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*Published in:*  
Physics Letters B

*DOI:*  
[10.1016/j.physletb.2016.02.075](https://doi.org/10.1016/j.physletb.2016.02.075)

*Publication date:*  
2016

*Document version*  
Publisher's PDF, also known as Version of record

*Citation for published version (APA):*  
Ambjorn, J., & Makeenko, Y. (2016). String theory as a Lilliputian world. *Physics Letters B*, 756, 142-146.  
<https://doi.org/10.1016/j.physletb.2016.02.075>



# String theory as a Lilliputian world



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## ARTICLE INFO

### Article history:

Received 22 January 2016

Received in revised form 25 February 2016

Accepted 29 February 2016

Available online 4 March 2016

Editor: N. Lambert

## ABSTRACT

Lattice regularizations of the bosonic string do not allow us to probe the tachyon. This has often been viewed as the reason why these theories have never managed to make any contact to standard continuum string theories when the dimension of spacetime is larger than two. We study the continuum string theory in large spacetime dimensions where simple mean field theory is reliable. By keeping carefully the cutoff we show that precisely the existence of a tachyon makes it possible to take a scaling limit which reproduces the lattice-string results. We compare this scaling limit with another scaling limit which reproduces standard continuum-string results. If the people working with lattice regularizations of string theories are akin to Gulliver they will view the standard string-world as a Lilliputian world no larger than a few lattice spacings.

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## 1. Introduction

A first quantization of the free particle using the path integral requires a regularization. A simple such regularization is to use a hypercubic  $D$ -dimensional lattice if the particle propagates in  $D$ -dimensional spacetime. The allowed worldlines for a particle propagating between two lattice points are link-paths connecting the two points and the action used is the length of the path, i.e. the number of links of the path multiplied with the link length  $a_\ell$ . This  $a_\ell$  is the UV cutoff of the path integral. The lattice regularization works nicely and the limit  $a_\ell \rightarrow 0$  can be taken such that one obtains the standard continuum propagator.

Similarly, the first quantization of the free bosonic string using the path integral requires a regularization. It seemed natural to repeat the successful story of the free particle and use a hypercubic  $D$ -dimensional lattice if the worldsheet of the string lived in  $D$ -dimensional spacetime, the worldsheet being a connected plaquette lattice surface [1]. The Nambu–Goto action, the area of the worldsheet, would then be the number of worldsheet plaquettes times  $a_\ell^2$ ,  $a_\ell$  again denoting the lattice spacing. However, in this case one could not reproduce the results obtained by standard canonical quantization of the string. First, one did not obtain the whole set of string masses, starting with the tachyon mass, but

only a single, positive mass state. Next, after having renormalized the bare string coupling constant to obtain a finite lowest mass state, this renormalization led to an infinite physical string tension for strings with extended boundaries.

Although it was not clear why the hypercubic lattice regularization did not work, the formalism known as dynamical triangulation (DT) was suggested as an alternative [2]. It discretized the independent intrinsic worldsheet geometry used in the Polyakov formulation of bosonic string theory [3] and the integration over these geometries was approximated by a summation over triangulations constructed from equilateral triangles with link lengths  $a_\ell$ , where  $a_\ell$  again was a UV cutoff. However, the results were identical to the hypercubic lattice results. In contrast to the hypercubic lattice model the DT model can be defined when the dimension of spacetime is less than two, where one encounters the so-called non-critical string theory. This string theory can be solved both using standard continuum quantization and using the DT-lattice regularization (and taking the limit  $a_\ell \rightarrow 0$ ). Agreement is found. Thus a lattice regularization is not incompatible with string theory as such. However, in the lattice regularized theories it is impossible to have a tachyonic lowest mass state and such states appear precisely when the dimension of spacetime exceeds two. It might explain the failure of lattice strings to connect to continuum bosonic string theory in dimensions  $D > 2$ .

