Relational algebra by way of adjunctions
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Relational Algebra by Way of Adjunctions

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(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)
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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation
  
  \[
  \left[ (\text{customer}.\text{name}, \text{invoice}.\text{amount})
  \mid \text{customer} \leftarrow \text{customers},
  \text{invoice} \leftarrow \text{invoices},
  \text{customer}.\text{cid} = \text{invoice}.\text{customer},
  \text{invoice}.\text{due} \leq \text{today} \right]
  \]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[ (A, \leq) \perp (B, \sqsubseteq) \]

means \( f \ b \leq a \iff b \sqsubseteq g \ a \)

For example,

\[ (\mathbb{R}, \leq_\mathbb{R}) \perp (\mathbb{Z}, \leq_\mathbb{Z}) \]

\[ (\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq) \]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives
\( n \times k \leq m \iff n \leq m \div k \), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $C$ consists of

- a set* $|C|$ of objects,
- a set* $C(X, Y)$ of arrows $X \to Y$ for each $X, Y : |C|$, 
- identity arrows $id_X : X \to X$ for each $X$ 
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$, 
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \cdots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a concrete category: roughly,

- the objects are sets with additional structure
- the arrows are structure-preserving mappings

Many useful categories are of this form.

For example, the category \( \text{CMon} \) has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \( h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon') \) as arrows:

\[
\begin{align*}
    h (m \otimes n) &= h m \oplus h n \\
    h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category \( \text{Set} \) has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor \( F : C \to D \) is an operation on both objects and arrows, preserving the structure: \( F f : F X \to F Y \) when \( f : X \to Y \), and

\[
F \ id_X = id_{F X} \\
F (f \cdot g) = F f \cdot F g
\]

For example, forgetful functor \( U : \text{CMon} \to \text{Set} \):

\[
U (M, \otimes, \epsilon) = M \\
U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'
\]

Conversely, \( \text{Free} : \text{Set} \to \text{CMon} \) generates the free commutative monoid (ie bags) on a set of elements:

\[
\text{Free} A = (\text{Bag} A, \cup, \emptyset) \\
\text{Free} (f : A \to B) = \text{map} f : \text{Bag} A \to \text{Bag} B
\]
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections. Given categories $\mathbf{C}, \mathbf{D}$, and functors $L : \mathbf{D} \to \mathbf{C}$ and $R : \mathbf{C} \to \mathbf{D}$, adjunction

$$\begin{array}{ccc}
\mathbf{C} & \perp & \mathbf{D} \\
\downarrow & & \downarrow \\
\mathbf{D} & \perp & \mathbf{C}
\end{array}$$

means* $[-] : \mathbf{C}(L X, Y) \simeq \mathbf{D}(X, R Y) : [-]$.

A familiar example is given by currying:

$$\begin{array}{ccc}
\text{Set} & \perp & \text{Set} \\
\downarrow & & \downarrow \\
\text{Set} & \perp & \text{Set}
\end{array}$$

with $\text{curry} : \text{Set}(X \times P, Y) \simeq \text{Set}(X, Y^P) : \text{curry}^\circ$

hence definitions and properties of $\text{apply} = \text{uncurry id}_{Y^P} : Y^P \times P \to Y$.
7. Products and coproducts

![Diagram](image)

with

\[
\text{fork} : \mathbf{Set}^2(\Delta A, (B, C)) \cong \mathbf{Set}(A, B \times C) : \text{fork}^\circ
\]

\[
\text{junc}^\circ : \mathbf{Set}(A + B, C) \cong \mathbf{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
dup = \text{fork \ id}_{A,A} : \mathbf{Set}(A, A \times A)
\]

\[
(fst, \ snd) = \text{fork}^\circ \ id_{B \times C} : \mathbf{Set}^2(\Delta(B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[ \text{CMon} \dashv \text{Set} \]

with \([ - ] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \approx \text{Set}(A, U (M, \otimes, \epsilon)) : [-] \)

Unit and counit:

\[ \text{single } A = [id_{\text{Free } A}] : A \rightarrow U (\text{Free } A) \]
\[ \text{reduce } M = [id_M] : \text{Free } (U M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \epsilon) \]

whence, for \( h : \text{Free } A \rightarrow M \) and \( f : A \rightarrow U M = M \),

\[ h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f \]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
## 9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>(\lfloor a \rfloor \to 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>(\lfloor a \rfloor \to a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \minBound, \max))</td>
<td>(\lfloor a \rfloor \to a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \maxBound, \min))</td>
<td>(\lfloor a \rfloor \to a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>(\lfloor a \rfloor \to a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>(\lfloor a \rfloor \to a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \to \mathbb{B}) \to \text{Bag } A \to \text{Bag } A
\]

\[guard \ p \ a = \text{if } p \ a \text{ then } \lfloor a \rfloor \text{ else } \emptyset\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a monad \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag}(\text{Bag} A) \to \text{Bag} A \\
\text{single} & : A \to \text{Bag} A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \}\).

In fact, for any adjunction \(L \dashv R\) between \(\mathbf{C}\) and \(\mathbf{D}\), we get a monad \((T, \mu, \eta)\) on \(\mathbf{D}\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu_A & = R \,[id_A] L : T(T A) \to T A \\
\eta_A & = [id_A] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the Reader monad in Haskell), so arise from an adjunction.

The laws of exponents arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \simeq 1$
- $\text{Map } 1 V \simeq V$
- $\text{Map } (K_1 + K_2) V \simeq \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \simeq \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \simeq 1$
- $\text{Map } K (V_1 \times V_2) \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\text{Rel} \downarrow \downarrow \text{Set}
\]

where \( J \) embeds, and \( E R : A \rightarrow \text{Set } B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[
\text{index} : \text{Bag}(K \times V) \simeq \text{Map } K(\text{Bag } V)
\]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[
x f \bowtie g y = \text{flatten}(\text{Map } K \text{ cp } (\text{merge}(\text{groupBy } f x, \text{groupBy } g y)))
\]

\[
\text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag } V \rightarrow \text{Map } K(\text{Bag } V)
\]

\[
\text{flatten} : \text{Map } K(\text{Bag } V) \rightarrow \text{Bag } V
\]
13. Pointed sets and finite maps

Model finite maps $\text{Map}_*$ not as partial functions, but total functions to a pointed codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$\begin{array}{ccc}
\text{Set}_* & \overset{\perp}{\longrightarrow} & \text{Set} \\
\downarrow & & \downarrow \quad \text{U} \\
\downarrow & & \downarrow \\
\text{Maybe} & \overset{\mapsto}{\longrightarrow} & \text{Set} \\
\end{array}$$

where $\text{Maybe } A \cong 1 + A$ adds a point, and $\text{U } (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

$$\text{index} : \text{Bag}_* (K \times V) \cong \text{Map}_* K (\text{Bag}_* V)$$
14. Graded monads

A catch: finite maps aren’t a monad, because

\[ \eta a = \lambda k \rightarrow a : A \rightarrow \text{Map } K A \]

in general yields an infinite map.

However, finite maps are a graded monad\(^*\): for monoid \((M, \otimes, \epsilon)\),

\[ \mu X : T_m (T_n X) \rightarrow T_{m \otimes n} X \]
\[ \eta X : X \rightarrow T_\epsilon X \]

satisfying the usual laws. These too arise from adjunctions\(^*\).

We use the monoid \((K, \times, 1)\) of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

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