Relational algebra by way of adjunctions
Gibbons, Jeremy; Henglein, Fritz; Hinze, Ralf; Wu, Nicolas

Publication date:
2016

Document version
Early version, also known as pre-print

Citation for published version (APA):
Relational Algebra by Way of Adjunctions

Jeremy Gibbons
(joint work with Fritz Henglein, Ralf Hinze, Nicolas Wu)
DBPL, October 2015
1. Summary

- bulk types (sets, bags, lists) are monads
- monads have nice mathematical foundations via adjunctions
- monads support comprehensions
- comprehension syntax provides a query notation

\[
\left\{ (\text{customer}.\text{name}, \text{invoice}.\text{amount}) \mid \text{customer} \leftarrow \text{customers}, \\
\text{invoice} \leftarrow \text{invoices}, \\
\text{customer}.\text{cid} = \text{invoice}.\text{customer}, \\
\text{invoice}.\text{due} \leq \text{today} \right\}
\]

- monad structure explains selection, projection
- less obvious how to explain join
2. Galois connections

Relating monotonic functions between two ordered sets:

\[(A, \leq) \perp (B, \subseteq)\]

means \(f b \leq a \iff b \subseteq g a\)

For example,

\[(\mathbb{R}, \leq_{\mathbb{R}}) \perp (\mathbb{Z}, \leq_{\mathbb{Z}})\]

\[(\mathbb{Z}, \leq) \perp (\mathbb{Z}, \leq)\]

“Change of coordinates” can sometimes simplify reasoning; eg rhs gives \(n \times k \leq m \iff n \leq m \div k\), and multiplication is easier to reason about than rounding division.
3. Category theory from ordered sets

A category $\mathbf{C}$ consists of

- a set* $|\mathbf{C}|$ of objects,
- a set* $\mathbf{C}(X, Y)$ of arrows $X \to Y$ for each $X, Y : |\mathbf{C}|$,
- identity arrows $\text{id}_X : X \to X$ for each $X$
- composition $f \cdot g : X \to Z$ of compatible arrows $g : X \to Y$ and $f : Y \to Z$,
- such that composition is associative, with identities as units.

Think of a directed graph, with vertices as objects and paths as arrows.

An ordered set $(A, \leq)$ is a degenerate category, with objects $A$ and a unique arrow $a \to b$ iff $a \leq b$.

\[ \ldots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \]

Many categorical concepts are generalisations from ordered sets.

*proviso...
4. Concrete categories

Ordered sets are a concrete category: roughly,

- the objects are sets with additional structure
- the arrows are structure-preserving mappings

Many useful categories are of this form.

For example, the category $\textbf{CMon}$ has commutative monoids $(M, \otimes, \epsilon)$ as objects, and homomorphisms $h : (M, \otimes, \epsilon) \rightarrow (M', \oplus, \epsilon')$ as arrows:

\[
\begin{align*}
    h (m \otimes n) &= h m \oplus h n \\
    h \epsilon &= \epsilon'
\end{align*}
\]

Trivially, category $\textbf{Set}$ has sets as objects, and total functions as arrows.
5. Functors

Categories are themselves structured objects...

A functor $F : C \to D$ is an operation on both objects and arrows, preserving the structure: $F f : F X \to F Y$ when $f : X \to Y$, and

$$F id_X = id_{F X}$$
$$F (f \cdot g) = F f \cdot F g$$

For example, forgetful functor $U : CMon \to Set$:

$$U (M, \otimes, \epsilon) = M$$
$$U (h : (M, \otimes, \epsilon) \to (M', \oplus, \epsilon')) = h : M \to M'$$

Conversely, $Free : Set \to CMon$ generates the free commutative monoid (ie bags) on a set of elements:

$$Free A = (Bag A, \cup, \emptyset)$$
$$Free (f : A \to B) = map f : Bag A \to Bag B$$
6. Adjunctions

Adjunctions are the categorical generalisation of Galois connections. Given categories $C, D,$ and functors $L : D \to C$ and $R : C \to D,$ adjunction

$$\begin{array}{cccccc}
& & L & & \Downarrow & \\
& C & \downarrow & D & \uparrow & \\
& & R & & \\
\end{array}
$$

means\* $[-] : C(LX, Y) \simeq D(X, RY) : [-]$.

