Relational algebra by way of adjunctions
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Publication date:
2016

Document version
Early version, also known as pre-print

Citation for published version (APA):
Relational Algebra by Way of Adjunctions

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DBPL, October 2015
1. Summary

- bulk types (sets, bags, lists) are **monads**
- monads have nice **mathematical foundations via adjunctions**
- monads support **comprehensions**
- comprehension syntax provides a **query notation**

```
[ (customer.name, invoice.amount)
| customer ← customers,
   invoice ← invoices,
   customer.cid = invoice.customer,
   invoice.due ≤ today ]
```

- monad structure explains **selection, projection**
- less obvious how to explain **join**
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \subseteq) \]

means \( f b \leq a \iff b \subseteq g a \)

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \]

\[(\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq) \]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \( n \times k \leq m \iff n \leq m \div k \), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set $|\mathbf{C}|$ of objects,
- a set $\mathbf{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathbf{C}|$,
- identity arrows $\text{id}_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \ldots \to -2 \to -1 \to 0 \to 1 \to 2 \to \ldots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category $\textbf{CMon}$ has commutative monoids $(M, \otimes, \epsilon)$ as objects, and homomorphisms $h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')$ as arrows:

\[
\begin{align*}
h (m \otimes n) &= h m \oplus h n \\
h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category $\textbf{Set}$ has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F id_X = id_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, forgetful functor $U : \text{CMon} \to \text{Set}$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, Free : $\text{Set} \to \text{CMon}$ generates the free commutative monoid (ie bags) on a set of elements:

$$\text{Free } A = (\text{Bag } A, \cup, \emptyset)$$
$$\text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $\mathcal{C}, \mathcal{D}$, and functors $L : \mathcal{D} \to \mathcal{C}$ and $R : \mathcal{C} \to \mathcal{D}$, adjunction $\mathcal{C} \perp \mathcal{D}$ means $[-] : \mathcal{C}(L X, Y) \simeq \mathcal{D}(X, R Y) : [-]$.

A familiar example is given by currying:

$\mathsf{Set} \perp \mathsf{Set}$ with $\text{curry} : \mathsf{Set}(X \times P, Y) \simeq \mathsf{Set}(X, Y^P) : \text{curry}^\circ$

hence definitions and properties of $\text{apply} = \text{uncurry} \ id_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

\[
\begin{array}{ccc}
\text{Set} & \\ & \perp & \text{Set}^2 \\
\perp & & \perp \\
\Delta & & \times
\end{array}
\]

with

\[
\text{fork} : \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) : \text{fork}^\circ
\]

\[
\text{junc}^\circ : \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
dup = \text{fork } \text{id}_{A,A} : \text{Set}(A, A \times A)
\]

\[
(fst, snd) = \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[ \text{CMon} \quad \perp \quad \text{Set} \]

with \([\cdot]:\text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \approx \text{Set}(A, U (M, \otimes, \epsilon)) : [\cdot] \]

Unit and counit:

\[
\begin{align*}
\text{single } A &= [id_{\text{Free } A}] : A \rightarrow U (\text{Free } A) \\
\text{reduce } M &= [id_M] : \text{Free } (U M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \epsilon)
\end{align*}
\]

whence, for \( h: \text{Free } A \rightarrow M \) and \( f : A \rightarrow U M = M \),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
# 9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>count</em></td>
<td>$\langle \mathbb{N}, 0, + \rangle$</td>
<td>${a} \mapsto 1$</td>
</tr>
<tr>
<td><em>sum</em></td>
<td>$\langle \mathbb{R}, 0, + \rangle$</td>
<td>${a} \mapsto a$</td>
</tr>
<tr>
<td><em>max</em></td>
<td>$\langle \mathbb{Z}, \text{minBound}, \text{max} \rangle$</td>
<td>${a} \mapsto a$</td>
</tr>
<tr>
<td><em>min</em></td>
<td>$\langle \mathbb{Z}, \text{maxBound}, \text{min} \rangle$</td>
<td>${a} \mapsto a$</td>
</tr>
<tr>
<td><em>all</em></td>
<td>$\langle \mathbb{B}, \text{True}, \land \rangle$</td>
<td>${a} \mapsto a$</td>
</tr>
<tr>
<td><em>any</em></td>
<td>$\langle \mathbb{B}, \text{False}, \lor \rangle$</td>
<td>${a} \mapsto a$</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

$$
guard : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A
$$

$$
guard p a = \text{if } p a \text{ then } \{a\} \text{ else } \emptyset
$$

Laws about selections follow from laws of homomorphisms (and of coproducts, since $\mathbb{B} = 1 + 1$).
10. Monads

Bags form a *monad* \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} & : \text{Bag} (\text{Bag} A) \to \text{Bag} A \\
\text{single} & : A \to \text{Bag} A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \}\). In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \lbrack id_A \rbrack L : T (T A) \to T A \\
\eta A & = \lbrack id_A \rbrack : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K \ V = V^K$. Maps $(-)^K$ from $K$ form a monad (the \textit{Reader} monad in Haskell), so arise from an adjunction. The \textit{laws of exponents} arise from this adjunction, and from those for products and coproducts:

\[
\begin{align*}
\text{Map } 0 \ V & \simeq 1 \\
\text{Map } 1 \ V & \simeq V \\
\text{Map } (K_1 + K_2) \ V & \simeq \text{Map } K_1 \ V \times \text{Map } K_2 \ V \\
\text{Map } (K_1 \times K_2) \ V & \simeq \text{Map } K_1 \ (\text{Map } K_2 \ V) \\
\text{Map } K \ 1 & \simeq 1 \\
\text{Map } K \ (V_1 \times V_2) & \simeq \text{Map } K \ V_1 \times \text{Map } K \ V_2 : \text{merge}
\end{align*}
\]
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \xymatrix{ & \perp \ar[dl]_J \ar[dr]^-E \ar@/_1.5pc/[rr] \ar@/^1.5pc/[rr] \cr \rightarrow \ar[ur] & \text{Set} } \]

where \( J \) embeds, and \( E \ R : A \rightarrow \text{Set} \ B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} \ (K \times V) \simeq \text{Map} \ K \ (\text{Bag} \ V) \]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[ x \ f \bowtie g \ y = \text{flatten} \ (\text{Map} \ K \ \text{cp} \ (\text{merge} \ (\text{groupBy} \ f \ x, \text{groupBy} \ g \ y))) \]

\( \text{groupBy} : (V \rightarrow K) \rightarrow \text{Bag} \ V \rightarrow \text{Map} \ K \ (\text{Bag} \ V) \)

\( \text{flatten} : \text{Map} \ K \ (\text{Bag} \ V) \rightarrow \text{Bag} \ V \)
13. Pointed sets and finite maps

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

\[
\begin{array}{ccc}
\text{Set}_* & \downarrow & \text{Set} \\
\text{Maybe} & \circlearrowleft & \\
\downarrow & & \uparrow \\
U & & \\
\end{array}
\]

where $\text{Maybe} A \simeq 1 + A$ adds a point, and $U (A, a) = A$ discards it.

In particular, $(\text{Bag} A, \emptyset)$ is a pointed set. Moreover, $\text{Bag} f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

\[
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \rightarrow a : A \rightarrow \text{Map} K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

$$\mu X : T_m (T_n X) \rightarrow T_{m \otimes n} X$$
$$\eta X : X \rightarrow T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *Monad comprehensions* for database queries
- Structure arising from *adjunctions*
- Equivalences from *universal properties*
- Fitting in *relational joins*, via indexing
- To do: calculating *query optimisations*

Thanks to EPSRC *Unifying Theories of Generic Programming* for funding.