Relational algebra by way of adjunctions
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Relational Algebra by Way of Adjunctions

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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation
  
  \[
  \left[ (\text{customer.name, invoice.amount}) \mid \text{customer} \leftarrow \text{customers}, \right.
  \left. \text{invoice} \leftarrow \text{invoices}, \right.
  \text{customer.cid} = \text{invoice.customer}, \\
  \text{invoice.due} \leq \text{today} \right]
  \]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[
(A, \leq) \perp (B, \subseteq) \quad \text{means} \quad f b \leq a \iff b \subseteq g a
\]

For example,

\[
(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \quad \text{inj} \quad (\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq) \quad \times k \quad (\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq) \quad \div k
\]

“Change of coordinates” can sometimes simplify reasoning; e.g. rhs gives
\[n \times k \leq m \iff n \leq m \div k,\]
and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set $|\mathbf{C}|$ of objects,
- a set $\mathbf{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathbf{C}|$,
- identity arrows $\text{id}_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category $\textbf{CMon}$ has commutative monoids $(M, \otimes, \epsilon)$ as objects, and homomorphisms $h: (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')$ as arrows:

\[
\begin{align*}
h (m \otimes n) &= h m \oplus h n \\
h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category $\textbf{Set}$ has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \rightarrow D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \rightarrow F Y$ when $f : X \rightarrow Y$, and

$$F \text{id}_X = \text{id}_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, forgetful functor $U : \text{CMon} \rightarrow \text{Set}$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')) = h : M \rightarrow M'$$

Conversely, Free : Set → CMon generates the free commutative monoid (ie bags) on a set of elements:

$$\text{Free } A = (\text{Bag } A, \cup, \emptyset)$$
$$\text{Free } (f : A \rightarrow B) = \text{map } f : \text{Bag } A \rightarrow \text{Bag } B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections. Given categories $\mathbf{C}, \mathbf{D}$, and functors $L : \mathbf{D} \to \mathbf{C}$ and $R : \mathbf{C} \to \mathbf{D}$, adjunction

\[
\begin{array}{ccc}
\mathbf{C} & \perp & \mathbf{D} \\
\downarrow & & \downarrow \\
\mathbf{R} & & \mathbf{L}
\end{array}
\]

means* $[-] : \mathbf{C}(L X, Y) \simeq \mathbf{D}(X, R Y) : [-]$.

A familiar example is given by currying:

\[
\begin{array}{ccc}
\mathbf{Set} & \perp & \mathbf{Set} \\
\downarrow & & \downarrow \\
(\cdot)^P & & (-)^P
\end{array}
\]

with $\text{curry} : \mathbf{Set}(X \times P, Y) \simeq \mathbf{Set}(X, Y^P) : \text{curry}$.

hence definitions and properties of $\text{apply} = \text{uncurry} \; id_{Y^P} : Y^P \times P \to Y$. 


7. Products and coproducts

\[
\begin{align*}
\text{Set} & \xrightarrow{\Delta} \text{Set}^2 & \xrightarrow{\Delta} \text{Set} \\
\text{Set} & \xleftarrow{\bigtriangleup} \text{Set} & \xleftarrow{\times} \text{Set}
\end{align*}
\]

with

\[
\begin{align*}
\text{fork} : \text{Set}^2(\Delta A, (B, C)) & \simeq \text{Set}(A, B \times C) : \text{fork}^\circ \\
\text{junc}^\circ : \text{Set}(A + B, C) & \simeq \text{Set}^2((A, B), \Delta C) : \text{junc}
\end{align*}
\]

hence

\[
\begin{align*}
\text{dup} & = \text{fork} \text{id}_{A,A} : \text{Set}(A, A \times A) \\
(fst, snd) & = \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C))
\end{align*}
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{align*}
\text{CMon} & \downarrow \quad \text{Set} \\
\downarrow \quad \text{U} & \\
\end{align*}
\]

with \([ - ] : \text{CMon} \to \text{Set} \)

\[
\text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \cong \text{Set}(A, U(M, \otimes, \epsilon)) : [-]
\]

Unit and counit:

\[
\begin{align*}
single A &= [id_{\text{Free } A}] : A \to U(\text{Free } A) \\
\text{reduce } M &= [id_M] : \text{Free } (U M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \( h : \text{Free } A \to M \) and \( f : A \to U M = M \),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U(h \cdot \text{single } A) = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>(ℕ, 0, +)</td>
<td>[a] \rightarrow 1</td>
</tr>
<tr>
<td>sum</td>
<td>(ℝ, 0, +)</td>
<td>[a] \rightarrow a</td>
</tr>
<tr>
<td>max</td>
<td>(ℤ, \text{minBound}, \text{max})</td>
<td>[a] \rightarrow a</td>
</tr>
<tr>
<td>min</td>
<td>(ℤ, \text{maxBound}, \text{min})</td>
<td>[a] \rightarrow a</td>
</tr>
<tr>
<td>all</td>
<td>(𝔹, \text{True}, \land)</td>
<td>[a] \rightarrow a</td>
</tr>
<tr>
<td>any</td>
<td>(𝔹, \text{False}, \lor)</td>
<td>[a] \rightarrow a</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action:

\[
guard : (A \rightarrow 𝔹) \rightarrow \text{Bag } A \rightarrow \text{Bag } A
\]

\[
guard p a = \text{if } p a \text{ then } \[a\] \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since 𝔹 = 1 + 1).
10. Monads

Bags form a *monad* \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag } A) \to \text{Bag } A \\
\text{single} & : A \to \text{Bag } A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \}^{\uparrow}\). In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \left[ id_A \right] L : T (T A) \to T A \\
\eta A & = \left[ id_A \right] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K \ V = V^K$. Maps $(-)^K$ from $K$ form a monad (the \textit{Reader} monad in Haskell), so arise from an adjunction.

The \textit{laws of exponents} arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 \ V \cong 1$
- $\text{Map } 1 \ V \cong V$
- $\text{Map } (K_1 + K_2) \ V \cong \text{Map } K_1 \ V \times \text{Map } K_2 \ V$
- $\text{Map } (K_1 \times K_2) \ V \cong \text{Map } K_1 \ (\text{Map } K_2 \ V)$
- $\text{Map } K \ 1 \cong 1$
- $\text{Map } K \ (V_1 \times V_2) \cong \text{Map } K \ V_1 \times \text{Map } K \ V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \xrightarrow{J} \downarrow \xleftarrow{E} \text{Set} \]

where \( J \) embeds, and \( E R : A \rightarrow \text{Set} B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V) \]

Together, \textit{index} and \textit{merge} give efficient relational joins:

\[ x f \bowtie g y = \text{flatten} \ (\text{Map} K \ cp \ (\text{merge} \ (\text{groupBy} f x, \text{groupBy} g y))) \]

\[ \text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag} V \rightarrow \text{Map} K (\text{Bag} V) \]

\[ \text{flatten} : \text{Map} K (\text{Bag} V) \rightarrow \text{Bag} V \]
13. Pointed sets and finite maps

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

![Diagram](attachment:image.png)

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $U (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

$$\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)$$
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \rightarrow a : A \rightarrow \text{Map } K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid $$(M, \otimes, \epsilon)$$,

$$\mu X : T_m (T_n X) \rightarrow T_{m \otimes n} X$$
$$\eta X : X \rightarrow T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid $$(\mathbb{K}, \times, 1)$$ of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

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