A familiar example is given by currying:

$$\begin{array}{cccccccc}
& & - \times P & & \Downarrow & & \\
& Set & \downarrow & Set & \uparrow & & \\
& & (-)^P & & \\
\end{array}
$$

with $\textit{curry} : \textit{Set}(X \times P, Y) \simeq \textit{Set}(X, Y^P) : \textit{curry}^\circ$.

hence definitions and properties of $\textit{apply} = \textit{uncurry} \textit{id}_{Y^P} : Y^P \times P \to Y$
7. Products and coproducts

\[
\begin{array}{ccc}
\text{Set} & \perp & \text{Set}^2 \\
\Delta & \text{×} & \perp \\
\end{array}
\]

with

\[
\text{fork}: \text{Set}^2(\Delta A, (B, C)) \simeq \text{Set}(A, B \times C) : \text{fork}^\circ
\]

\[
\text{junc}^\circ : \text{Set}(A + B, C) \simeq \text{Set}^2((A, B), \Delta C) : \text{junc}
\]

hence

\[
dup = \text{fork} \ id_{A,A} : \text{Set}(A, A \times A)
\]

\[
(fst, snd) = \text{fork}^\circ \ id_{B \times C} : \text{Set}^2(\Delta(B, C), (B, C))
\]

give tupling and projection. Dually for sums and injections, and generally for any arity—even zero.
8. Free commutative monoids

Adjunctions often capture embedding/projection pairs:

\[ \text{CMon} \downarrow \text{Set} \]

\[ \text{with } [-] : \text{CMon}(\text{Free } A, (M, \otimes, \epsilon)) \]
\[ \simeq \text{Set}(A, U (M, \otimes, \epsilon)) : [-] \]

Unit and counit:

\[ \text{single } A = [id_{\text{Free } A}] : A \rightarrow U (\text{Free } A) \]
\[ \text{reduce } M = [id_M] : \text{Free } (U M) \rightarrow M \quad \text{-- for } M = (M, \otimes, \epsilon) \]

whence, for \( h : \text{Free } A \rightarrow M \) and \( f : A \rightarrow U M = M \),

\[ h = \text{reduce } M \cdot \text{Free } f \iff U h \cdot \text{single } A = f \]

ie 1-to-1 correspondence between homomorphisms from the free commutative monoid (bags) and their behaviour on singletons.
9. Aggregation

Aggregations are bag homomorphisms:

<table>
<thead>
<tr>
<th>aggregation</th>
<th>monoid</th>
<th>action on singletons</th>
</tr>
</thead>
<tbody>
<tr>
<td>count</td>
<td>((\mathbb{N}, 0, +))</td>
<td>({a} \rightarrow 1)</td>
</tr>
<tr>
<td>sum</td>
<td>((\mathbb{R}, 0, +))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>max</td>
<td>((\mathbb{Z}, \text{minBound}, \max))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>min</td>
<td>((\mathbb{Z}, \text{maxBound}, \min))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>all</td>
<td>((\mathbb{B}, \text{True}, \land))</td>
<td>({a} \rightarrow a)</td>
</tr>
<tr>
<td>any</td>
<td>((\mathbb{B}, \text{False}, \lor))</td>
<td>({a} \rightarrow a)</td>
</tr>
</tbody>
</table>

Selection is a homomorphism, to bags, using action

\[
guard : (A \rightarrow \mathbb{B}) \rightarrow \text{Bag} \ A \rightarrow \text{Bag} \ A
\]

\[
guard \ p \ a = \text{if} \ p \ a \ \text{then} \ \{a\} \ \text{else} \ \emptyset
\]

Laws about selections follow from laws of homomorphisms (and of coproducts, since \(\mathbb{B} = 1 + 1\)).
10. Monads

Bags form a monad \((\text{Bag}, \text{union}, \text{single})\) with

\[
\text{Bag} = \mathcal{U} \cdot \text{Free}
\]

\(\text{union} : \text{Bag} (\text{Bag} A) \rightarrow \text{Bag} A\)

\(\text{single} : A \rightarrow \text{Bag} A\)

which justifies the use of comprehension notation \(\{ f \ a \ b \mid a \leftarrow x, b \leftarrow g a \}\).