The purpose of this article is to highlight the role of the tachyon in connecting the continuum bosonic string theory to lattice strings. In [4] we showed how one could technically make

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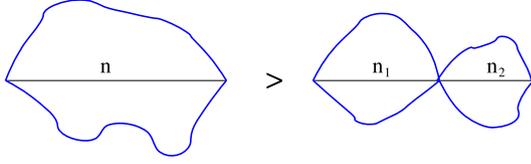


Fig. 1. Illustration of why  $G(n) > G(n_1)G(n_2)$ ,  $n = n_1 + n_2$ .

such a connection. However, the role played in the connection by the tachyon was not emphasized to the extent it deserves.

Before starting the discussion let us make clear why there are no tachyons in, say, a hypercubic lattice string theory [1,5]. Consider the two point function for a closed bosonic string on the lattice. We have an entrance loop of minimal length, say four links spanning a plaquette (not belonging to the string worldsheet) and a similar exit loop, separated by  $n$  lattice spacings. The genus zero closed string two-point function  $G(n)$  is the sum over all plaquette cylinder lattice surfaces  $F$  with these two boundary loops and with  $F$  assigned the weight  $e^{-\mu N(F)}$ ,  $N(F)$  being the number of plaquettes of  $F$ ,  $\mu$  the (dimensionless) string tension and  $\mu N(F)$  the Nambu–Goto action associated with  $F$ . Clearly this sum is larger than the sum over surfaces where the surfaces are constraint to meet at a “bottleneck” (again a single plaquette not belonging to  $F$ ) separated by  $n_1$  lattice spacing from the entrance loop,  $n_1 < n$  (see Fig. 1), i.e.

$$G(n) \geq G(n_1) G(n - n_1), \quad (1)$$

and thus  $-\log G(n)$  is a subadditive function. Further it can be shown that  $G(n) \rightarrow 0$  for  $n \rightarrow \infty$ . According to Feteke’s lemma this implies that  $(-\log G(n))/n$  converges to a real non-negative number for  $n \rightarrow \infty$  and consequently the lowest mass cannot be tachyonic. Thus in the Euclidean lattice formulation one cannot probe the tachyonic instability which can be seen in the Minkowskian lattice computation [6].

Let us list some other results obtained in the hypercubic lattice string theory, formulated in dimensionless lattice units. The theory has a critical point  $\mu_c$ , such that the partition function is defined for  $\mu > \mu_c$  and the scaling limit (where one can attempt to define a continuum theory) is obtained for  $\mu \rightarrow \mu_c$ . One finds (up to subleading corrections)

$$G_\mu(n) \sim e^{-m(\mu)n}, \quad m(\mu) \sim (\mu - \mu_c)^{1/4}, \quad (2)$$

for  $\mu \rightarrow \mu_c$ . Here  $G_\mu(n)$  is the two-point function defined above and  $m(\mu)$  is the positive mass mentioned above. Scaling to a continuum theory is now done by introducing a dimensionful lattice spacing  $a_\ell$  and by requiring that the two-point function survives when the lattice spacing  $a_\ell \rightarrow 0$ . Thus we write

$$L = n \cdot a_\ell, \quad m(\mu)n = m_{\text{ph}}L, \quad \mu \rightarrow \mu_c. \quad (3)$$

This determines  $a_\ell$  as a function of  $\mu$

$$m_{\text{ph}}a_\ell(\mu) = m(\mu) \sim (\mu - \mu_c)^{1/4}. \quad (4)$$

The problem with the lattice string theory is that the so-called effective string tension  $\sigma(\mu)$  does not scale to zero for  $\mu \rightarrow \mu_c$  [7]. The effective string tension is defined as follows (for a closed string): compactify one of the lattice directions to  $m$  links and insist that the string wraps around this dimension once. We still assume that the string propagates  $n$  links in one of the other lattice directions. Again one can show that the corresponding partition function  $G_\mu(m, n)$  falls off exponentially with the minimal lattice area  $m \times n$  spanned by the string worldsheet:

$$G_\mu(m, n) \sim e^{-\sigma(\mu)mn}. \quad (5)$$

However,  $\sigma(\mu)$  does not scale to zero for  $\mu \rightarrow \mu_c$ :

$$\sigma(\mu) = \sigma(\mu_c) + c(\mu - \mu_c)^{1/2}, \quad \sigma(\mu_c) > 0. \quad (6)$$

