Relational algebra by way of adjunctions

Gibbons, Jeremy; Henglein, Fritz; Hinze, Ralf; Wu, Nicolas

Publication date:
2016

Document version
Early version, also known as pre-print

Citation for published version (APA):
Relational Algebra by Way of Adjunctions

Jeremy Gibbons
(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)
DBPL, October 2015
1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation
  \[
  \[( \text{customer.name, invoice.amount}) \\
  | \text{customer} \leftarrow \text{customers}, \\
  \text{invoice} \leftarrow \text{invoices}, \\
  \text{customer.cid} = \text{invoice.customer}, \\
  \text{invoice.due} \leq \text{today} \]
  \]
- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq) \quad \text{means} \quad f b \leq a \iff b \sqsubseteq g a\]

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

\[(\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq)\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives

\[n \times k \leq m \iff n \leq m \div k,\]

and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathcal{C}$ consists of

- a set* $|\mathcal{C}|$ of objects,
- a set* $\mathcal{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathcal{C}|$,
- identity arrows $\text{id}_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \ldots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a concrete category: roughly,

- the objects are sets with additional structure
- the arrows are structure-preserving mappings

Many useful categories are of this form.

For example, the category $\textbf{CMon}$ has commutative monoids $(M, \otimes, \epsilon)$ as objects, and homomorphisms $h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')$ as arrows:

\[
\begin{align*}
h (m \otimes n) &= h m \oplus h n \\
h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category $\textbf{Set}$ has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A *functor* $F : C \rightarrow D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \rightarrow F Y$ when $f : X \rightarrow Y$, and

\[
F \ id_X = id_{F X} \\
F (f \cdot g) = F f \cdot F g
\]

For example, *forgetful* functor $U : C\text{Mon} \rightarrow \text{Set}$:

\[
U (M, \otimes, \epsilon) = M \\
U (h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')) = h : M \rightarrow M'
\]

Conversely, $\text{Free} : \text{Set} \rightarrow C\text{Mon}$ generates the *free* commutative monoid (ie bags) on a set of elements:

\[
\text{Free } A = (\text{Bag } A, \cup, \emptyset) \\
\text{Free } (f : A \rightarrow B) = map f : \text{Bag } A \rightarrow \text{Bag } B
\]
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $\mathcal{C}, \mathcal{D}$, and functors $L : \mathcal{D} \to \mathcal{C}$ and $R : \mathcal{C} \to \mathcal{D}$, adjunction

$$\begin{array}{c}
\mathcal{C} \\
\downarrow L \\
\downarrow \Downarrow \\
\mathcal{D} \\
\uparrow R \\
\uparrow \Uparrow
\end{array}$$

means\* $[-] : \mathcal{C}(L X, Y) \simeq \mathcal{D}(X, R Y) : [-]$

A familiar example is given by currying:

$$\begin{array}{c}
\text{Set} \\
\downarrow - \times P \\
\downarrow \Downarrow \\
\text{Set} \\
\uparrow (-)^P
\end{array}$$

with $\text{curry} : \text{Set}(X \times P, Y) \simeq \text{Set}(X, Y^P) : \text{curry}^\circ$

hence definitions and properties of $\text{apply} = \text{uncurry \ id}_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

\[ \begin{array}{c}
\text{Set} \\
\Delta \\
\downarrow \\
\text{Set}^2 \\
\Delta \\
\downarrow \\
\text{Set} \\
\times \\
\end{array} \]

with

\[
\text{fork} : \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) : \text{fork}^\circ
\]

\[
\text{junc}^\circ : \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
dup = \text{fork } \text{id}_{A,A} : \text{Set}(A, A \times A)
\]

\[
(fst, snd) = \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\text{CMon} \quad \bot \quad \text{Set}
\]

\[
\begin{align*}
\text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) & \cong \text{Set}(A, U(M, \otimes, \epsilon)) : [-] \\
\end{align*}
\]