In fact, for any adjunction \(L \dashv R\) between \(C\) and \(D\), we get a monad \((T, \mu, \eta)\) on \(D\), where

\[
T = R \cdot L
\]

\[\mu A = R \lceil id_A \rceil L : T (T A) \rightarrow T A\]

\[\eta A = \lfloor id_A \rfloor : A \rightarrow T A\]
11. Maps

Database indexes are essentially maps $\text{Map } K V = V^K$. Maps $(-)^K$ from $K$ form a monad (the $\text{Reader}$ monad in Haskell), so arise from an adjunction.

The $\text{laws of exponents}$ arise from this adjunction, and from those for products and coproducts:

- $\text{Map } 0 V \approx 1$
- $\text{Map } 1 V \approx V$
- $\text{Map } (K_1 + K_2) V \approx \text{Map } K_1 V \times \text{Map } K_2 V$
- $\text{Map } (K_1 \times K_2) V \approx \text{Map } K_1 (\text{Map } K_2 V)$
- $\text{Map } K 1 \approx 1$
- $\text{Map } K (V_1 \times V_2) \approx \text{Map } K V_1 \times \text{Map } K V_2 : \text{merge}$
12. Indexing

Relations are in 1-to-1 correspondence with set-valued functions:

\[ \text{Rel} \xrightarrow{J} \downarrow \xrightarrow{\perp} \text{Set} \xleftarrow{E} \]

where \( J \) embeds, and \( E \ R : A \to \text{Set} \ B \) for \( R : A \sim B \).

Moreover, the correspondence remains valid for bags:

\[ \text{index} : \text{Bag} \ (K \times V) \simeq \text{Map} \ K \ (\text{Bag} \ V) \]

Together, \( \text{index} \) and \( \text{merge} \) give efficient relational joins:

\[ x \ f \land g \ y = \text{flatten} \ (\text{Map} \ K \ \text{cp} \ (\text{merge} \ (\text{groupBy} \ f \ x, \text{groupBy} \ g \ y))) \]

\( \text{groupBy} : (V \to K) \to \text{Bag} \ V \to \text{Map} \ K \ (\text{Bag} \ V) \)

\( \text{flatten} : \text{Map} \ K \ (\text{Bag} \ V) \to \text{Bag} \ V \)
13. Pointed sets and finite maps

Model *finite maps* \( \text{Map}_* \) not as partial functions, but *total* functions to a *pointed* codomain \((A, a)\), i.e. a set \( A \) with a distinguished element \( a : A \).

Pointed sets and point-preserving functions form a category \( \text{Set}_* \).

There is an adjunction to \( \text{Set} \), via

\[
\begin{array}{ccc}
\text{Set}_* & \perp & \text{Set} \\
\text{Maybe} & \dashv & \text{U} \\
\end{array}
\]

where \( \text{Maybe} A \cong 1 + A \) adds a point, and \( \text{U} (A, a) = A \) discards it.

In particular, \((\text{Bag} A, \emptyset)\) is a pointed set. Moreover, \( \text{Bag} f \) is point-preserving, so we get a functor \( \text{Bag}_* : \text{Set} \to \text{Set}_* \).

Indexing remains an isomorphism:

\[
\text{index} : \text{Bag}_* (K \times V) \cong \text{Map}_* K (\text{Bag}_* V)
\]
14. Graded monads

A catch: finite maps aren’t a monad, because

\[ \eta \ a = \lambda k \rightarrow a : A \rightarrow \text{Map} \ K \ A \]

in general yields an infinite map.

However, finite maps are a \textit{graded monad}*: for monoid \((M, \otimes, \epsilon)\),

\[ \mu \ X : T_m \ (T_n \ X) \rightarrow T_{m \otimes n} \ X \]
\[ \eta \ X : X \rightarrow T_\epsilon \ X \]

satisfying the usual laws. These too arise from adjunctions*.

We use the monoid \((\mathbb{K}, \times, 1)\) of finite key types under product.
15. Conclusions

- *monad comprehensions* for database queries
- structure arising from *adjunctions*
- equivalences from *universal properties*
- fitting in *relational joins*, via indexing
- to do: calculating *query optimisations*

Thanks to EPSRC *Unifying Theories of Generic Programming* for funding.