Thus the physical, effective string tension  $K_{\text{ph}}$  (defined analogously to the physical mass  $m_{\text{ph}}$  in eq. (4))

$$\sigma(\mu) = K_{\text{ph}} a_\ell^2(\mu) \quad (7)$$

scales to infinity for  $\mu \rightarrow \mu_c$  and  $G_\mu(m, n)$  has no continuum limit.

## 2. The bosonic string at large $D$

Let us consider the closed bosonic string at large  $D$ . We choose the large  $D$  limit because it allows us to perform a reliable mean field calculation, as noticed already a long time ago [8]. In order to make contact to lattice results we want to control the appearance of the tachyon. We do that by compactifying one of the dimensions such that it has length  $\beta$  and by insisting that the worldsheet wraps once around this compactified dimension. With this setup there will be no tachyons if only  $\beta$  is larger than the cutoff, as we will show. Also, this setup allows us to define the physical, effective string tension precisely as we did it for the lattice string theory.

We use the Nambu–Goto action (to make the situation analogous to the hypercubic lattice) and deal with the area-action of the embedded surface by using a Lagrange multiplier  $\lambda^{ab}$  and an independent intrinsic metric  $\rho_{ab}$  ( $\rho := \det \rho_{ab}$ ):

$$K_0 \int d^2\omega \sqrt{\det \partial_a X \cdot \partial_b X} = K_0 \int d^2\omega \sqrt{\rho} + \frac{K_0}{2} \int d^2\omega \lambda^{ab} (\partial_a X \cdot \partial_b X - \rho_{ab}). \quad (8)$$

We choose the world-sheet parameters  $\omega_1$  and  $\omega_2$  inside an  $\omega_L \times \omega_\beta$  rectangle in the parameter space and find the classical solution

$$X_{\text{cl}}^1 = \frac{L}{\omega_L} \omega_1, \quad X_{\text{cl}}^2 = \frac{\beta}{\omega_\beta} \omega_2, \quad X_{\text{cl}}^\perp = 0, \quad (9a)$$

$$[\rho_{ab}]_{\text{cl}} = \text{diag} \left( \frac{L^2}{\omega_L^2}, \frac{\beta^2}{\omega_\beta^2} \right), \quad (9b)$$

$$\lambda_{\text{cl}}^{ab} = \text{diag} \left( \frac{\beta \omega_L}{L \omega_\beta}, \frac{L \omega_\beta}{\beta \omega_L} \right) = \rho_{\text{cl}}^{ab} \sqrt{\rho_{\text{cl}}}, \quad (9c)$$

minimizing the action (8).

Quantization is performed using the path integral. We integrate out the quantum fluctuations of the  $X$  fields by performing a split  $X^\mu = X_{\text{cl}}^\mu + X_q^\mu$ , where  $X_{\text{cl}}^\mu$  is given by eq. (9a), and then performing the Gaussian path integral over  $X_q^\mu$ . We fix the gauge to the so-called static gauge  $X_q^1 = X_q^2 = 0$ .<sup>1</sup> The number of fluctuating  $X$ ’s is then  $d = D - 2$ . We thus obtain the effective action, governing the fields  $\lambda^{ab}$  and  $\rho_{ab}$ ,

$$S_{\text{eff}} = K_0 \int d^2\omega \sqrt{\rho} + \frac{K_0}{2} \int d^2\omega \lambda^{ab} (\partial_a X_{\text{cl}} \cdot \partial_b X_{\text{cl}} - \rho_{ab}) + \frac{d}{2} \text{tr} \log \mathcal{O}, \quad \mathcal{O} := -\frac{1}{\sqrt{\rho}} \partial_a \lambda^{ab} \partial_b. \quad (10)$$

<sup>1</sup> Gauge fixing will in general produce a ghost determinant. To leading order in  $D$  this determinant can be ignored, but it may have to be included in a  $1/D$ -expansion.

