Relational algebra by way of adjunctions

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Relational Algebra by Way of Adjunctions

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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
\begin{array}{l}
\left[ \left( \text{customer}.\text{name}, \text{invoice}.\text{amount} \right) \\
| \text{customer} \leftarrow \text{customers}, \\
\quad \text{invoice} \leftarrow \text{invoices}, \\
\quad \text{customer}.\text{cid} = \text{invoice}.\text{customer}, \\
\quad \text{invoice}.\text{due} \leq \text{today} \right]
\end{array}
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \sqsubseteq)\]

means \(f b \leq a \iff b \sqsubseteq g a\)

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \(n \times k \leq m \iff n \leq m \div k\), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category \( C \) consists of

- a set* \( |C| \) of objects,
- a set* \( C(X, Y) \) of arrows \( X \to Y \) for each \( X, Y : |C| \),
- identity arrows \( id_X : X \to X \) for each \( X \)
- composition \( f \cdot g : X \to Z \) of compatible arrows \( g : X \to Y \) and \( f : Y \to Z \),
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set \((A, \leq)\) is a degenerate category, with objects \( A \) and a unique arrow \( a \to b \) iff \( a \leq b \).

\[ \cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to \cdots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a \textit{concrete category}: roughly,

- the objects are \textit{sets with additional structure}
- the arrows are \textit{structure-preserving mappings}

Many useful categories are of this form.

For example, the category \textbf{CMon} has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
    h (m \otimes n) &= h m \oplus h n \\
    h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category \textbf{Set} has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F \ id_X = id_{F \ X}$$
$$F (f \cdot g) = F \ f \cdot F \ g$$

For example, forgetful functor $U : \text{CMon} \to \text{Set}$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, $\text{Free} : \text{Set} \to \text{CMon}$ generates the free commutative monoid (ie bags) on a set of elements:

$$\text{Free} A = (\text{Bag} A, \cup, \emptyset)$$
$$\text{Free} (f : A \to B) = \text{map} f : \text{Bag} A \to \text{Bag} B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $\mathcal{C}, \mathcal{D}$, and functors $L : \mathcal{D} \to \mathcal{C}$ and $R : \mathcal{C} \to \mathcal{D}$, adjunction

$$\mathcal{C} \perp \mathcal{D}$$

means* $[-] : \mathcal{C}(L X, Y) \simeq \mathcal{D}(X, R Y) : [-]$

A familiar example is given by currying:

$$\mathsf{Set} \perp \mathsf{Set}$$

with $\text{curry} : \mathsf{Set}(X \times P, Y) \simeq \mathsf{Set}(X, Y^P) : \text{curry}^\circ$

hence definitions and properties of $\text{apply} = \text{uncurry } \text{id}_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

\[
\begin{array}{c}
\text{Set} \\
\downarrow \delta \\
\text{Set} \quad \downarrow \times \\
\text{Set} \\
\end{array}
\quad \quad
\begin{array}{c}
\text{Set}^2 \\
\downarrow \delta \\
\text{Set} \\
\end{array}
\begin{array}{c}
\downarrow \times \\
\delta \\
\end{array}
\begin{array}{c}
\text{Set} \\
\end{array}
\]

with

\[
fork : \text{Set}^2(\delta A, (B, C)) \cong \text{Set}(A, B \times C) : \text{fork}^\circ
\]

\[
junc^\circ : \text{Set}(A + B, C) \cong \text{Set}^2((A, B), \delta C) : junc
\]

hence

\[
dup = fork \ id_{A,A} : \text{Set}(A, A \times A)
\]

\[
(fst, snd) = fork^\circ id_{B \times C} : \text{Set}^2(\delta (B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\text{CMon} \quad \bot \quad \text{Set}
\]

with \( [-] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \)

\[\approx \text{Set}(A, \text{U} (M, \otimes, \epsilon)) : [-] \]

Unit and counit:

\[
single A = [id_{\text{Free } A}] : A \rightarrow \text{U} (\text{Free } A)
\]
\[
reduce M = [id_M] : \text{Free} (\text{U } M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \epsilon)
\]

whence, for \( h : \text{Free } A \rightarrow M \) and \( f : A \rightarrow \text{U } M = M \),

\[
h = reduce M \cdot \text{Free } f \iff \text{U } h \cdot single A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>(\llbracket a \rrbracket \rightarrow 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>(\llbracket a \rrbracket \rightarrow a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>(\llbracket a \rrbracket \rightarrow a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>(\llbracket a \rrbracket \rightarrow a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>(\llbracket a \rrbracket \rightarrow a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>(\llbracket a \rrbracket \rightarrow a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
\text{guard} : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag} A \rightarrow \text{Bag} A
\]

\[
\text{guard } p \ a = \text{if } p \ a \text{ then } \llbracket a \rrbracket \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a monad \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag} A) \to \text{Bag} A \\
\text{single} & : A \to \text{Bag} A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g a \}\).

In fact, for any adjunction \(L \dashv R\) between \(\mathbf{C}\) and \(\mathbf{D}\), we get a monad \((T, \mu, \eta)\) on \(\mathbf{D}\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \ [id_A] \ L : T (T A) \to T A \\
\eta A & = [id_A] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K \times V \simeq V^K$. Maps $(-)^K$ from $K$ form a monad (the Reader monad in Haskell), so arise from an adjunction.

The laws of exponents arise from this adjunction, and from those for products and coproducts:

\[
\begin{align*}
\text{Map } 0 & \simeq 1 \\
\text{Map } 1 & \simeq V \\
\text{Map } (K_1 + K_2) & \simeq \text{Map } K_1 \times \text{Map } K_2 \times V \\
\text{Map } (K_1 \times K_2) & \simeq \text{Map } K_1 \text{ (Map } K_2 \times V) \\
\text{Map } K & \simeq 1 \\
\text{Map } K (V_1 \times V_2) & \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}
\end{align*}
\]
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \xrightarrow{J} \text{Set} \xleftarrow{E} \text{Set} \]

where \( J \) embeds, and \( E \) \( R : A \rightarrow \text{Set} \ B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V) \]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[ x f \bowtie g y = \text{flatten} (\text{Map} K \ \text{cp} (\text{merge} (\text{groupBy} f x, \text{groupBy} g y))) \]

\( \text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag} V \rightarrow \text{Map} K (\text{Bag} V) \)

\( \text{flatten} : \text{Map} K (\text{Bag} V) \rightarrow \text{Bag} V \)
13. Pointed sets and finite maps

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

\[
\begin{array}{ccc}
\text{Set}_* & \perp & \text{Set} \\
\text{Maybe} & \downarrow & \downarrow \text{U} \\
\end{array}
\]

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $\text{U } (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

\[
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

\[ \eta a = \lambda k \to a : A \to \text{Map} K A \]

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

\[ \mu X : T_m (T_n X) \to T_{m \otimes n} X \]
\[ \eta X : X \to T_\epsilon X \]

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

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