Relational algebra by way of adjunctions
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Relational Algebra by Way of Adjunctions

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1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

```
[ (customer.name, invoice.amount)
| customer ← customers,
  invoice ← invoices,
  customer.cid = invoice.customer,
  invoice.due ≤ today ]
```

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[ (A, \leq) \perp (B, \sqsubseteq) \]

means \( f b \leq a \iff b \sqsubseteq g a \)

For example,

\[ (\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \]

\[ (\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq) \]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \( n \times k \leq m \iff n \leq m \div k \), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A *category* $\mathbf{C}$ consists of

- a set* $|\mathbf{C}|$ of *objects*,
- a set* $\mathbf{C}(X, Y)$ of *arrows* $X \to Y$ for each $X, Y : |\mathbf{C}|$,
- *identity* arrows $\text{id}_X : X \to X$ for each $X$
- *composition* $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to \cdots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a \textit{concrete category}: roughly,

\begin{itemize}
  \item the objects are \textit{sets with additional structure}
  \item the arrows are \textit{structure-preserving mappings}
\end{itemize}

Many useful categories are of this form.

For example, the category \textbf{CMon} has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
  h (m \otimes n) &= h m \oplus h n \\
  h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category \textbf{Set} has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor \( F : C \to D \) is an operation on both objects and arrows, preserving the structure: \( F f : F X \to F Y \) when \( f : X \to Y \), and

\[
F \ id_X = id_{F X} \\
F (f \cdot g) = F f \cdot F g
\]

For example, forgetful functor \( U : \text{CMon} \to \text{Set} \):

\[
U (M, \otimes, \epsilon) = M \\
U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'
\]

Conversely, \( \text{Free} : \text{Set} \to \text{CMon} \) generates the free commutative monoid (ie bags) on a set of elements:

\[
\text{Free } A = (\text{Bag } A, \cup, \emptyset) \\
\text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B
\]
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections. Given categories $C, D$, and functors $L : D \to C$ and $R : C \to D$, adjunction

\[ C \perp D \]

means* \( [-] : C(L X, Y) \simeq D(X, R Y) : [-] \)

A familiar example is given by currying:

\[ \text{Set} \perp \text{Set} \]

with \( \text{curry} : \text{Set}(X \times P, Y) \simeq \text{Set}(X, Y^P) : \text{curry}^\circ \)

hence definitions and properties of \( \text{apply} = \text{uncurry} \ \text{id}_{Y^P} : Y^P \times P \to Y \)
7. Products and coproducts

with

\[ \text{fork} : \mathbf{Set}^2(\Delta A, (B, C)) \cong \mathbf{Set}(A, B \times C) : \text{fork}^\circ \]
\[ \text{junc}^\circ : \mathbf{Set}(A + B, C) \cong \mathbf{Set}^2(((A, B), \Delta C) : \text{junc} \]

hence

\[ \text{dup} \quad = \quad \text{fork id}_{A,A} : \mathbf{Set}(A, A \times A) \]
\[ (\text{fst}, \text{snd}) \quad = \quad \text{fork}^\circ \text{id}_{B \times C} : \mathbf{Set}^2(\Delta(B, C), (B, C)) \]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{align*}
\text{CMon} & \quad \bot \quad \text{Set} \quad \text{with} \quad [-] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \\
& \cong \text{Set}(A, U (M, \otimes, \epsilon)) : [-]
\end{align*}
\]

Unit and counit:

\[
\begin{align*}
single A &= [id_{\text{Free } A}] : A \to U (\text{Free } A) \\
reduce M &= [id_M] : \text{Free } (U M) \to M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \( h : \text{Free } A \to M \) and \( f : A \to U M = M \),

\[
h = reduce M \cdot \text{Free } f \iff U h \cdot single A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>( (\mathbb{N}, 0, +) )</td>
<td>{a} \to 1</td>
</tr>
<tr>
<td>sum</td>
<td>( (\mathbb{R}, 0, +) )</td>
<td>{a} \to a</td>
</tr>
<tr>
<td>max</td>
<td>( (\mathbb{Z}, \text{minBound}, \text{max}) )</td>
<td>{a} \to a</td>
</tr>
<tr>
<td>min</td>
<td>( (\mathbb{Z}, \text{maxBound}, \text{min}) )</td>
<td>{a} \to a</td>
</tr>
<tr>
<td>all</td>
<td>( (\mathbb{B}, \text{True}, \land) )</td>
<td>{a} \to a</td>
</tr>
<tr>
<td>any</td>
<td>( (\mathbb{B}, \text{False}, \lor) )</td>
<td>{a} \to a</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \to \mathbb{B}) \to \text{Bag} \ A \to \text{Bag} \ A
\]

\[
guard \ p \ a = \text{if } p \ a \ \text{then } \{a\} \ \text{else } \emptyset
\]

Laws about selections follow from laws of homomorphisms
(and of coproducts, since \( \mathbb{B} = 1 + 1 \)).
10. Monads

Bags form a monad \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} : \text{Bag} \ (\text{Bag} \ A) & \to \text{Bag} \ A \\
\text{single} : A & \to \text{Bag} \ A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \rfloor\).

In fact, for any adjunction \(L \dashv R\) between \(\mathbf{C}\) and \(\mathbf{D}\), we get a monad \((T, \mu, \eta)\) on \(\mathbf{D}\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \ [ \text{id}_A ] \ L : T \ (T \ A) \to T \ A \\
\eta A & = [ \text{id}_A ] : A \to T \ A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(\cdot)^K$ from $K$ form a monad (the *Reader* monad in Haskell), so arise from an adjunction.

The *laws of exponents* arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \cong 1$
- $\text{Map } 1 V \cong V$
- $\text{Map } (K_1 + K_2) V \cong \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \cong \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \cong 1$
- $\text{Map } K (V_1 \times V_2) \cong \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \xrightarrow{\bot} \text{Set} \xleftarrow{\text{E}} \text{J} \]

where J embeds, and E \( R : A \rightarrow \text{Set} \ B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V) \]

Together, \text{index} and \text{merge} give efficient relational joins:

\[ x_{f \rhd g} y = \text{flatten} (\text{Map} K \ \text{cp} (\text{merge} (\text{groupBy} f x, \text{groupBy} g y))) \]

\[ \text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag} V \rightarrow \text{Map} K (\text{Bag} V) \]

\[ \text{flatten} : \text{Map} K (\text{Bag} V) \rightarrow \text{Bag} V \]
13. Pointed sets and finite maps

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$
\begin{align*}
\text{Set}_* & \xrightarrow{\perp} \text{Set} \\
\text{Maybe} & \xleftarrow{\perp} \text{Set}_*
\end{align*}
$$

where $\text{Maybe} A \simeq 1 + A$ adds a point, and $U (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

$$
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
$$
14. Graded monads

A catch: finite maps aren’t a monad, because

\[ \eta a = \lambda k \to a : A \to \text{Map} K A \]

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

\[ \mu X : T_m (T_n X) \to T_{m \otimes n} X \]
\[ \eta X : X \to T\epsilon X \]

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *Monad comprehensions* for database queries
- Structure arising from *adjunctions*
- Equivalences from *universal properties*
- Fitting in *relational joins*, via indexing
- To do: calculating *query optimisations*

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