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Publication date:
2016

Document version
Early version, also known as pre-print

Citation for published version (APA):
Gibbons, J., Henglein, F., Hinze, R., & Wu, N. (2016). *Relational algebra by way of adjunctions*. Abstract from 15th International Symposium on Database Programming Languages, Pittsburgh, United States.



Relational Algebra by Way of Adjunctions

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(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)

DBPL, October 2015

1. Summary

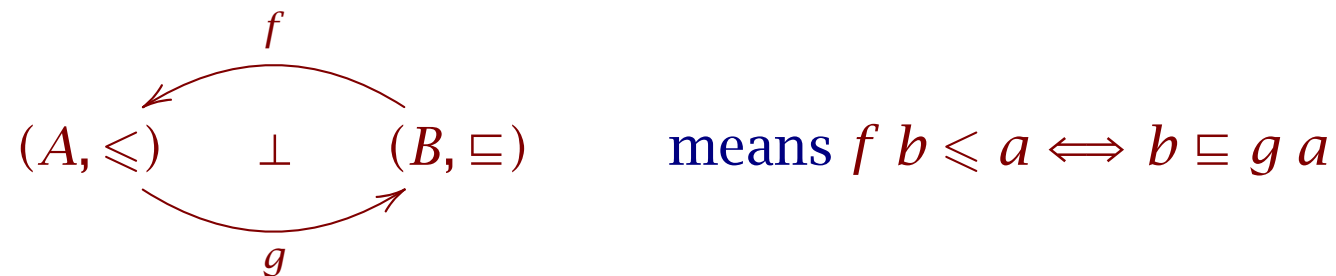
- bulk types (sets, bags, lists) are *monads*
- monads have nice *mathematical foundations via adjunctions*
- monads support *comprehensions*
- comprehension syntax provides a *query* notation

[(*customer.name, invoice.amount*)
| *customer* ← *customers*,
invoice ← *invoices*,
customer.cid = *invoice.customer*,
invoice.due ≤ *today*]

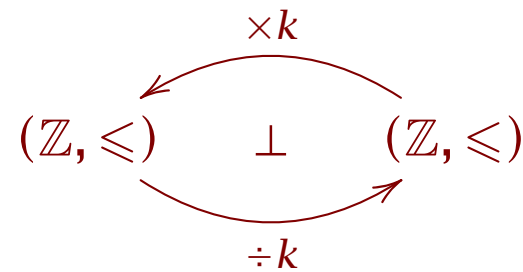
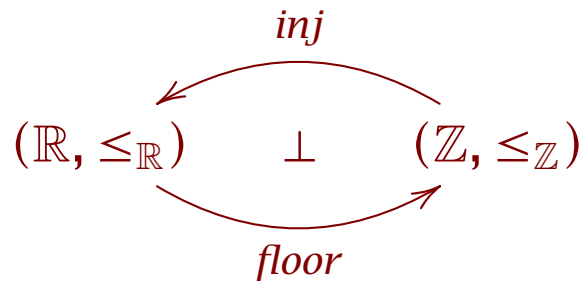
- monad structure explains *selection, projection*
- less obvious how to explain *join*

2. Galois connections

Relating monotonic functions between two ordered sets:



For example,



“Change of coordinates” can sometimes simplify reasoning; eg rhs gives $n \times k \leq m \iff n \leq m \div k$, and multiplication is easier to reason about than rounding division.

3. Category theory from ordered sets

A *category* \mathbf{C} consists of

- a set* $|\mathbf{C}|$ of *objects*,
- a set* $\mathbf{C}(X, Y)$ of *arrows* $X \rightarrow Y$ for each $X, Y : |\mathbf{C}|$,
- *identity* arrows $id_X : X \rightarrow X$ for each X
- *composition* $f \cdot g : X \rightarrow Z$ of compatible arrows $g : X \rightarrow Y$ and $f : Y \rightarrow Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set (A, \leq) is a degenerate category, with objects A and a unique arrow $a \rightarrow b$ iff $a \leq b$.



Many categorical concepts are generalisations from ordered sets.

*proviso...

4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category **CMon** has commutative monoids (M, \otimes, ϵ) as objects, and homomorphisms $h: (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')$ as arrows:

$$\begin{aligned} h(m \otimes n) &= h m \oplus h n \\ h \epsilon &= \epsilon' \end{aligned}$$

Trivially, category **Set** has sets as objects, and total functions as arrows.

5. Functors

Categories are themselves structured objects...

