Relational algebra by way of adjunctions
Gibbons, Jeremy; Henglein, Fritz; Hinze, Ralf; Wu, Nicolas

Publication date:
2016

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Relational Algebra by Way of Adjunctions

Jeremy Gibbons
(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)
DBPL, October 2015
1. Summary

- bulk types (sets, bags, lists) are *monads*
- monads have nice *mathematical foundations via adjunctions*
- monads support *comprehensions*
- comprehension syntax provides a *query notation*

\[
[ (\text{customer}.\text{name}, \text{invoice}.\text{amount}) \\
| \text{customer} \leftarrow \text{customers}, \\
\text{invoice} \leftarrow \text{invoices}, \\
\text{customer}.\text{cid} = \text{invoice}.\text{customer}, \\
\text{invoice}.\text{due} \leq \text{today} ]
\]

- monad structure explains *selection, projection*
- less obvious how to explain *join*
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq)\] means \(f b \leq a \iff b \sqsubseteq g a\)

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \(n \times k \leq m \iff n \leq m \div k\), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set* $|\mathbf{C}|$ of objects,
- a set* $\mathbf{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathbf{C}|$,
- identity arrows $\text{id}_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[\ldots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots\]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a **concrete category**: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category **CMon** has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
    h (m \otimes n) &= h m \oplus h n \\
    h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category **Set** has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F \ id_X = id_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, forgetful functor $U : \text{CMon} \to \text{Set}$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, $\text{Free} : \text{Set} \to \text{CMon}$ generates the free commutative monoid (ie bags) on a set of elements:

$$\text{Free } A = (\text{Bag } A, \cup, \emptyset)$$
$$\text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $C$, $D$, and functors $L : D \to C$ and $R : C \to D$, adjunction

\[
\begin{array}{ccc}
  \downarrow & \Leftrightarrow & \downarrow \\
  L & & R \\
  C & & D
\end{array}
\]

means

\[
[-] : C(L X, Y) \simeq D(X, R Y) : [-]
\]

A familiar example is given by currying:

\[
\begin{array}{ccc}
  \downarrow & \Leftrightarrow & \downarrow \\
  - \times P & & (\cdot)^P \\
  \text{Set} & & \text{Set}
\end{array}
\]

with $curry : \text{Set}(X \times P, Y) \simeq \text{Set}(X, Y^P) : curry^\circ$

hence definitions and properties of $apply = uncurry \ id_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

with

\[
\text{fork} : \text{Set}^2(\Delta A, (B, C)) \cong \text{Set}(A, B \times C) : \text{fork}^\circ \\
\text{junc}^\circ : \text{Set}(A + B, C) \cong \text{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
\text{dup} = \text{fork} \ id_{A,A} : \text{Set}(A, A \times A) \\
(fst, snd) = \text{fork}^\circ \ id_{B \times C} : \text{Set}^2(\Delta (B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{array}{c}
\text{CMon} \downarrow \text{Set} \\
\text{Free} \\
\end{array}
\]

\[ \text{with } [-] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \cong \text{Set}(A, U (M, \otimes, \epsilon)) : [-] \]

Unit and counit:

\[
\begin{align*}
\text{single } A &= [id_{\text{Free } A}] : A \to U (\text{Free } A) \\
\text{reduce } M &= [id_M] : \text{Free } (U M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \( h : \text{Free } A \to M \) and \( f : A \to U M = M \),

\[ h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f \]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>$(\mathbb{N}, 0, +)$</td>
<td>${ a } \mapsto 1$</td>
</tr>
<tr>
<td>sum</td>
<td>$(\mathbb{R}, 0, +)$</td>
<td>${ a } \mapsto a$</td>
</tr>
<tr>
<td>max</td>
<td>$(\mathbb{Z}, \text{minBound}, \text{max})$</td>
<td>${ a } \mapsto a$</td>
</tr>
<tr>
<td>min</td>
<td>$(\mathbb{Z}, \text{maxBound}, \text{min})$</td>
<td>${ a } \mapsto a$</td>
</tr>
<tr>
<td>all</td>
<td>$(\mathbb{B}, \text{True}, \land)$</td>
<td>${ a } \mapsto a$</td>
</tr>
<tr>
<td>any</td>
<td>$(\mathbb{B}, \text{False}, \lor)$</td>
<td>${ a } \mapsto a$</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

