Relational algebra by way of adjunctions

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Publication date: 2016

Document version: Early version, also known as pre-print

Citation for published version (APA):
Relational Algebra by Way of Adjunctions

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DBPL, October 2015
1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
[ (\text{customer}.*\text{name}, \text{invoice}.*\text{amount}) \\
\mid \text{customer} \leftarrow \text{customers}, \\
\text{invoice} \leftarrow \text{invoices}, \\
\text{customer}.*\text{cid} = \text{invoice}.*\text{customer}, \\
\text{invoice}.*\text{due} \leq \text{today} ]
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[ (A, \leq) \perp (B, \sqsubseteq) \]

means \( f b \leq a \iff b \sqsubseteq g a \)

For example,

\[ (\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}}) \]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \( n \times k \leq m \iff n \leq m \div k \), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category \( \mathbf{C} \) consists of

- a set* \( |\mathbf{C}| \) of objects,
- a set* \( \mathbf{C}(X, Y) \) of arrows \( X \to Y \) for each \( X, Y : |\mathbf{C}| \),
- identity arrows \( \text{id}_X : X \to X \) for each \( X \)
- composition \( f \cdot g : X \to Z \) of compatible arrows \( g : X \to Y \) and \( f : Y \to Z \),
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set \((A, \leq)\) is a degenerate category, with objects \( A \) and a unique arrow \( a \to b \) iff \( a \leq b \).

\[ \cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to \cdots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a *concrete category*: roughly,

- the objects are *sets with additional structure*
- the arrows are *structure-preserving mappings*

Many useful categories are of this form.

For example, the category $\text{CMon}$ has commutative monoids $(M, \otimes, \epsilon)$ as objects, and homomorphisms $h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')$ as arrows:

\[
\begin{align*}
    h (m \otimes n) &= h m \oplus h n \\
    h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category $\text{Set}$ has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A *functor* $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

\[
F \text{id}_X = \text{id}_{F X} \\
F (f \cdot g) = F f \cdot F g
\]

For example, *forgetful* functor $U : \text{CMon} \to \text{Set}$:

\[
U (M, \otimes, \epsilon) = M \\
U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'
\]

Conversely, $\text{Free} : \text{Set} \to \text{CMon}$ generates the *free* commutative monoid (ie bags) on a set of elements:

\[
\text{Free } A = (\text{Bag } A, \cup, \emptyset) \\
\text{Free } (f : A \to B) = \text{map } f : \text{Bag } A \to \text{Bag } B
\]
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections. Given categories $C, D$, and functors $L : D \to C$ and $R : C \to D$, adjunction

\[ C \bot D \]

\[ \begin{array}{c}
\downarrow L \\
\downarrow R
\end{array} \]

means\* $[-] : C(L X, Y) \simeq D(X, R Y) : [-]$.

A familiar example is given by currying:

\[ \begin{array}{c}
\downarrow (- \times P) \\
\downarrow (\cdot)^P
\end{array} \]

\[ \text{Set} \bot \text{Set} \]

with $\text{curry} : \text{Set}(X \times P, Y) \simeq \text{Set}(X, Y^P) : \text{curry}^\circ$

hence definitions and properties of $\text{apply} = \text{uncurry \, id}_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

with

\[ \text{fork} : \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) : \text{fork}^\circ \]
\[ \text{junc}^\circ : \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc} \]

hence

\[ \text{dup} = \text{fork id}_{A,A} : \text{Set}(A, A \times A) \]
\[ (\text{fst}, \text{snd}) = \text{fork}^\circ \text{id}_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C)) \]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[
\begin{array}{ccc}
\text{CMon} & \perp & \text{Set} \\
\oplus & \rightarrow & \rightarrow \downarrow \rightarrow \\
\text{Free} & \Downarrow & \text{Set} \\
\end{array}
\]

with \([-]\) : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \cong \text{Set}(A, U (M, \otimes, \epsilon)) : [-]

Unit and counit:

