Relational algebra by way of adjunctions
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Relational Algebra by Way of Adjunctions

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(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)
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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
\begin{align*}
[ & (\text{customer.name, invoice.amount}) \\
| & \text{customer} \leftarrow \text{customers}, \\
& \text{invoice} \leftarrow \text{invoices}, \\
& \text{customer.cid} = \text{invoice.customer}, \\
& \text{invoice.due} \leq \text{today} ]
\end{align*}
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq)\]

means \(f(b) \leq a \iff b \sqsubseteq g(a)\)

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

\[\mathbb{Z}, \leq \perp \mathbb{Z}, \leq\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \(n \times k \leq m \iff n \leq m \div k\), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A **category** \( \mathcal{C} \) consists of

- a set* \( |\mathcal{C}| \) of **objects**, 
- a set* \( \mathcal{C}(X, Y) \) of **arrows** \( X \to Y \) for each \( X, Y : |\mathcal{C}| \), 
- **identity** arrows \( \text{id}_X : X \to X \) for each \( X \), 
- **composition** \( f \cdot g : X \to Z \) of compatible arrows \( g : X \to Y \) and \( f : Y \to Z \),
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set \((A, \leq)\) is a degenerate category, with objects \( A \) and a unique arrow \( a \to b \) iff \( a \leq b \).

\[ \cdots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a concrete category: roughly,

- the objects are sets with additional structure
- the arrows are structure-preserving mappings

Many useful categories are of this form.

For example, the category $\text{CMon}$ has commutative monoids $(M, \otimes, \epsilon)$ as objects, and homomorphisms $h: (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')$ as arrows:

$$h (m \otimes n) = h m \oplus h n$$
$$h \epsilon = \epsilon'$$

Trivially, category $\text{Set}$ has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor \( F : C \to D \) is an operation on both objects and arrows, preserving the structure: \( F f : F X \to F Y \) when \( f : X \to Y \), and

\[
F \ id_X = id_{F X} \quad F (f \cdot g) = F f \cdot F g
\]

For example, forgetful functor \( U : \text{CMon} \to \text{Set} \):

\[
U (M, \otimes, \epsilon) = M \quad U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'
\]

Conversely, \( \text{Free} : \text{Set} \to \text{CMon} \) generates the free commutative monoid (ie bags) on a set of elements:

\[
\text{Free } A = (\text{Bag } A, \cup, \emptyset) \quad \text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B
\]
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $\mathcal{C}, \mathcal{D}$, and functors $L: \mathcal{D} \to \mathcal{C}$ and $R: \mathcal{C} \to \mathcal{D}$, adjunction

\[ \begin{array}{c}
\mathcal{C} \\
\downarrow \downarrow \\
\mathcal{D} \\
\downarrow \downarrow \\
\mathcal{C}
\end{array} \]

means $[-]: \mathcal{C}(L X, Y) \simeq \mathcal{D}(X, R Y): [-]$.

A familiar example is given by currying:

\[ \begin{array}{c}
\text{Set} \\
\downarrow \downarrow \\
\text{Set}
\end{array} \]

with $\text{curry} : \text{Set}(X \times P, Y) \simeq \text{Set}(X, Y^P) : \text{curry}^\circ$

hence definitions and properties of $\text{apply} = \text{uncurry \ id}_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

\[
\begin{array}{ccc}
\text{Set} & \perp & \text{Set}^2 \\
\Delta & \Rightarrow & \Rightarrow & \Delta
\end{array}
\]

with

\[
\begin{align*}
\text{fork} : & \text{Set}^2(\Delta A, (B, C)) \cong \text{Set}(A, B \times C) : \text{fork}^\circ \\
\text{junc}^\circ : & \text{Set}(A + B, C) \cong \text{Set}^2((A, B), \Delta C) : \text{junc}
\end{align*}
\]

hence

\[
\begin{align*}
\text{dup} & = \text{fork id}_{A,A} : \text{Set}(A, A \times A) \\
(fst, snd) & = \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta (B, C), (B, C))
\end{align*}
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\text{CMon} \quad \bot \quad \text{Set}
\]

with \([-\,]\) : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \approx \text{Set}(A, U (M, \otimes, \epsilon)) : [-\,]