A *functor* $F : \mathbf{C} \rightarrow \mathbf{D}$ is an operation on both objects and arrows, preserving the structure: $F f : F X \rightarrow F Y$ when $f : X \rightarrow Y$, and

$$F id_X = id_{F X}$$

$$F (f \cdot g) = F f \cdot F g$$

For example, *forgetful* functor $U : \mathbf{CMon} \rightarrow \mathbf{Set}$:

$$U (M, \otimes, \epsilon) = M$$

$$U (h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')) = h : M \rightarrow M'$$

Conversely, $\mathbf{Free} : \mathbf{Set} \rightarrow \mathbf{CMon}$ generates the *free* commutative monoid (ie bags) on a set of elements:

$$\mathbf{Free} A = (\mathbf{Bag} A, \uplus, \emptyset)$$

$$\mathbf{Free} (f : A \rightarrow B) = \mathit{map} f : \mathbf{Bag} A \rightarrow \mathbf{Bag} B$$

6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories \mathbf{C}, \mathbf{D} , and functors $L: \mathbf{D} \rightarrow \mathbf{C}$ and $R: \mathbf{C} \rightarrow \mathbf{D}$, adjunction

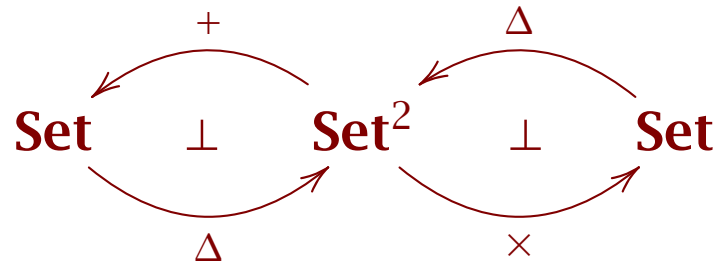
$$\begin{array}{ccc}
 & L & \\
 \mathbf{C} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathbf{D} \\
 & R &
 \end{array}
 \quad \text{means}^* \quad [-]: \mathbf{C}(L X, Y) \simeq \mathbf{D}(X, R Y) : [-]$$

A familiar example is given by *currying*:

$$\begin{array}{ccc}
 & - \times P & \\
 \mathbf{Set} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathbf{Set} \\
 & (-)^P &
 \end{array}
 \quad \text{with } \mathit{curry}: \mathbf{Set}(X \times P, Y) \simeq \mathbf{Set}(X, Y^P) : \mathit{curry}^\circ$$

hence definitions and properties of $\mathit{apply} = \mathit{uncurry} \mathit{id}_{Y^P}: Y^P \times P \rightarrow Y$

7. Products and coproducts



with

$$\begin{aligned}
 \mathit{fork} &: \mathbf{Set}^2(\Delta A, (B, C)) \simeq \mathbf{Set}(A, B \times C) & : \mathit{fork}^\circ \\
 \mathit{junc}^\circ &: \mathbf{Set}(A + B, C) \simeq \mathbf{Set}^2((A, B), \Delta C) & : \mathit{junc}
 \end{aligned}$$

hence

$$\begin{aligned}
 \mathit{dup} &= \mathit{fork} \mathit{id}_{A,A} : \mathbf{Set}(A, A \times A) \\
 (\mathit{fst}, \mathit{snd}) &= \mathit{fork}^\circ \mathit{id}_{B \times C} : \mathbf{Set}^2(\Delta(B, C), (B, C))
 \end{aligned}$$

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.

8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

$$\begin{array}{ccc}
 & \text{Free} & \\
 & \curvearrowright & \\
 \mathbf{CMon} & \perp & \mathbf{Set} \\
 & \curvearrowleft & \\
 & \mathbf{U} &
 \end{array}
 \quad \text{with } [-] : \mathbf{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \\
 \simeq \mathbf{Set}(A, \mathbf{U}(M, \otimes, \epsilon)) \quad : [-]$$

Unit and counit:

$$\mathit{single } A = [id_{\text{Free } A}] : A \rightarrow \mathbf{U}(\text{Free } A)$$

$$\mathit{reduce } M = [id_M] : \text{Free}(\mathbf{U} M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \epsilon)$$

whence, for $h : \text{Free } A \rightarrow M$ and $f : A \rightarrow \mathbf{U} M = M$,

$$h = \mathit{reduce } M \cdot \text{Free } f \iff \mathbf{U} h \cdot \mathit{single } A = f$$

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.

9. Aggregation

Aggregations are bag homomorphisms:

aggregation	monoid	action on singletons
<i>count</i>	$(\mathbb{N}, 0, +)$	$\{a\} \mapsto 1$
<i>sum</i>	$(\mathbb{R}, 0, +)$	$\{a\} \mapsto a$
<i>max</i>	$(\mathbb{Z}, \text{minBound}, \text{max})$	$\{a\} \mapsto a$
<i>min</i>	$(\mathbb{Z}, \text{maxBound}, \text{min})$	$\{a\} \mapsto a$
<i>all</i>	$(\mathbb{B}, \text{True}, \wedge)$	$\{a\} \mapsto a$
<i>any</i>	$(\mathbb{B}, \text{False}, \vee)$	$\{a\} \mapsto a$

Selection is a homomorphism, to bags, using action

$$\begin{aligned} \text{guard} &: (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A \\ \text{guard } p \ a &= \mathbf{if } p \ a \ \mathbf{then } \{a\} \ \mathbf{else } \emptyset \end{aligned}$$

Laws about selections follow from laws of homomorphisms (and of coproducts, since $\mathbb{B} = 1 + 1$).

