Relational algebra by way of adjunctions
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Publication date:
2016

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Relational Algebra by Way of Adjunctions

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DBPL, October 2015
1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
[ (\text{customer.name}, \text{invoice.amount}) \\
| \text{customer} \leftarrow \text{customers}, \\
\quad \text{invoice} \leftarrow \text{invoices}, \\
\quad \text{customer.cid} = \text{invoice.customer}, \\
\quad \text{invoice.due} \leq \text{today} ]
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[
(A, \leq) \perp (B, \sqsubseteq) \quad \text{means} \quad f b \leq a \iff b \sqsubseteq g a
\]

For example,

\[
(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \quad \text{and} \quad (\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq)
\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives
\[
n \times k \leq m \iff n \leq m \div k,
\]
and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set* $|\mathbf{C}|$ of objects,
- a set* $\mathbf{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathbf{C}|$,
- identity arrows $\text{id}_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

$$\cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a concrete category: roughly,

- the objects are sets with additional structure
- the arrows are structure-preserving mappings

Many useful categories are of this form.

For example, the category \textbf{CMon} has commutative monoids \((M, \otimes, \epsilon)\) as objects, and homomorphisms \(h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')\) as arrows:

\[
\begin{align*}
    h (m \otimes n) &= h m \oplus h n \\
    h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category \textbf{Set} has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F \text{id}_X = \text{id}_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, *forgetful* functor $U : \text{CMon} \to \text{Set}$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, $\text{Free} : \text{Set} \to \text{CMon}$ generates the *free* commutative monoid (ie bags) on a set of elements:

$$\text{Free} A = (\text{Bag} A, \cup, \emptyset)$$
$$\text{Free} (f : A \to B) = \text{map } f : \text{Bag} A \to \text{Bag} B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections.

Given categories $\mathbf{C}, \mathbf{D}$, and functors $L : \mathbf{D} \to \mathbf{C}$ and $R : \mathbf{C} \to \mathbf{D}$, adjunction

$\begin{array}{ccc}
\mathbf{C} & \perp & \mathbf{D} \\
\downarrow L & & \downarrow R \\
\mathbf{C} & \perp & \mathbf{D}
\end{array}$

means* $[-] : \mathbf{C}(L X, Y) \simeq \mathbf{D}(X, R Y) : [-]$.

A familiar example is given by currying:

$\begin{array}{ccc}
\mathbf{Set} & \perp & \mathbf{Set} \\
\downarrow - \times P & & \downarrow (-)^P \\
\mathbf{Set} & \perp & \mathbf{Set}
\end{array}$

with $\text{curry} : \mathbf{Set}(X \times P, Y) \simeq \mathbf{Set}(X, Y^P) : \text{curry}^\circ$.

hence definitions and properties of $\text{apply} = \text{uncurry} \ id_{Y^P} : Y^P \times P \to Y$.
7. Products and coproducts

\[
\begin{array}{ccc}
\Delta & \rightarrow & \leftarrow \\
\downarrow & & & \downarrow \\
\downarrow & & & \downarrow \\
\downarrow & & & \downarrow \\
\Delta & \rightarrow & \leftarrow \\
\end{array}
\]

with

\[
\text{fork} : \mathsf{Set}^2(\Delta A, (B, C)) \cong \mathsf{Set}(A, B \times C) : \text{fork}^\circ
\]
\[
\text{junc}^\circ : \mathsf{Set}(A + B, C) \cong \mathsf{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
dup = \text{fork id}_{A,A} : \mathsf{Set}(A, A \times A)
\]
\[
(fst, snd) = \text{fork}^\circ \text{id}_{B \times C} : \mathsf{Set}^2(\Delta (B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\text{CMon} \downarrow \text{Set} \quad \quad \text{with} \quad \text{[-]} : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \\
\cong \text{Set}(A, \mathcal{U}(M, \otimes, \epsilon)) \quad : \text{[-]}
\]

Unit and counit:

\[
\text{single } A = [\text{id}_{\text{Free } A}] : A \to \mathcal{U}(\text{Free } A) \\
\text{reduce } M = [\text{id}_M] : \text{Free } (\mathcal{U} M) \to M \quad \quad \text{-- for } M = (M, \otimes, \epsilon)
\]

whence, for \( h : \text{Free } A \to M \) and \( f : A \to \mathcal{U} M = M \),

\[
h = \text{reduce } M \cdot \text{Free } f \iff \mathcal{U} h \cdot \text{single } A = f
\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
# 9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>(\emptyset \rightarrow 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>(\emptyset \rightarrow a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>(\emptyset \rightarrow a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>(\emptyset \rightarrow a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>(\emptyset \rightarrow a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>(\emptyset \rightarrow a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A
\]

\[
guard p a = \text{if } p a \text{ then } \emptyset \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a *monad* \((\text{Bag}, \text{union}, \text{single})\) with

\[
\begin{align*}
\text{Bag} & = U \cdot \text{Free} \\
\text{union} &: \text{Bag} \ (\text{Bag} \ A) \to \text{Bag} \ A \\
\text{single} &: \ A \to \text{Bag} \ A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \}\). In fact, for any adjunction \(\mathbf{L} \dashv \mathbf{R}\) between \(\mathbf{C}\) and \(\mathbf{D}\), we get a monad \((T, \mu, \eta)\) on \(\mathbf{D}\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu \ A & = R \ [id_A] \ L : T \ (T \ A) \to T \ A \\
\eta \ A & = [id_A] : A \to T \ A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the *Reader* monad in Haskell), so arise from an adjunction.

The *laws of exponents* arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \simeq 1$
- $\text{Map } 1 V \simeq V$
- $\text{Map } (K_1 + K_2) V \simeq \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \simeq \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \simeq 1$
- $\text{Map } K (V_1 \times V_2) \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\begin{array}{c}
\text{Rel} \\
\downarrow \\
\text{Set}
\end{array}
\]

where \( J \) embeds, and \( E R : A \to \text{Set} B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[
\text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V)
\]

Together, \text{index} and \text{merge} give efficient relational joins:

\[
x f \bowtie g y = \text{flatten} (\text{Map} K cp (\text{merge} (\text{groupBy} f x, \text{groupBy} g y)))
\]

\[
\text{groupBy} : (V \to K) \to \text{Bag} V \to \text{Map} K (\text{Bag} V)
\]

\[
\text{flatten} : \text{Map} K (\text{Bag} V) \to \text{Bag} V
\]
13. Pointed sets and finite maps

Model finite maps $\text{Map}_*$ not as partial functions, but total functions to a pointed codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$\begin{array}{c}
\text{Set}_* \\
\downarrow \\
\text{Set} \\
\downarrow \\
\text{Set}_*
\end{array}
$$

where $\text{Maybe } A \simeq 1 + A$ adds a point, and $U (A, a) = A$ discards it.

In particular, $(\text{Bag } A, \emptyset)$ is a pointed set. Moreover, $\text{Bag } f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

$$\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)$$
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta a = \lambda k \to a : A \to \text{Map} \ K \ A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

$$\mu X : T_m (T_n X) \to T_{m \otimes n} X$$
$$\eta X : X \to T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *Monad comprehensions* for database queries
- Structure arising from *adjunctions*
- Equivalences from *universal properties*
- Fitting in *relational joins*, via indexing
- To do: calculating *query optimisations*

Thanks to EPSRC *Unifying Theories of Generic Programming* for funding.