- \(\text{single } A = [id_{\text{Free } A}] : A \rightarrow U (\text{Free } A)\)
- \(\text{reduce } M = [id_M] : \text{Free } (U M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \epsilon)\)

whence, for \(h : \text{Free } A \rightarrow M\) and \(f : A \rightarrow U M = M\),

\[h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f\]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \rightarrow 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \text{max}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \text{min}))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>({a} \rightarrow a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag } A \rightarrow \text{Bag } A \\
guard p a = \text{if } p a \text{ then } \{a\} \text{ else } \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a *Monad* \((\text{Bag, union, single})\) with

\[
\begin{align*}
\text{Bag} & = \text{U} \cdot \text{Free} \\
\text{union} & : \text{Bag (Bag } A) \rightarrow \text{Bag } A \\
\text{single} & : A \rightarrow \text{Bag } A
\end{align*}
\]

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g \ a \} \).

In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
\begin{align*}
T & = R \cdot L \\
\mu A & = R \left[ id_A \right] L : T (T A) \rightarrow T A \\
\eta A & = [id_A] : A \rightarrow T A
\end{align*}
\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the $\text{Reader}$ monad in Haskell), so arise from an adjunction.

The *laws of exponents* arise from this adjunction, and from those for products and coproducts:

\[
\begin{align*}
\text{Map } 0 V & \simeq 1 \\
\text{Map } 1 V & \simeq V \\
\text{Map } (K_1 + K_2) V & \simeq \text{Map } K_1 V \times \text{Map } K_2 V \\
\text{Map } (K_1 \times K_2) V & \simeq \text{Map } K_1 (\text{Map } K_2 V) \\
\text{Map } K 1 & \simeq 1 \\
\text{Map } K (V_1 \times V_2) & \simeq \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}
\end{align*}
\]
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[
\begin{array}{ccc}
\text{Rel} & \xrightarrow{J} & \text{Set} \\
\downarrow & & \downarrow \\
\bot & \xleftarrow{E} & B
\end{array}
\]

where \( J \) embeds, and \( E R : A \to \text{Set} B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[
\text{index} : \text{Bag} (K \times V) \simeq \text{Map} K (\text{Bag} V)
\]

Together, \textit{index} and \textit{merge} give efficient relational joins:

\[
x f \Join_J g y = \text{flatten} (\text{Map} K \ \text{cp} (\text{merge} (\text{groupBy} f x, \text{groupBy} g y)))
\]

\[
\text{groupBy} : (V \to K) \to \text{Bag} V \to \text{Map} K (\text{Bag} V)
\]

\[
\text{flatten} : \text{Map} K (\text{Bag} V) \to \text{Bag} V
\]
13. **Pointed sets and finite maps**

Model *finite maps* $\text{Map}_*$ not as partial functions, but *total* functions to a *pointed* codomain $(A, a)$, i.e. a set $A$ with a distinguished element $a : A$.

Pointed sets and point-preserving functions form a category $\text{Set}_*$. There is an adjunction to $\text{Set}$, via

$$
\begin{array}{ccc}
\text{Set}_* & \Downarrow & \text{Set} \\
\uparrow \text{Maybe} & & \uparrow \text{U} \\
\end{array}
$$

where $\text{Maybe} A \simeq 1 + A$ adds a point, and $\text{U} (A, a) = A$ discards it.

In particular, $(\text{Bag} A, \emptyset)$ is a pointed set. Moreover, $\text{Bag} f$ is point-preserving, so we get a functor $\text{Bag}_* : \text{Set} \to \text{Set}_*$.

Indexing remains an isomorphism:

$$
\text{index} : \text{Bag}_* (K \times V) \simeq \text{Map}_* K (\text{Bag}_* V)
$$
14. Graded monads

A catch: finite maps aren’t a monad, because

$$\eta \ a = \lambda k \to a : A \to \text{Map} \ K \ A$$

in general yields an infinite map.

However, finite maps are a graded monad*: for monoid \((M, \otimes, \epsilon)\),

$$\mu \ X : T_m (T_n X) \to T_{m \otimes n} X$$

$$\eta \ X : X \to T_\epsilon X$$

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

Thanks to EPSRC *Unifying Theories of Generic Programming* for funding.