Relational algebra by way of adjunctions
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Relational Algebra by Way of Adjunctions

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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
\begin{align*}
[ (\text{customer}.\text{name}, \text{invoice}.\text{amount}) \\
| \text{customer} \leftarrow \text{customers}, \\
\text{invoice} \leftarrow \text{invoices}, \\
\text{customer}.\text{cid} = \text{invoice}.\text{customer}, \\
\text{invoice}.\text{due} \leq \text{today} ]
\end{align*}
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[
(A, \leq) \perp (B, \sqsubseteq) \quad \text{means } f(b) \leq a \iff b \sqsubseteq g(a)
\]

For example,

\[
(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \\
(\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq)
\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \(n \times k \leq m \iff n \leq m \div k\), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set* $|\mathbf{C}|$ of objects,
- a set* $\mathbf{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathbf{C}|$,
- identity arrows $\text{id}_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to \cdots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category \( \textbf{CMon} \) has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \( h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon') \) as arrows:

\[
\begin{align*}
    h (m \otimes n) &= h m \oplus h n \\
    h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category \( \textbf{Set} \) has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F \ id_X = id_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, forgetful functor $U : \text{CMon} \to \text{Set}$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, $\text{Free} : \text{Set} \to \text{CMon}$ generates the free commutative monoid (ie bags) on a set of elements:

$$\text{Free} A = (\text{Bag} A, \cup, \emptyset)$$
$$\text{Free} (f : A \to B) = \text{map} f : \text{Bag} A \to \text{Bag} B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories \( \mathbf{C}, \mathbf{D} \), and functors \( L : \mathbf{D} \to \mathbf{C} \) and \( R : \mathbf{C} \to \mathbf{D} \), adjunction

\[
\begin{array}{ccc}
\mathbf{C} & \Downarrow & \mathbf{D} \\
\circlearrowleft & & \circlearrowright \\
R & & L
\end{array}
\]

means*

\[ [-] : \mathbf{C}(L X, Y) \simeq \mathbf{D}(X, R Y) : [-] \]

A familiar example is given by **currying**:

\[
\begin{array}{ccc}
\mathbf{Set} & \Downarrow & \mathbf{Set} \\
\circlearrowleft & & \circlearrowright \\
\circlearrowleft & & \circlearrowright \\
(-)P & & (-)P \\
\end{array}
\]

with \( \text{curry} : \mathbf{Set}(X \times P, Y) \simeq \mathbf{Set}(X, Y^P) : \text{curry}^\circ \)

hence definitions and properties of \( \text{apply} = \text{uncurry} \ id_{Y^P} : Y^P \times P \to Y \)
7. Products and coproducts

\[
\begin{array}{c}
\text{Set} \\
\downarrow \quad \Delta \quad \downarrow \\
\downarrow \quad \Delta \quad \downarrow \\
\text{Set} \\
\end{array}
\]

with

\[
\text{fork} : \text{Set}^2(\Delta A, (B, C)) \cong \text{Set}(A, B \times C) : \text{fork}^\circ \\
\text{junc}^\circ : \text{Set}(A + B, C) \cong \text{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
dup = \text{fork id}_{A,A} : \text{Set}(A, A \times A) \\
(fst, snd) = \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta (B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{array}{c}
\text{CMon} \\
\downarrow \\
\text{Set}
\end{array}
\quad \Rightarrow
\begin{array}{c}
\text{with } [-] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \\
\cong \text{Set}(A, U (M, \otimes, \epsilon)) : [-]
\end{array}
\]

Unit and counit:

- \(\text{single } A = [id_{\text{Free } A}] : A \to U (\text{Free } A)\)
- \(\text{reduce } M = [id_M] : \text{Free } (U M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)\)

whence, for \(h : \text{Free } A \to M\) and \(f : A \to U M = M\),

\[h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>$(\mathbb{N}, 0, +)$</td>
<td>$\lfloor a \rfloor \rightarrow 1$</td>
</tr>
<tr>
<td>sum</td>
<td>$(\mathbb{R}, 0, +)$</td>
<td>$\lfloor a \rfloor \rightarrow a$</td>
</tr>
<tr>
<td>max</td>
<td>$(\mathbb{Z}, \text{minBound}, \text{max})$</td>
<td>$\lfloor a \rfloor \rightarrow a$</td>
</tr>
<tr>
<td>min</td>
<td>$(\mathbb{Z}, \text{maxBound}, \text{min})$</td>
<td>$\lfloor a \rfloor \rightarrow a$</td>
</tr>
<tr>
<td>all</td>
<td>$(\mathbb{B}, \text{True}, \wedge)$</td>
<td>$\lfloor a \rfloor \rightarrow a$</td>
</tr>
<tr>
<td>any</td>
<td>$(\mathbb{B}, \text{False}, \lor)$</td>
<td>$\lfloor a \rfloor \rightarrow a$</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

$\text{guard} : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A$

$\text{guard } p \ a = \text{if } p \ a \ \text{then } \lfloor a \rfloor \ \text{else } \emptyset$

Laws about selections follow from laws of homomorphisms (and of coproducts, since $\mathbb{B} = 1 + 1$).
10. Monads

Bags form a monad \((\text{Bag, union, single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag } A) \to \text{Bag } A \\
\text{single} & : A \to \text{Bag } A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \} \).

In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \left[ id_A \right] L : T (T A) \to T A \\
\eta A & = [ id_A ] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the *Reader* monad in Haskell), so arise from an adjunction.

The *laws of exponents* arise from this adjunction, and from those for products and coproducts:

\[
\begin{align*}
\text{Map } 0 V & \approx 1 \\
\text{Map } 1 V & \approx V \\
\text{Map } (K_1 + K_2) V & \approx \text{Map } K_1 V \times \text{Map } K_2 V \\
\text{Map } (K_1 \times K_2) V & \approx \text{Map } K_1 (\text{Map } K_2 V) \\
\text{Map } K 1 & \approx 1 \\
\text{Map } K (V_1 \times V_2) & \approx \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}
\end{align*}
\]
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \rightarrow\!
\left\downarrow \text{Set} \right\rightarrow\!
\text{E} \rightarrow\!
\left\uparrow \text{J} \right\rightarrow\!
\text{Rel} \]

where \( \text{J} \) embeds, and \( \text{E} \ R : A \rightarrow \text{Set} \ B \) for \( R : A \sim B \).
Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} \ (K \times V) \simeq \text{Map} \ K \ (\text{Bag} \ V) \]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[ x \ f \bowtie g \ y = \text{flatten} \ (\text{Map} \ K \ \text{cp} \ (\text{merge} \ (\text{groupBy} \ f \ x, \text{groupBy} \ g \ y))) \]

\[ \text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag} \ V \rightarrow \text{Map} \ K \ (\text{Bag} \ V) \]

\[ \text{flatten} \ : \text{Map} \ K \ (\text{Bag} \ V) \rightarrow \text{Bag} \ V \]
13. Pointed sets and finite maps

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$
\begin{array}{ccc}
\text{Set}_* & \dashv & \bot \\
\nearrow & \quad & \nearrow \\
\text{Set} & \quad & \text{Set}_*
\end{array}
$$

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $U (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \rightarrow \text{Set}_*$.

Indexing remains an isomorphism:

$$
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
$$
14. Graded monads

A catch: finite maps aren’t a monad, because

\[ \eta a = \lambda k \rightarrow a : A \rightarrow \text{Map } K A \]

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

\[
\begin{align*}
\mu X &: T_m (T_n X) \rightarrow T_{m \otimes n} X \\
\eta X &: X \rightarrow T_\epsilon X 
\end{align*}
\]

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

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