Relational algebra by way of adjunctions

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Relational Algebra by Way of Adjunctions

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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
\left\{ (\text{customer}.\text{name}, \text{invoice}.\text{amount}) \mid \text{customer} \leftarrow \text{customers}, \\
\text{invoice} \leftarrow \text{invoices}, \\
\text{customer}.\text{cid} = \text{invoice}.\text{customer}, \\
\text{invoice}.\text{due} \leq \text{today} \right\}
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq) \quad \text{means} \quad f b \leq a \iff b \sqsubseteq g a\]

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \quad \text{and} \quad (\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq)\]

“Change of coordinates” can sometimes simplify reasoning; e.g., rhs gives $n \times k \leq m \iff n \leq m \div k$, and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathcal{C}$ consists of

- a set* $|\mathcal{C}|$ of objects,
- a set* $\mathcal{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathcal{C}|$,
- identity arrows $id_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \ldots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category $\textbf{CMon}$ has commutative monoids $(M, \otimes, \epsilon)$ as objects, and homomorphisms $h: (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')$ as arrows:

\[
\begin{aligned}
h (m \otimes n) &= h m \oplus h n \\
h \epsilon &= \epsilon'
\end{aligned}
\]

Trivially, category $\textbf{Set}$ has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A *functor* $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

\[
F \ id_X = id_{F X} \\
F (f \cdot g) = F f \cdot F g
\]

For example, *forgetful* functor $U : \text{CMon} \to \text{Set}$:

\[
U (M, \otimes, \epsilon) = M \\
U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'
\]

Conversely, $\text{Free} : \text{Set} \to \text{CMon}$ generates the *free* commutative monoid (ie bags) on a set of elements:

\[
\text{Free } A = (\text{Bag } A, \cup, \emptyset) \\
\text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B
\]
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories \( \mathbf{C}, \mathbf{D} \), and functors \( L : \mathbf{D} \to \mathbf{C} \) and \( R : \mathbf{C} \to \mathbf{D} \), adjunction

\[
\begin{array}{ccc}
\mathbf{C} & \perp & \mathbf{D} \\
\xrightarrow{L} & & \xleftarrow{R}
\end{array}
\]

means\(^*\) \( [-] : \mathbf{C}(L X, Y) \cong \mathbf{D}(X, R Y) : [-] \)

A familiar example is given by \textit{currying}:

\[
\begin{array}{ccc}
\text{Set} & \perp & \text{Set} \\
\xleftarrow{(-)P} & & \xrightarrow{- \times P}
\end{array}
\]

with \( \text{curry} : \text{Set}(X \times P, Y) \cong \text{Set}(X, Y^P) : \text{curry}^\circ \)

hence definitions and properties of \( \text{apply} = \text{uncurry } \text{id}_{Y^P} : Y^P \times P \to Y \)
7. Products and coproducts

\[
\begin{array}{ccc}
\text{Set} & \downarrow & \text{Set}^2 \\
\Delta & \swarrow & \nwarrow \\
\text{Set} & \downarrow & \text{Set}
\end{array}
\]

with

\[
\text{fork} : \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) : \text{fork}^\circ
\]

\[
\text{junc}^\circ : \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
dup = \text{fork}\ id_{A,A} : \text{Set}(A, A \times A)
\]

\[
(fst, snd) = \text{fork}^\circ id_{B \times C} : \text{Set}^2(\Delta (B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\text{CMon} \downarrow \text{Set} \quad \quad \text{with } [-] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \\
\approx \text{Set}(A, U (M, \otimes, \epsilon)) : [-]
\]

Unit and counit:

\[
single A = [id_{\text{Free } A}] : A \rightarrow U (\text{Free } A)
\]

\[
\text{reduce } M = [id_M] : \text{Free } (U M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \epsilon)
\]

whence, for \( h : \text{Free } A \rightarrow M \) and \( f : A \rightarrow U M = M \),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \rightarrow 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>({a} \rightarrow a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A
\]

\[
guard p a = \text{if } p a \text{ then } \{a\} \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a \textit{monad} \((\text{Bag}, \text{union}, \text{single})\) with

\begin{align*}
\text{Bag} &= U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag} A) \to \text{Bag} A \\
\text{single} & : A \to \text{Bag} A
\end{align*}

which justifies the use of comprehension notation \(\{ f a b \mid a \leftarrow x, b \leftarrow g a \}^\cdot\).

In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\begin{align*}
T &= R \cdot L \\
\mu A &= R \lceil id_A \rceil L : T (T A) \to T A \\
\eta A &= \lfloor id_A \rfloor : A \to T A
\end{align*}
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the $\text{Reader}$ monad in Haskell), so arise from an adjunction.

The *laws of exponents* arise from this adjunction, and from those for products and coproducts:

$\text{Map } 0 V \simeq 1$

$\text{Map } 1 V \simeq V$

$\text{Map } (K_1 + K_2) V \simeq \text{Map } K_1 V \times \text{Map } K_2 V$

$\text{Map } (K_1 \times K_2) V \simeq \text{Map } K_1 (\text{Map } K_2 V)$

$\text{Map } K 1 \simeq 1$

$\text{Map } K (V_1 \times V_2) \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\text{Rel} \xrightarrow{J} \downarrow \xrightarrow{E} \text{Set}
\]

where \(J\) embeds, and \(E R : A \to \text{Set} B\) for \(R : A \sim B\).

Moreover, the correspondence remains valid for bags:

\[
\text{index} : \text{Bag} \,(K \times V) \simeq \text{Map} K \,(\text{Bag} V)
\]

Together, \textit{index} and \textit{merge} give efficient relational joins:

\[
x f \Join g y = \text{flatten} \,(\text{Map} K \,cp \,(\text{merge} \,(\text{groupBy} f x, \text{groupBy} g y)))
\]

\textit{groupBy} : \((V \to K) \to \text{Bag} V \to \text{Map} K \,(\text{Bag} V)

\textit{flatten} : \text{Map} K \,(\text{Bag} V) \to \text{Bag} V
13. Pointed sets and finite maps

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$
\begin{array}{ccc}
\text{Set}_* & \vdash & \downarrow \\
\text{Maybe} & \leftrightarrow & \text{Set} \\
\downarrow & & \uparrow \\
\text{U} & \leftrightarrow & 
\end{array}
$$

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $\text{U} (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

$$
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
$$
14. Graded monads

A catch: finite maps aren’t a monad, because

\[ \eta \ a = \lambda k \rightarrow a : A \rightarrow \text{Map} \ K \ A \]

in general yields an infinite map.

However, finite maps are a graded monad\(^*\): for monoid \((M, \otimes, \epsilon)\),

\[ \mu \ X : T_m (T_n X) \rightarrow T_{m \otimes n} X \]
\[ \eta \ X : X \rightarrow T_\epsilon X \]

satisfying the usual laws. These too arise from adjunctions\(^*\).

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

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