The operator  $-\mathcal{O}$  reproduces the usual 2d Laplacian for  $\lambda^{ab} = \rho^{ab} \sqrt{\rho}$ , and we use the proper-time regularization of the trace

$$\text{tr} \log \mathcal{O} = - \int_{a^2}^{\infty} \frac{d\tau}{\tau} \text{tr} e^{-\tau \mathcal{O}}, \quad a^2 \equiv \frac{1}{4\pi \Lambda^2}. \quad (11)$$

In the mean field approximation which becomes exact at large  $D$  we disregard fluctuations of  $\lambda^{ab}$  and  $\rho_{ab}$  about the saddle-point values  $\bar{\lambda}^{ab}$  and  $\bar{\rho}_{ab}$ , i.e. simply substitute them by the mean values, which are the minima of the effective action (10). Integration around the saddle-point values will produce corrections which are subleading in  $1/D$ .

For diagonal and constant  $\bar{\lambda}^{ab}$  and  $\bar{\rho}_{ab}$  the explicit formula for the determinant is well-known and for  $L \gg \beta$  we obtain<sup>2</sup>

$$S_{\text{eff}} = \frac{K_0}{2} \left( \bar{\lambda}^{11} \frac{L^2}{\omega_L^2} + \bar{\lambda}^{22} \frac{\beta^2}{\omega_\beta^2} + 2\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}} - \bar{\lambda}^{11} \bar{\rho}_{11} - \bar{\lambda}^{22} \bar{\rho}_{22} \right) \omega_\beta \omega_L - \frac{\pi d}{6} \sqrt{\frac{\bar{\lambda}^{22}}{\bar{\lambda}^{11}}} \frac{\omega_L}{\omega_\beta} - \frac{d\sqrt{\bar{\rho}_{11}\bar{\rho}_{22}} \omega_\beta \omega_L}{2\sqrt{\bar{\lambda}^{11}\bar{\lambda}^{22}}} \Lambda^2. \quad (12)$$

The minimum of the effective action (12) is reached at

$$\bar{\rho}_{11} = \frac{L^2 \left( \beta^2 - \frac{\beta_0^2}{2C} \right)}{\omega_L^2 \left( \beta^2 - \frac{\beta_0^2}{C} \right)} \frac{C}{2C-1},$$

$$\bar{\rho}_{22} = \frac{1}{\omega_\beta^2} \left( \beta^2 - \frac{\beta_0^2}{2C} \right) \frac{C}{2C-1}, \quad (13)$$

$$\bar{\lambda}^{ab} = C \bar{\rho}^{ab} \sqrt{\bar{\rho}}, \quad C = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{d\Lambda^2}{2K_0}}, \quad (14)$$

where

$$\beta_0^2 = \frac{\pi d}{3K_0}. \quad (15)$$

Equations (13) and (14) generalize the classical solution (9). Note that  $C$  as given in (14) takes values between 1 and 1/2. This will play a crucial role in what follows. Also note that if one were performing a perturbative expansion in  $1/K_0$ , both  $C$  and eqs. (13) and (14) would start out with their classical values.

