Relational algebra by way of adjunctions
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Relational Algebra by Way of Adjunctions

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(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)
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1. Summary

- bulk types (sets, bags, lists) are *monads*
- monads have nice *mathematical foundations via adjunctions*
- monads support *comprehensions*
- comprehension syntax provides a *query* notation

\[
\left[ (\text{customer}.\text{name}, \text{invoice}.\text{amount}) \\
| \text{customer} \leftarrow \text{customers}, \\
\text{invoice} \leftarrow \text{invoices}, \\
\text{customer}.\text{cid} = \text{invoice}.\text{customer}, \\
\text{invoice}.\text{due} \leq \text{today} \right]
\]

- monad structure explains *selection, projection*
- less obvious how to explain *join*
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq)\] means \(f b \leq a \iff b \sqsubseteq g a\)

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \(n \times k \leq m \iff n \leq m \div k\), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $C$ consists of

- a set* $|C|$ of objects,
- a set* $C(X, Y)$ of arrows $X \to Y$ for each $X, Y : |C|$, 
- identity arrows $id_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

$$\ldots \Rightarrow -2 \Rightarrow -1 \Rightarrow 0 \Rightarrow 1 \Rightarrow 2 \Rightarrow \ldots$$

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a \textit{concrete category}: roughly,

- the objects are \textit{sets with additional structure}
- the arrows are \textit{structure-preserving mappings}

Many useful categories are of this form.

For example, the category \textbf{CMon} has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
    h (m \otimes n) &= h m \oplus h n \\
    h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category \textbf{Set} has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F \text{id}_X = \text{id}_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, forgetful functor $U : \text{CMon} \to \text{Set}$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, Free : \text{Set} \to \text{CMon} generates the free commutative monoid (ie bags) on a set of elements:

$$\text{Free } A = (\text{Bag } A, \uplus, \emptyset)$$
$$\text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $C, D$, and functors $L : D \to C$ and $R : C \to D$, adjunction

\[
\begin{array}{c}
C & \dashv & D \\
\downarrow & & \downarrow \\
\uparrow & & \uparrow \\
R & & L
\end{array}
\]

means $\ast : C(LX,Y) \simeq D(X,RY) : \ast$

A familiar example is given by currying:

\[
\begin{array}{c}
\text{Set} & \dashv & \text{Set} \\
\downarrow & & \downarrow \\
\uparrow & & \uparrow \\
(-)^P & & (-\times P)
\end{array}
\]

with $\text{curry} : \text{Set}(X \times P,Y) \simeq \text{Set}(X,Y^P) : \text{curry}^\circ$

hence definitions and properties of $\text{apply} = \text{uncurry \ id}_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

\[
\begin{array}{ccc}
\text{Set} & \langle \Delta \rangle & \text{Set} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\text{Set} & \langle \times \rangle & \text{Set} \\
\end{array}
\]

with

\[
\begin{align*}
\text{fork} : \text{Set}^2(\Delta A, (B, C)) & \cong \text{Set}(A, B \times C) : \text{fork}^\circ \\
\text{junc}^\circ : \text{Set}(A + B, C) & \cong \text{Set}^2((A, B), \Delta C) : \text{junc}
\end{align*}
\]

hence

\[
\begin{align*}
dup & = \text{fork id}_{A,A} : \text{Set}(A, A \times A) \\
(fst, snd) & = \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C))
\end{align*}
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\text{CMon} \quad \dashv \quad \text{Set}
\]

\[
\text{with } [-] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \approx \text{Set}(A, U (M, \otimes, \epsilon)) : [-]
\]

Unit and counit:

\[
single A = [\text{id}_{\text{Free } A}] : A \rightarrow U (\text{Free } A)
\]

\[
\text{reduce } M = [\text{id}_M] : \text{Free } (U M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \epsilon)
\]

whence, for \( h : \text{Free } A \rightarrow M \) and \( f : A \rightarrow U M = M \),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \rightarrow 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \wedge))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \vee))</td>
<td>({a} \rightarrow a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A
\]

\[
guard p a = \text{if } p a \text{ then } \{a\} \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a *monad* \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} &= \mathbb{U} \cdot \text{Free} \\
\text{union} : \text{Bag} (\text{Bag } A) &\to \text{Bag } A \\
\text{single} : A &\to \text{Bag } A
\end{align*}
\]

which justifies the use of comprehension notation \(\{f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a\} \).  

In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
\begin{align*}
T &= R \cdot L \\
\mu A &= R [\text{id}_A] \ L : T (T A) \to T A \\
\eta A &= [\text{id}_A] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the *Reader* monad in Haskell), so arise from an adjunction. The *laws of exponents* arise from this adjunction, and from those for products and coproducts:

\[
\begin{align*}
\text{Map } 0 V & \approx 1 \\
\text{Map } 1 V & \approx V \\
\text{Map } (K_1 + K_2) V & \approx \text{Map } K_1 V \times \text{Map } K_2 V \\
\text{Map } (K_1 \times K_2) V & \approx \text{Map } K_1 (\text{Map } K_2 V) \\
\text{Map } K 1 & \approx 1 \\
\text{Map } K (V_1 \times V_2) & \approx \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}
\end{align*}
\]
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\text{Rel} \quad \perp \quad \text{Set}
\]

where \( J \) embeds, and \( E R : A \rightarrow \text{Set} \ B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[\text{index} : \text{Bag} \ (K \times V) \simeq \text{Map} \ K \ (\text{Bag} \ V)\]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[x \ f \triangleleft g \ y = \text{flatten} \ (\text{Map} \ K \ cp \ (\text{merge} \ (\text{groupBy} \ f \ x, \text{groupBy} \ g \ y)))\]

\[\text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag} \ V \rightarrow \text{Map} \ K \ (\text{Bag} \ V)\]

\[\text{flatten} \quad : \text{Map} \ K \ (\text{Bag} \ V) \rightarrow \text{Bag} \ V\]
13. Pointed sets and finite maps

Model finite maps $\text{Map}_*$ not as partial functions, but total functions to a pointed codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$\begin{array}{ccc}
\text{Set}_* & \perp & \text{Set} \\
\downarrow \text{Maybe} & & \downarrow \text{U} \\
\text{Set}_* & \perp & \text{Set}
\end{array}$$

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $\text{U } (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_*: \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

$$\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)$$
14. Graded monads

A catch: finite maps aren’t a monad, because

\[ \eta a = \lambda k \to a : A \to \text{Map} \, K \, A \]

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

\[ \mu X : T_m(nX) \to T_{m\otimes n}X \]
\[ \eta X : X \to T_\epsilon X \]

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

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