10. Monads

Bags form a *monad* (Bag , *union*, *single*) with

$$\text{Bag} = \mathbf{U} \cdot \text{Free}$$

$$\text{union} : \text{Bag} (\text{Bag } A) \rightarrow \text{Bag } A$$

$$\text{single} : A \rightarrow \text{Bag } A$$

which justifies the use of comprehension notation $\{f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a\}$.

In fact, for any adjunction $\mathbf{L} \dashv \mathbf{R}$ between \mathbf{C} and \mathbf{D} , we get a monad (\mathbf{T}, μ, η) on \mathbf{D} , where

$$\mathbf{T} = \mathbf{R} \cdot \mathbf{L}$$

$$\mu \ A = \mathbf{R} [id_A] \ \mathbf{L} : \mathbf{T} (\mathbf{T} \ A) \rightarrow \mathbf{T} \ A$$

$$\eta \ A = [id_A] : A \rightarrow \mathbf{T} \ A$$

11. Maps

Database indexes are essentially maps $\text{Map } K \ V = V^K$. Maps $(-)^K$ from K form a monad (the *Reader* monad in Haskell), so arise from an adjunction.

The *laws of exponents* arise from this adjunction, and from those for products and coproducts:

$$\text{Map } 0 \ V \quad \simeq \ 1$$

$$\text{Map } 1 \ V \quad \simeq \ V$$

$$\text{Map } (K_1 + K_2) \ V \simeq \text{Map } K_1 \ V \times \text{Map } K_2 \ V$$

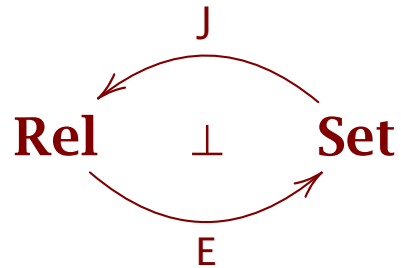
$$\text{Map } (K_1 \times K_2) \ V \simeq \text{Map } K_1 \ (\text{Map } K_2 \ V)$$

$$\text{Map } K \ 1 \quad \simeq \ 1$$

$$\text{Map } K \ (V_1 \times V_2) \simeq \text{Map } K \ V_1 \times \text{Map } K \ V_2 : \textit{merge}$$

12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:



where J embeds, and $E R: A \rightarrow \text{Set } B$ for $R: A \sim B$.

Moreover, the correspondence remains valid for bags:

$$\text{index}: \text{Bag } (K \times V) \simeq \text{Map } K \text{ (Bag } V)$$

Together, *index* and *merge* give efficient relational joins:

$$x \bowtie_g y = \text{flatten } (\text{Map } K \text{ cp } (\text{merge } (\text{groupBy } f \ x, \text{groupBy } g \ y)))$$

$$\text{groupBy}: (V \rightarrow K) \rightarrow \text{Bag } V \rightarrow \text{Map } K \text{ (Bag } V)$$

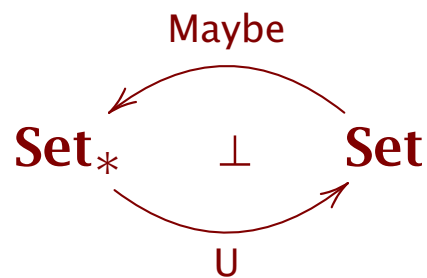
$$\text{flatten} \quad : \text{Map } K \text{ (Bag } V) \rightarrow \text{Bag } V$$

13. Pointed sets and finite maps

Model *finite maps* \mathbf{Map}_* not as partial functions, but *total* functions to a *pointed* codomain (A, a) , i.e. a set A with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category \mathbf{Set}_* .

There is an adjunction to \mathbf{Set} , via



where $\mathbf{Maybe} A \simeq 1 + A$ adds a point, and $\mathbf{U} (A, a) = A$ discards it.

In particular, $(\mathbf{Bag} A, \emptyset)$ is a pointed set. Moreover, $\mathbf{Bag} f$ is point-preserving, so we get a functor $\mathbf{Bag}_* : \mathbf{Set} \rightarrow \mathbf{Set}_*$.

Indexing remains an isomorphism:

$$\mathit{index} : \mathbf{Bag}_* (K \times V) \simeq \mathbf{Map}_* K (\mathbf{Bag}_* V)$$

14. Graded monads

A catch: finite maps aren't a monad, because

$$\eta a = \lambda k \rightarrow a : A \rightarrow \text{Map } K A$$

in general yields an infinite map.

However, finite maps are a *graded monad*^{*}: for monoid (M, \otimes, ϵ) ,

$$\mu X : T_m (T_n X) \rightarrow T_{m \otimes n} X$$

$$\eta X : X \rightarrow T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions^{*}.

We use the monoid $(\mathbb{K}, \times, 1)$ of finite key types under product.

15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

Thanks to EPSRC *Unifying Theories of Generic Programming* for funding.