Substituting the solution (13)–(14) into eq. (12), we obtain

$$S_{\text{eff}}^{\text{s.p.}} = K_0 C L \sqrt{\beta^2 - \beta_0^2 / C} \quad (16)$$

for the saddle-point value of the effective action, which is nothing but  $L$  times the mass of string ground state. Further, we find that the average area  $\mathcal{A} = \langle \text{Area} \rangle$  of the surface which appears in the path integral is

$$\mathcal{A} = \int d^2\omega \sqrt{\bar{\rho}_{11}\bar{\rho}_{22}} = L \frac{(\beta^2 - \beta_0^2 / 2C)}{\sqrt{\beta^2 - \beta_0^2 / C}} \frac{C}{(2C-1)}. \quad (17)$$

All these results are just a repetition of the original Alvarez computation [8], except that he used  $\omega_L = L$ ,  $\omega_\beta = \beta$  and, more

importantly, he used the zeta-function regularization where formally our cutoff  $\Lambda$  is put to zero. As we will see, maintaining a real cutoff  $a$  will be important, so let us discuss its relation to a spacetime cutoff like the hypercubic  $a_\ell$  mentioned above.

The cutoff  $a$  refers to the operator  $\mathcal{O}$  defined in parameter space and provides a cutoff of the eigenvalues of  $\mathcal{O}$ . However, the eigenvalues (which are invariant under change of parametrization) are linked to the target space distances  $L$  and  $\beta$  and *not* to the parameters  $\omega_L$  and  $\omega_\beta$ . Effectively  $a$  acts as a cutoff on the worldsheet, measured in length units from target space, in agreement with the way we introduced  $\rho_{ab}$  and  $\lambda^{ab}$  in the first place. We can write symbolically

$$(\Delta s)^2 = \rho_{ab} \Delta \omega^a \Delta \omega^b = \Delta X \cdot \Delta X, \quad (18)$$

where  $\Delta s \sim a$  and  $\Delta \omega \sim a/\sqrt{\rho}$ , which reflects to what extent the eigenfunctions of  $\mathcal{O}$  which are not suppressed by the proper-time cutoff  $a$  can resolve points on the worldsheet. This is true semi-classically where the worldsheet is just the minimal surface (9) and it will be true when we consider genuine quantum surfaces which are much larger. In the latter case eq. (18) has to be averaged over the quantum fluctuations, so  $\rho_{ab}$  will change accordingly (see eq. (13)), such that the allowed eigenfunctions still can resolve these larger surfaces down to order  $a$ , measured in target space length units.

Formulas (16) and (17) are our main results, valid for  $L \gg \beta$  in the mean field or large  $D$  approximation. The term  $-\beta_0^2/C$  appearing under the square root in eq. (16) is a manifestation of the closed string tachyon as first pointed out in [9,10] and this minus sign will be essential for the limit we take in the next Section and which will reproduce the lattice string scaling.

### 3. The lattice-like scaling limit

Equation (14) shows that the bare string tension  $K_0$  needs to be renormalized in order for  $C$  to remain real since this requirement forces

$$K_0 > 2d\Lambda^2 = \frac{d}{2\pi a^2}. \quad (19)$$

Also,  $C$  is clearly constraint to take values between 1 and 1/2 when  $K_0$  is decreasing from infinity to  $2d\Lambda^2$ . We also require that  $S_{\text{eff}}$  is real. This is ensured for all allowed values of  $K_0$  if

$$\beta^2 > \beta_{\text{min}}^2 = \frac{(2\pi a)^2}{3}. \quad (20)$$

For  $\beta \geq \beta_{\text{min}}$  we have no tachyonic modes, precisely the scenario needed if we should have a chance to make contact to lattice string theory.

At first glance it seems impossible to obtain a finite  $S_{\text{eff}}$  by renormalizing  $K_0$  in (16), since  $K_0$  is of order  $\Lambda^2$ . However, let us try to imitate as closely as possible the calculation of the two-point function on the lattice by choosing, for a fixed cutoff  $a$  (or  $\Lambda$ ),  $\beta$  as small as possible without entering into the tachyonic regime of  $S_{\text{eff}}$ , i.e. by choosing  $\beta = \beta_{\text{min}}$ . With this choice we obtain

$$S_{\text{eff}}^{\text{s.p.}} = \sqrt{\frac{\pi}{3}} \frac{K_0 C L}{\Lambda} \sqrt{2C-1}. \quad (21)$$

