Relational algebra by way of adjunctions

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Relational Algebra by Way of Adjunctions

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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
[ (customer.name, invoice.amount) \\
| customer ← customers, \\
| invoice ← invoices, \\
| customer.cid = invoice.customer, \\
| invoice.due ≤ today ]
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq) \text{ means } f(b) \leq a \iff b \sqsubseteq g(a)\]

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \(n \times k \leq m \iff n \leq m \div k\), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set* $|\mathbf{C}|$ of objects,
- a set* $\mathbf{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathbf{C}|$,
- identity arrows $\text{id}_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,

such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

... $\to -2 \to -1 \to 0 \to 1 \to 2 \to \ldots$

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a concrete category: roughly,

- the objects are sets with additional structure
- the arrows are structure-preserving mappings

Many useful categories are of this form.

For example, the category $\textbf{CMon}$ has commutative monoids $(M, \otimes, \epsilon)$ as objects, and homomorphisms $h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')$ as arrows:

$$h(m \otimes n) = h m \oplus h n$$
$$h \epsilon = \epsilon'$$

Trivially, category $\textbf{Set}$ has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F \ id_X = \ id_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, forgetful functor $U : \text{CMon} \to \text{Set}$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, $\text{Free} : \text{Set} \to \text{CMon}$ generates the free commutative monoid (ie bags) on a set of elements:

$$\text{Free } A = (\text{Bag } A, \cup, \emptyset)$$
$$\text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B$$
6. Adjunctions

*Adjunctions* are the categorical generalisation of Galois connections. Given categories $\mathcal{C}, \mathcal{D}$, and functors $L : \mathcal{D} \to \mathcal{C}$ and $R : \mathcal{C} \to \mathcal{D}$, adjunction $\mathcal{C} \perp \mathcal{D}$ means $\hom{\mathcal{C}}{L X, Y} \simeq \hom{\mathcal{D}}{X, R Y} : [-]$.

A familiar example is given by *currying*:

$$\hom{\mathsf{Set}}{- \times P, Y} \simeq \hom{\mathsf{Set}}{X, Y^P} : \text{curry} \circ \text{id}_{Y^P} : Y^P \times P \to Y$$
7. Products and coproducts

with

\[ \text{fork} : \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) : \text{fork}^\circ \]

\[ \text{junc}^\circ : \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc} \]

hence

\[ \text{dup} = \text{fork} \ id_{A,A} : \text{Set}(A, A \times A) \]

\[ (\text{fst}, \text{snd}) = \text{fork}^\circ \ id_{B \times C} : \text{Set}^2(\Delta (B, C), (B, C)) \]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{array}{c}
\text{CMon} \\
\Rightarrow \\
\downarrow \\
\Rightarrow \\
\text{Set}
\end{array}
\quad \text{with } [-] : \text{CMon}(\text{Free } A, (M, \otimes, \varepsilon)) \cong \text{Set}(A, U (M, \otimes, \varepsilon)) \quad : [-]
\]

Unit and counit:

\[
single A = [id_{\text{Free } A}] : A \rightarrow U (\text{Free } A)
\]

\[
\text{reduce } M = [id_M] : \text{Free } (U M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \varepsilon)
\]

whence, for \( h : \text{Free } A \rightarrow M \) and \( f : A \rightarrow U M = M \),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>$(\mathbb{N}, 0, +)$</td>
<td>${a} \rightarrow 1$</td>
</tr>
<tr>
<td>sum</td>
<td>$(\mathbb{R}, 0, +)$</td>
<td>${a} \rightarrow a$</td>
</tr>
<tr>
<td>max</td>
<td>$(\mathbb{Z}, \text{minBound}, \text{max})$</td>
<td>${a} \rightarrow a$</td>
</tr>
<tr>
<td>min</td>
<td>$(\mathbb{Z}, \text{maxBound}, \text{min})$</td>
<td>${a} \rightarrow a$</td>
</tr>
<tr>
<td>all</td>
<td>$(\mathbb{B}, \text{True}, \land)$</td>
<td>${a} \rightarrow a$</td>
</tr>
<tr>
<td>any</td>
<td>$(\mathbb{B}, \text{False}, \lor)$</td>
<td>${a} \rightarrow a$</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
\text{guard} : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A
\]

\[
\text{guard } p \ a = \text{if } p \ a \text{ then } \{a\} \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since $\mathbb{B} = 1 + 1$).
10. Monads

Bags form a *monad* \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag} A) \to \text{Bag} A \\
\text{single} & : A \to \text{Bag} A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \}\).

In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \ [ id_A ] L : T (T A) \to T A \\
\eta A & = [ id_A ] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the \textit{Reader} monad in Haskell), so arise from an adjunction.

The \textit{laws of exponents} arise from this adjunction, and from those for products and coproducts:

$$\begin{align*}
\text{Map } 0 V & \simeq 1 \\
\text{Map } 1 V & \simeq V \\
\text{Map } (K_1 + K_2) V & \simeq \text{Map } K_1 V \times \text{Map } K_2 V \\
\text{Map } (K_1 \times K_2) V & \simeq \text{Map } K_1 (\text{Map } K_2 V) \\
\text{Map } K 1 & \simeq 1 \\
\text{Map } K (V_1 \times V_2) & \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}
\end{align*}$$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \xrightarrow{J} \bot \xrightarrow{E} \text{Set} \]

where \( J \) embeds, and \( E \ R : A \to \text{Set} \ B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} (K \times V) \simeq \text{Map} \ K \ (\text{Bag} \ V) \]

Together, \text{index} and \text{merge} give efficient relational joins:

\[ x \ f \triangleleft g \ y = \text{flatten} \ (\text{Map} \ K \ cp \ (\text{merge} \ (\text{groupBy} \ f \ x, \text{groupBy} \ g \ y))) \]

\[ \text{groupBy} : (V \to K) \to \text{Bag} \ V \to \text{Map} \ K \ (\text{Bag} \ V) \]

\[ \text{flatten} : \text{Map} \ K \ (\text{Bag} \ V) \to \text{Bag} \ V \]
13. Pointed sets and finite maps

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

\[
\begin{array}{ccc}
\text{Set}_* & \perp & \text{Set} \\
\downarrow \text{U} & & \downarrow \text{Maybe}
\end{array}
\]

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $\text{U } (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

\[
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \to a : A \to \text{Map} \ K \ A$$

in general yields an infinite map.

However, finite maps are a *graded monad*: for monoid \((M, \otimes, \epsilon)\),

$$\mu X : T_m (T_n X) \to T_{m \otimes n} X$$
$$\eta X : X \to T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- monad comprehensions for database queries
- structure arising from adjunctions
- equivalences from universal properties
- fitting in relational joins, via indexing
- to do: calculating query optimisations

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