$$\text{guard} : (A \to \mathbb{B}) \to \text{Bag } A \to \text{Bag } A$$

$$\text{guard } p \ a = \text{if } p \ a \text{ then } \{ a \} \text{ else } \emptyset$$

Laws about selections follow from laws of homomorphisms (and of coproducts, since $\mathbb{B} = 1 + 1$).
10. Monads

Bags form a *monad* \((Bag, \text{union}, \text{single})\) with

\[
\text{Bag} = U \cdot \text{Free} \\
\text{union} : \text{Bag} (\text{Bag} A) \to \text{Bag} A \\
\text{single} : A \to \text{Bag} A
\]

which justifies the use of comprehension notation \(\{f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a\}\).

In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
T = R \cdot L \\
\mu A = R \ [id_A] \ L : T (T A) \to T A \\
\eta A = [id_A] : A \to T A
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the Reader monad in Haskell), so arise from an adjunction.

The laws of exponents arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \simeq 1$
- $\text{Map } 1 V \simeq V$
- $\text{Map } (K_1 + K_2) V \simeq \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \simeq \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \simeq 1$
- $\text{Map } K (V_1 \times V_2) \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \xrightarrow{J} \text{Set} \xleftarrow{E} \text{Set} \]

where \( J \) embeds, and \( E R : A \rightarrow \text{Set} B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V) \]

Together, \textit{index} and \textit{merge} give efficient relational joins:

\[ x f \bowtie g y = \text{flatten} (\text{Map} K \text{ cp} (\text{merge} (\text{groupBy} f x, \text{groupBy} g y))) \]

\[ \text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag} V \rightarrow \text{Map} K (\text{Bag} V) \]

\[ \text{flatten} : \text{Map} K (\text{Bag} V) \rightarrow \text{Bag} V \]
13. Pointed sets and finite maps

Model *finite maps* \( \text{Map}_\ast \) not as partial functions, but *total* functions to a *pointed* codomain \((A, a)\), i.e. a set \(A\) with a distinguished element \(a : A\).

Pointed sets and point-preserving functions form a category \(\text{Set}_\ast\).

There is an adjunction to \(\text{Set}\), via

\[
\begin{array}{ccc}
\text{Set}_\ast & \dashv & \text{Set} \\
\downarrow \quad \text{Maybe} & & \quad \uparrow \\
\downarrow \quad \text{U} & & \quad \\
\end{array}
\]

where \(\text{Maybe} A \cong 1 + A\) adds a point, and \(\text{U} (A, a) = A\) discards it.

In particular, \((\text{Bag} A, \emptyset)\) is a pointed set. Moreover, \(\text{Bag} f\) is point-preserving, so we get a functor \(\text{Bag}_\ast : \text{Set} \to \text{Set}_\ast\).

Indexing remains an isomorphism:

\[
\text{index} : \text{Bag}_\ast (K \times V) \cong \text{Map}_\ast K (\text{Bag}_\ast V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta_a = \lambda k \rightarrow a : A \rightarrow \text{Map } K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid $$(M, \otimes, \epsilon)$$,

$$\mu_X : T_m (T_n X) \rightarrow T_{m \otimes n} X$$
$$\eta_X : X \rightarrow T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid $$(\mathbb{K}, \times, 1)$$ of finite key types under product.
15. Conclusions

- *Monad comprehensions* for database queries
- Structure arising from *adjunctions*
- Equivalences from *universal properties*
- Fitting in *relational joins*, via indexing
- To do: calculating *query optimisations*

Thanks to EPSRC *Unifying Theories of Generic Programming* for funding.