Only if  $\sqrt{2C-1} \sim 1/\Lambda$  can we obtain a finite limit for  $\Lambda \rightarrow \infty$ . Thus we are forced to renormalize  $K_0$  as follows

$$K_0 = 2d\Lambda^2 + \frac{\tilde{K}_{\text{ph}}^2}{2d\Lambda^2} \quad (22)$$

<sup>2</sup> The averaged over quantum fluctuations induced metric  $\langle \partial_a X \cdot \partial_b X \rangle$  which equals  $\bar{\rho}_{ab}$  at large  $D$  depends in fact on  $\omega_1$  near the boundaries, but this is not essential for  $L \gg \beta$  [4].

where  $\tilde{K}_{\text{ph}}$  is finite in the limit  $\Lambda \rightarrow \infty$ . With this renormalization we find

$$S_{\text{eff}}^{\text{s.p.}} = m_{\text{ph}} L, \quad m_{\text{ph}}^2 = \frac{\pi d}{6} \tilde{K}_{\text{ph}}. \quad (23)$$

Note that  $m_{\text{ph}}$  and  $\tilde{K}_{\text{ph}}$  are proportional to  $d$  as to be expected in a large  $d$  limit.

Since the partition function in this case has the interpretation as a kind of the two-point function for a string propagating a distance  $L$ , we have the following leading  $L$  behavior of the two-point function

$$G(L) \sim e^{-S_{\text{eff}}^{\text{s.p.}}} = e^{-m_{\text{ph}} L}, \quad (24)$$

where the mass  $m_{\text{ph}}$  is a tunable parameter. Note that we have the classical value  $C = 1$  and a semiclassical expansion in  $1/K_0$  interpolating between  $C = 1$  and the quantum value  $C = 1/2$ , which is very similar to the situation for the free particle where a semiclassical expansion in the inverse bare mass interpolates between the classical and quantum cases.

In the scaling limit (22) we can calculate the average area  $\mathcal{A}$  of a surface using (17):

$$\mathcal{A} \propto \frac{L}{m_{\text{ph}}^3 a^2}. \quad (25)$$

It diverges when the cutoff  $a \rightarrow 0$ . This is to be expected. The quantum fluctuations of the worldsheet is included in the effective action (12) and the same thing will happen if we consider a free particle propagating a distance  $L$  and integrate out the quantum fluctuations. The average length  $\ell$  of a quantum path in the path integral will be

$$\ell \propto \frac{L}{m_{\text{ph}} a}. \quad (26)$$

We can express (25) and (26) in dimensionless units

$$n_L = \frac{L}{a}, \quad n_{\mathcal{A}} = \frac{\mathcal{A}}{a^2}, \quad n_\ell = \frac{\ell}{a} \quad (27)$$

to have

$$n_{\mathcal{A}} \propto \frac{1}{m_{\text{ph}}^3 L^3} n_L^4, \quad n_\ell \propto \frac{1}{m_{\text{ph}} L} n_L^2. \quad (28)$$

These formulas tell us the Hausdorff dimension of a quantum surface is  $d_H = 4$  and the Hausdorff dimension of a quantum path of a particle is  $d_H = 2$  in the scaling limit where  $m_{\text{ph}} L$  is kept fixed while the cutoff  $a \rightarrow 0$ .

Let us now discuss how we define the physical string tension. With the given boundary conditions the string extends over the minimal area  $A_{\text{min}} = \beta L$  and we write the partition function as

$$Z(K_0, L, \beta) = e^{-S_{\text{eff}}^{\text{s.p.}}(K_0, L, \beta)} = e^{-K_{\text{ph}} A_{\text{min}} + \mathcal{O}(L, \beta)}. \quad (29)$$

This is precisely the way one would define the physical (renormalized) string tension in a lattice gauge theory via the correlator of two periodic Wilson lines of length  $\beta$  separated by the distance  $L \gg \beta \gg a$ , where  $a$  is the lattice spacing. This is also the way the physical string tension is defined in lattice string theories as discussed in the Introduction.

