Relational algebra by way of adjunctions
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Relational Algebra by Way of Adjunctions

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(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)
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1. Summary

- bulk types (sets, bags, lists) are *monads*
- monads have nice *mathematical foundations via adjunctions*
- monads support *comprehensions*
- comprehension syntax provides a *query notation*

```
[ (customer.name, invoice.amount)
| customer ← customers,
  invoice ← invoices,
  customer.cid = invoice.customer,
  invoice.due ≤ today ]
```

- monad structure explains *selection, projection*
- less obvious how to explain *join*
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq) \text{ means } f b \leq a \iff b \sqsubseteq g a\]

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \text{ and } (\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq)\]

“Change of coordinates” can sometimes simplify reasoning; e.g., rhs gives
\[n \times k \leq m \iff n \leq m \div k,\] and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $C$ consists of

- a set* $|C|$ of objects,
- a set* $C(X, Y)$ of arrows $X \to Y$ for each $X, Y : |C|$, 
- identity arrows $id_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \ldots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category **CMon** has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h: (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
h (m \otimes n) &= h m \oplus h n \\
h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category **Set** has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A **functor** $F : C \rightarrow D$ is an operation on both objects and arrows, preserving the structure: $F \ f : F \ X \rightarrow F \ Y$ when $f : X \rightarrow Y$, and

$$
F \ id_X = id_{F \ X}
$$

$$
F \ (f \cdot g) = F \ f \cdot F \ g
$$

For example, **forgetful** functor $U : CMon \rightarrow Set$:

$$
U \ (M, \otimes, \epsilon) = M
$$

$$
U \ (h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')) = h : M \rightarrow M'
$$

Conversely, $Free : Set \rightarrow CMon$ generates the **free** commutative monoid (ie bags) on a set of elements:

$$
Free \ A = (Bag \ A, \cup, \emptyset)
$$

$$
Free \ (f : A \rightarrow B) = map \ f : Bag \ A \rightarrow Bag \ B
$$
### 6. Adjunctions

**Adjunctions** are the categorical generalisation of Galois connections.

Given categories $C, D$, and functors $L : D \to C$ and $R : C \to D$, adjunction

\[
\begin{array}{c}
\text{C} \\
\downarrow \\
\text{D} \\
\downarrow \\
\text{D} \\
\uparrow \\
\text{C}
\end{array}
\]

means\(^\ast\) $[-] : C(LX, Y) \simeq D(X, RY) : [-]$.

A familiar example is given by *currying*:

\[
\begin{array}{c}
\text{Set} \\
\downarrow \\
\text{Set} \\
\downarrow \\
\text{Set} \\
\uparrow \\
\text{Set}
\end{array}
\]

with $\text{curry} : \text{Set}(X \times P, Y) \simeq \text{Set}(X, Y^P) : \text{curry}^\circ$.

hence definitions and properties of $\text{apply} = \text{uncurry} \ id_{Y^P} : Y^P \times P \to Y$. 
7. Products and coproducts

with

\[ \text{fork} : \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) \quad : \text{fork}^\circ \]
\[ \text{junc}^\circ : \text{Set}(A + B, C) \quad \simeq \text{Set}^2((A, B), \Delta C) : \text{junc} \]

hence

\[ \text{dup} = \text{fork} \, \text{id}_{A,A} : \text{Set}(A, A \times A) \]
\[ (\text{fst}, \text{snd}) = \text{fork}^\circ \, \text{id}_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C)) \]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{array}{ccc}
\text{CMon} & \downarrow & \text{Set} \\
\uparrow \quad \text{Free} & \quad \uparrow \quad \perp \\
\uparrow \quad U & \quad \downarrow \\
\end{array}
\]

with \([\cdot] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \cong \text{Set}(A, U (M, \otimes, \epsilon)) : [\cdot]\)

Unit and counit:

\[
\begin{align*}
\text{single } A & = [id_{\text{Free } A}] : A \to U (\text{Free } A) \\
\text{reduce } M & = [id_M] : \text{Free } (U M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \(h : \text{Free } A \to M\) and \(f : A \to U M = M\),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \mapsto 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \mapsto a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>({a} \mapsto a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>({a} \mapsto a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \wedge))</td>
<td>({a} \mapsto a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>({a} \mapsto a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \to \mathbb{B}) \to \text{Bag } A \to \text{Bag } A
\]

\[
guard p a = \text{if } p a \text{ then } \{a\} \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a monad \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag} A) \to \text{Bag} A \\
\text{single} & : A \to \text{Bag} A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f a b \mid a \leftarrow x, b \leftarrow g a \} \).

In fact, for any adjunction \(L \dashv R\) between \(\mathbf{C}\) and \(\mathbf{D}\), we get a monad \((T, \mu, \eta)\) on \(\mathbf{D}\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R [id_A] L : T (T A) \to T A \\
\eta A & = [id_A] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the \textit{Reader} monad in Haskell), so arise from an adjunction. The \textit{laws of exponents} arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \cong 1$
- $\text{Map } 1 V \cong V$
- $\text{Map } (K_1 + K_2) V \cong \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \cong \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \cong 1$
- $\text{Map } K (V_1 \times V_2) \cong \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\text{Rel} \xrightarrow{\downarrow} \text{Set} \\
\text{E} \quad \text{J}
\]

where \( J \) embeds, and \( E \) \( R : A \rightarrow \text{Set} \ B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[\text{index} : \text{Bag} (K \times V) \cong \text{Map} K (\text{Bag} V)\]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[x \ _f \bowtie \ _g \ y = \text{flatten} (\text{Map} K \ cp (\text{merge} (\text{groupBy} f x, \text{groupBy} g y)))\]

\(\text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag} V \rightarrow \text{Map} K (\text{Bag} V)\)

\(\text{flatten} : \text{Map} K (\text{Bag} V) \rightarrow \text{Bag} V\)
13. Pointed sets and finite maps

Model finite maps $\text{Map}_*$ not as partial functions, but total functions to a pointed codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$
\begin{array}{ccc}
\text{Set}_* & \downarrow & \text{Set} \\
\downarrow & & \downarrow \\
\text{Maybe} & \rightleftharpoons & \text{U}
\end{array}
$$

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $\text{U} (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_*: \text{Set} \rightarrow \text{Set}_*$.

Indexing remains an isomorphism:

$$ index : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V) $$
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \to a : A \to \text{Map } K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

$$\mu X : T_m (T_n X) \to T_{m \otimes n} X$$
$$\eta X : X \to T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *Monad comprehensions* for database queries
- Structure arising from *adjunctions*
- Equivalences from *universal properties*
- Fitting in *relational joins*, via indexing
- To do: calculating *query optimisations*

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