Relational algebra by way of adjunctions

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Relational Algebra by Way of Adjunctions

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(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)
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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation
  
  \[
  \left[(\text{customer}.\text{name}, \text{invoice}.\text{amount})
  \mid \text{customer} \leftarrow \text{customers},
  \text{invoice} \leftarrow \text{invoices},
  \text{customer}.\text{cid} = \text{invoice}.\text{customer},
  \text{invoice}.\text{due} \leq \text{today}\right]
  \]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq)\]

means \[f b \leq a \iff b \sqsubseteq g a\]

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

\[(\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq)\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \[n \times k \leq m \iff n \leq m \div k\], and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathcal{C}$ consists of

- a set* $|\mathcal{C}|$ of objects,
- a set* $\mathcal{C}(X, Y)$ of arrows $X \rightarrow Y$ for each $X, Y : |\mathcal{C}|$,  
- identity arrows $\text{id}_X : X \rightarrow X$ for each $X$
- composition $f \cdot g : X \rightarrow Z$ of compatible arrows $g : X \rightarrow Y$ and $f : Y \rightarrow Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \rightarrow b$ iff $a \leq b$.

\[ \ldots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \]

Many categorical concepts are generalisations from ordered sets.

*proviso…
4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category \( \textbf{CMon} \) has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
h (m \otimes n) &= h m \oplus h n \\
h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category \( \textbf{Set} \) has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor \( F : C \to D \) is an operation on both objects and arrows, preserving the structure: \( F f : F X \to F Y \) when \( f : X \to Y \), and

\[
\begin{align*}
F \ id_X &= id_{F X} \\
F (f \cdot g) &= F f \cdot F g
\end{align*}
\]

For example, forgetful functor \( U : \text{CMon} \to \text{Set} \):

\[
\begin{align*}
U (M, \otimes, \epsilon) &= M \\
U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) &= h : M \to M'
\end{align*}
\]

Conversely, Free : \text{Set} \to \text{CMon} generates the free commutative monoid (ie bags) on a set of elements:

\[
\begin{align*}
\text{Free } A &= (\text{Bag } A, \cup, \emptyset) \\
\text{Free } (f : A \to B) &= \text{map } f : \text{Bag } A \to \text{Bag } B
\end{align*}
\]
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections. Given categories $C, D,$ and functors $L : D \to C$ and $R : C \to D,$ adjunction

\[
\begin{array}{ccc}
C & \perp & D \\
\downarrow L & & \downarrow R \\
\end{array}
\]

means $\ast \lfloor - \rfloor : C(L X, Y) \cong D(X, R Y) : \lfloor - \rfloor$

A familiar example is given by currying:

\[
\begin{array}{ccc}
\text{Set} & \perp & \text{Set} \\
\downarrow \times P & & \downarrow (-)^P \\
\end{array}
\]

with $\text{curry} : \text{Set}(X \times P, Y) \cong \text{Set}(X, Y^P) : \text{curry}^\circ$

hence definitions and properties of apply $= \text{uncurry} \ id_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

\[
\begin{array}{c}
\text{Set} \xleftarrow{\Delta} \text{Set}^2 \xrightarrow{\Delta} \text{Set} \\
\text{Set}^2 \xleftarrow{\times} \text{Set} \xrightarrow{\times} \text{Set}^2
\end{array}
\]

with

\[
\text{fork} : \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) \quad : \text{fork}^\circ
\]

\[
\text{junc}^\circ : \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
\text{dup} = \text{fork \ id}_{A,A} : \text{Set}(A, A \times A)
\]

\[
(fst, snd) = \text{fork}^\circ \ id_{B \times C} : \text{Set}^2(\Delta (B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{array}{ccc}
\text{CMon} & \Downarrow & \text{Set} \\
\downarrow & & \downarrow \\
\text{Free} & \rightarrow & \text{Set} \\
\end{array}
\]

\[
\text{CMon} \quad \cong \quad \text{Set}
\]

with \([\cdot] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \rightarrow \text{Set}(A, \text{U} (M, \otimes, \epsilon)) : [\cdot]
\]

Unit and counit:

\[
single A = [id_{\text{Free } A}] : A \rightarrow \text{U} (\text{Free } A)
\]

\[
\text{reduce } M = [id_M] : \text{Free} (\text{U} M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \epsilon)
\]

whence, for \(h : \text{Free } A \rightarrow M\) and \(f : A \rightarrow \text{U} M = M\),

\[
h = \text{reduce } M \cdot \text{Free } f \iff \text{U} h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \to 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \to a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>({a} \to a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>({a} \to a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>({a} \to a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>({a} \to a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \to \mathbb{B}) \to \text{Bag } A \to \text{Bag } A
\]

\[
guard p a = \text{if } p a \text{ then } \{a\} \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms
(and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a \textit{monad} \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag} \ A) \to \text{Bag} \ A \\
\text{single} & : A \to \text{Bag} \ A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \ | \ a \leftarrow x, b \leftarrow g \ a \} \).

In fact, for any adjunction \(L \dashv R\) between \(\mathbf{C}\) and \(\mathbf{D}\), we get a monad \((T, \mu, \eta)\) on \(\mathbf{D}\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu \ A & = R \left[ id_A \right] \ L : T (T \ A) \to T \ A \\
\eta \ A & = \left[ id_A \right] : A \to T \ A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(\cdot)^K$ from $K$ form a monad (the *Reader* monad in Haskell), so arise from an adjunction.

The *laws of exponents* arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \cong 1$
- $\text{Map } 1 V \cong V$
- $\text{Map } (K_1 + K_2) V \cong \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \cong \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \cong 1$
- $\text{Map } K (V_1 \times V_2) \cong \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\text{Rel} \quad \perp \quad \text{Set}
\]

where \( J \) embeds, and \( E \ R : A \to \text{Set} B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[
\text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V)
\]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[
x f \bowtie g y = \text{flatten} (\text{Map} K \ \text{cp} (\text{merge} (\text{groupBy} f \ x, \text{groupBy} g \ y)))
\]

\[
\text{groupBy} : (V \to K) \to \text{Bag} V \to \text{Map} K (\text{Bag} V)
\]

\[
\text{flatten} : \text{Map} K (\text{Bag} V) \to \text{Bag} V
\]
13. Pointed sets and finite maps

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$
\begin{align*}
\begin{array}{ccc}
\text{Set}_* & \downarrow & \text{Set} \\
\downarrow & & \downarrow \\
\text{Maybe} & \rightarrow & \top \\
\downarrow & & \downarrow \\
\text{U} & \rightarrow & \rightarrow
\end{array}
\end{align*}
$$

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $\text{U } (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \rightarrow \text{Set}_*$.

Indexing remains an isomorphism:

$$
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
$$
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \to a : A \to \text{Map } K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

$$\mu X : T_m (T_n X) \to T_{m \otimes n} X$$

$$\eta X : X \to T_{\epsilon} X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

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