From the explicit form of  $S_{\text{eff}}^{\text{s.p.}}$  given in (16) we have from (22):

$$K_{\text{ph}} = K_0 C = d \Lambda^2 + \frac{1}{2} \tilde{K}_{\text{ph}} + \mathcal{O}(1/\Lambda^2). \quad (30)$$

Thus the physical string tension as defined above diverges as the cutoff  $\Lambda$  is taken to infinity. However, the first correction is finite and behaves as we would have liked  $K_{\text{ph}}$  to behave, namely as  $\tilde{K}_{\text{ph}} \propto m_{\text{ph}}^2/d$ .

We have thus reproduced the scenario from the lattice strings: it is possible by a renormalizing of the bare coupling constant ( $K_0 = 1/(2\pi\alpha'_0)$ ) to define a two-point function with a positive, finite mass. In the limit where the cutoff  $a \rightarrow 0$  the Hausdorff dimension of the ensemble of quantum surfaces is  $d_H = 4$ , but then the effective string tension defined as in eq. (29) will be infinite. In addition the relation (30) is precisely the relation (6) from the lattice string theories. To make this explicit let us introduce dimensionless variables

$$\mu = K_0 a^2, \quad \mu_c = \frac{d}{2\pi}, \quad n = \frac{L}{a}, \quad \sigma(\mu) = K_{\text{ph}} a^2. \quad (31)$$

Then the renormalization of  $K_0$ , eq. (22), can be inverted to define the cutoff  $a$  in terms of  $\mu - \mu_c$  and it becomes identical to eq. (4). Similarly eqs. (22) and (23) can now be written as

$$m(\mu)n = m_{\text{ph}} L, \quad \sigma(\mu) = \sigma(\mu_c) + c(\mu - \mu_c)^{1/2}, \quad (32)$$

where  $c = 1/(2\sqrt{\mu_c})$  and

$$m(\mu) \sim (\mu - \mu_c)^{1/4}, \quad \sigma(\mu_c) = \frac{\mu_c}{2} > 0. \quad (33)$$

Thus one obtains identical scaling formulas by continuum renormalization and by lattice renormalization.

#### 4. Scaling to the standard string theory limit

When we integrated out the quantum fluctuations of the worldsheet we made decomposition  $X^\mu = X_{\text{cl}}^\mu + X_{\text{q}}^\mu$ , where the parameters  $L$  and  $\beta$  refer to the “background” fields  $X_{\text{cl}}^\mu$ . In standard quantum field theory we usually have to perform a renormalization of the background field to obtain a finite effective action. It is possible to do the same here by rescaling

$$X_{\text{cl}}^\mu = Z^{1/2} X_R^\mu, \quad Z = (2C - 1)/C. \quad (34)$$

Notice that the field renormalization  $Z$  has a standard perturbative expansion

$$Z = 1 - \frac{d\Lambda^2}{2K_0} + \mathcal{O}(K_0^{-2}) \quad (35)$$

in terms of the coupling constant  $K_0^{-1}$ , which in perturbation theory is always assumed to be small, even compared to the cutoff.

However, in the limit  $C \rightarrow 1/2$  it has dramatic effects since, working with renormalized lengths  $L_R$  and  $\beta_R$  defined as in (34):

$$L_R = \sqrt{\frac{C}{2C-1}} L, \quad \beta_R = \sqrt{\frac{C}{2C-1}} \beta, \quad (36)$$

we now obtain for the effective action

$$S_{\text{eff}} = K_R L_R \sqrt{\beta_R^2 - \frac{\pi d}{3K_R}}, \quad K_R = K_0(2C - 1) \equiv \tilde{K}_{\text{ph}}. \quad (37)$$

The renormalized coupling constant  $K_R$  indeed makes  $S_{\text{eff}}$  finite and is identical to the  $\tilde{K}_{\text{ph}}$  defined in (22). In fact the renormalization  $K_R = (2C - 1)K_0$  is identical to the renormalization (22) for  $\Lambda \rightarrow \infty$ . If we view  $L_R$  and  $\beta_R$  as representing physical distances, eq. (37) tells us that we have a renormalized, finite string tension  $\tilde{K}_{\text{ph}}$  in the scaling limit and even more, (37) is the Alvarez-Arvis continuum string theory formula [8,9].

