Relational algebra by way of adjunctions
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Publication date:
2016

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Relational Algebra by Way of Adjunctions

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(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)
DBPL, October 2015
1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
[ (\text{customer}.\text{name}, \text{invoice}.\text{amount}) \\
| \text{customer} \leftarrow \text{customers}, \\
\quad \text{invoice} \leftarrow \text{invoices}, \\
\quad \text{customer}.\text{cid} = \text{invoice}.\text{customer}, \\
\quad \text{invoice}.\text{due} \leq \text{today} ]
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[ (A, \leq) \perp (B, \sqsubseteq) \quad \text{means} \quad f(b) \leq a \iff b \sqsubseteq g(a) \]

For example,

\[ (\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \]

\[ (\mathbb{Z}, \leq_{\mathbb{Z}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \]

“Change of coordinates” can sometimes simplify reasoning; e.g., rhs gives

\[ n \times k \leq m \iff n \leq m \div k, \]

and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set* $\mid \mathbf{C} \mid$ of objects,
- a set* $\mathbf{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : \mid \mathbf{C} \mid$,
- identity arrows $\text{id}_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \ldots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a **concrete category**: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category **CMon** has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
h (m \otimes n) &= h m \oplus h n \\
h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category **Set** has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F id_X = id_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, forgetful functor $U : CMon \to Set$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, Free : Set $\to$ CMon generates the free commutative monoid (ie bags) on a set of elements:

$$Free A = (Bag A, \cup, \emptyset)$$
$$Free (f : A \to B) = map f : Bag A \to Bag B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $\mathcal{C}, \mathcal{D}$, and functors $L : \mathcal{D} \to \mathcal{C}$ and $R : \mathcal{C} \to \mathcal{D}$, adjunction

$\mathcal{C} \perp \mathcal{D}$

means

$$[-] : \mathcal{C}(L X, Y) \simeq \mathcal{D}(X, R Y) : [-]$$

A familiar example is given by currying:

$\mathsf{Set} \perp \mathsf{Set}$

with $\text{curry} : \mathsf{Set}(X \times P, Y) \simeq \mathsf{Set}(X, Y^P) : \text{curry}^\circ$

hence definitions and properties of $\text{apply} = \text{uncurry } \text{id}_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

\[
\begin{array}{c}
\text{Set} & \perp & \text{Set}^2 & \perp & \text{Set} \\
\Delta & \downarrow & \Delta & \downarrow & \times
\end{array}
\]

with

\[
\begin{align*}
\text{fork} &: \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) : \text{fork}^\circ \\
\text{junc}^\circ &: \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc}
\end{align*}
\]

hence

\[
\begin{align*}
\text{dup} &= \text{fork} \text{id}_{A,A} : \text{Set}(A, A \times A) \\
(fst, snd) &= \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta (B, C), (B, C))
\end{align*}
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

```
\[ \text{CMon} \downarrow \text{Set} \]
```

with \([\cdot]: \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \cong \text{Set}(A, U(M, \otimes, \epsilon)) : [\cdot] \]

Unit and counit:

- **single** \(A\) = \(\lfloor id_{\text{Free } A} \rfloor : A \to U(\text{Free } A)\)
- **reduce** \(M\) = \(\lceil id_M \rceil : \text{Free } (U M) \to M -- \text{for } M = (M, \otimes, \epsilon)\)

whence, for \(h: \text{Free } A \to M \) and \(f: A \to U M = M\),

\[
h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \rightarrow 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>({a} \rightarrow a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[ \text{guard} : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag} \ A \rightarrow \text{Bag} \ A \]

\[ \text{guard} \ p \ a = \text{if} \ p \ a \ \text{then} \ \{a\} \ \text{else} \ \emptyset \]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a monad \((\text{Bag, union, single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} : \text{Bag} (\text{Bag} A) & \to \text{Bag} A \\
\text{single} : A & \to \text{Bag} A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g a \}\).

In fact, for any adjunction \(L \dashv R\) between \(\mathcal{C}\) and \(\mathcal{D}\), we get a monad \((T, \mu, \eta)\) on \(\mathcal{D}\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \left[ id_A \right] L : T (T A) \to T A \\
\eta A & = [id_A] : A \to T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the $\text{Reader}$ monad in Haskell), so arise from an adjunction. The $\textit{laws of exponents}$ arise from this adjunction, and from those for products and coproducts:

\[
\begin{align*}
\text{Map } 0 V & \cong 1 \\
\text{Map } 1 V & \cong V \\
\text{Map } (K_1 + K_2) V & \cong \text{Map } K_1 V \times \text{Map } K_2 V \\
\text{Map } (K_1 \times K_2) V & \cong \text{Map } K_1 (\text{Map } K_2 V) \\
\text{Map } K 1 & \cong 1 \\
\text{Map } K (V_1 \times V_2) & \cong \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}
\end{align*}
\]
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\begin{array}{ccc}
\text{Rel} & \downarrow & \text{Set} \\
\text{J} & \circlearrowleft & \\
\text{E} & \circlearrowright & \\
\end{array}
\]

where \( J \) embeds, and \( E R : A \to \text{Set} B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[
\text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V)
\]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[
x f \bowtie g y = \text{flatten} (\text{Map} K \ cp (\text{merge} (\text{groupBy} f x, \text{groupBy} g y)))
\]

\( \text{groupBy} : (V \to K) \to \text{Bag} V \to \text{Map} K (\text{Bag} V) \)

\( \text{flatten} : \text{Map} K (\text{Bag} V) \to \text{Bag} V \)
13. Pointed sets and finite maps

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$\begin{array}{ccc}
\text{Set}_* & \dashv & \text{Set} \\
\downarrow \text{Maybe} & & \downarrow \text{U} \\
\text{Set}_* & \dashv & \text{Set}
\end{array}$$

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $\text{U } (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

$$\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)$$
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \to a : A \to \text{Map} K A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

$$\mu X : T_m (T_n X) \to T_{m \otimes n} X$$

$$\eta X : X \to T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

Thanks to EPSRC *Unifying Theories of Generic Programming* for funding.