead, we found that optimists (who buy) buy less, while pessimists (who sell) sell more when L increases—so the price had to equilibrate against the direction of the information.

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