The background field renormalization makes the average area  $\mathcal{A}$  of the worldsheet finite. If the scaling (34) for  $X_{\text{cl}}^\mu$  and (37) for  $K_0$  is inserted in the expression (17) for  $\mathcal{A}$  we obtain

$$\mathcal{A} = L_R \frac{\left(\beta_R^2 - \frac{\pi d}{6K_R}\right)}{\sqrt{\beta_R^2 - \frac{\pi d}{3K_R}}}, \quad (38)$$

which is cutoff independent and thus finite when the cutoff is removed. The area is simply the minimal area for  $\beta_R^2 \gg \pi d/(3K_R)$  and diverges when  $\beta_R^2 \rightarrow \pi d/(3K_R)$ .

## 5. Discussion

As mentioned in the Introduction the fact that lattice string theories seemingly are unable to produce anything resembling ordinary bosonic string theory has often been “blamed” on the absence of a tachyonic mass in these regularized theories, but it is of course difficult to study the role of the tachyon in a theory where it is absent. Here we have addressed the problem from the continuum string theory point of view by repeating the old calculation [8], while keeping a dimensionful cutoff  $a$  explicitly. In the continuum calculation the tachyonic term  $-\beta_0^2/C$  appears in formula (16), and if this term was not negative it would be impossible to find a renormalization of  $K_0$  which reproduces the lattice string scenario.

Somewhat surprising the same renormalization of  $K_0$  can produce a completely different scaling limit (the conventional string theory limit) provided we are allowed to perform a “background” renormalization of the coordinates  $X_{cl}^\mu$ . It is seemingly difficult to reconcile the two scaling limits. In the limit where the cutoff  $a \rightarrow 0$  we can write (36) as

$$L = a \cdot \sqrt{2K_R/\mu_c} L_R, \quad \beta = a \cdot \sqrt{2K_R/\mu_c} \beta_R. \quad (39)$$

Thus the scaling limit where  $K_R$ ,  $L_R$  and  $\beta_R$  are finite as  $a \rightarrow 0$  is a limit where  $L$  and  $\beta$  are of the order of the cutoff  $a$ . From the point of view of the hypercubic lattice theory we have the lattice cutoff  $a_\ell$  which acts simultaneously as a cutoff in the target space where the string is propagating and as a cutoff on the worldsheet of string. In the lattice world (“Gulliver’s world”) everything is defined in terms of  $a_\ell$  and the lattice scaling is such that Gulliver’s  $L \gg a_\ell$ . Since we have argued that one essentially can identify the proper-time cutoff  $a$  with a minimal distance  $a$  in  $R^D$  similar to  $a_\ell$ , the conventional string limit where  $L_R$  and  $\beta_R$  are kept fixed be-

comes a “Lilliputian world” since  $L$  and  $\beta$  are then of the order  $a_\ell$  from Gulliver’s perspective. Gulliver’s tools are too coarse to deal with the Lilliputian world.

## Acknowledgements

We thank Poul Olesen, Peter Orland and Arkady Tseytlin for valuable communication. The authors acknowledge support by the ERC-Advance grant 291092, “Exploring the Quantum Universe” (EQU). Y.M. thanks the Theoretical Particle Physics and Cosmology group at the Niels Bohr Institute for the hospitality. The authors also thank Dr. Isabella Wheeler for encouragement to use the Lilliputian analogy.

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