Unit and counit:

\[
\begin{align*}
\text{single } A &= [id_{\text{Free } A}] : A \to U (\text{Free } A) \\
\text{reduce } M &= [id_M] : \text{Free } (U M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \(h : \text{Free } A \to M\) and \(f : A \to U M = M\),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \to 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \to a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \minBound, \max))</td>
<td>({a} \to a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \maxBound, \min))</td>
<td>({a} \to a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \wedge))</td>
<td>({a} \to a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>({a} \to a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
\text{guard} : (A \to \mathbb{B}) \to \text{Bag A} \to \text{Bag A}
\]

\[
\text{guard } p \ a = \text{if } p \ a \text{ then } \{a\} \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a monad \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag } A) \to \text{Bag } A \\
\text{single} & : A \to \text{Bag } A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \} \).

In fact, for any adjunction \(L \dashv R\) between \(\mathbf{C}\) and \(\mathbf{D}\), we get a monad \((T, \mu, \eta)\) on \(\mathbf{D}\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \left[ id_A \right] L : T (T A) \to T A \\
\eta A & = \left[ id_A \right] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K \ V = V^K$. Maps $(-)^K$ from $K$ form a monad (the Reader monad in Haskell), so arise from an adjunction.

The laws of exponents arise from this adjunction, and from those for products and coproducts:

\[
\begin{align*}
\text{Map } 0 \ V & \approx 1 \\
\text{Map } 1 \ V & \approx V \\
\text{Map } (K_1 + K_2) \ V & \approx \text{Map } K_1 \ V \times \text{Map } K_2 \ V \\
\text{Map } (K_1 \times K_2) \ V & \approx \text{Map } K_1 (\text{Map } K_2 \ V) \\
\text{Map } K \ 1 & \approx 1 \\
\text{Map } K (V_1 \times V_2) & \approx \text{Map } K \ V_1 \times \text{Map } K \ V_2 : \text{merge}
\end{align*}
\]
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \overset{J}{\cong} \text{Set} \]

where \( J \) embeds, and \( \text{E} R : A \to \text{Set} B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} (K \times V) \cong \text{Map} K (\text{Bag} V) \]

Together, \textit{index} and \textit{merge} give efficient relational joins:

\[ x \ f \bowtie g \ y = \text{flatten} (\text{Map} K \ cp (\text{merge} (\text{groupBy} f \ x, \text{groupBy} g \ y))) \]

\[ \text{groupBy} : (V \to K) \to \text{Bag} \ V \to \text{Map} K (\text{Bag} V) \]

\[ \text{flatten} : \text{Map} K (\text{Bag} V) \to \text{Bag} V \]
13. Pointed sets and finite maps

Model \textit{finite maps} \( \text{Map}_* \) not as partial functions, but \textit{total} functions to a \textit{pointed} codomain \((A, a)\), i.e. a set \(A\) with a distinguished element \(a : A\).

Pointed sets and point-preserving functions form a category \(\text{Set}_*\). There is an adjunction to \(\text{Set}\), via

\[
\begin{array}{ccc}
\text{Set}_* & \cong & \text{Set} \\
\downarrow & & \downarrow \\
\text{U} & & \text{U}
\end{array}
\]

where \(\text{Maybe } A \cong 1 + A\) adds a point, and \(\text{U } (A, a) = A\) discards it.

In particular, \((\text{Bag } A, \emptyset)\) is a pointed set. Moreover, \(\text{Bag } f\) is point-preserving, so we get a functor \(\text{Bag}_* : \text{Set} \to \text{Set}_*\).

Indexing remains an isomorphism:

\[
\text{index} : \text{Bag}_* (K \times V) \cong \text{Map}_* K (\text{Bag}_* V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \to a : A \to \text{Map } K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid $$(M, \otimes, \epsilon)$$,

$$\mu X : T_m (T_n X) \to T_{m \otimes n} X$$
$$\eta X : X \to T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid $$(\mathbb{K}, \times, 1)$$ of finite key types under product.
15. Conclusions

- *Monad comprehensions* for database queries
- Structure arising from *adjunctions*
- Equivalences from *universal properties*
- Fitting in *relational joins*, via indexing
- To do: calculating *query optimisations*

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