Unit and counit:

\[
\begin{align*}
single A & = [id_{\text{Free } A}] : A \to U(\text{Free } A) \\
reduce M & = [id_M] : \text{Free } (U M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \( h : \text{Free } A \to M \) and \( f : A \to U M = M \),

\[
h = reduce M \cdot \text{Free } f \iff U h \cdot single A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \mapsto 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \mapsto a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>({a} \mapsto a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>({a} \mapsto a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>({a} \mapsto a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>({a} \mapsto a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \to \mathbb{B}) \to \text{Bag } A \to \text{Bag } A
\]

\[
guard p a = \text{if } p a \text{ then } \{a\} \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms
(and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a *monad* \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag} A) \to \text{Bag} A \\
\text{single} & : A \to \text{Bag} A
\end{align*}
\]

which justifies the use of comprehension notation \(\{f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a\}\).

In fact, for any adjunction \(L \dashv R\) between \(\mathbf{C}\) and \(\mathbf{D}\), we get a monad \((T, \mu, \eta)\) on \(\mathbf{D}\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \cdot [id_A] \cdot L : T (T A) \to T A \\
\eta A & = [id_A] : A \to T A
\end{align*}
\]
Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the $\text{Reader}$ monad in Haskell), so arise from an adjunction. The \textit{laws of exponents} arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \simeq 1$
- $\text{Map } 1 V \simeq V$
- $\text{Map } (K_1 + K_2) V \simeq \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \simeq \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \simeq 1$
- $\text{Map } K (V_1 \times V_2) \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \overset{J}{\longrightarrow} \text{Set} \]

where \( J \) embeds, and \( E R : A \to \text{Set} B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V) \]

Together, \textit{index} and \textit{merge} give efficient relational joins:

\[
x f \bowtie g y = \text{flatten} (\text{Map} K cp (\text{merge} (\text{groupBy} f x, \text{groupBy} g y)))
\]

\[
\text{groupBy} : (V \to K) \to \text{Bag} V \to \text{Map} K (\text{Bag} V)
\]

\[
\text{flatten} : \text{Map} K (\text{Bag} V) \to \text{Bag} V
\]
13. Pointed sets and finite maps

Model \textit{finite maps} \( \text{Map}_* \) not as partial functions, but \textit{total} functions to a \textit{pointed} codomain \((A, a)\), i.e. a set \( A \) with a distinguished element \( a : A \).

Pointed sets and point-preserving functions form a category \( \text{Set}_* \).

There is an adjunction to \( \text{Set} \), via

\[
\begin{array}{ccc}
\text{Set}_* & \dashv & \text{Set} \\
\downarrow \text{Maybe} & & \downarrow \text{U} \\
\text{Set}_* & \rightarrow & \text{Set}
\end{array}
\]

where \( \text{Maybe} \ A \cong 1 + A \) adds a point, and \( \text{U} (A, a) = A \) discards it.

In particular, \((\text{Bag} \ A, \emptyset)\) is a pointed set. Moreover, \( \text{Bag} \ f \) is point-preserving, so we get a functor \( \text{Bag}_*: \text{Set} \rightarrow \text{Set}_* \).

Indexing remains an isomorphism:

\[
\text{index} : \text{Bag}_* (K \times V) \cong \text{Map}_* K (\text{Bag}_* V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \to a : A \to \text{Map } K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid $$(M, \otimes, \epsilon)$$,

$$\mu X : T_m (T_n X) \to T_{m \otimes n} X$$
$$\eta X : X \to T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid $$(\mathbb{K}, \times, 1)$$ of finite key types under product.
15. Conclusions

- *Monad comprehensions* for database queries
- Structure arising from *adjunctions*
- Equivalences from *universal properties*
- Fitting in *relational joins*, via indexing
- To do: calculating *query optimisations*

Thanks to EPSRC *Unifying Theories of Generic Programming* for funding.